Left-Leaning Red-Black Trees

Robert Sedgewick
Princeton University

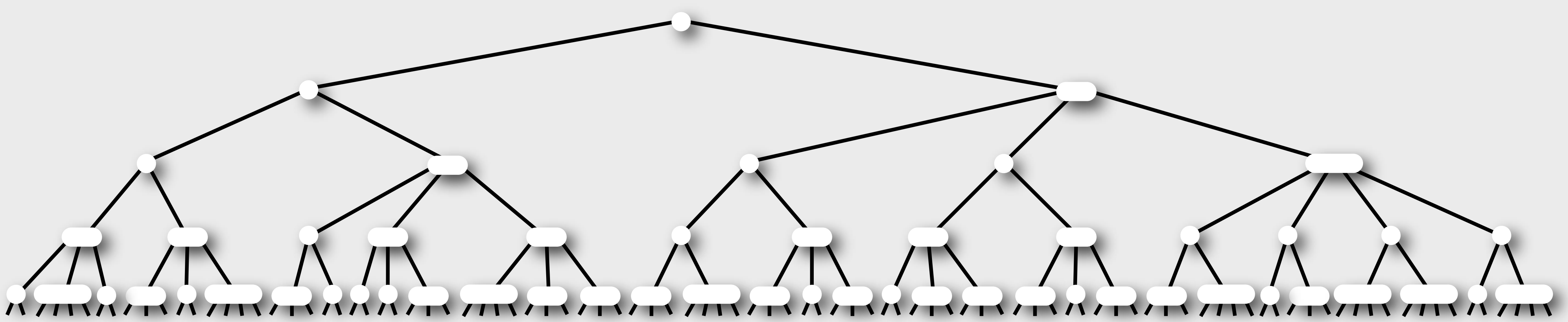
Introduction

2-3-4 Trees

Red-Black Trees

Left-Leaning RB Trees

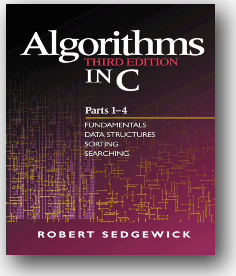
Deletion

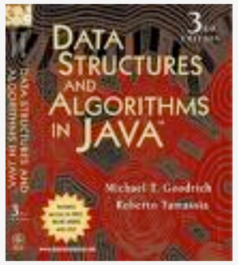


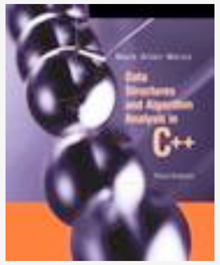
Red-black trees

are now found throughout our computational infrastructure

Textbooks on algorithms







. . .

Library search function in many programming environments

. . .

Popular culture (stay tuned)

Worth revisiting?

Red-black trees

are now found throughout our computational infrastructure

Typical:

> ya thanks,
> i got the idea
> but is there some other place on the web where only the algorithms
> used by STL is
> explained. (that is the underlying data structures etc.) without
> explicit reference to the code (as it is pretty confusing if I try to
> read through).
>
> thanks[/color]

The standard does not specify which algorithms the STL must use. Implementers are free to choose which ever algorithm or data structure that fulfils the functional and efficiency requirements of the standard.

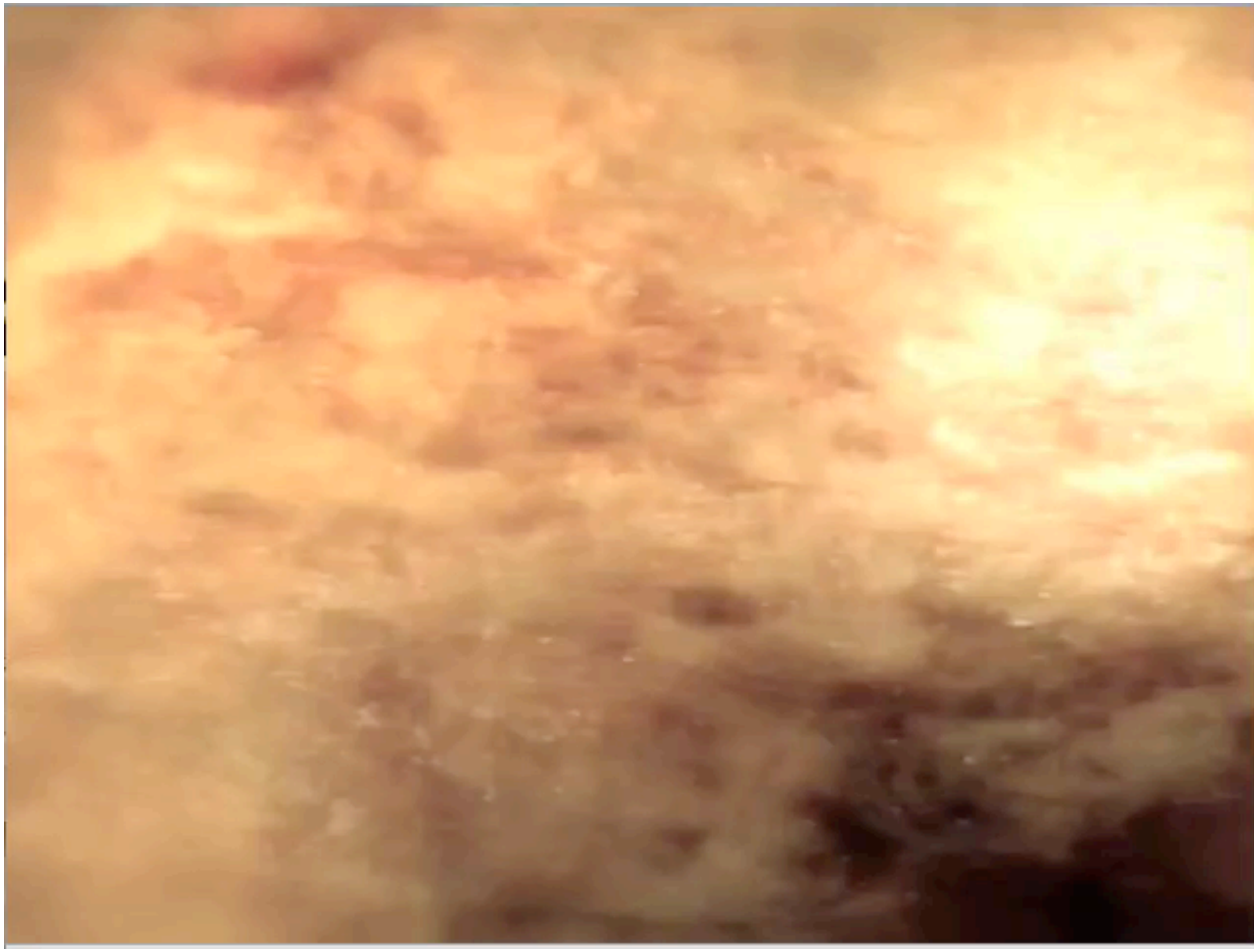
There are some common choices however. For instance every implementation of map, multimap, set and multiset that I have ever seen uses a structure called a red black tree. Typing 'red black tree algorithm' in google produces a number of likely looking links.

john

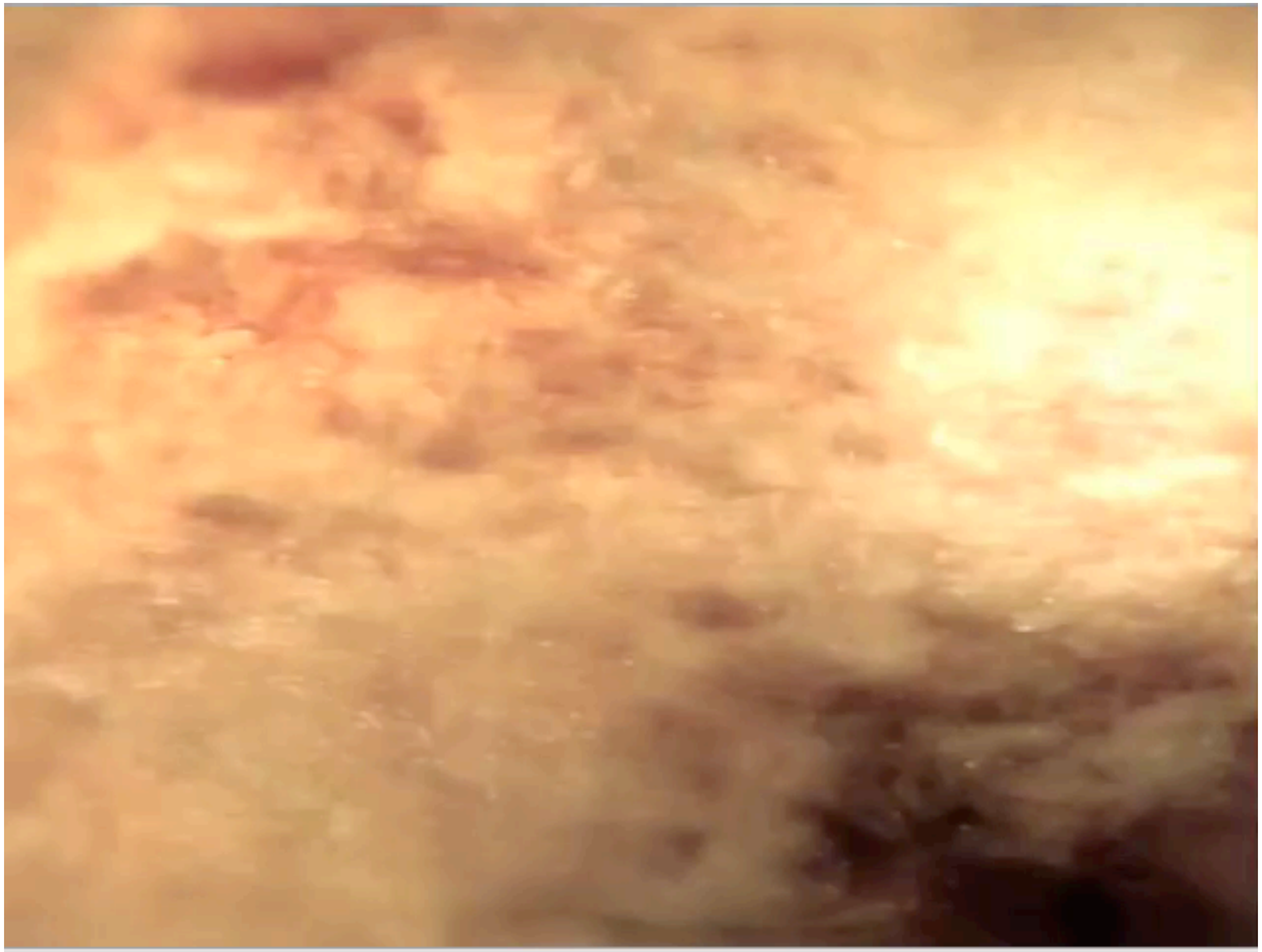
Digression:

Red-black trees are found in popular culture??

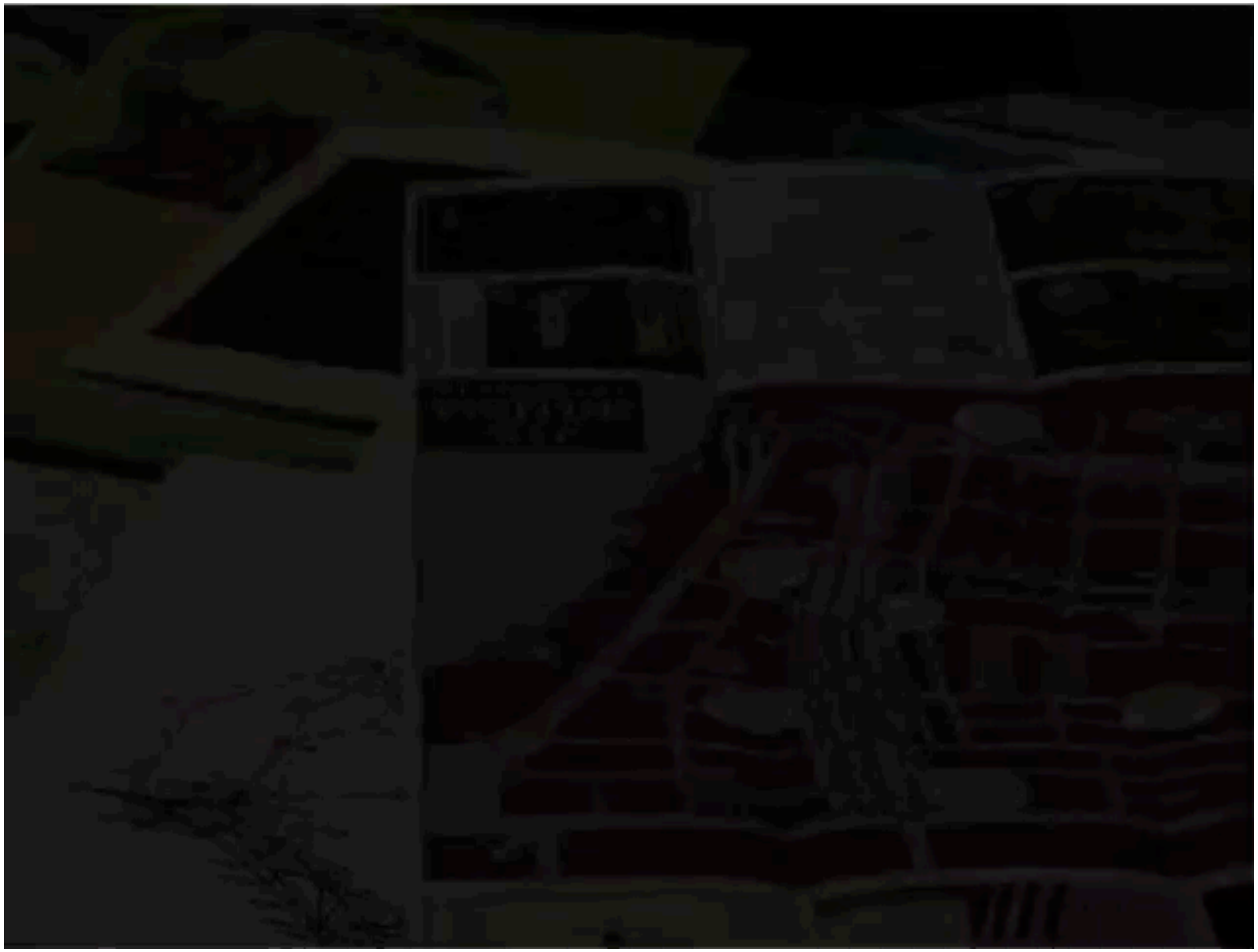
Mystery: black door?



Mystery: red door?



An explanation ?



Primary goals

Red-black trees (Guibas-Sedgewick, 1978)

- reduce code complexity
- minimize or eliminate space overhead
- unify balanced tree algorithms
- single top-down pass (for concurrent algorithms)
- find version amenable to average-case analysis

Current implementations

- maintenance
- migration
- space not so important (??)
- guaranteed performance
- support full suite of operations

Worth revisiting ?

Primary goals

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Current implementations

- maintenance
- migration
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- guaranteed performance
- support full suite of operations

Worth revisiting ? YES. Code complexity is out of hand.

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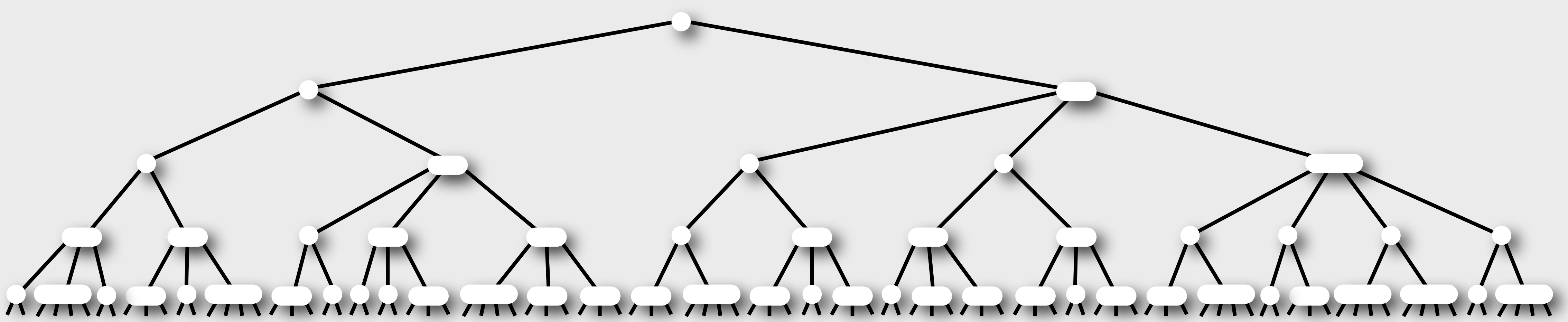


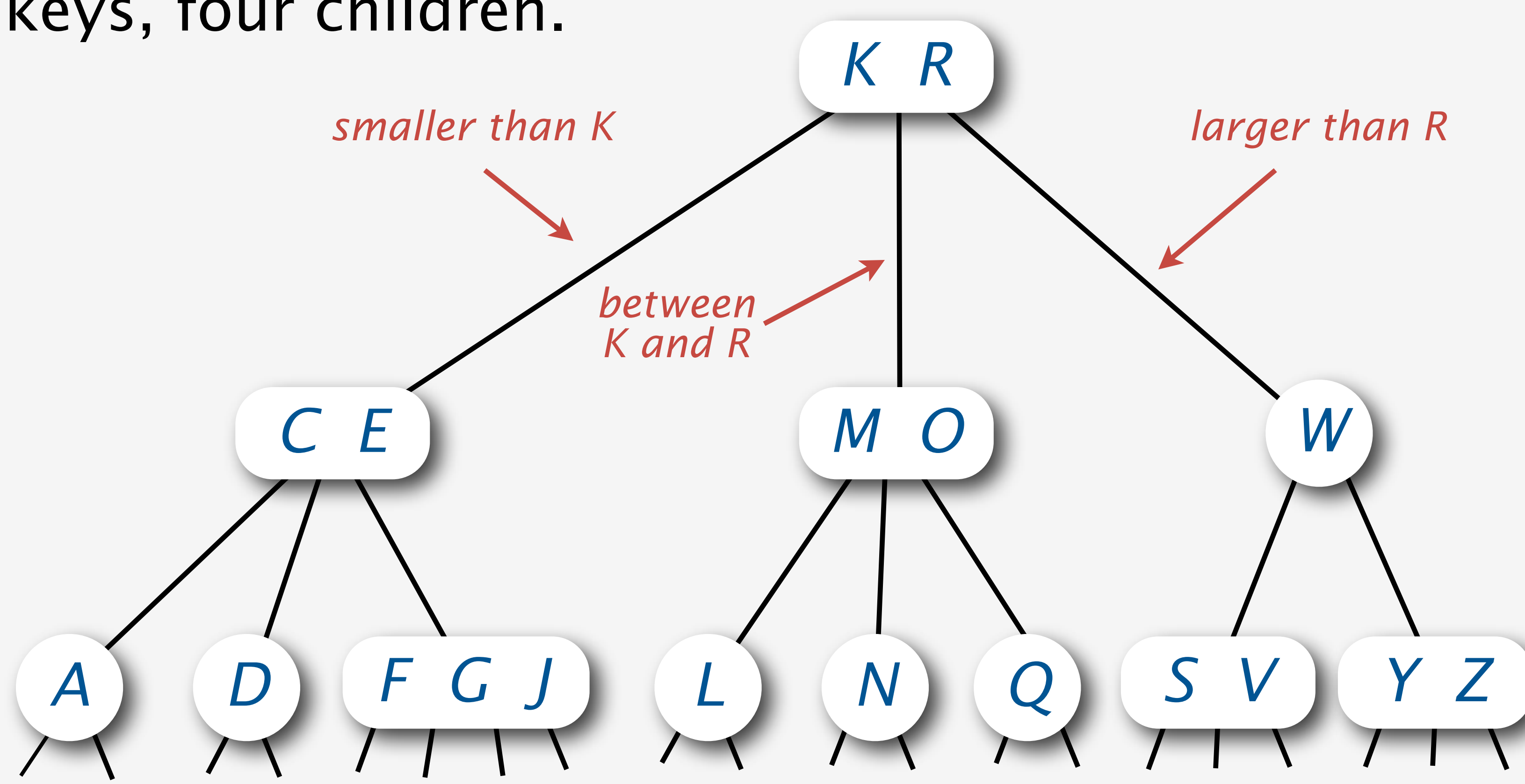
2-3-4 Tree

Generalize BST node to allow multiple keys.
Keep tree in perfect balance.

Perfect balance. Every path from root to leaf has same length.

Allow 1, 2, or 3 keys per node.

- 2-node: one key, two children.
- 3-node: two keys, three children.
- 4-node: three keys, four children.



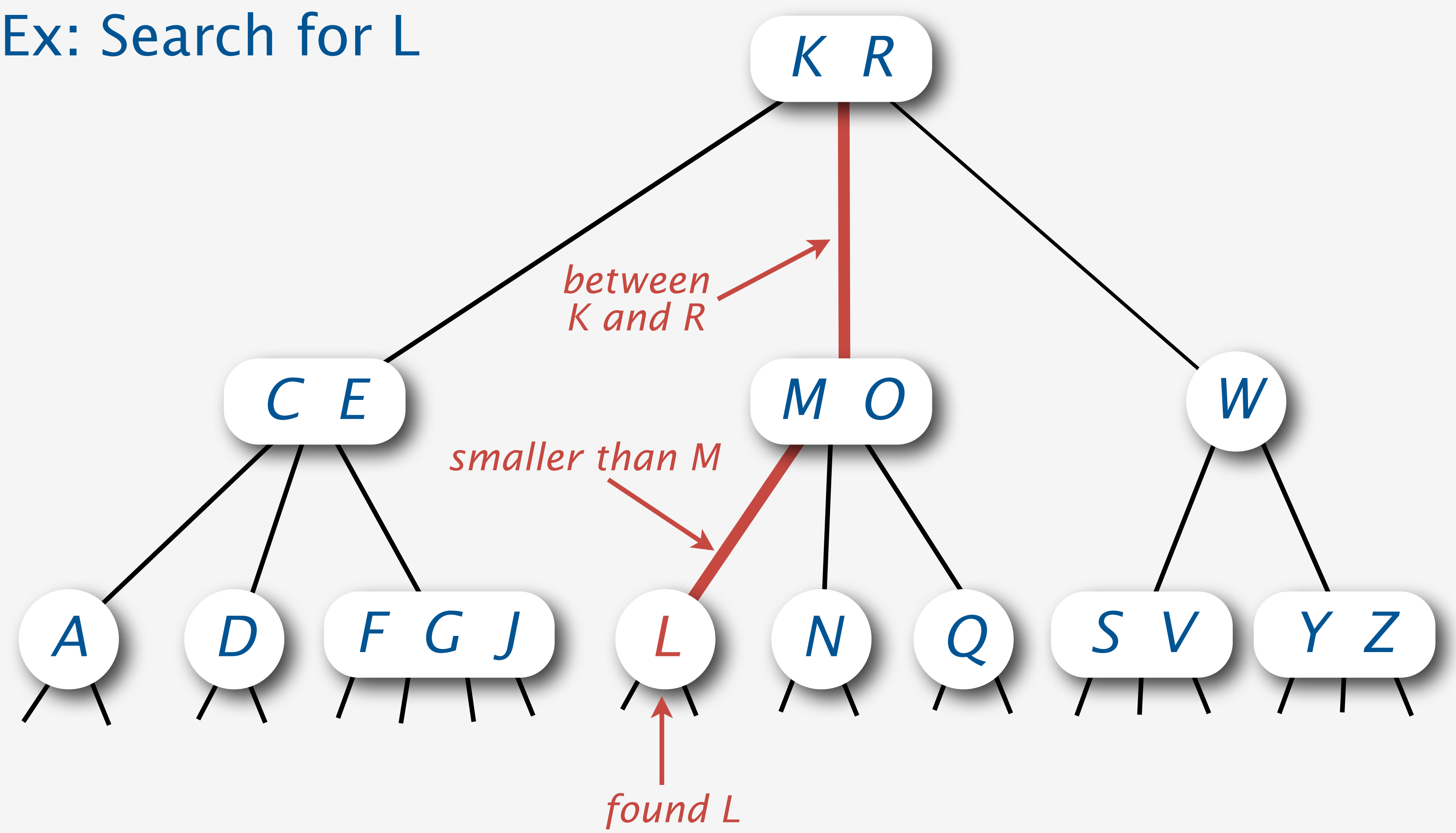
Search in a 2-3-4 Tree

Compare node keys against search key to guide search.

Search.

- Compare search key against keys in node.
- Find interval containing search key.
- Follow associated link (recursively).

Ex: Search for L



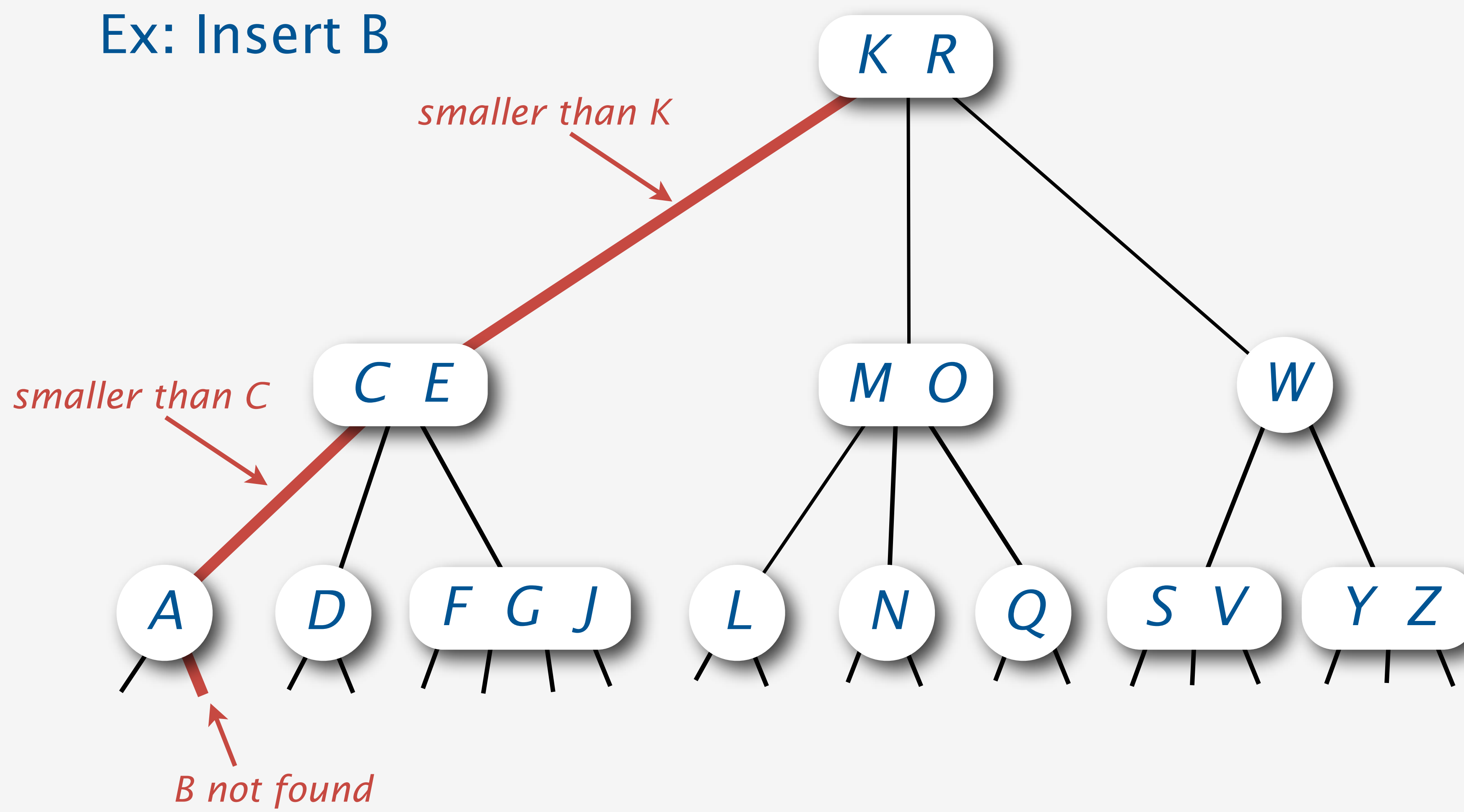
Insertion in a 2-3-4 Tree

Add new keys at the bottom of the tree.

Insert.

- Search to bottom for key.

Ex: Insert B



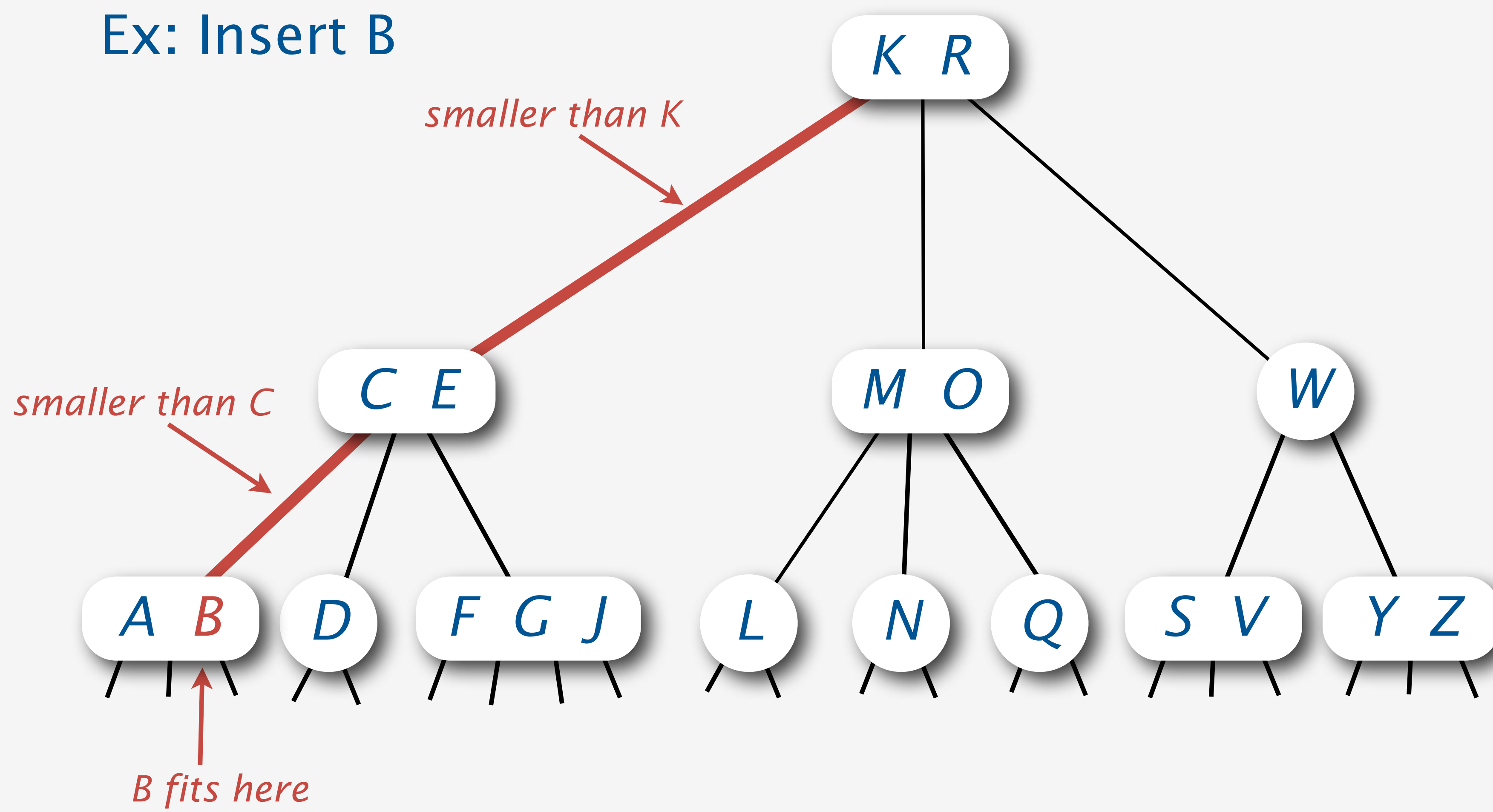
Insertion in a 2-3-4 Tree

Add new keys at the bottom of the tree.

Insert.

- Search to bottom for key.
- 2-node at bottom: convert to a 3-node.

Ex: Insert B



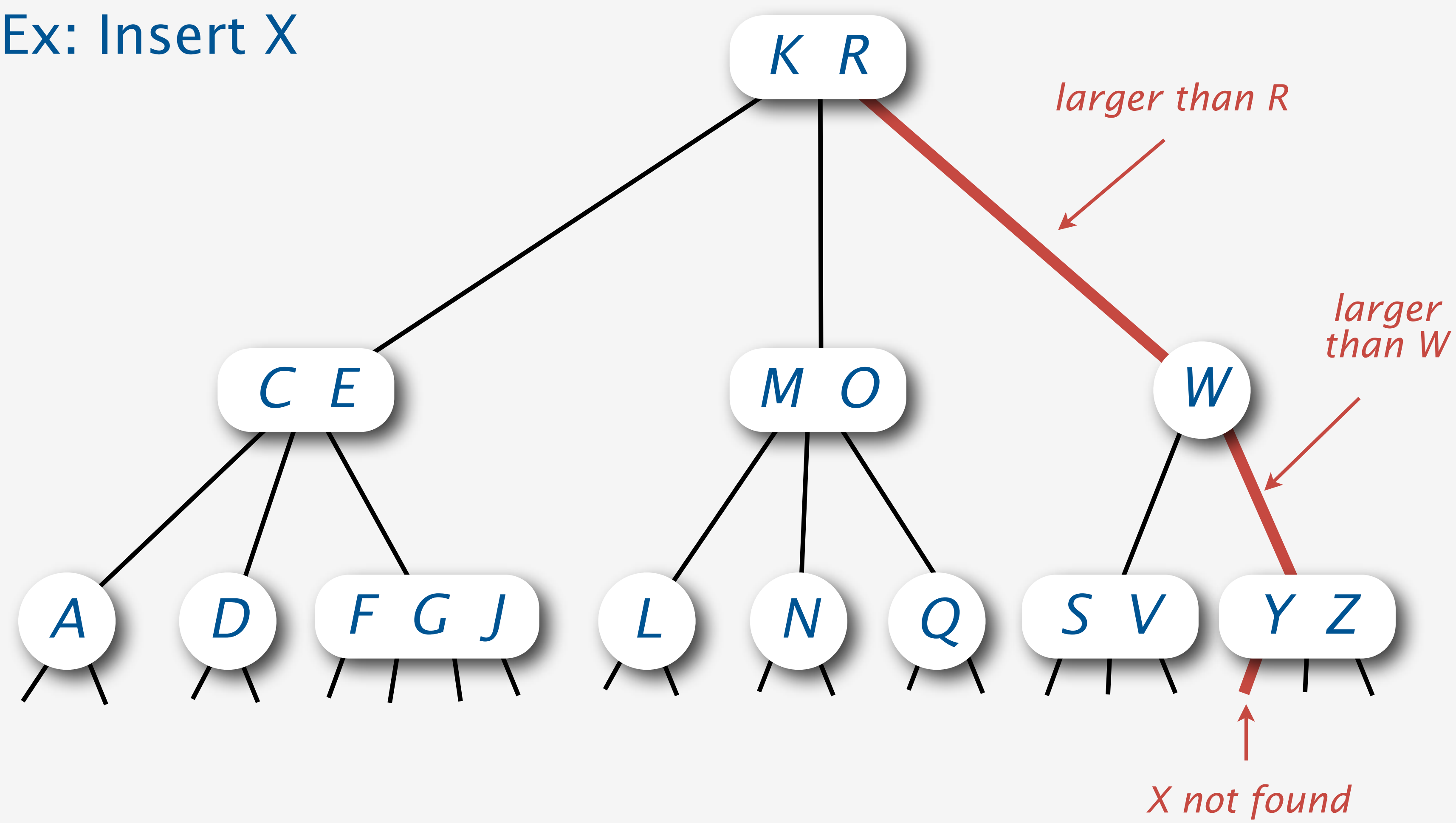
Insertion in a 2-3-4 Tree

Add new keys at the bottom of the tree.

Insert.

- Search to bottom for key.

Ex: Insert X



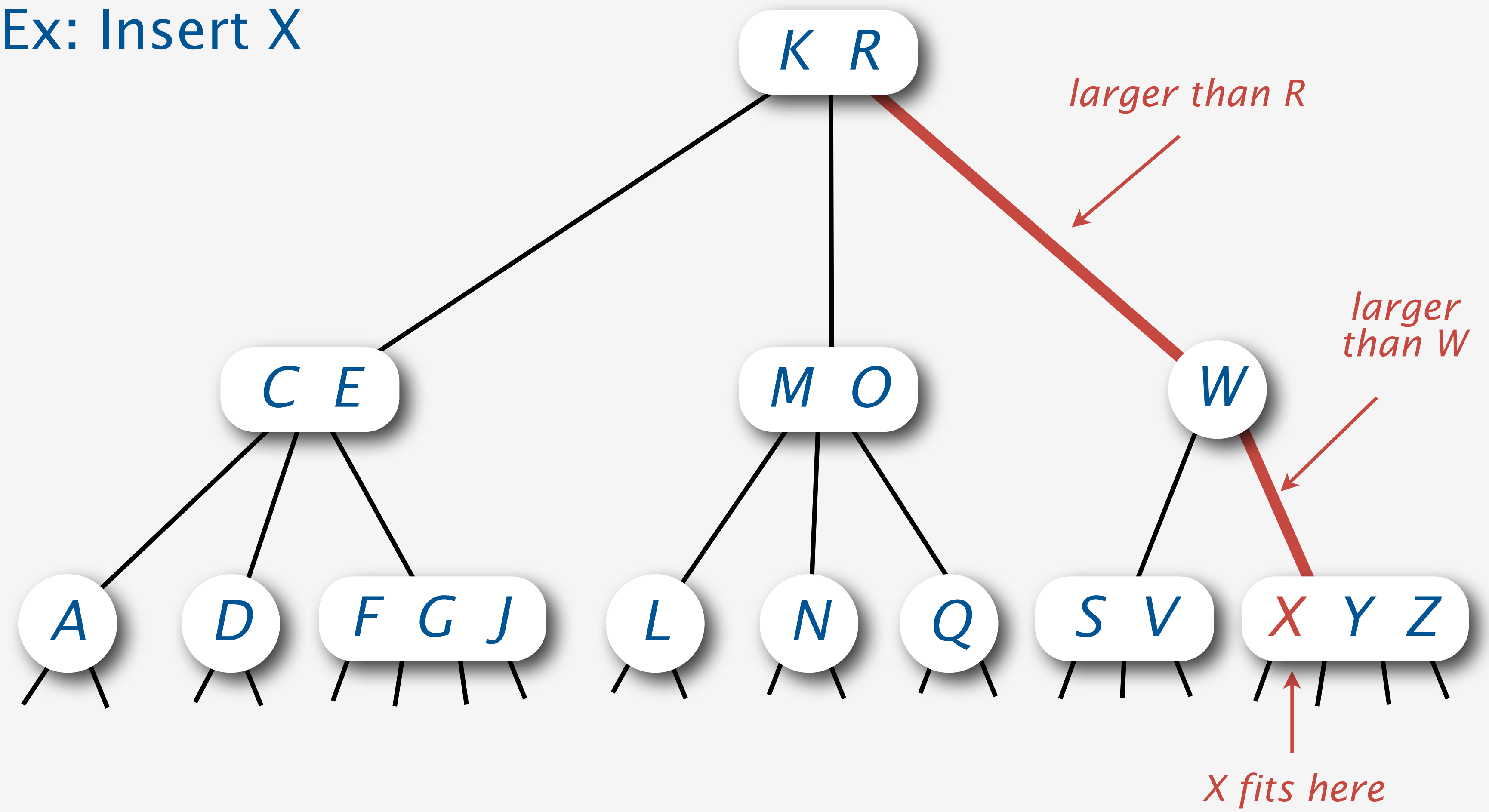
Insertion in a 2-3-4 Tree

Add new keys at the bottom of the tree.

Insert.

- Search to bottom for key.
- 3-node at bottom: convert to a 4-node.

Ex: Insert X



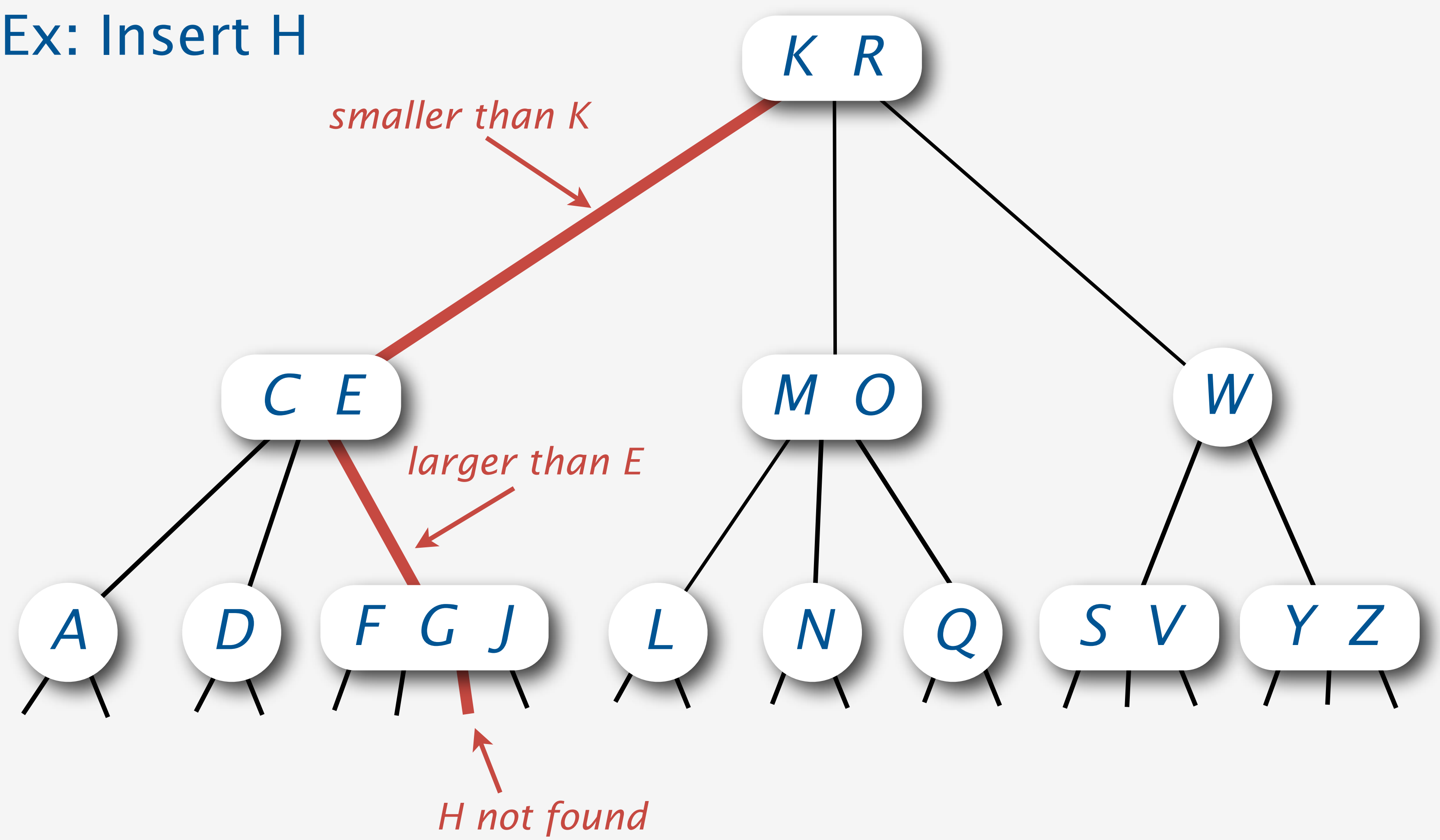
Insertion in a 2-3-4 Tree

Add new keys at the bottom of the tree.

Insert.

- Search to bottom for key.

Ex: Insert H



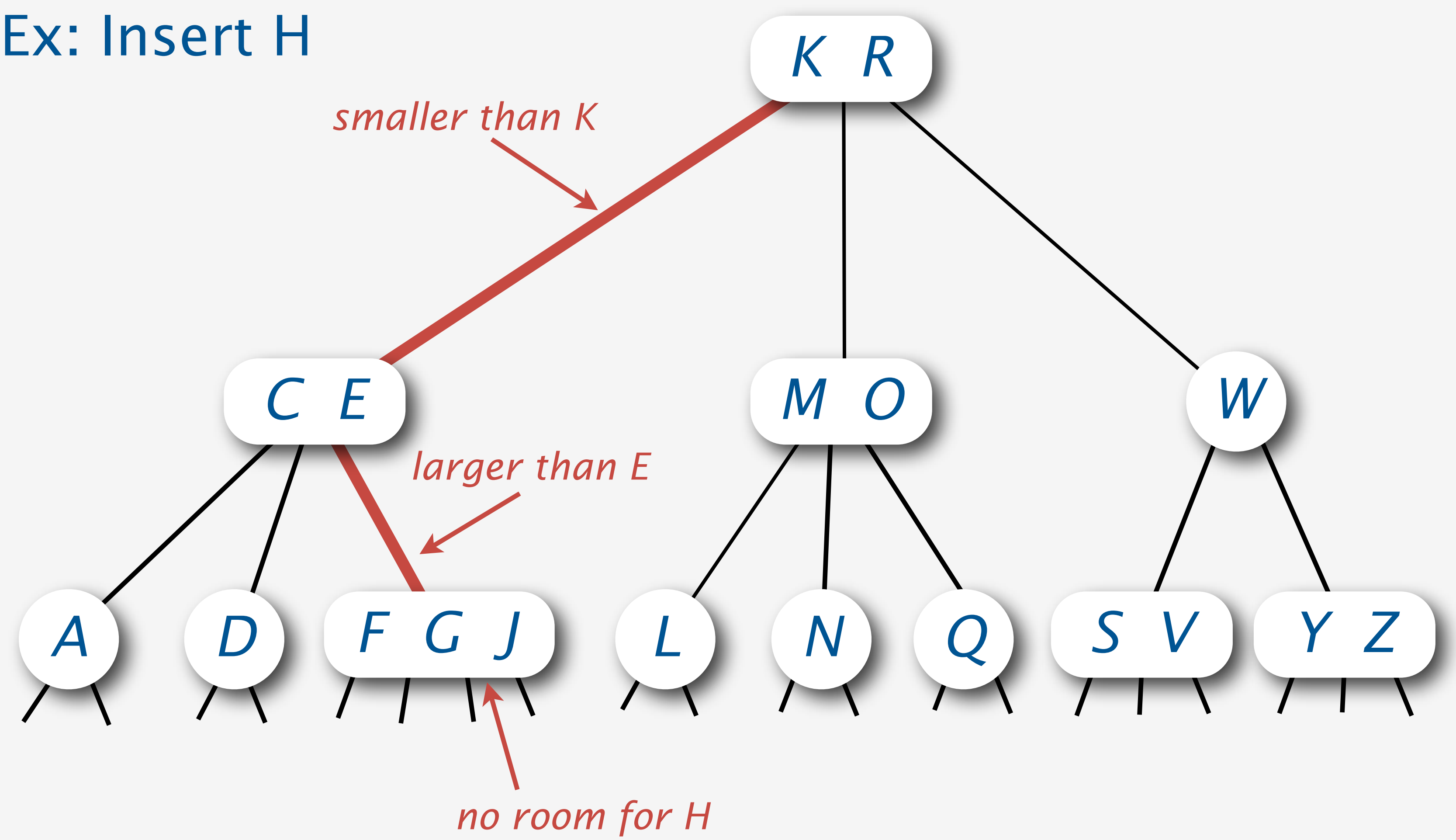
Insertion in a 2-3-4 Tree

Add new keys at the bottom of the tree.

Insert.

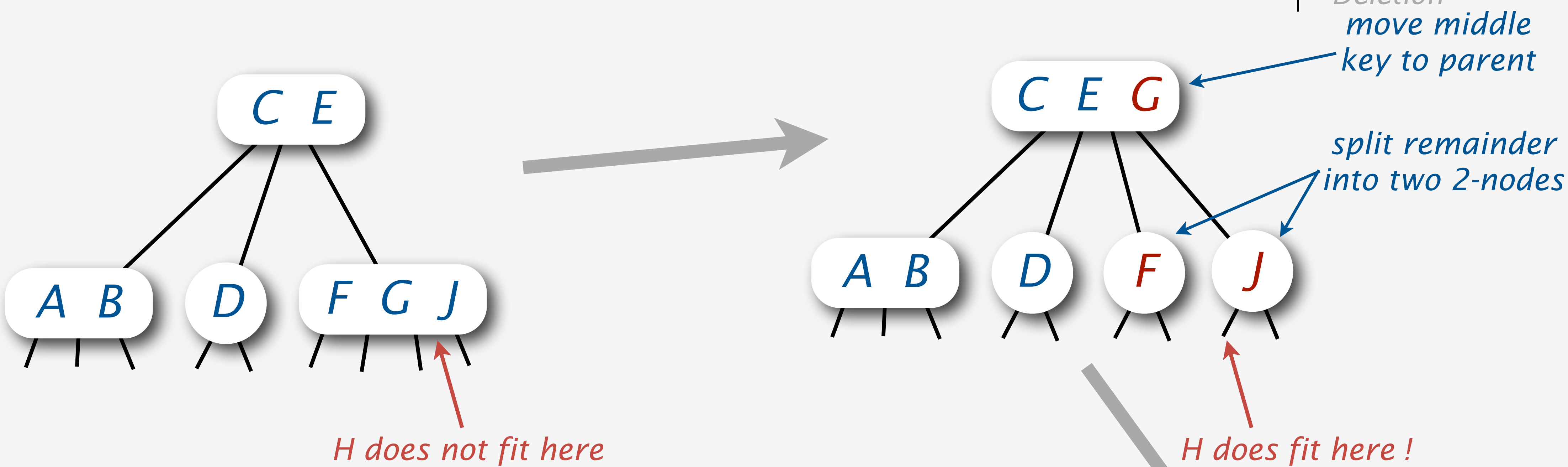
- Search to bottom for key.
- 2-node at bottom: convert to a 3-node.
- 3-node at bottom: convert to a 4-node.
- 4-node at bottom: no room for new key.

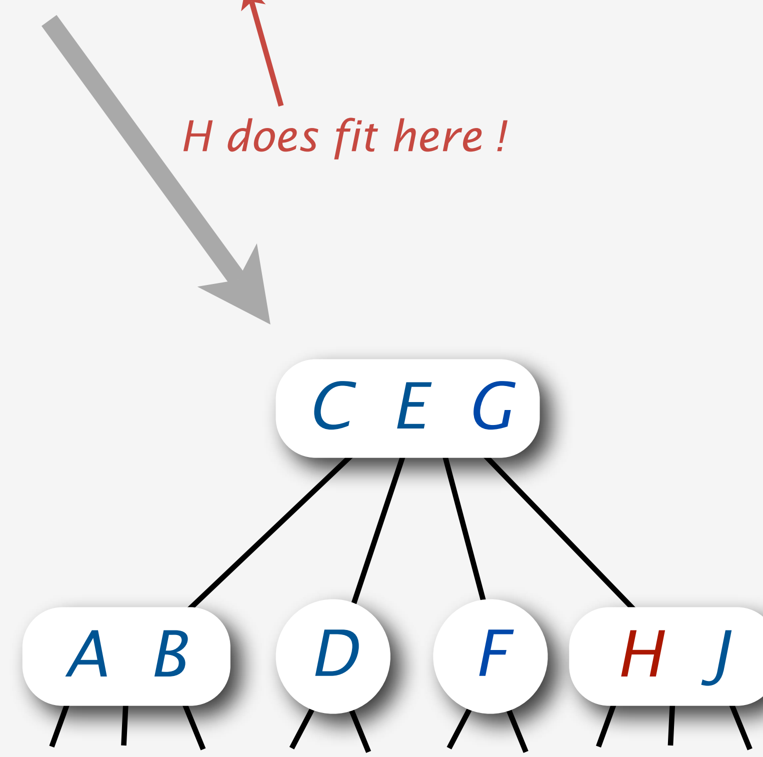
Ex: Insert H



Splitting 4-nodes in a 2-3-4 tree

is an effective way to make room for insertions





Problem: Doesn't work if parent is a 4-node

Bottom-up solution (Bayer, 1972)

- Use same method to split parent
- Continue up the tree while necessary

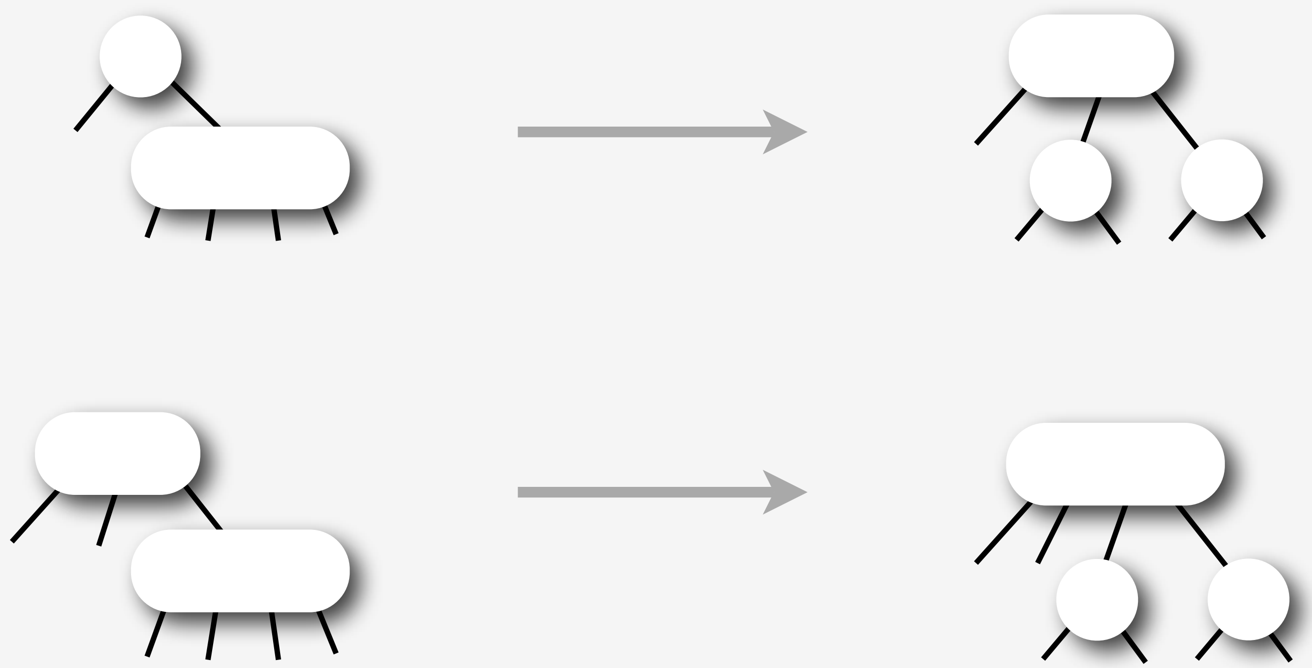
Top-down solution (Guibas-Sedgewick, 1978)

- Split 4-nodes on the way down
- Insert at bottom

Splitting 4-nodes on the way down

ensures that the "current" node is not a 4-node

Transformations to split 4-nodes:



local transformations that work anywhere in the tree

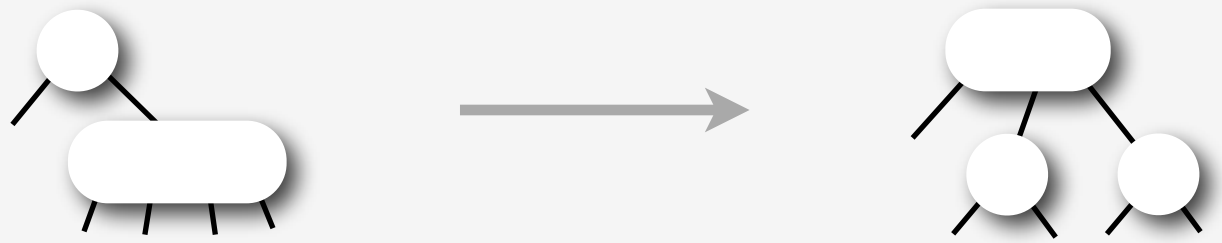
Invariant: "Current" node is not a 4-node

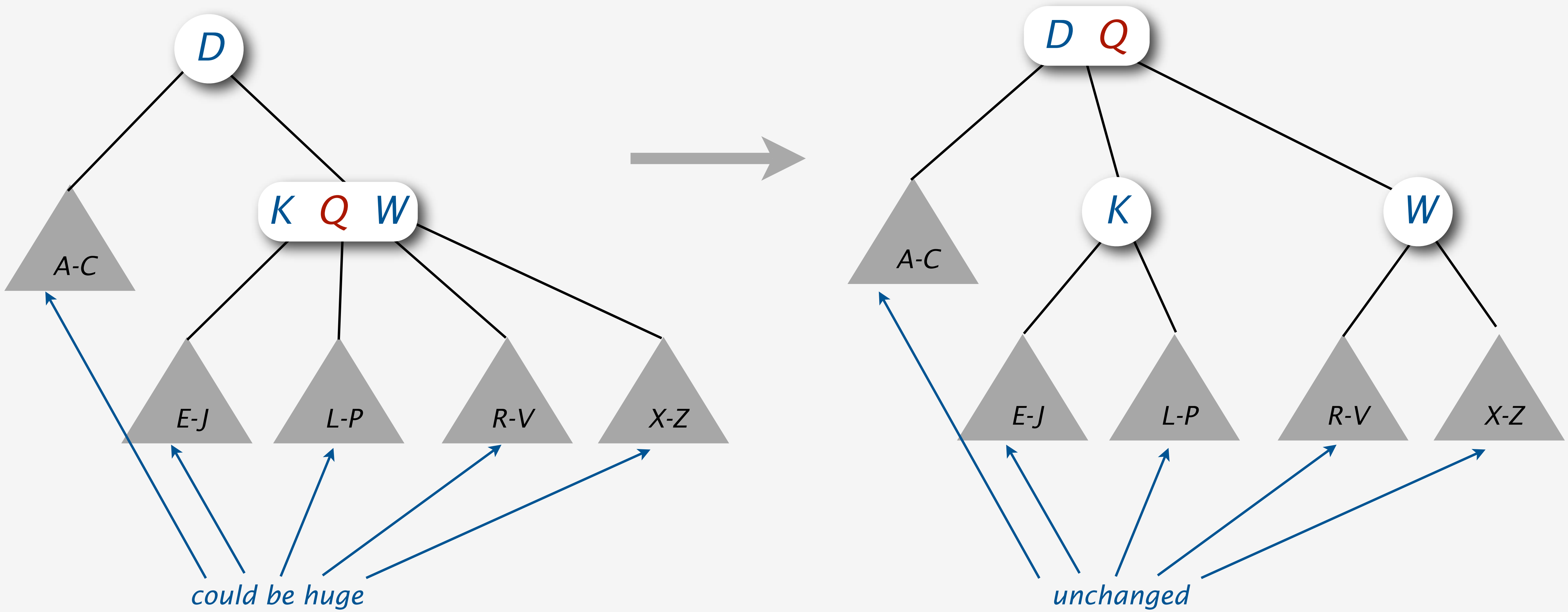
Consequences:

- 4-node below a 4-node case never happens
- Bottom node reached is always a 2-node or a 3-node

Splitting a 4-node below a 2-node

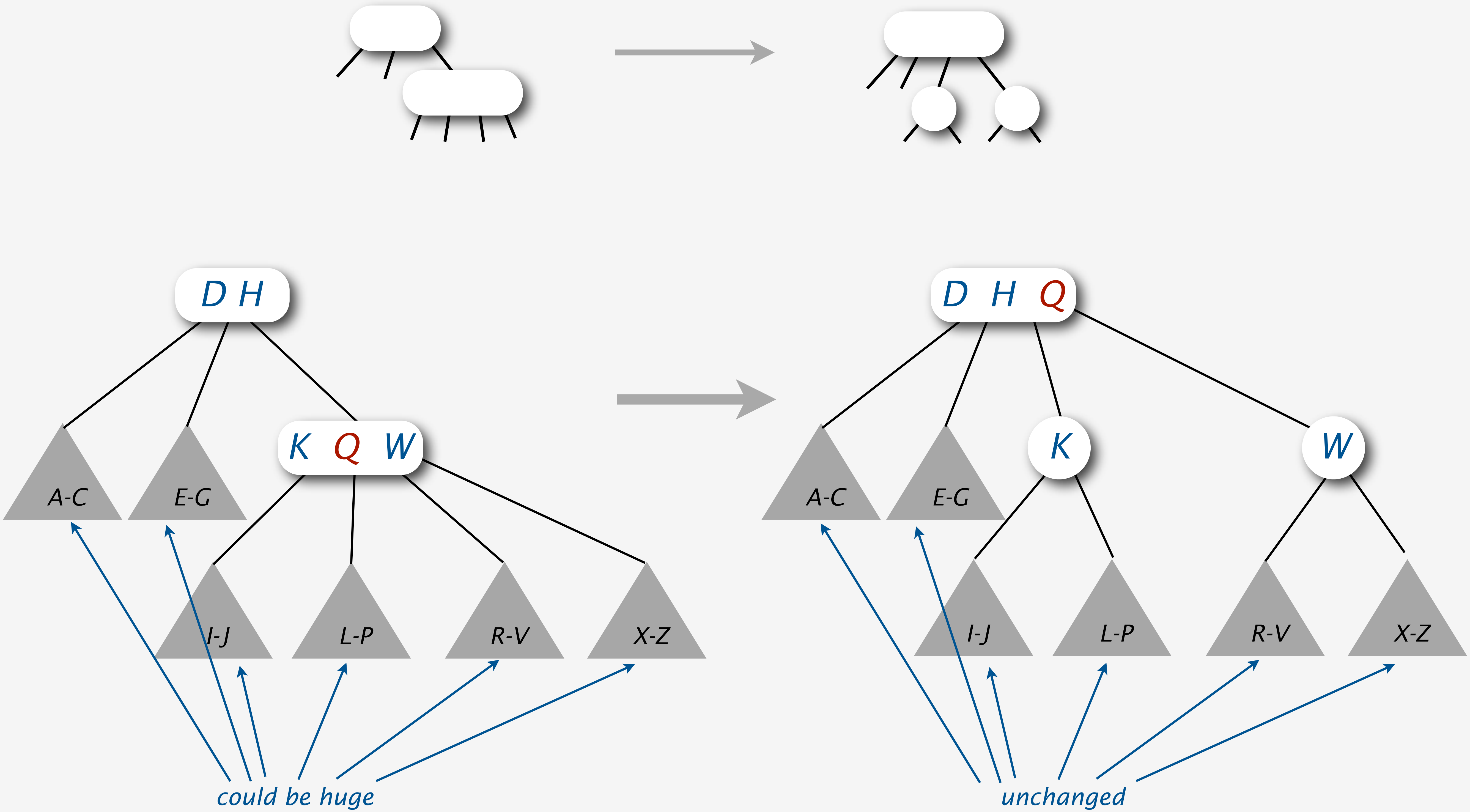
is a local transformation that works anywhere in the tree





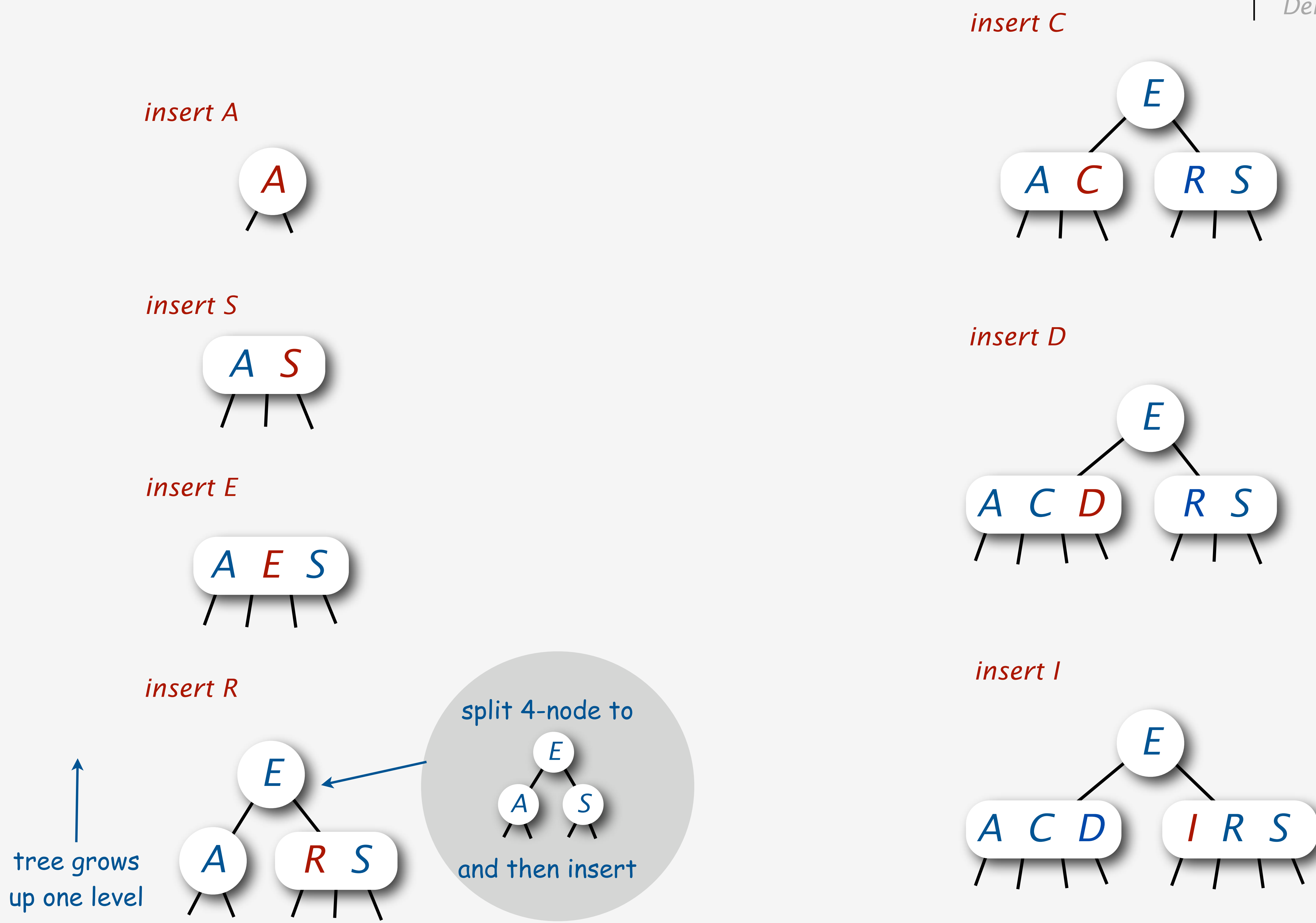
Splitting a 4-node below a 3-node

is a local transformation that works anywhere in the tree



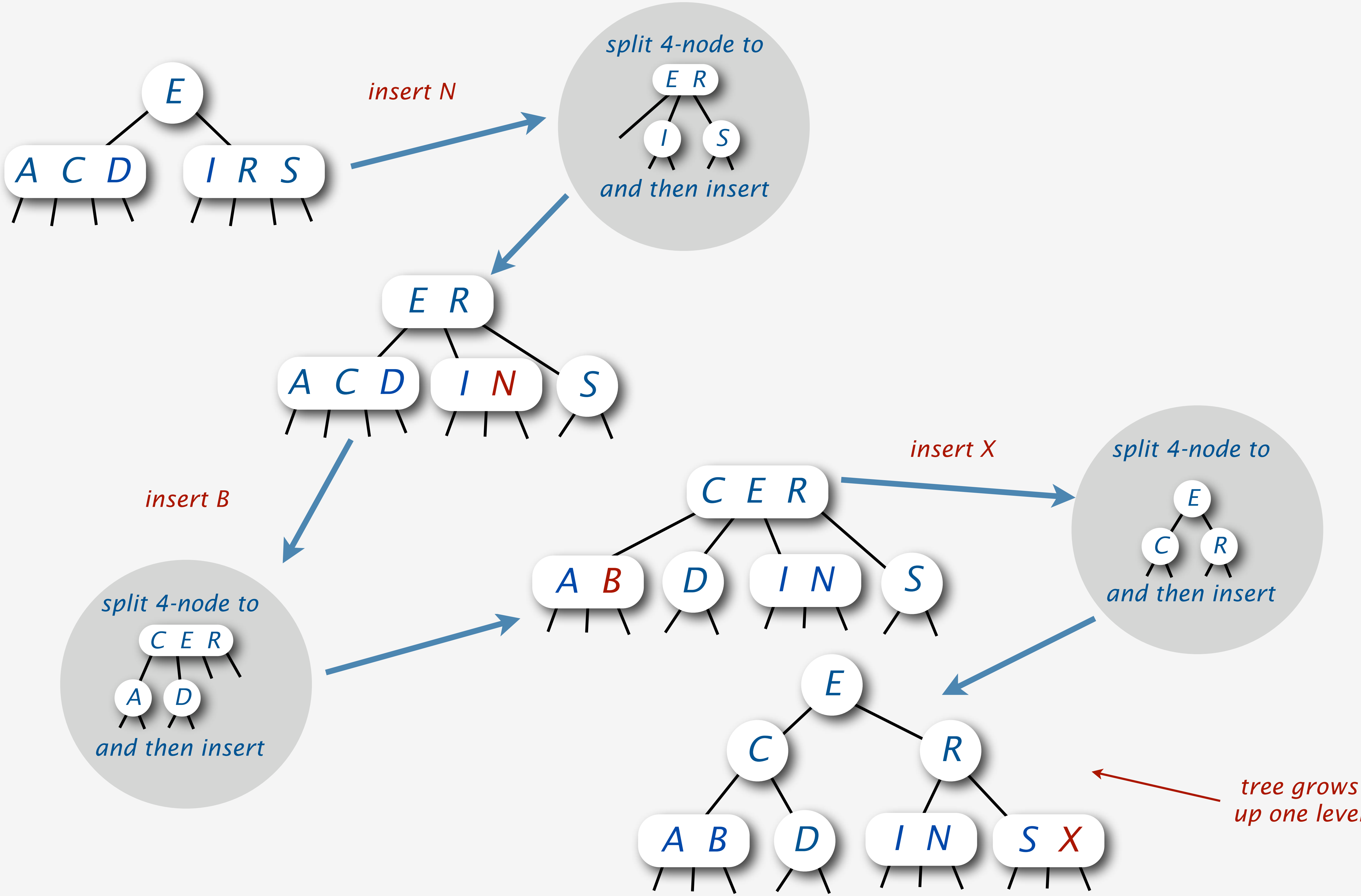
Growth of a 2-3-4 tree

happens upwards from the bottom



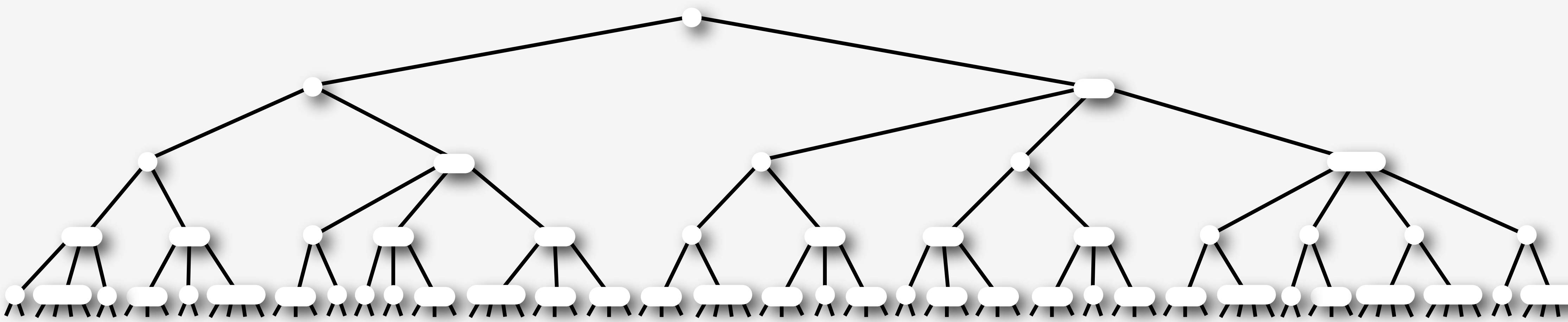
Growth of a 2-3-4 tree (continued)

happens upwards from the bottom



Balance in 2-3-4 trees

Key property: All paths from root to leaf are the same length



Tree height.

- Worst case: lg N [all 2-nodes]
- Best case: log4 N = 1/2 lg N [all 4-nodes]
- Between 10 and 20 for 1 million nodes.
- Between 15 and 30 for 1 billion nodes.

Guaranteed logarithmic performance for both search and insert.

Direct implementation of 2-3-4 trees

is complicated because of code complexity.

Maintaining multiple node types is cumbersome.

- Representation?
- Need multiple compares to move down in tree.
- Large number of cases for splitting.
- Need to convert 2-node to 3-node and 3-node to 4-node.

fantasy code

```
private void insert(Key key, Val val)
{
   Node x = root;
   while (x.getChild(key) != null)
   {
      x = x.getChild(key);
      if (x.is4Node()) x.split();
   }
   if      (x.is2Node()) x.make3Node(key, val);
   else if (x.is3Node()) x.make4Node(key, val);
   return x;
}
```

Bottom line: Could do it, but stay tuned for an easier way.

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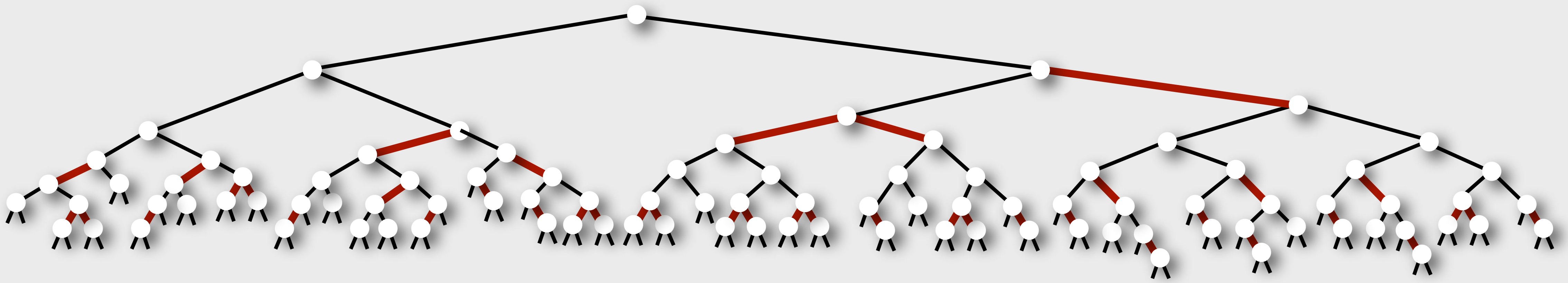
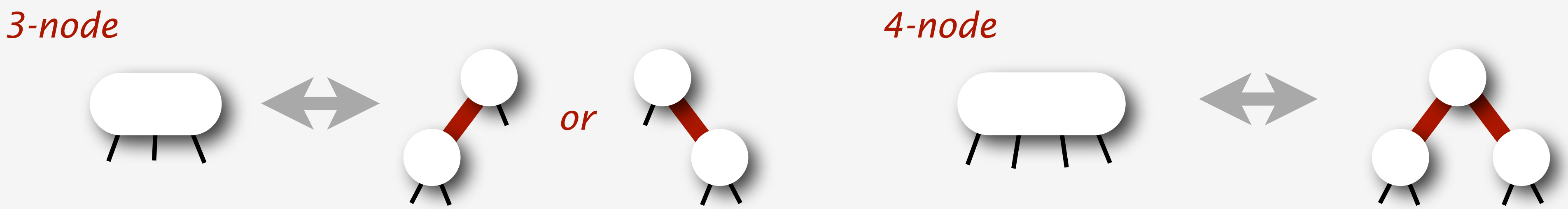


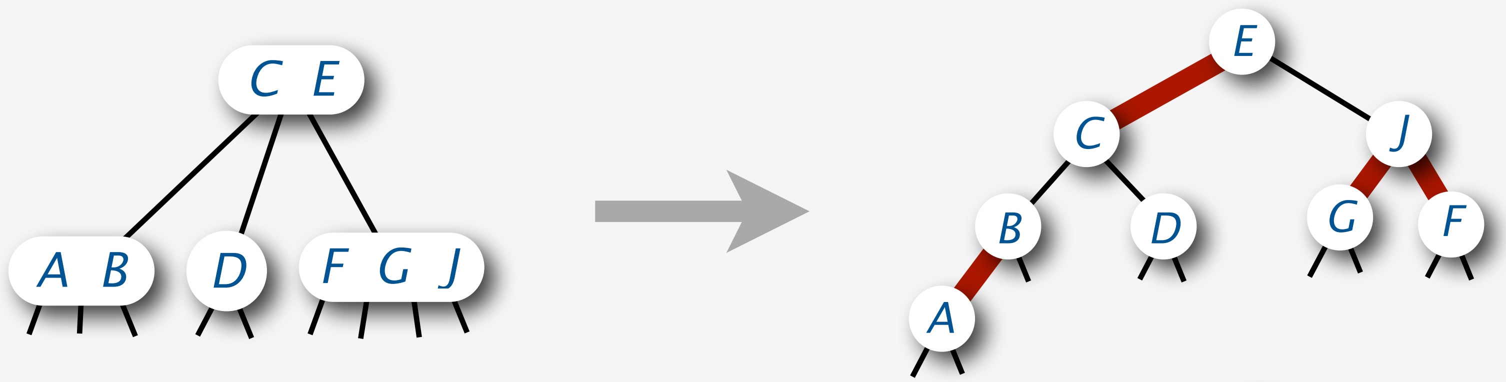
Red-black trees (Guibas-Sedgewick, 1978)

1. Represent 2-3-4 tree as a BST.
2. Use "internal" edges for 3- and 4- nodes.

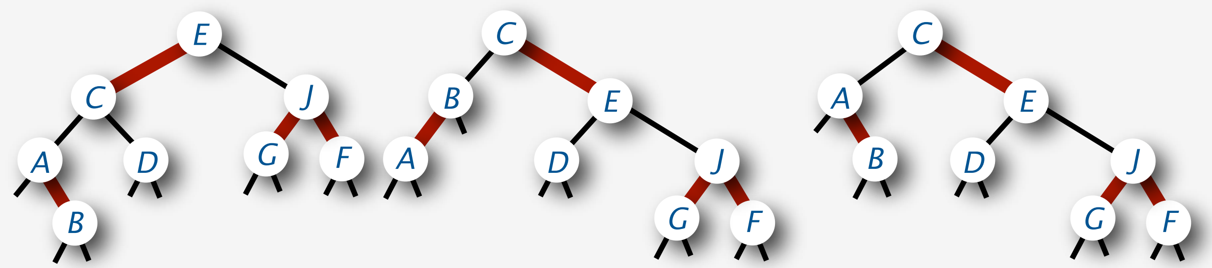


Key Properties

- elementary BST search works
- easy to maintain a correspondence with 2-3-4 trees (and several other types of balanced trees)



Note: correspondence is not 1-1.
(3-nodes can lean either way)



Many variants studied (details omitted.)

NEW VARIANT (this talk): Left-leaning red-black trees

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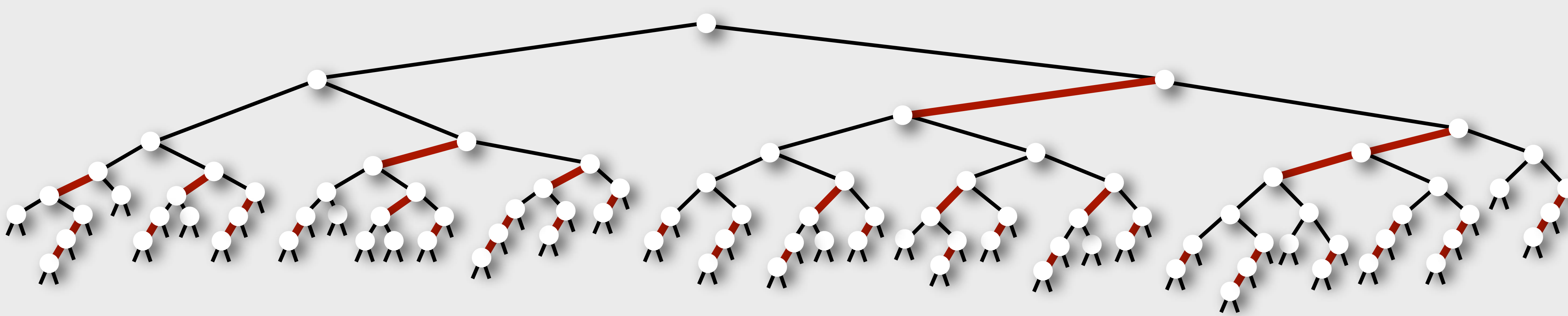
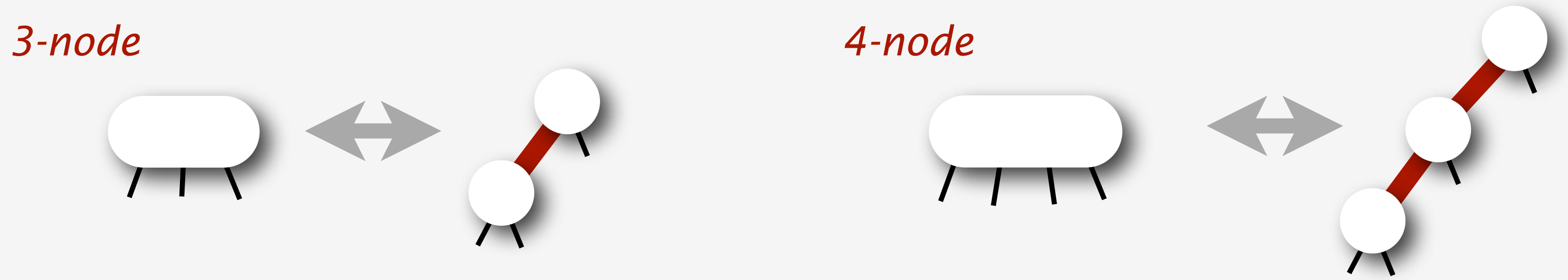


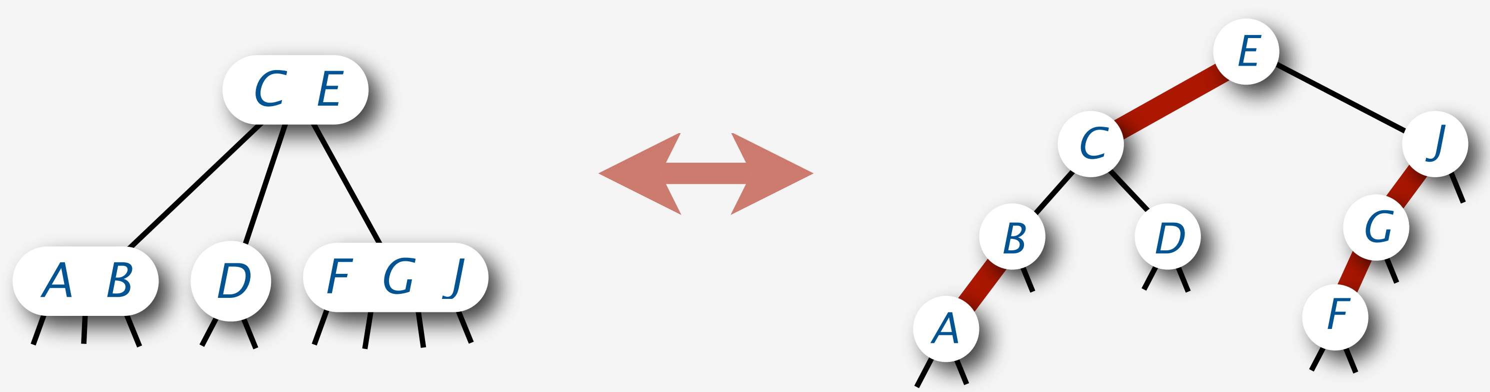
Left-leaning red-black trees

1. Represent 2-3-4 tree as a BST.
2. Use "internal" left-leaning edges for 3- and 4- nodes.



Key Properties

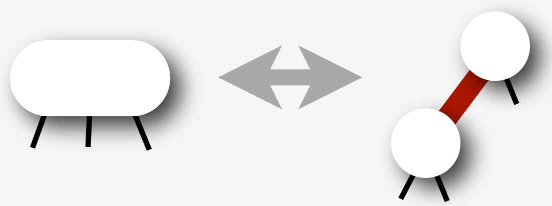
- elementary BST search works
- easy-to-maintain 1-1 correspondence with 2-3-4 trees

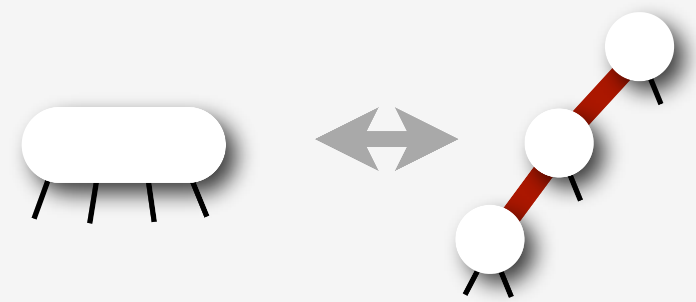


Left-leaning red-black trees

1. Represent 2-3-4 tree as a BST.
2. Use "internal" left-leaning edges for 3- and 4- nodes.

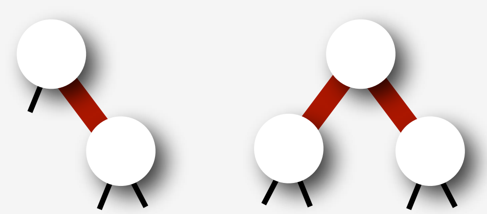
3-node



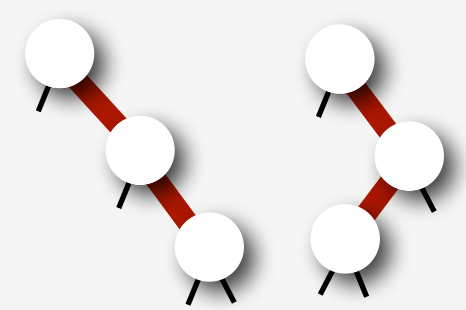


Disallowed

- right-leaning edges

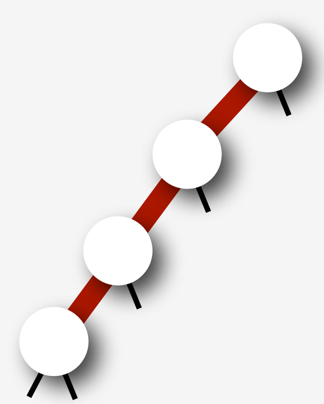


standard red-black trees allow these two



single-rotation trees allow these two

- three reds in a row



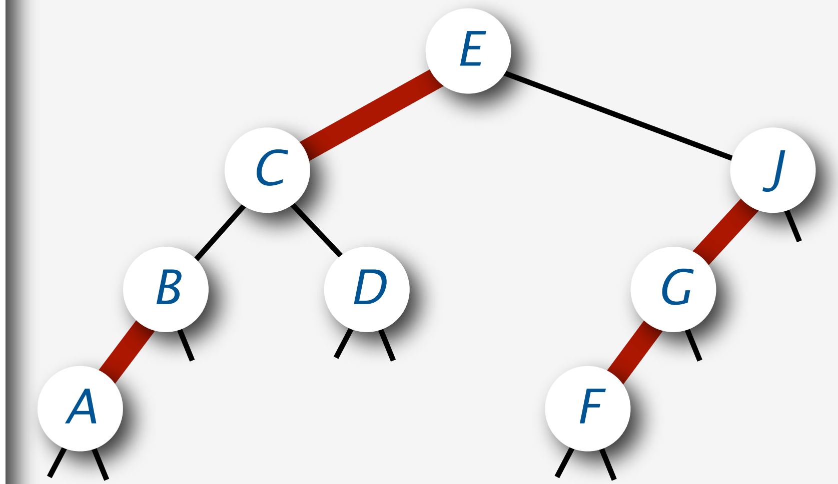
Java data structure for red-black trees

adds one bit for color to elementary BST data structure

```
public class BST<Key extends Comparable<Key>, Value>
{
    private static final boolean RED   = true;
    private static final boolean BLACK = false;
    private Node root;

    private class Node
    {
        Key key;
        Value val;
        Node left, right;
        boolean color;
        Node(Key key, Value val, boolean color)
        {
            this.key   = key;
            this.val = val;
            this.color = color;
        }
    }

   public Value get(Key key)
   // Search method.

   public void put(Key key, Value val)
   // Insert method.
}
```

constants

color of incoming link



helper method to test node color

```
private boolean isRed(Node x)
{
   if (x == null) return false;
   return (x.color == RED);
}
```

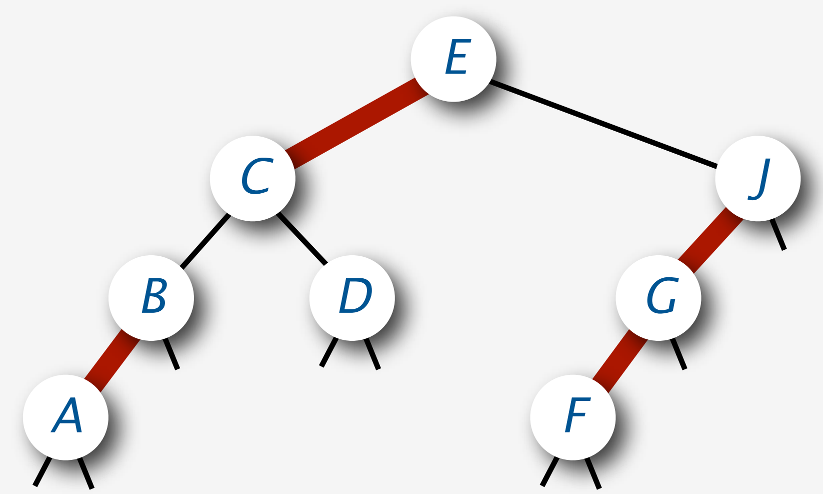
Search implementation for red-black trees

is the same as for elementary BSTs

(but typically runs faster because of better balance in the tree).

BST (and LLRB tree) search implementation

```
public Value get(Key key)
{
   Node x = root;
   while (x != null)
   {
      int cmp = key.compareTo(x.key);
      if (cmp == 0)      return x.val;
      else if (cmp < 0) x = x.left;
      else if (cmp > 0) x = x.right;
   }
   return null;
}
```



Note: Other BST methods also work

- order statistics
- iteration

Ex: Find the minimum key

```
public Key min()
{
   Node x = root;
   while (x != null) x = x.left;
   if (x == null) return null;
   else           return x.key;
}
```

Insert implementation for LLRB trees

is best expressed in a recursive implementation

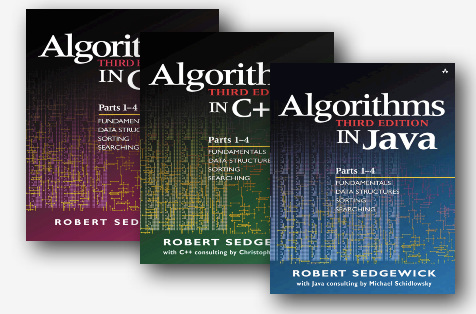
Recursive insert() *implementation for elementary BSTs*

```
private Node insert(Node h, Key key, Value val)
{
   if (h == null)
      return new Node(key, val);

   int cmp = key.compareTo(h.key);
   if (cmp == 0) h.val = val;
   else if (cmp < 0)
      h.left = insert(h.left, key, val);
   else
      h.right = insert(h.right, key, val);

   return h;
}
```

associative model (no duplicate keys)

Nonrecursive

Recursive



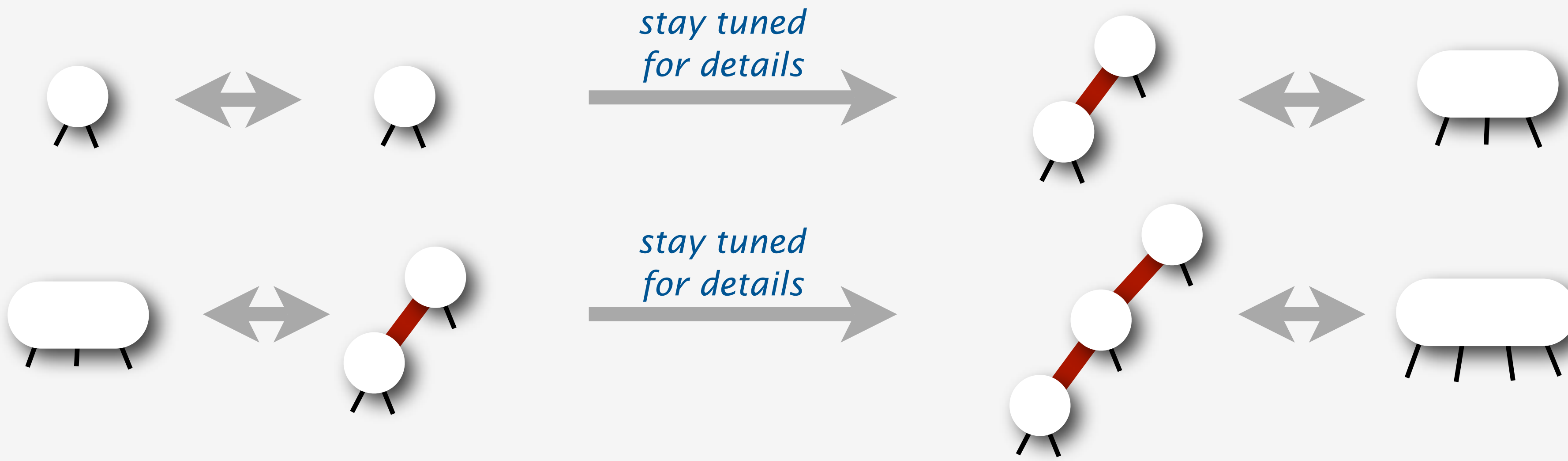
Note: effectively travels down the tree and then up the tree.

- simplifies correctness proof
- simplifies code for balanced BST implementations
- could remove recursion to get single-pass algorithm

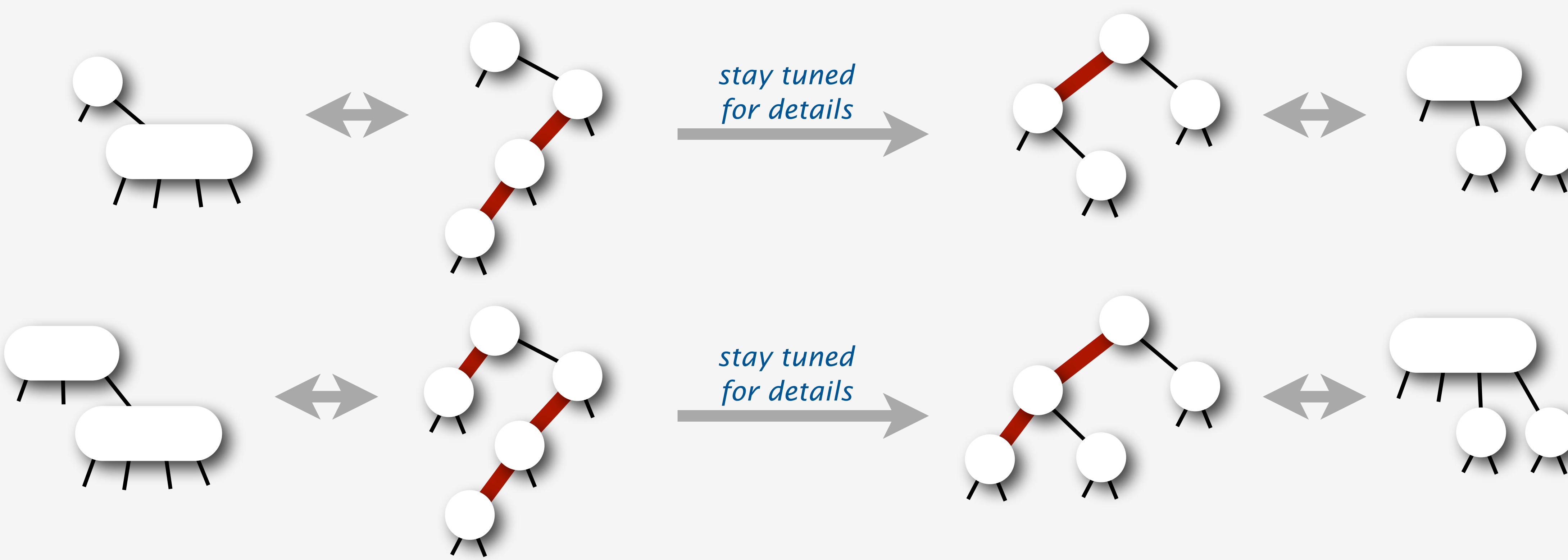
Insert implementation for LLRB trees

follows directly from 1-1 correspondence with 2-3-4 trees

1. If key found on recursive search, reset value, as usual.
2. If key not found, insert at the bottom.



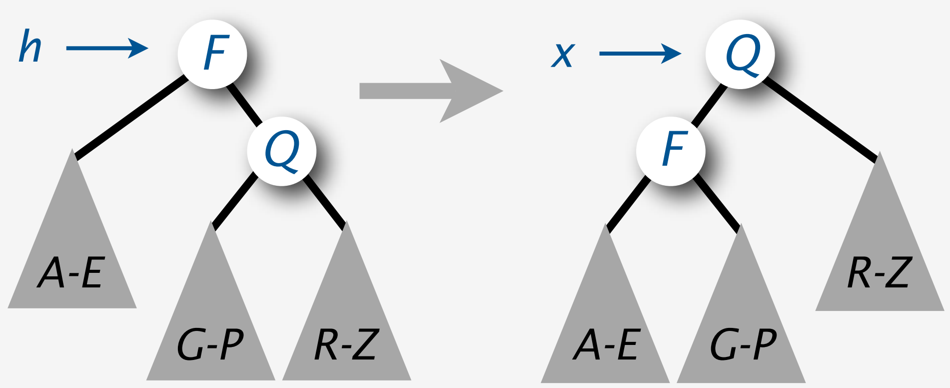
3. Split 4-nodes on the way down



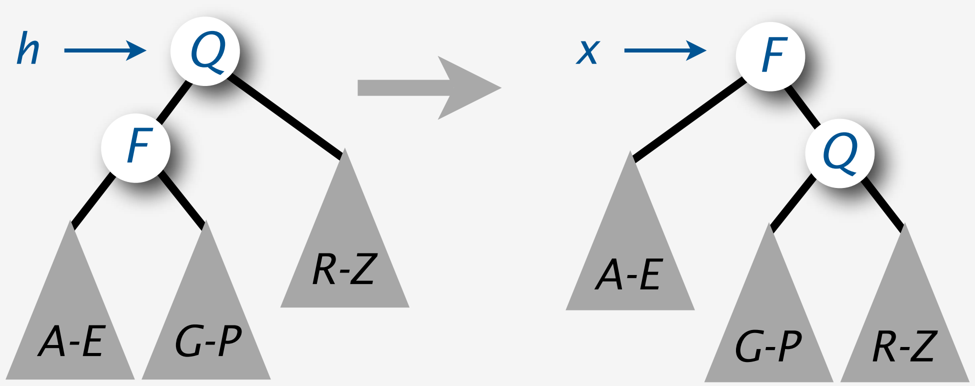
Balanced tree code

is based on local transformations known as rotations

```
private Node rotL(Node h)
{
   Node x = h.right;
   h.right = x.left;
   x.left = h;
   return x;
}
```



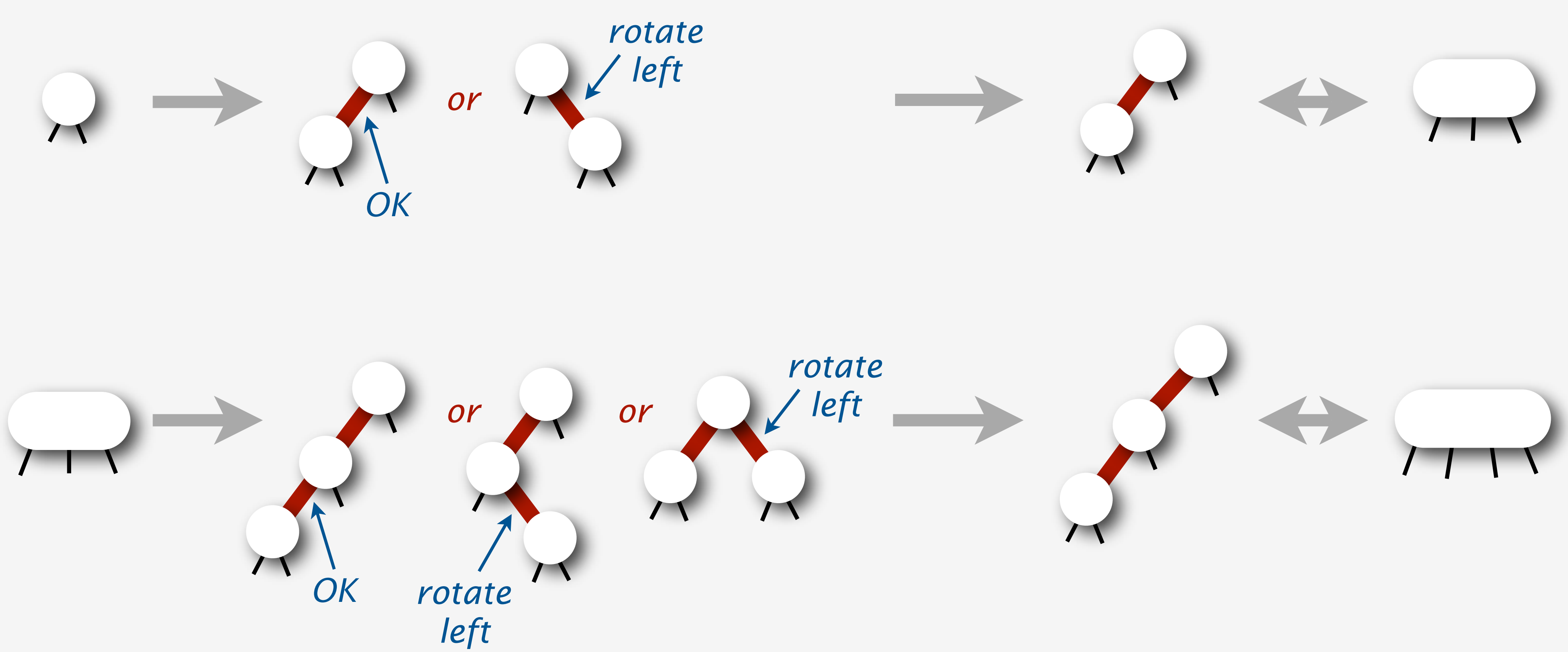

```
private Node rotR(Node h)
{
   Node x = h.left;
   h.left = x.right;
   x.right = h;
   return x;
}
```



Insert a new node at the bottom in a LLRB tree

follows directly from 1-1 correspondence with 2-3-4 trees

1. Add new node as usual, with red link to glue it to node above
2. Rotate left if necessary to make link lean left

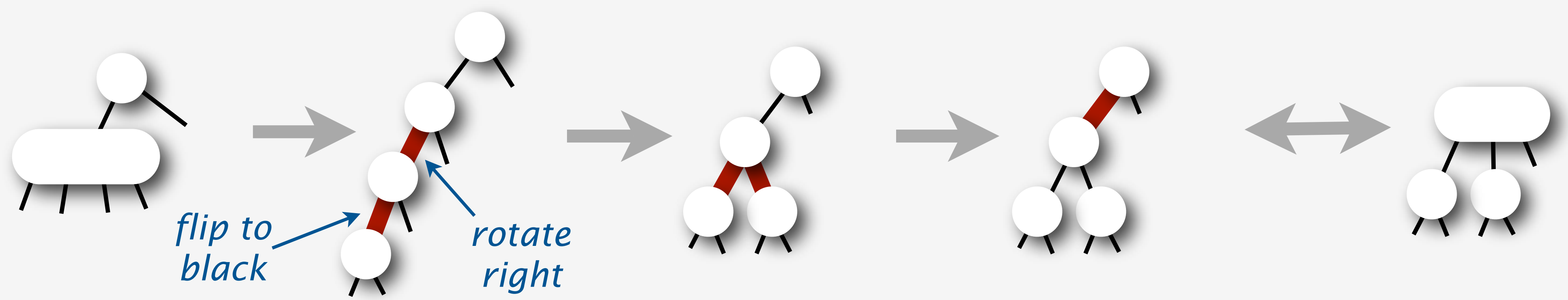


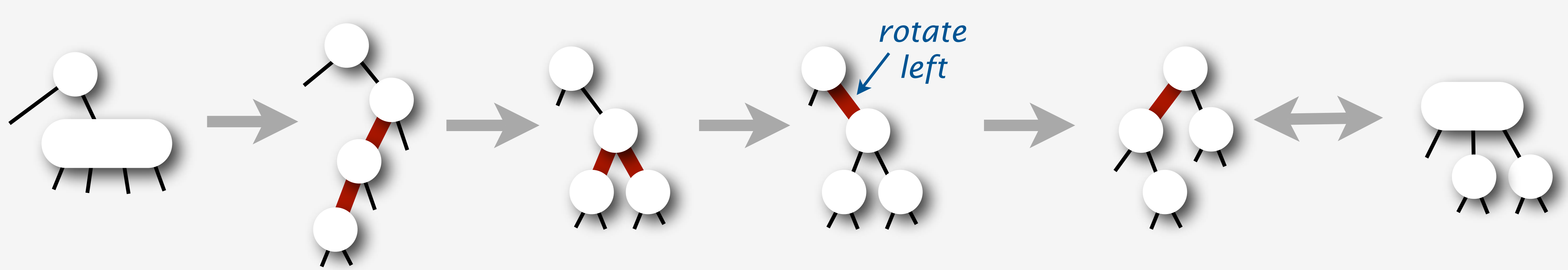
Splitting a 4-node in a LLRB tree

follows directly from 1-1 correspondence with 2-3-4 trees

1. Rotate right to balance the 4-node
2. Flip colors to pass red link up one level
3. Rotate left if necessary to make link lean left

Parent is a 2-node: two cases

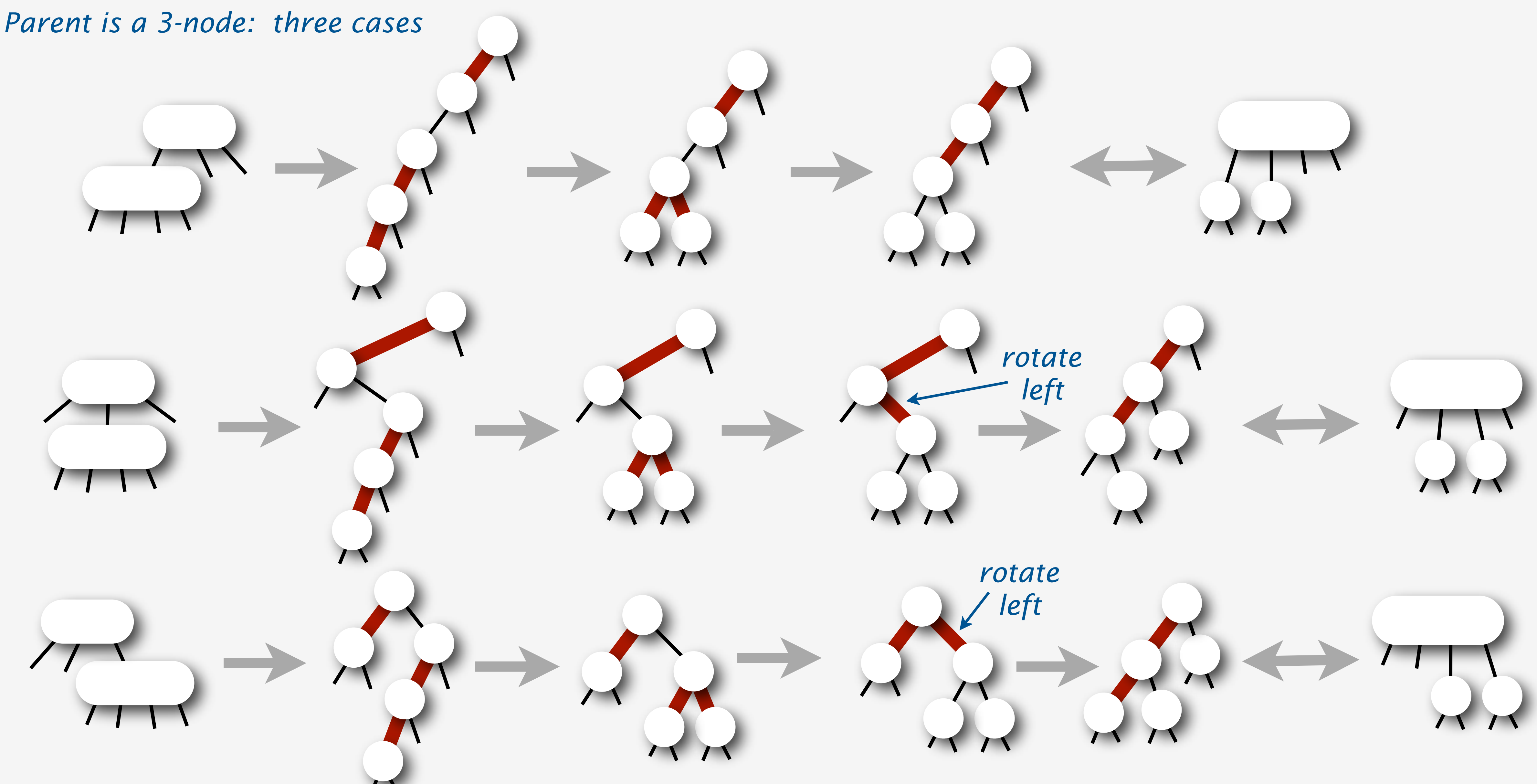




Splitting a 4-node in a LLRB tree

follows directly from 1-1 correspondence with 2-3-4 trees

1. Rotate right to balance the 4-node
2. Flip colors to pass red link up one level
3. Rotate left if necessary to make link lean left

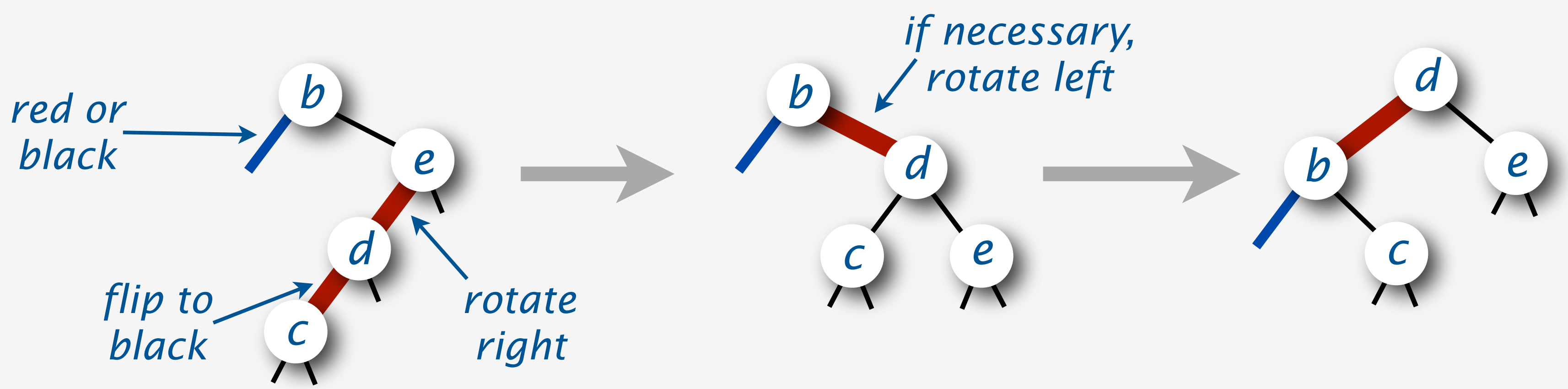


Splitting a 4-node in a LLRB tree

follows directly from 1-1 correspondence with 2-3-4 trees

1. Rotate right to balance the 4-node
2. Flip colors to pass red link up one level
3. Rotate left if necessary to make link lean left

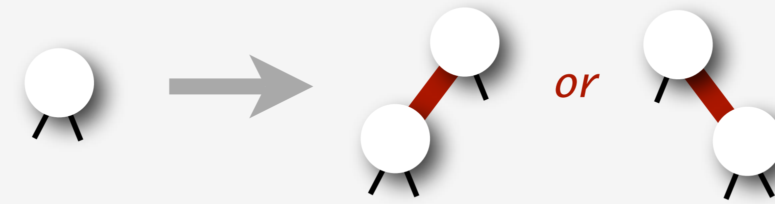
Key point: The transformations are all the same.



Inserting and splitting nodes in LLRB trees

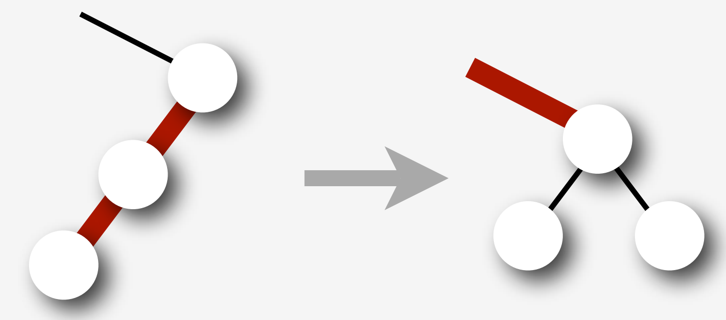
are easier when left rotates are done on the way up the tree.

Search as usual



- if key found reset value, as usual
- if key not found insert a new red node at the bottom [might be right-leaning red link]

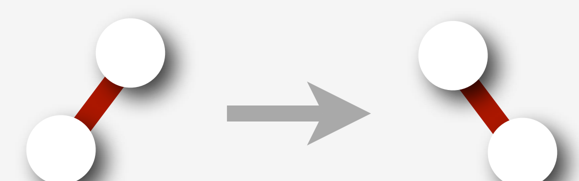
Split 4-nodes on the way down the tree.



- right-rotate and flip color
- might leave right-leaning link higher up in the tree

NEW TRICK: enforce left-leaning condition on the way up the tree.

- left-rotate any right-leaning link on search path
- trivial with recursion (do it after recursive calls)
- no other right-leaning links elsewhere

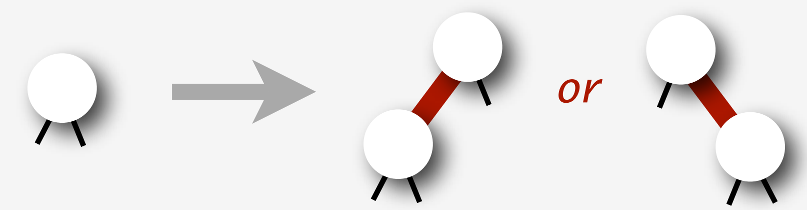


Insert code for LLRB trees

is based on three simple operations.

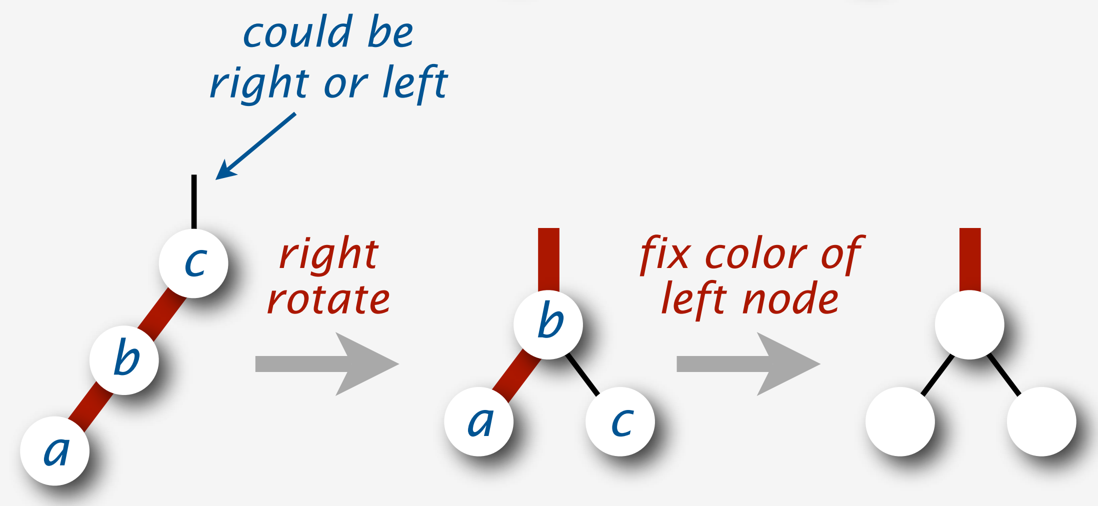
1. Insert a new node at the bottom.

```
if (h == null)
      return new Node(key, value, RED);
```



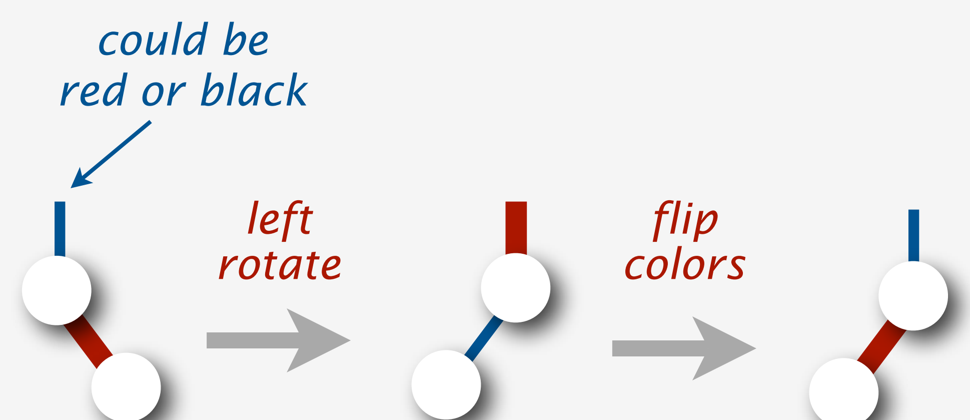
2. Split a 4-node.

```
private Node splitFourNode(Node h)
{
   x = rotR(h);
   x.left.color  = BLACK;
   return x;
}
```



3. Enforce left-leaning condition.

```
private Node leanLeft(Node h)
{
   x = rotL(h);
   x.color      = x.left.color;
   x.left.color = RED;
   return x;
}
```



Insert implementation for LLRB trees

is a few lines of code added to elementary BST insert

```
private Node insert(Node h, Key key, Value val)
{
   if (h == null)
      return new Node(key, val, RED);

   if (isRed(h.left))
      if (isRed(h.left.left))
         h = splitFourNode(h);

   int cmp = key.compareTo(h.key);
   if (cmp == 0) h.val = val;
   else if (cmp < 0)
      h.left = insert(h.left, key, val);
   else
      h.right = insert(h.right, key, val);

   if (isRed(h.right))
      h = leanLeft(h);

   return h;
}
```

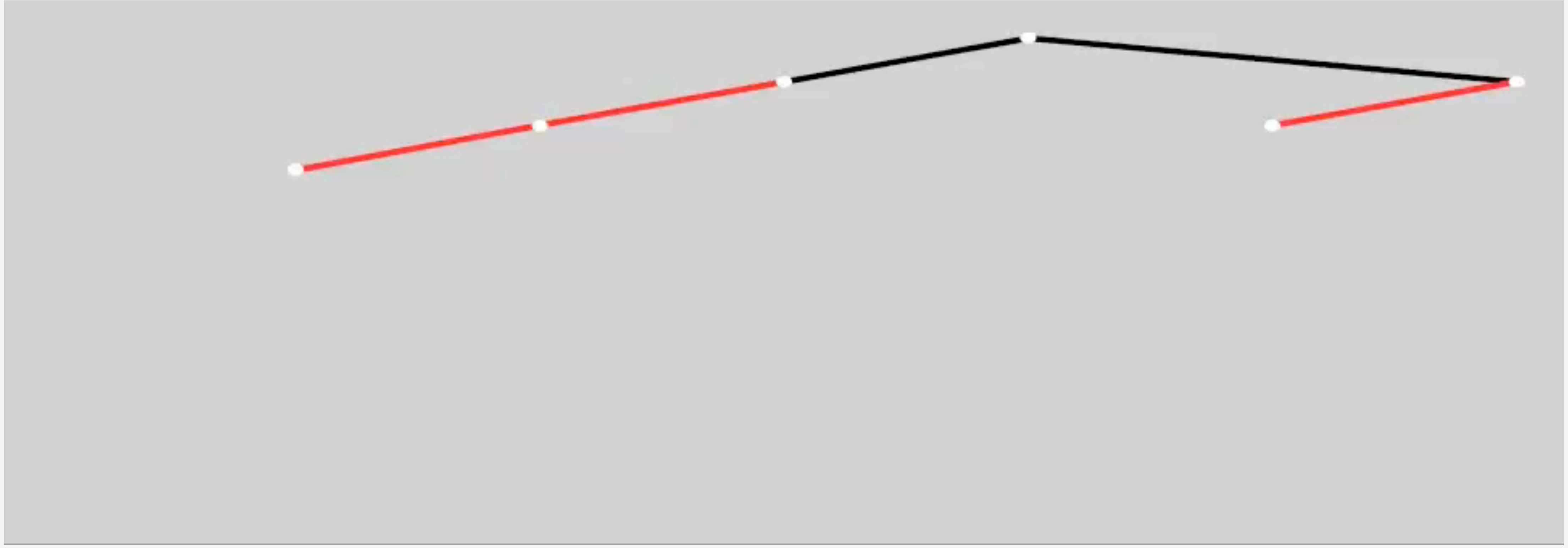
insert at the bottom

split 4-nodes on the way down

standard BST insert code

fix right-leaning reds on the way up

LLRB insert movie



Why revisit red-black trees?

Take your pick:


```
private Node insert(Node x, Key key, Value val, boolean sw)
{
   if (x == null)
      return new Node(key, value, RED);
   int cmp = key.compareTo(x.key);

   if (isRed(x.left) && isRed(x.right))
   {
      x.color = RED;
      x.left.color  = BLACK;
      x.right.color = BLACK;
   }
   if (cmp == 0) x.val = val;
   else if (cmp < 0))
   {
     x.left = insert(x.left, key, val, false);
     if (isRed(x) && isRed(x.left) && sw)
        x = rotR(x);
     if (isRed(x.left) && isRed(x.left.left))
      {
         x = rotR(x);
         x.color = BLACK; x.right.color = RED;
      }
   }
   else // if (cmp > 0)
   {
      x.right = insert(x.right, key, val, true);
      if (isRed(h) && isRed(x.right) && !sw)
         x = rotL(x);
      if (isRed(h.right) && isRed(h.right.right))
      {
         x = rotL(x);
         x.color = BLACK; x.left.color = RED;
      }
   }
   return x;
}
```

very tricky

```
private Node insert(Node h, Key key, Value val)
{
    int cmp = key.compareTo(h.key);
    if (h == null)
      return new Node(key, val, RED);
    if (isRed(h.left))
      if (isRed(h.left.left))
      {
         h = rotR(h);
         h.left.color  = BLACK;
      }
   if (cmp == 0) x.val = val;
   else if (cmp < 0)
      h.left = insert(h.left, key, val);
   else
      h.right = insert(h.right, key, val);
   if (isRed(h.right))
   {
      h = rotL(h);
      h.color      = h.left.color;
      h.left.color = RED;
   }
   return h;
}
```

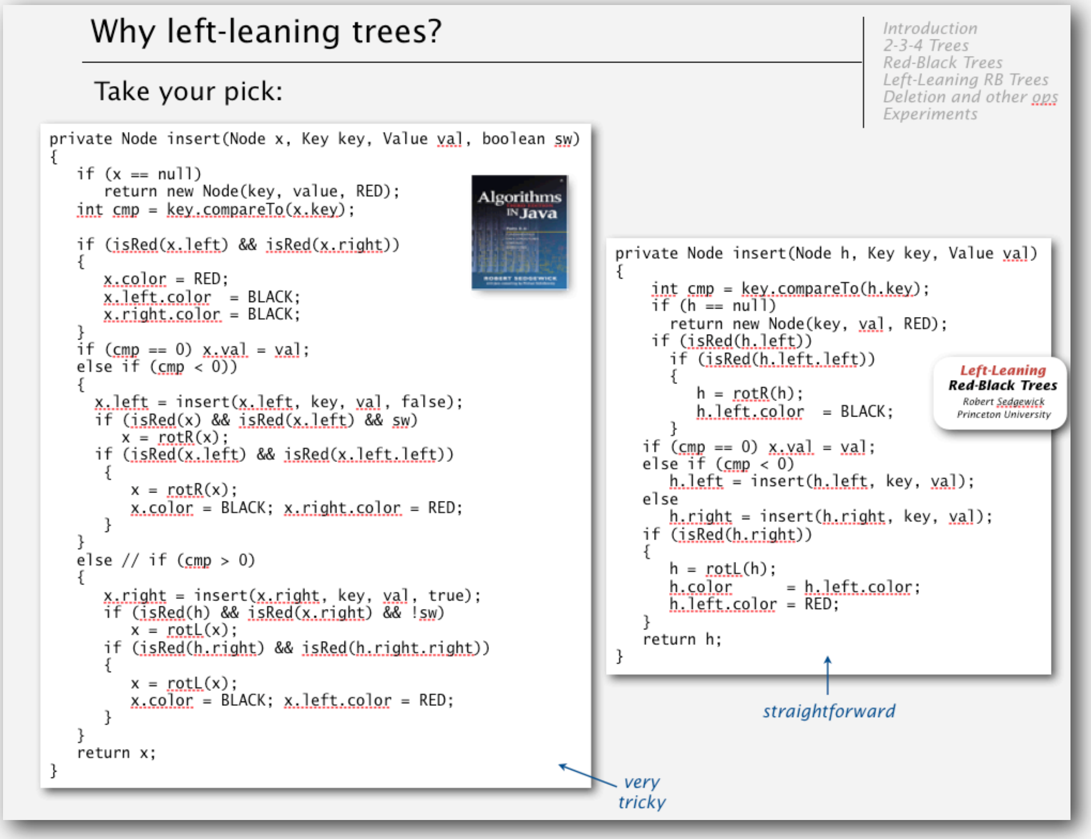
Left-Leaning Red-Black Trees
Robert Sedgewick
Princeton University

straightforward

Why revisit red-black trees?

Take your pick:

wrong scale!



Why left-leaning trees?

Take your pick:

```
private Node insert(Node x, Key key, Value val, boolean sw)
{
   if (x == null)
      return new Node(key, value, RED);
   int cmp = key.compareTo(x.key);

   if (isRed(x.left) && isRed(x.right))
   {
      x.color = RED;
      x.left.color  = BLACK;
      x.right.color = BLACK;
   }
   if (cmp == 0) x.val = val;
   else if (cmp < 0))
   {
     x.left = insert(x.left, key, val, false);
     if (isRed(x) && isRed(x.left) && sw)
        x = rotR(x);
     if (isRed(x.left) && isRed(x.left.left))
      {
         x = rotR(x);
         x.color = BLACK; x.right.color = RED;
      }
   }
   else // if (cmp > 0)
   {
      x.right = insert(x.right, key, val, true);
      if (isRed(h) && isRed(x.right) && !sw)
         x = rotL(x);
      if (isRed(h.right) && isRed(h.right.right))
      {
         x = rotL(x);
         x.color = BLACK; x.left.color = RED;
      }
   }
   return x;
}
```

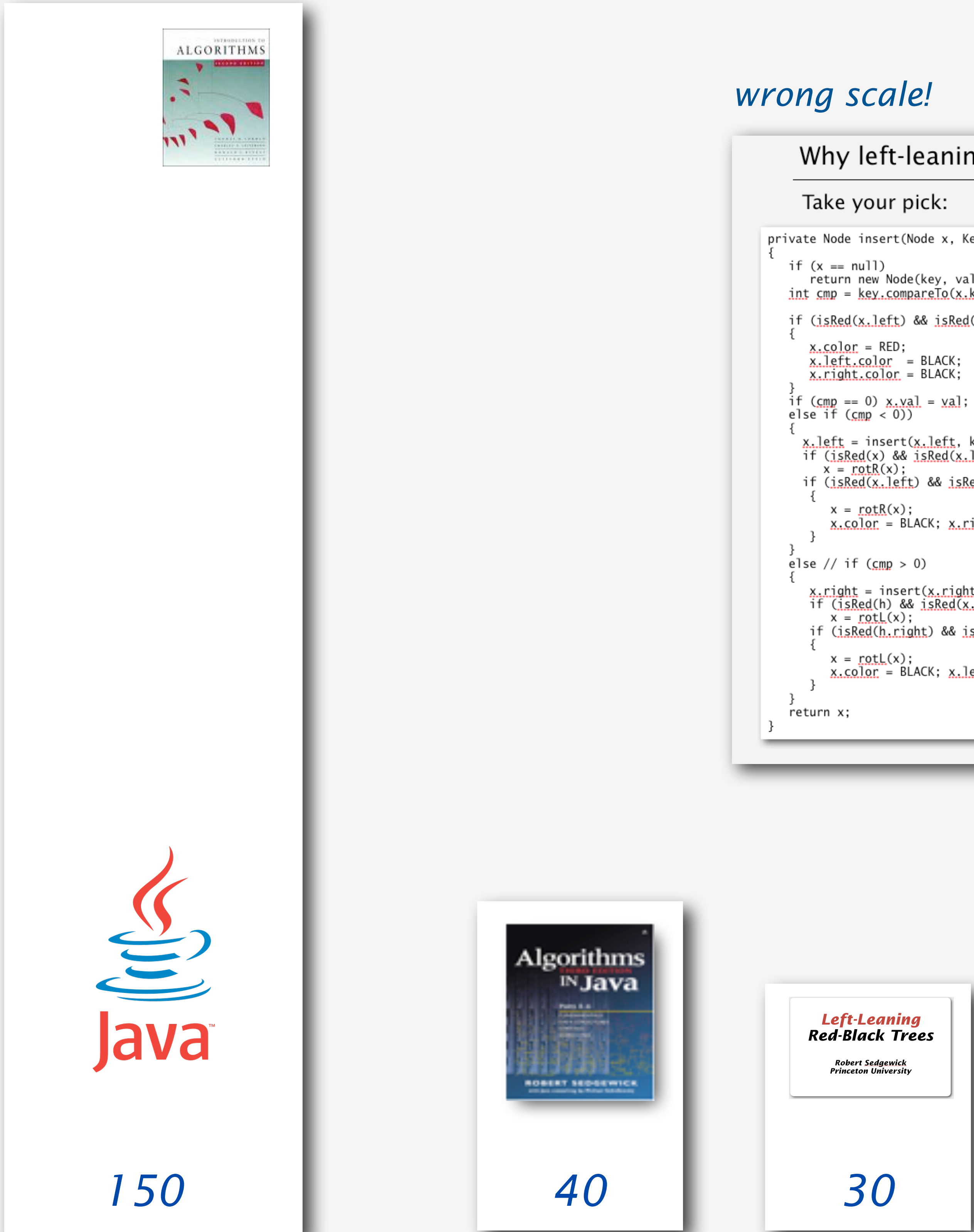
very tricky

```
private Node insert(Node h, Key key, Value val)
{
    int cmp = key.compareTo(h.key);
    if (h == null)
      return new Node(key, val, RED);
    if (isRed(h.left))
      if (isRed(h.left.left))
      {
         h = rotR(h);
         h.left.color  = BLACK;
      }
   if (cmp == 0) x.val = val;
   else if (cmp < 0)
      h.left = insert(h.left, key, val);
   else
      h.right = insert(h.right, key, val);
   if (isRed(h.right))
   {
      h = rotL(h);
      h.color      = h.left.color;
      h.left.color = RED;
   }
   return h;
}
```

straightforward

TreeMap.java

Adapted from CLR by experienced professional programmers (2004)

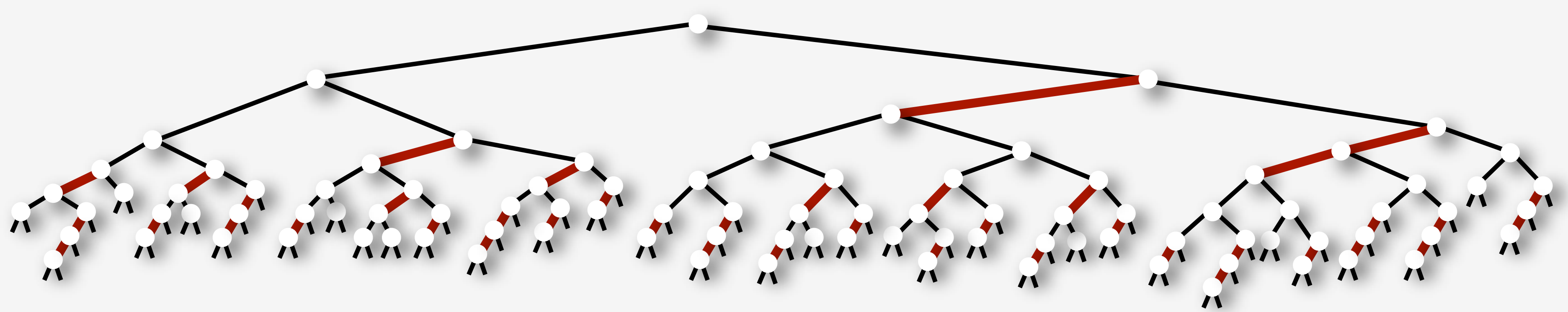


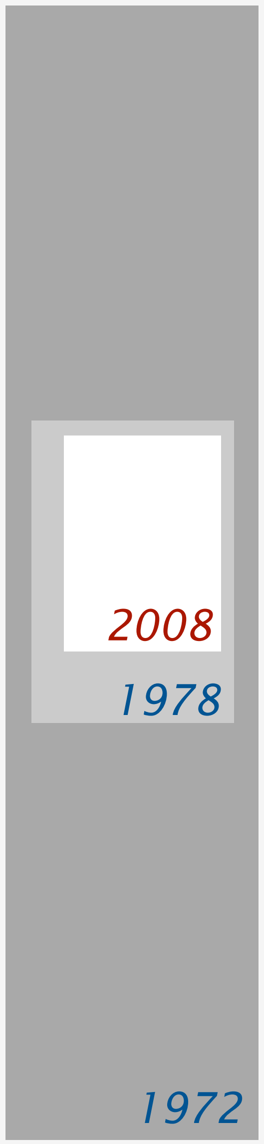
lines of code for insert (lower is better!)

Why revisit red-black trees?

LLRB implementation is far simpler than previous attempts.

- left-leaning restriction reduces number of cases
- recursion gives two (easy) chances to fix each node
- short inner loop more than compensates for slight increase in height





Improves widely used algorithms

- AVL, 2-3, and 2-3-4 trees
- red-black trees

Same ideas simplify implementation of other operations

- delete min, max
- arbitrary delete

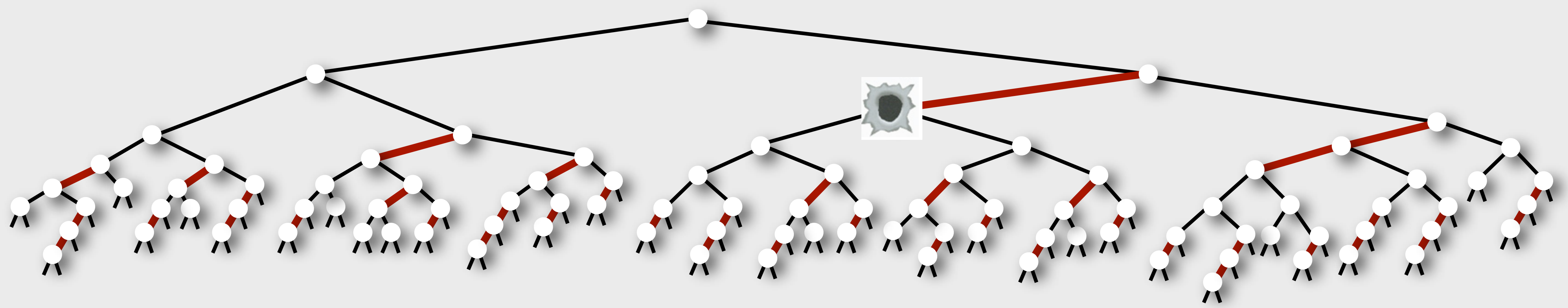
Introduction

2-3-4 Trees

Red-Black Trees

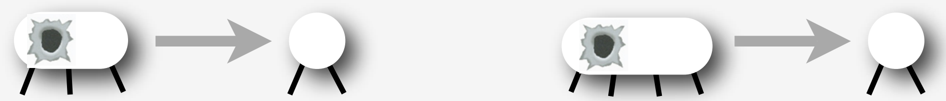
Left-Leaning RB Trees

Deletion

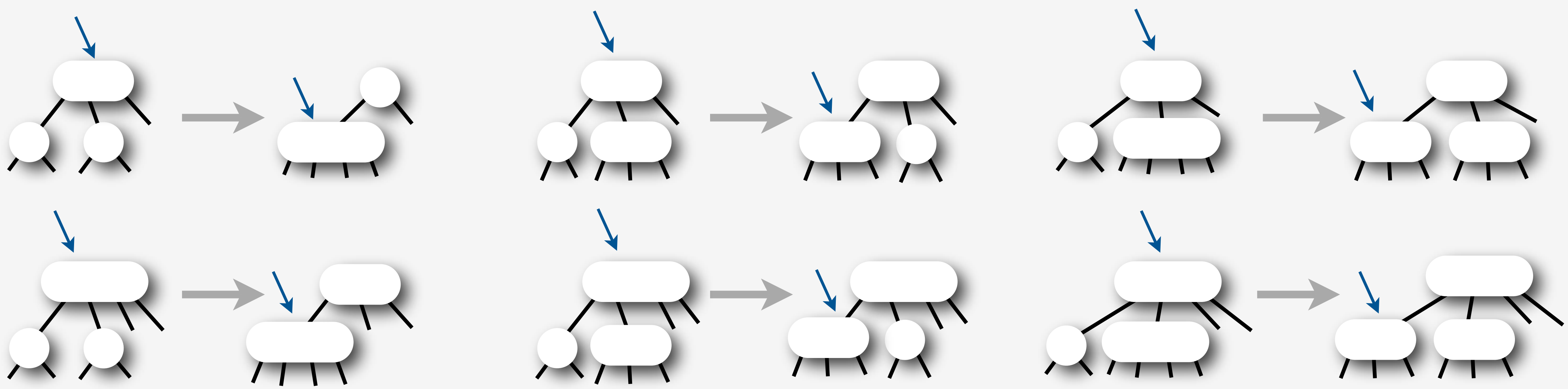


Warmup 1: delete the minimum

1. Search down the left spine of the tree.
2. If search ends in a 3-node or 4-node: just remove it.



3. Removing a 2-node would destroy balance
   - transform tree on the way down the search path
   - Invariant: current node is not a 2-node

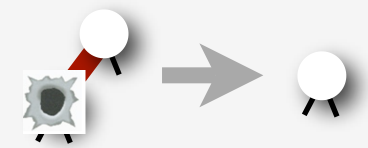


Note: LLRB representation reduces number of cases (as for insert)

Warmup 1: delete the minimum

Carry a red link down the left spine of the tree.

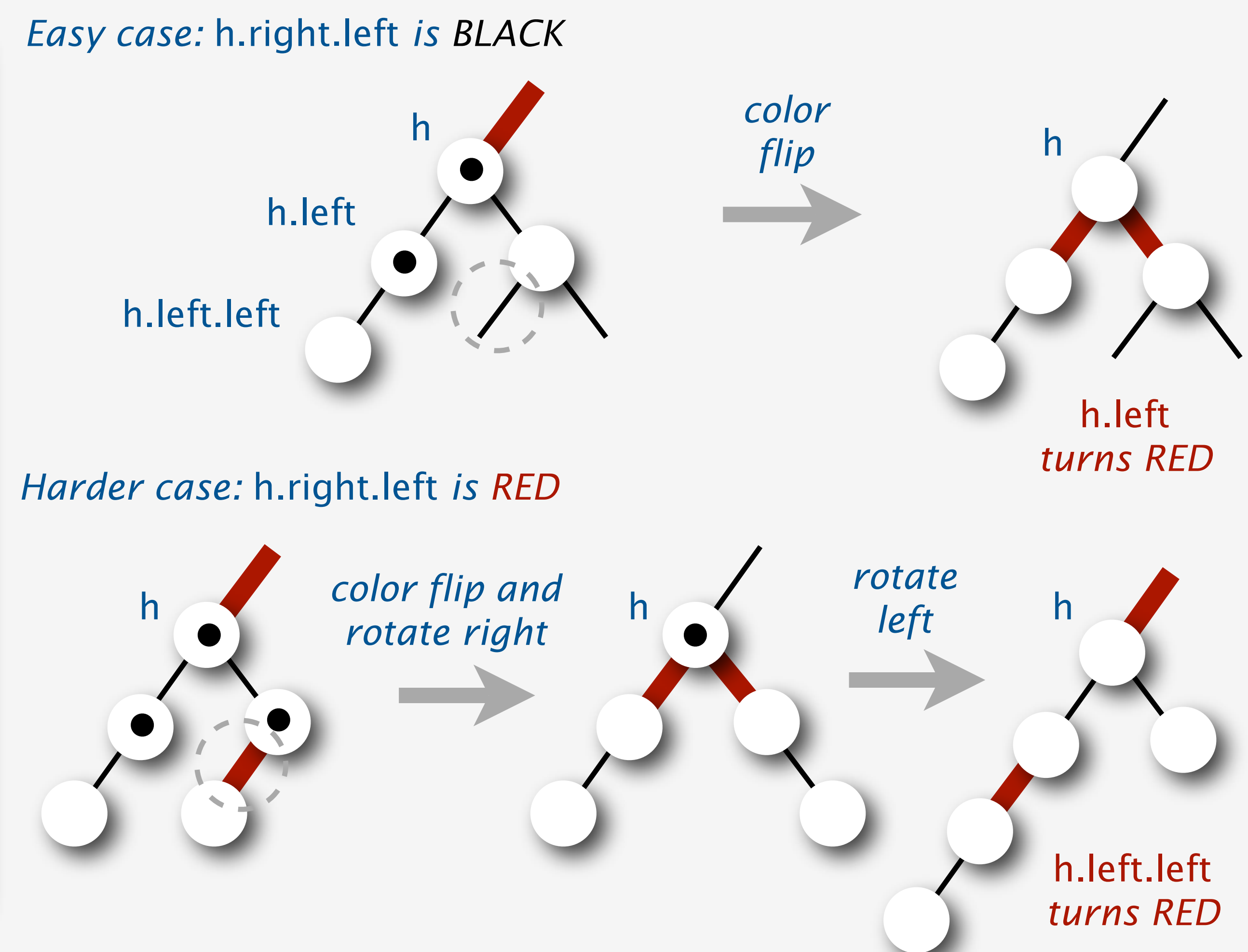
Invariant: either h or h.left is RED

Implication: deletion easy at bottom 

Need to adjust tree only when h.left and h.left.left are both BLACK

Two cases, depending on color of h.right.left

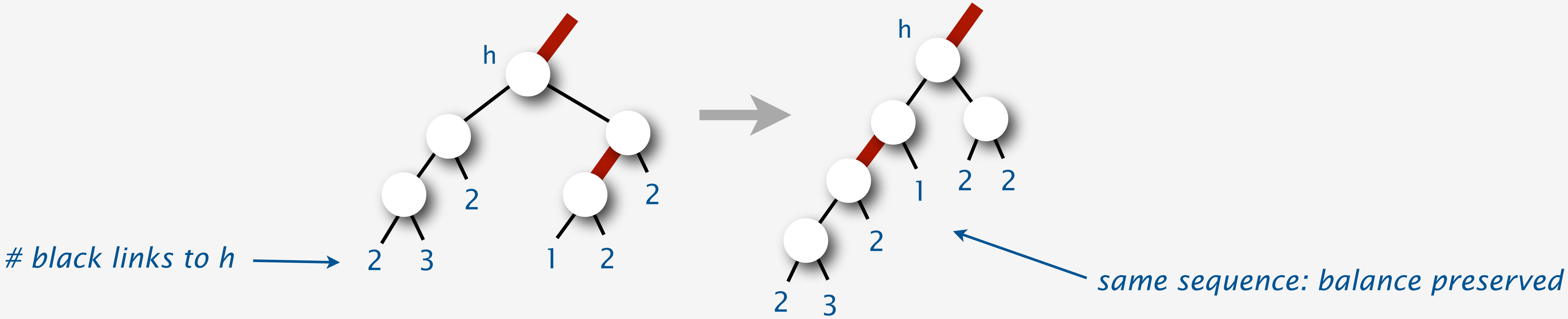
```
private Node moveRedLeft(Node h)
{
   h.color      = BLACK;
   h.left.color = RED;
   if (isRed(h.right.left))
   {
      h.right = rotR(h.right);
      h = rotL(h);
   }
   else h.right.color = RED;

   return h;
}
```



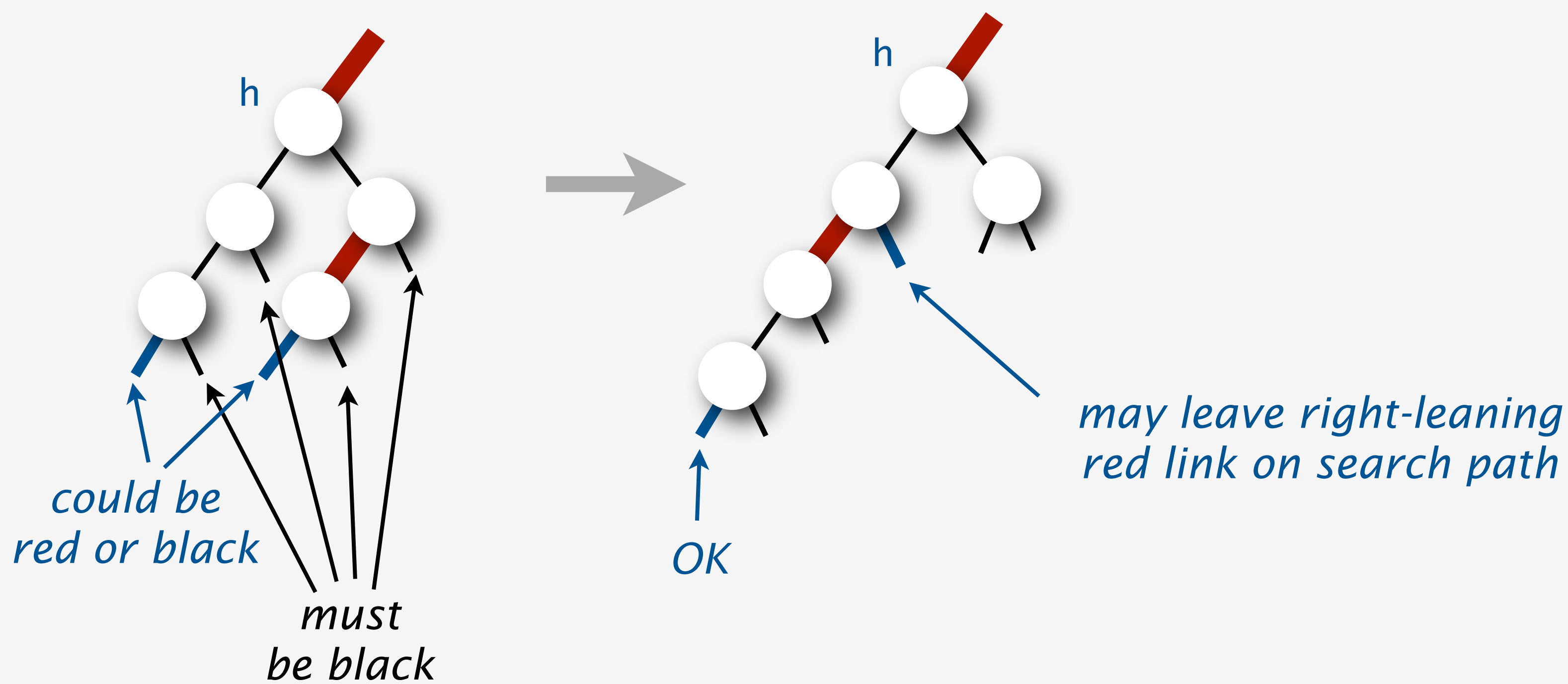
Leaving right red links on the search path

simplifies the code, complicates the proof.

1. Does each transformation preserve balance?



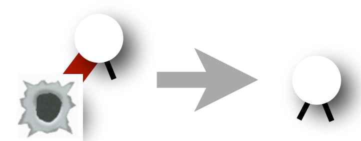
2. Does each transformation preserve correspondence with 2-3-4 trees?



deleteMin() implementation for LLRB trees

is otherwise a few lines of code

```
public void deleteMin()
{
  root = deleteMin(root);
  root.color = BLACK;
}

private Node deleteMin(Node h)
{
   if (h.left == null)
      return null;
```



remove node on bottom level
(h must be RED *by invariant)*

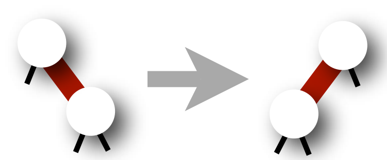
```
   if (!isRed(h.left) && !isRed(h.left.left))
      h = moveRedLeft(h);
```

push red link down if necessary

```
   h.left = deleteMin(h.left);
```

move down one level

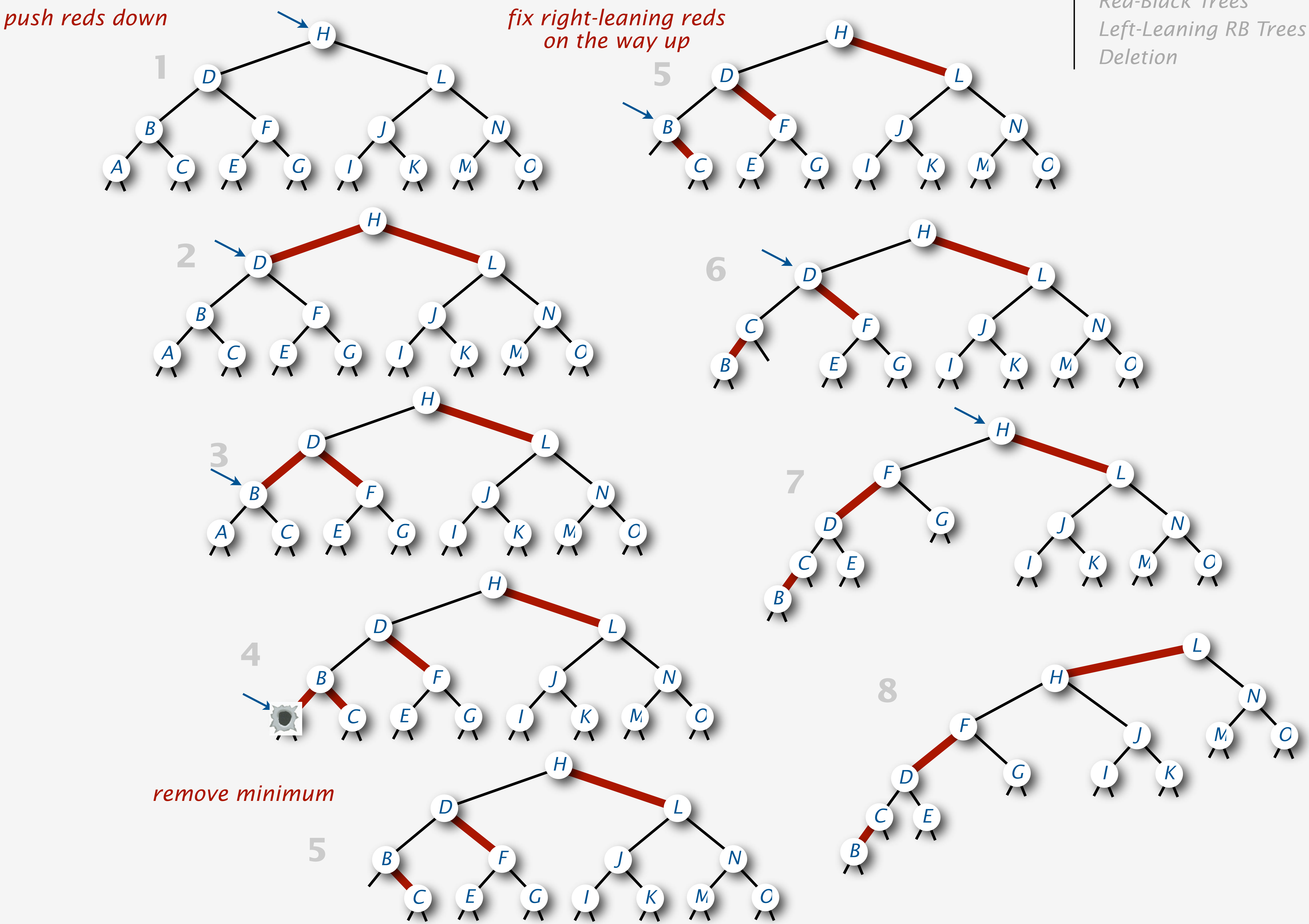
```
   if (isRed(h.right))
      h = leanLeft(h);
```



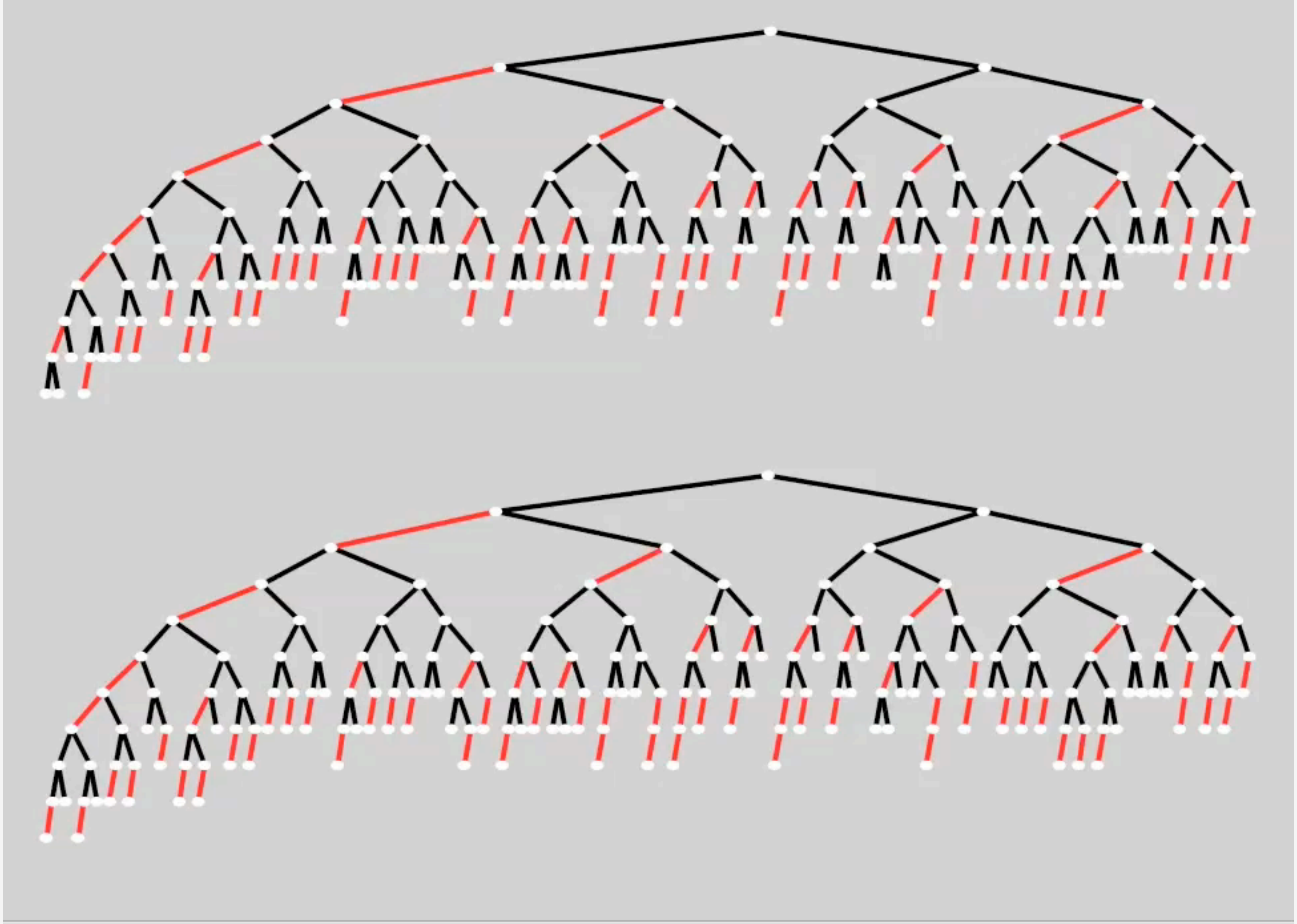
fix right-leaning red links
on the way up the tree

```
   return h;
}
```

deleteMin() example



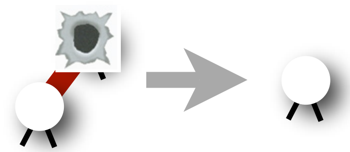
LLRB deleteMin() movie

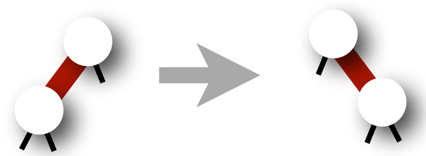


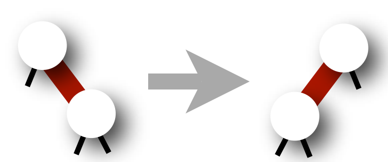
Warmup 2: delete the maximum

is similar, but slightly different (since trees lean left).

```
private Node deleteMax(Node h)
{
   if (h.right == null)
   {
      if (h.left != null)
         h.left.color = BLACK;
      return h.left;
   }

   if (isRed(h.left))
      h = leanRight(h);

   if (!isRed(h.right)
      && !isRed(h.right.left))
         h = moveRedRight(h);

   h.right = deleteMax(h.right);

   if (isRed(h.right))
      h = leanLeft(h);

   return h;
}
```

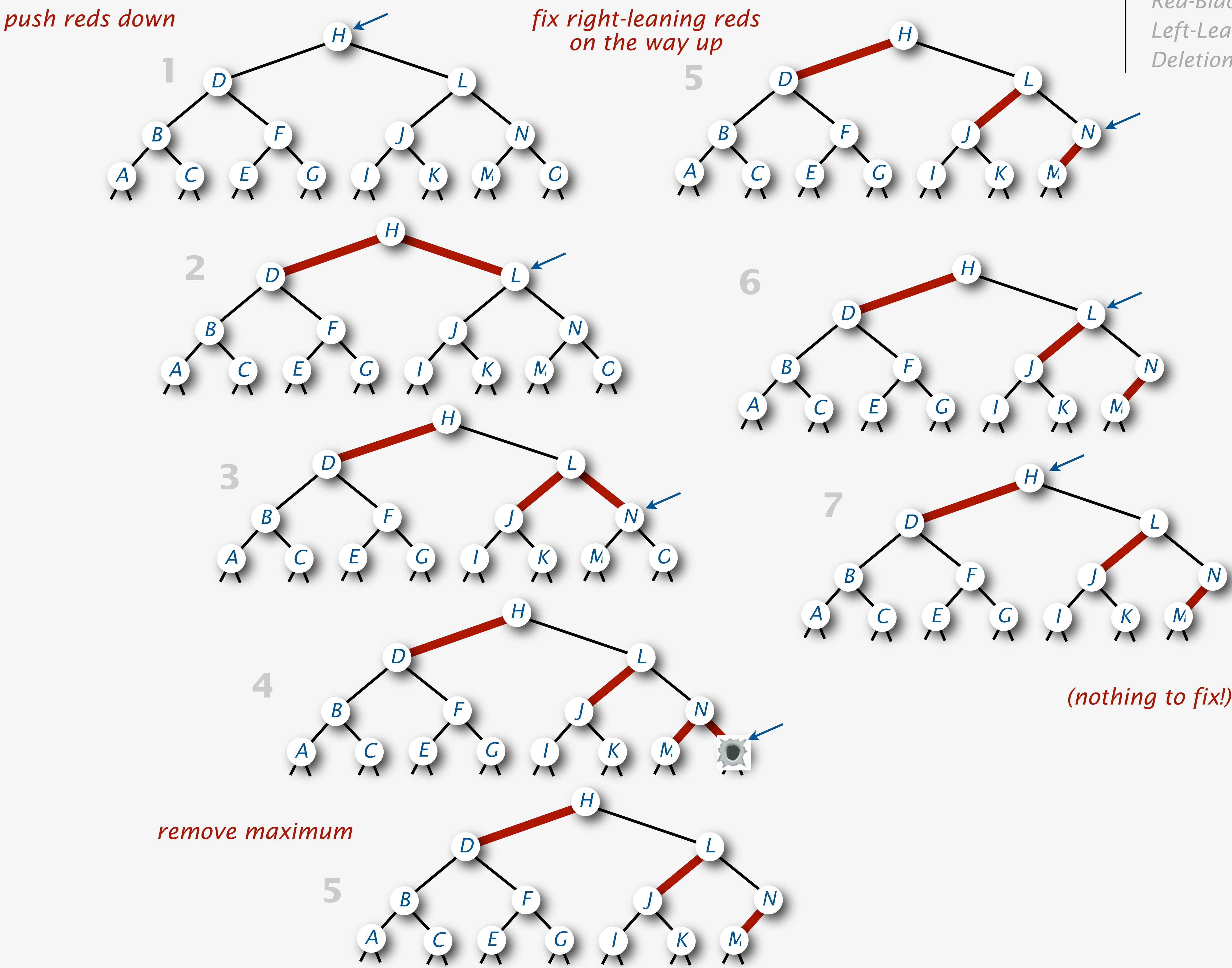




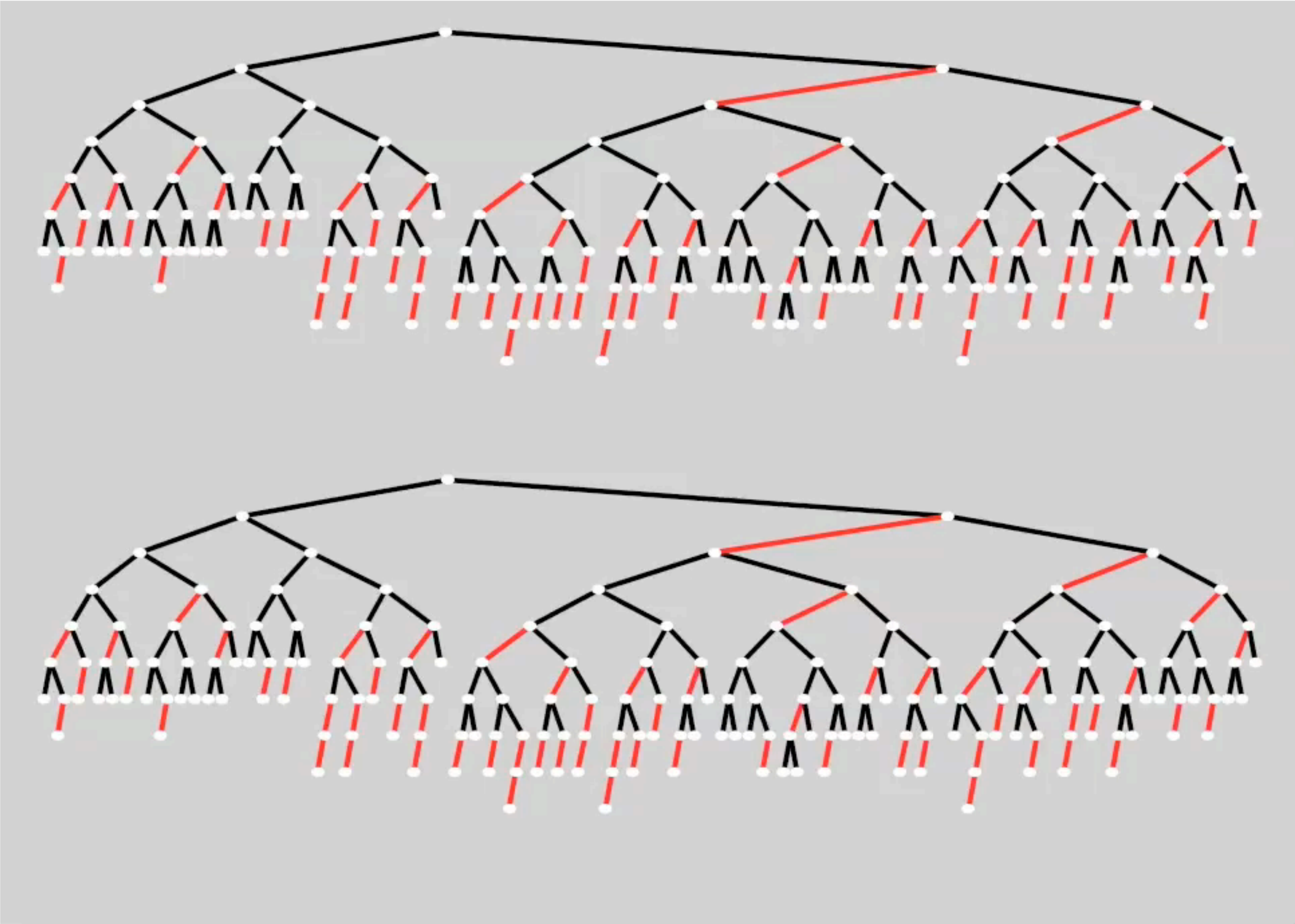


```
private Node moveRedRight(Node h)
{
   h.color       = BLACK;
   h.right.color = RED;
   if (isRed(h.left.left))
   {
      h = rotR(h);
      h.color = RED;
      h.left.color = BLACK;
   }
   else h.left.color = RED;
   return h;
}
```

deleteMax() example



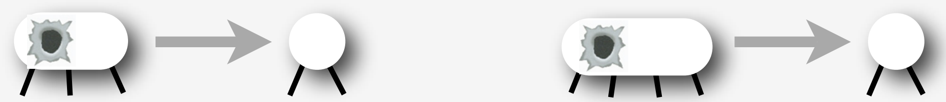
LLRB deleteMax() movie



Deleting an arbitrary node

involves the same general strategy.

1. Search down the left spine of the tree.
2. If search ends in a 3-node or 4-node: just remove it.



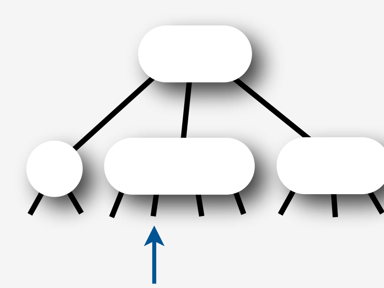
3. Removing a 2-node would destroy balance
   - transform tree on the way down the search path
   - Invariant: current node is not a 2-node

Difficulty:

- Far too many cases!
- LLRB representation dramatically reduces the number of cases.

Q: How many possible search paths in two levels ?

A: $9 * 6 + 27 * 9 + 81 * 12 = 1269$ (! !)

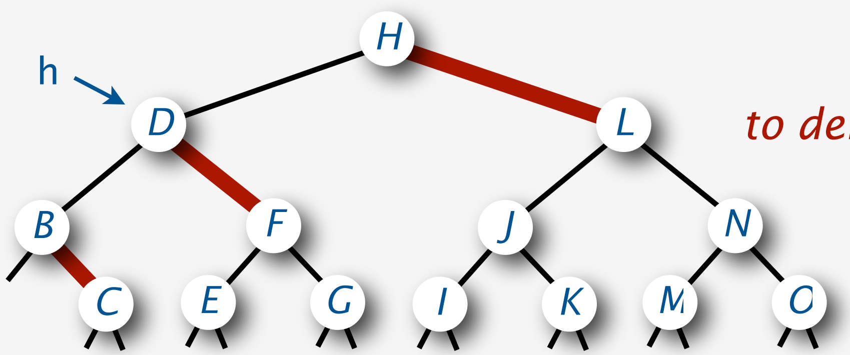


Deleting an arbitrary node

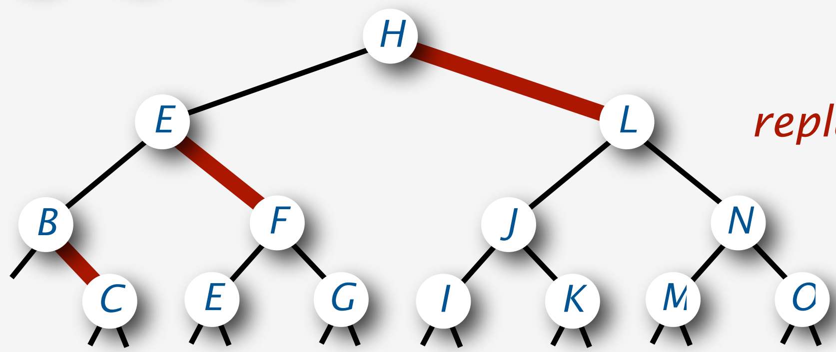
reduces to `deleteMin()`

A standard trick:

```
h.key   = min(h.right);
h.value = get(h.right, h.key);
h.right = deleteMin(h.right);
```



to delete D

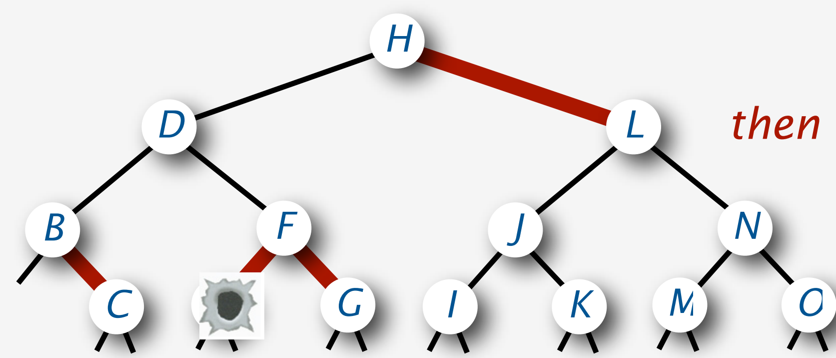


replace its key, value with those of its successor

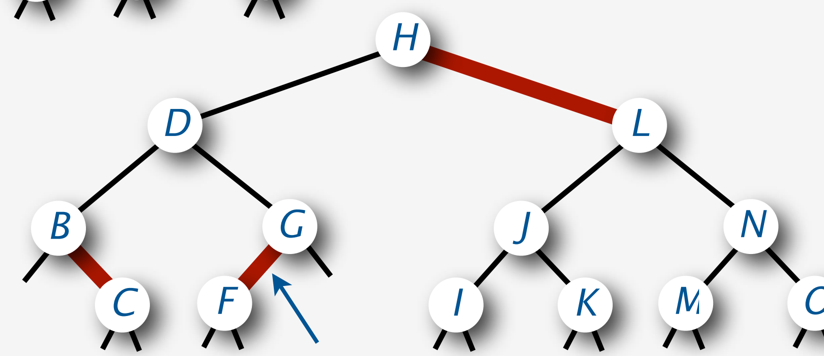
then delete the successor

deleteMin(right child of D)

flip colors, delete node



fix right-leaning red link

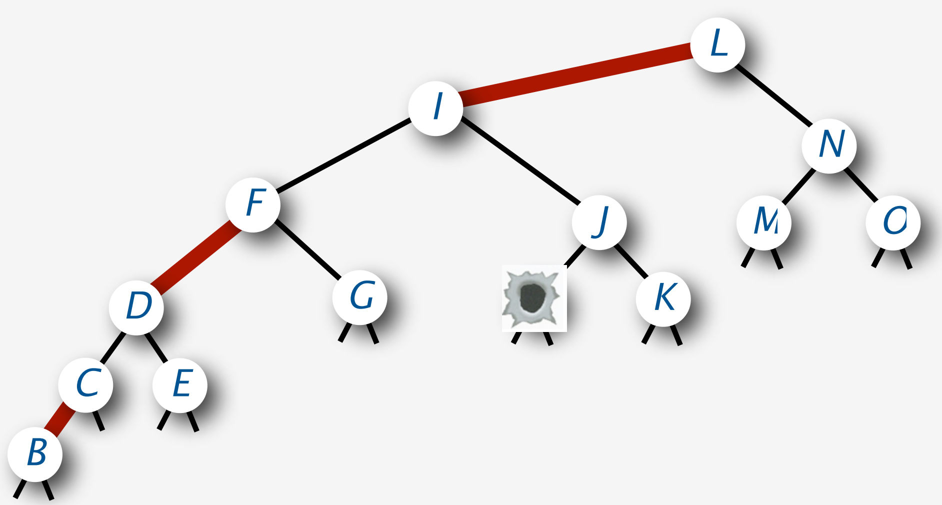


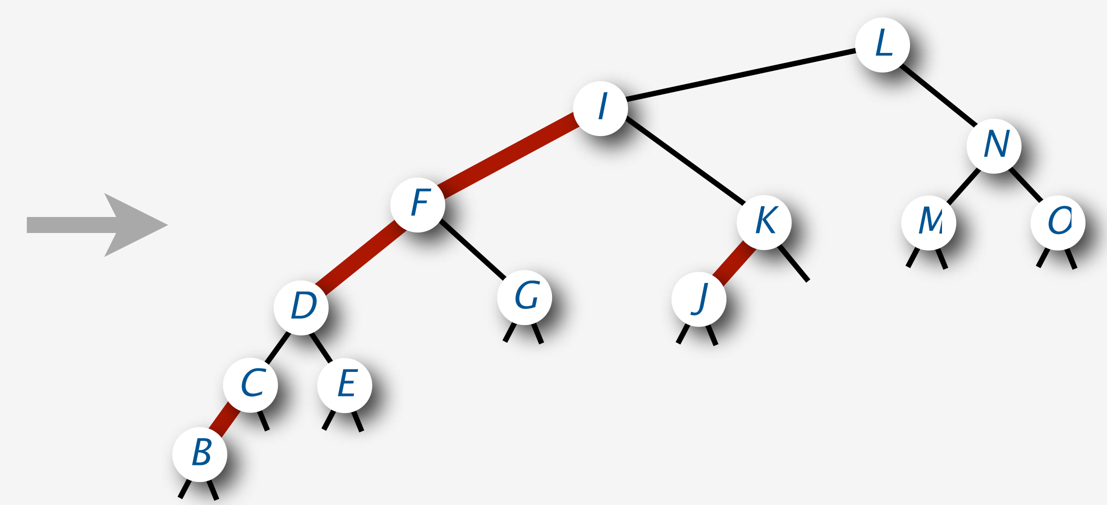
Deleting an arbitrary node at the bottom

can be implemented with the same helper methods used for `deleteMin()` and `deleteMax()`.

Invariant: h or one of its children is RED

- search path goes left: use `moveRedLeft()`.
- search path goes right: use `moveRedRight()`.
- delete node at bottom
- fix right-leaning reds on the way up





A few loose ends remain . . . et voilà! (see next page)

delete() implementation for LLRB trees


```
private Node delete(Node h, Key key)
{
   int cmp = key.compareTo(h.key);
   if (cmp < 0)
   {
      if (!isRed(h.left) && !isRed(h.left.left))
          h = moveRedLeft(h);
      h.left =  delete(h.left, key);
   }
   else
   {
      if (isRed(h.left)) h = leanRight(h);

      if (cmp == 0 && (h.right == null))
         return null;

      if (!isRed(h.right) && !isRed(h.right.left))
         h = moveRedRight(h);

      if (cmp == 0)
      {
         h.key = min(h.right);
         h.value = get(h.right, h.key);
         h.right = deleteMin(h.right);
      }
      else h.right = delete(h.right, key);
   }
   if (isRed(h.right)) h = leanLeft(h);

   return h;
}
```

LEFT

push red right if necessary

move down (left)

RIGHT or EQUAL

rotate to push red right

EQUAL (at bottom)
delete node

push red right if necessary

EQUAL (not at bottom)

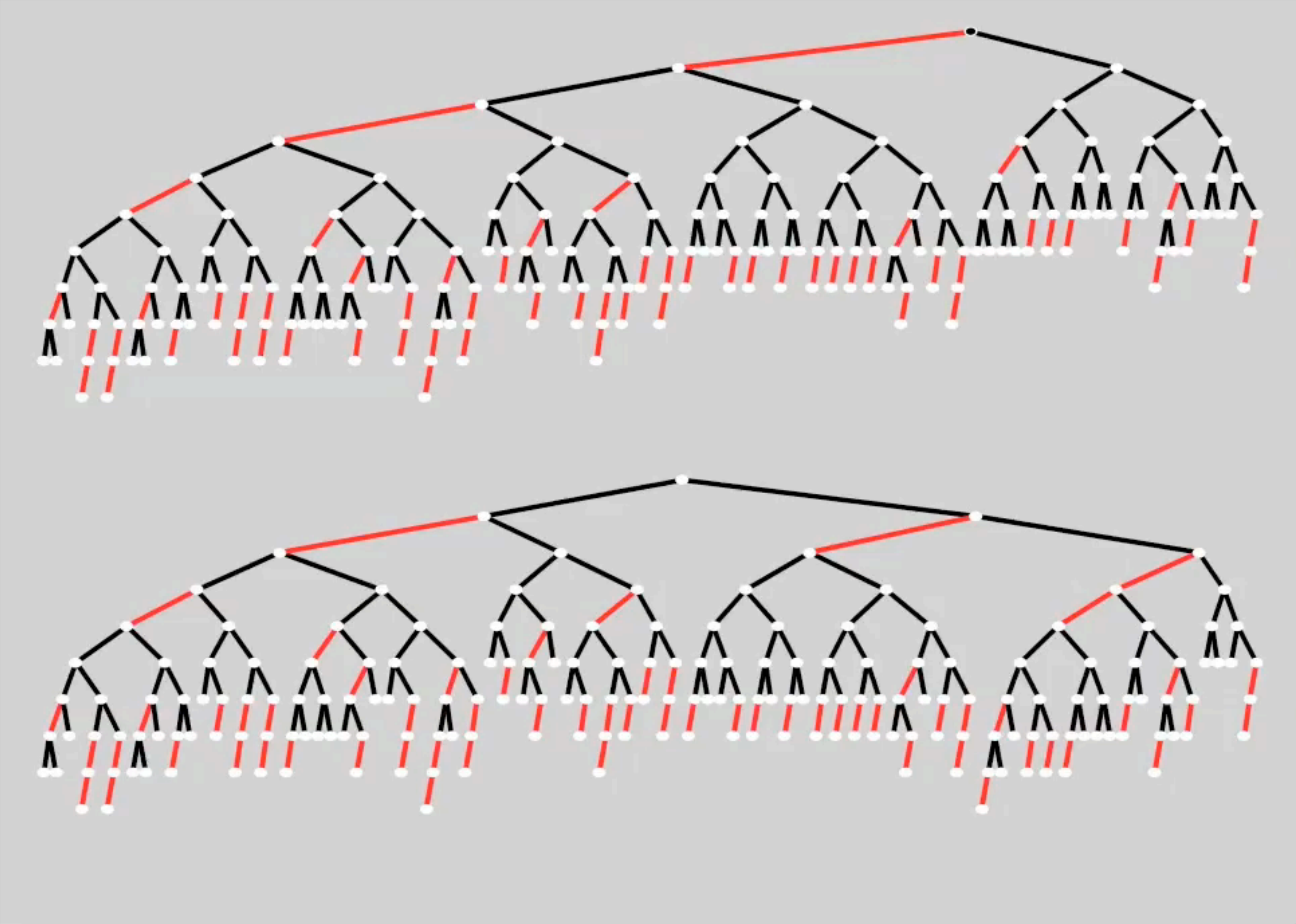
replace current node with successor key, value

delete successor

move down (right)

Fix right-leaning red links on the way up the tree

LLRB delete() movie



Alternatives

Red-black-tree implementations in widespread use:

- are based on pseudocode with “case bloat”
- use parent pointers (!)
- 400+ lines of code for core algorithms

Left-leaning red-black trees

- you just saw all the code
- single pass (remove recursion if concurrency matters)
- <80 lines of code for core algorithms
- less code implies faster insert, delete
- less code implies easier maintenance and migration