A SHORT PROOF THAT ADDITIVE, MEASURABLE FUNCTIONS ARE LINEAR

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For notation, we use λ to mean Lebesgue measure on $\mathbb{R}$.

Lemma 0.1. *If A, B are sets of positive measure, then there exists $q \in \mathbb{Q}$ such that $\lambda((A+q)\cap B) > 0$.*

Proof. By the Lebesgue density thereom we get $\Delta > 0$, $a \in A$ and $b \in B$ such that $\lambda(A \cap [a-\delta, a+\delta]), \lambda(B \cap [b-\delta, b+\delta]) \geq 3\delta/2$ for all $\delta \leq \Delta$. If we assume that $\lambda((A+q)\cap B) = 0$ for all $q \in \mathbb{Q}$ then for any $q \in \mathbb{Q}$ such that $a+q$ is close to b, we get

$$\begin{aligned}
2\Delta &\geq \lambda(((A+q)\cap[a+q-\Delta, a+q+\Delta]) \cup (B \cap [a+q-\Delta, a+q+\Delta])) \\
&\geq \lambda(A \cap [a-\Delta, a+\Delta]) + \lambda(B \cap [b-(\Delta-|b-(a+q)|), b+(\Delta-|b-(a+q)|)]) \\
&\geq (3/2)\Delta + (3/2)\Delta - (3/2)|b-(a+q)|
\end{aligned}$$

Since we can choose q to make the last term as small as we please, we get a contradiction. □

Lemma 0.2. *If A is a set with $\lambda(A) > 0$ which has the property that $A + q = A$ for every $q \in Q$ then $\lambda(A^c) = 0$.*

Proof. Notice that for all q, $(A+q)\cap A^c = \emptyset$. Hence, $\lambda(A^c) > 0$ would contradict the lemma above. □

Theorem 0.3. *If $f : \mathbb{R} \to \mathbb{R}$ is additive and measurable then f is linear.*

Proof. Let $g(x) = f(x) - f(1)x$ and we are finished if we show that g is 0. Notice that g is linear and measurable as well. We first show that g takes the value 0 on the rational numbers. Notice that $g(1) = 0$. Now, take a rational number $\frac{p}{q}$ and we get that:

$$0 = pg(1) = g(1) + \cdots + g(1) = g(p) = g\left(p/q\right) + \cdots + g\left(p/q\right) = qg\left(p/q\right)$$

Hence $g(p/q) = 0$. An important consequence of this is that $g(x+r) = g(x)$ whenever $r \in \mathbb{Q}$.

Now we show that g is essentially bounded. Suppose this isn't true. Then for every M, the set $A_M = \{x : |g(x)| \geq M\}$ has positive measure. Since, for any $q \in \mathbb{Q}$, $g(x+q) = g(x)$, then $A_M = A_M + q$. By the lemma above, $\lambda(A_M^c) = 0$. Notice that this holds for all M. However, since $g : \mathbb{R} \to \mathbb{R}$ is necessarily finite, then $\mathbb{R} = \cup_n A_n^c$. Hence $0 = \lim \lambda(A_n^c) = \lambda(\mathbb{R}) = \infty$. This is clearly a contradiction.

Now, we know that g is integrable on compact sets and that it is periodic. Hence, for any $y \in \mathbb{R}$ we have

$$\int_0^1 g(x)dx = \int_{0-y}^{1-y} g(x+y)dx = \int_0^1 g(x+y)dx = \int_0^1 g(x)dx + \int_0^1 g(y)dx = g(y) + \int_0^1 g(x)dx$$

This clearly implies $g(y) = 0$ for all y, finishing the proof. □