LaSalle's Invariance Principle

Lecture 23
Math 634
10/22/99

Linearization versus Lyapunov Functions

In the previous two lectures, we have talked about two different tools that can be used to prove that an equilibrium point x_0 of an autonomous system

$$\dot{x} = f(x) \tag{1}$$

is asymptotically stable: linearization and Lyapunov's direct method. One might ask which of these methods is better. Certainly, linearization seems easier to apply because of its straightforward nature: Compute the eigenvalues of $Df(x_0)$. The direct method requires you to find an appropriate Lyapunov function, which doesn't seem so straightforward. But, in fact, anytime linearization works, a simple Lyapunov function works, as well.

To be more precise, suppose $x_0 = 0$ and all the eigenvalues of $A := Df(0)$ have negative real part. Pick an inner product $\langle \cdot, \cdot \rangle$ and induced norm $\| \cdot \|$ such that, for some $c > 0$,

$$\langle x, Ax \rangle \leq -c\|x\|^2$$

for all $x \in \mathbb{R}^n$. Pick $r > 0$ small enough that $\|f(x) - Ax\| \leq (c/2)\|x\|$ whenever $\|x\| \leq r$, let

$$\mathcal{D} = \left\{ x \in \mathbb{R}^n \;\middle|\; \|x\| \leq r \right\},$$

and define $V : \mathbb{R} \times \mathcal{D} \to \mathbb{R}$ by the formula $V(t, x) = \|x\|^2$. Since $\| \cdot \|$ is a norm, V is positive definite. Also

$$\begin{aligned}\dot{V}(t, x) &= 2\langle x, f(x)\rangle = 2(\langle x, Ax\rangle + \langle x, f(x) - Ax\rangle) \\ &\leq 2(-c\|x\|^2 + \|x\|\|f(x) - Ax\|) \leq -c\|x\|^2,\end{aligned}$$

so $\dot{V}$ is negative definite.

On the other hand, there are very simple examples to illustrate that the direct method works in some cases where linearization doesn't. For example, consider $\dot{x} = -x^3$ on $\mathbb{R}$. The equilibrium point at the origin is not hyperbolic, so linearization fails to determine stability, but it is easy to check that x^2 is positive definite and has a negative definite orbital derivative, thus ensuring the asymptotic stability of 0.

A More Complicated Example

The previous example is so simple that it might make one question whether the direct method is of any use on problems where stability cannot be determined by linearization *or* by inspection. Thus, let's consider something more complicated. Consider the planar system

$$\begin{cases} \dot{x} = -y - x^3 \\ \dot{y} = x^5. \end{cases}$$

The origin is a nonhyperbolic equilibrium point, with 0 being the only eigenvalue, so the principle of linearized stability is of no use. A sketch of the phase portrait indicates that orbits circle the origin in the counterclockwise direction, but it is not obvious whether they spiral in, spiral out, or move on closed curves.

The simplest potential Lyapunov function that often turns out to be useful is the square of the standard Euclidean norm, which in this case is $V := x^2 + y^2$. The orbital derivative is

$$\dot{V} = 2x\dot{x} + 2y\dot{y} = 2x^5y - 2xy - 2x^4. \tag{2}$$

For some points (x, y) near the origin (*e.g.*, (δ, δ)) $\dot{V} < 0$, while for other points near the origin (*e.g.*, $(\delta, -\delta)$) $\dot{V} > 0$, so this function doesn't seem to be of much use.

Sometimes when the square of the standard Euclidean norm doesn't work, some other homogeneous quadratic function does. Suppose we try $V := x^2 + \alpha xy + \beta y^2$, with α and β to be determined. Then

$$\begin{aligned} \dot{V} &= (2x + \alpha y)\dot{x} + (\alpha x + 2\beta y)\dot{y} = -(2x + \alpha y)(y + x^3) + (\alpha x + 2\beta y)x^5 \\ &= -2x^4 + \alpha x^6 - 2xy - \alpha x^3 y + 2\beta x^5 y - \alpha y^2. \end{aligned}$$

Setting $(x, y) = (\delta, -\delta^2)$ for δ positive and small, we see that $\dot{V}$ is not going to be negative semidefinite, no matter what we pick α and β to be.

If these quadratic functions don't work, maybe something customized for the particular equation might. Note that the right-hand side of the first equation in (2) sort of suggests that x^3 and y should be treated as quantities of the same order of magnitude. Let's try $V := x^6 + \alpha y^2$, for some $\alpha > 0$ to be determined. Clearly, V is positive definite, and

$$\dot{V} = 6x^5\dot{x} + 2\alpha y\dot{y} = (2\alpha - 6)x^5y - 6x^8.$$

If $\alpha \neq 3$, then $\dot{V}$ is of opposite signs for $(x, y) = (\delta, \delta)$ and for $(x, y) = (\delta, -\delta)$ when δ is small. Hence, we should set $\alpha = 3$, yielding $\dot{V} = -6x^8 \leq 0$. Thus V is positive definite and $\dot{V}$ is negative semidefinite, implying that the origin is Lyapunov stable.

Is the origin asymptotically stable? Perhaps we can make a minor modification to the preceding formula for V so as to make $\dot{V}$ strictly negative in a deleted neighborhood of the origin without destroying the positive definiteness of V. If we added a small quantity whose orbital derivative was strictly negative when $x = 0$ and $|y|$ is small and positive, this might work. Experimentation suggests that a positive multiple of xy^3 might work, since this quantity changes from positive to negative as we cross the y-axis in the counterclockwise direction. Also, it is at least of higher order than $3y^2$ near the origin, so it has the potential of preserving the positive definiteness of V.

In fact, we claim that $V := x^6 + xy^3 + 3y^2$ is positive definite with negative definite orbital derivative near 0. A handy inequality, sometimes called Young's inequality, that can be used in verifying this claim (and in other circumstances, as well) is given in the following lemma.

Lemma (Young's Inequality) *If $a, b \geq 0$, then*

$$ab \leq \frac{a^p}{p} + \frac{b^q}{q}, \tag{3}$$

for every pair of numbers $p, q \in (1, \infty)$ satisfying

$$\frac{1}{p} + \frac{1}{q} = 1. \tag{4}$$

Proof. Assume that (4) holds. Clearly (3) holds if $b = 0$, so assume that $b > 0$, and fix it. Define $g : [0, \infty)$ by the formula

$$g(x) := \frac{x^p}{p} + \frac{b^q}{q} - xb.$$

Note that g is continuous, and $g'(x) = x^{p-1} - b$ for every $x \in (0, \infty)$. Since $\lim_{x\downarrow 0} g'(x) = -b < 0$, $\lim_{x\uparrow\infty} g'(x) = \infty$, and g' is increasing on $(0, \infty)$, we know that g has a unique minimizer at $x_0 = b^{1/(p-1)}$. Thus, for every $x \in [0, \infty)$ we see, using (4), that

$$g(x) \geq g(b^{1/(p-1)}) = \frac{b^{p/(p-1)}}{p} + \frac{b^q}{q} - b^{p/(p-1)} = \left(\frac{1}{p} + \frac{1}{q} - 1\right) b^q = 0.$$

In particular, $g(a) \geq 0$, so (3) holds. $\square$

Now, let $V = x^6 + xy^3 + 3y^2$. Applying Young's inequality with $a = |x|$, $b = |y|^3$, $p = 6$, and $q = 6/5$, we see that

$$|xy^3| = |x||y|^3 \leq \frac{|x|^6}{6} + \frac{5|y|^{18/5}}{6} \leq \frac{1}{6}x^6 + \frac{5}{6}y^2$$

if $|y| \leq 1$, so

$$V \geq \frac{5}{6}x^6 + \frac{13}{6}y^2$$

if $|y| \leq 1$. Also,

$$\begin{aligned}\dot{V} &= -6x^8 + y^3\dot{x} + 3xy^2\dot{y} = -6x^8 - y^3(y + x^3) + 3x^6y^2 \\ &= -6x^8 - x^3y^3 + 3x^6y^2 - y^4.\end{aligned}$$

Applying Young's inequality to the two mixed terms in this orbital derivative, we have

$$|-x^3y^3| = |x|^3|y|^3 \leq \frac{3|x|^8}{8} + \frac{5|y|^{24/5}}{8} \leq \frac{3}{8}x^8 + \frac{5}{8}y^4$$

if $|y| \leq 1$, and

$$|3x^6y^2| = 3|x|^6|y|^2 \leq 3\left[\frac{3|x|^8}{4} + \frac{|y|^8}{4}\right] = \frac{9}{4}x^8 + \frac{3}{4}y^8 \leq \frac{9}{4}x^8 + \frac{3}{64}y^4$$

if $|y| \leq 1/2$. Thus,

$$\dot{V} \leq -\frac{27}{8}x^8 - \frac{21}{64}y^4$$

if $|y| \leq 1/2$, so, in a neighborhood of 0, V is positive definite and $\dot{V}$ is negative definite, which implies that 0 is asymptotically stable.

LaSalle's Invariance Principle

We would have saved ourselves a lot of work on the previous example if we could have just stuck with the moderately simple function $V = x^6 + 3y^2$, even though its orbital derivative was only negative semidefinite. Notice that the set of points where $\dot{V}$ was 0 was small (the y-axis) and at most of those points the vector field was not parallel to the set. LaSalle's Invariance Principle, which we shall state and prove for the autonomous system

$$\dot{x} = f(x), \tag{5}$$

allows us to use such a V to prove asymptotic stability.

Theorem (LaSalle's Invariance Principle) *Suppose there is a neighborhood $\mathcal{D}$ of 0 and a continuously differentiable (time-independent) positive definite function $V : \mathcal{D} \to \mathbb{R}$ whose orbital derivative $\dot{V}$ (w.r.t. (5)) is negative semidefinite. Let $\mathcal{I}$ be the union of all complete orbits contained in*

$$\left\{x \in \mathcal{D} \;\middle|\; \dot{V}(x) = 0\right\}.$$

Then there is a neighborhood $\mathcal{U}$ of 0 such that for every $x_0 \in \mathcal{U}$, $\omega(x_0) \subseteq \mathcal{I}$.

Before proving this, we note that when applying it to $V = x^6 + 3y^2$ in the previous example, the set $\mathcal{I}$ is a singleton containing the origin and, since $\mathcal{D}$ can be assumed to be compact, each solution beginning in $\mathcal{U}$ actually converges to 0 as $t \uparrow \infty$.

Proof of LaSalle's Invariance Principle. Let φ be the flow generated by (5). By a previous theorem, 0 must be Lyapunov stable, so we can pick a neighborhood $\mathcal{U}$ of 0 such that $\varphi(t, x) \in \mathcal{D}$ for every $x_0 \in \mathcal{U}$ and every $t \geq 0$.

Let $x_0 \in \mathcal{U}$ and $y \in \omega(x_0)$ be given. By the negative semidefiniteness of $\dot{V}$, we know that $V(\varphi(t, x_0))$ is a nonincreasing function of t. By the positive definiteness of V, we know that $V(\varphi(t, x_0))$ remains nonnegative, so it must approach some constant $c \geq 0$ as $t \uparrow \infty$. By continuity of V, $V(z) = c$ for every $z \in \omega(x_0)$. Since $\omega(x_0)$ is invariant, $V(\varphi(t, y)) = c$ for every $t \in \mathbb{R}$. The definition of orbital derivative then implies that $\dot{V}(\varphi(t, y)) = 0$ for every $t \in \mathbb{R}$. Hence, $y \in \mathcal{I}$. $\square$

Exercise 15 Show that $(x(t), y(t)) = (0, 0)$ is an asymptotically stable solution of

$$\begin{cases} \dot{x} = -x^3 + 2y^3 \\ \dot{y} = -2xy^2. \end{cases}$$