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FRACTIONAL q-DERIVATIVE AND GENERALIZED M-SERIES

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Abstract -This paper is devoted to fractional q-derivative of special functions. To begin with the theorem on term by term q-fractional differentiation has been derived. The result is an extension of an earlier result due to Yadav and Purohit [8] and Sharma, Jain and Ali [9]. As a special case, of fractional q-differentiation of Generalized M-series has been obtained.

Keywords and Phrases—Fractional integral and derivative operators, Fractional q-derivative, Generalized M-series and Special functions

Mathematics Subject Classification— Primary33A30, Secondary 33A25, 83C99

DEFINITION:

1.1. Q-ANALOGUE OF DIFFERENTIAL OPERATOR

Al-Salam [3], has given the q-analogue of differential operator as

$$D_q f(x) = \frac{f(xq) - f(x)}{x(q-1)} \tag{1.1}$$

This is an inverse of the q-integral operator defined as

$$\int_x^\infty f(t)\, d(t:q) = x(1-q) \sum_{k=1}^{\infty} q^{-k} f(xq^{-k}) \tag{1.2}$$

Where $\mathbf{0} < |\boldsymbol{q}| < 1$

1.2. FRACTIONAL Q-DERIVATIVE OF ORDER α:

The fractional q-derivative of order $\boldsymbol{\alpha}$ is defined as

$$\boldsymbol{D^{\alpha}_{x,q} f(x) = \frac{1}{\Gamma_q(-\alpha)} \int_0^x (x - yq)_{-\alpha-1} f(y) d(y;q)} \tag{1.2.1}$$

Where Re $(\boldsymbol{\alpha}) < 0$

As a particular case of (1.2.1), we have

$$\boldsymbol{D^{\alpha}_{x,q} x^{\mu-1} = \frac{\Gamma_q(\mu)}{\Gamma_q(\mu-\alpha)} x^{\mu-\alpha-1}} \tag{1.2.2}$$

2. MAIN RESULTS

In this section we drive the results on term by term q-fractional differentiation of a power series. As particular case we will the fractional q-differentiation of the Generalized M-Series and exponential series.

THEOREM 1: If the series ${}_pM_q^{\alpha,\beta}(z)$ converges absolutely for $|\boldsymbol{q}| < \rho$ then

$$\boldsymbol{D^{\mu}_{z,q}}\left\{\boldsymbol{z^{\lambda-1}} \sum_{k=o}^{\infty} \frac{(a_1)_k \;.\;.\;.\;.\;.\;(a_p)_k}{(b_1)_k \;.\;.\;.\;.\;.\;(b_q)_k} \frac{z^k}{\Gamma(\alpha k + \beta)}\right\} = \sum_{k=o}^{\infty} \frac{(a_1)_k \;.\;.\;.\;.\;.\;(a_p)_k}{(b_1)_k \;.\;.\;.\;.\;.\;(b_q)_k \Gamma(\alpha k + \beta)} \boldsymbol{D^{\mu}_{z,q}} z^{k+\lambda-1} \tag{2.1}$$

Where Re $(\boldsymbol{\lambda}) > 0$, Re $(\boldsymbol{\mu}) < 0$, 0< $|\boldsymbol{q}| < 1$

PROOF: Starting From the left side and using equation (1.2.1), we have

$$\boldsymbol{D^{\mu}_{z,q}}\left\{\boldsymbol{z^{\lambda-1}} \sum_{k=o}^{\infty} \frac{(a_1)_k \;.\;.\;.\;.\;.\;(a_p)_k}{(b_1)_k \;.\;.\;.\;.\;.\;(b_q)_k} \frac{z^k}{\Gamma(\alpha k + \beta)}\right\}$$

$$= \frac{\boldsymbol{1}}{\boldsymbol{\Gamma_q(-\mu)}} \int_{\boldsymbol{0}}^{\boldsymbol{z}} \boldsymbol{(z - yq)_{-\mu-1}\, y^{\lambda-1}} \sum_{k=o}^{\infty} \frac{(a_1)_k \;.\;.\;.\;.\;.\;(a_p)_k}{(b_1)_k \;.\;.\;.\;.\;.\;(b_q)_k} \frac{y^k}{\Gamma(\alpha k + \beta)} \boldsymbol{d(y;q)}$$

$$= \frac{z^{\lambda-\mu-1}}{\Gamma_q(-\mu)} \int_0^1 (1-tq)_{-\mu-1}\, t^{\lambda-1} \sum_{k=o}^{\infty} \frac{(a_1)_k \;.\;.\;.\;.\;.(a_p)_k}{(b_1)_k \;.\;.\;.\;.\;.(b_q)_k} \frac{z^k t^k}{\Gamma(\alpha k+\beta)} d(t;q) \qquad (2.2)$$

Now the following observation are made

(i) $\sum_{k=o}^{\infty} \frac{(a_1)_k \;.\;.\;.\;.\;.(a_p)_k}{(b_1)_k \;.\;.\;.\;.\;.(b_q)_k} \frac{z^k t^k}{\Gamma(\alpha k+\beta)}$ converges absolutely and therefore uniformly on domain of x over the region of integration.

(ii) $\int_0^1 \left| (1-tq)_{-\mu-1} t^{\lambda-1} \right| d(t;q)$ is convergent,

Provided Re $(\lambda) > 0$, Re $(\mu) < 0$, $0 < |q| < 1$

Therefore the order of integration and summation can be interchanged in (2.2) to obtain.

$$= \frac{z^{\lambda-\mu-1}}{\Gamma_q(-\mu)} \sum_{k=o}^{\infty} \frac{(a_1)_k \;.\;.\;.\;.\;.(a_p)_k}{(b_1)_k \;.\;.\;.\;.\;.(b_q)_k} \frac{z^k}{\Gamma(\alpha k+\beta)} \int_0^1 (1-tq)_{-\mu-1}\, t^{\lambda+k-1} d(t;q)$$

$$= \sum_{k=o}^{\infty} \frac{(a_1)_k \;.\;.\;.\;.\;.(a_p)_k}{(b_1)_k \;.\;.\;.\;.\;.(b_q)_k} \frac{1}{\Gamma(\alpha k+\beta)} D^{\mu}_{z,q} z^{k+\lambda-1}$$

Hence the statement (2.1) is proved.

3. SOME SPECIAL CASES:

(i) If we take $\alpha = 0, \beta = 0$ in equation (2.1) it becomes the fractional q-derivative of power series.

$$D^{\mu}_{z,q}\left\{ z^{\lambda-1} \sum_{k=o}^{\infty} \frac{(a_1)_k \;.\;.\;.\;.\;.(a_p)_k}{(b_1)_k \;.\;.\;.\;.\;.(b_q)_k} z^k \right\} = \sum_{k=o}^{\infty} \frac{(a_1)_k \;.\;.\;.\;.\;.(a_p)_k}{(b_1)_k \;.\;.\;.\;.\;.(b_q)_k} D^{\mu}_{z,q}\{z^{k+\lambda-1}\} \qquad (3.1)$$

This equation (3.1) is known result given by Yadav and Purohit [8] and Ali, Jain and Sharma [9].

(ii) When $\alpha = 1, \beta = 1$ and no upper or lower parameter in(5), we have

$$D^{\mu}_{z,q}\left\{ z^{\lambda-1} \sum_{k=o}^{\infty} \frac{z^k}{\Gamma(k+1)} \right\} = \sum_{k=o}^{\infty} \frac{1}{k!} D^{\mu}_{z,q}\{z^{k+\lambda-1}\} \qquad (3.2)$$

Equivalently,

$$D^{\mu}_{z,q}\{ z^{\lambda-1} e^z \} = \sum_{k=o}^{\infty} \frac{1}{k!} D^{\mu}_{z,q}\{z^{k+\lambda-1}\} \qquad (3.3)$$

Thus the equation reduces to fractional q-derivative of exponential function.

(iii) If no upper or lower parameter, we have

$$D^{\mu}_{z,q}\left\{ z^{\lambda-1} \sum_{k=o}^{\infty} \frac{z^k}{\Gamma(\alpha k+\beta)} \right\} = \sum_{k=o}^{\infty} \frac{1}{\Gamma(\alpha k+\beta)} D^{\mu}_{z,q} z^{k+\lambda-1} \qquad (3.4)$$

or

$$D^{\mu}_{z,q}\{ z^{\lambda-1} E_{\alpha,\beta}(z) \} = \sum_{k=o}^{\infty} \frac{1}{\Gamma(\alpha k+\beta)} D^{\mu}_{z,q}\{z^{k+\lambda-1}\} \qquad (3.5)$$

Hence the series convert in fractional q-derivative of Mittag-Lefller function. Thus it is the complete analysis of the statement (2.1).

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