Left-Leaning Red-Black Trees

Robert Sedgewick

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Original version: Data structures seminar at Dagstuhl (Feb 2008)

- red-black trees made simpler (!)
- full delete() implementation

This version: Analysis of Algorithms meeting at Maresias (Apr 2008)

- back to balanced 4-nodes
- back to 2-3 trees (!)
- scientific analysis

Addendum: observations developed after talk at Maresias

Java code at www.cs.princeton.edu/~rs/talks/LLRB/Java

Movies at www.cs.princeton.edu/~rs/talks/LLRB/movies

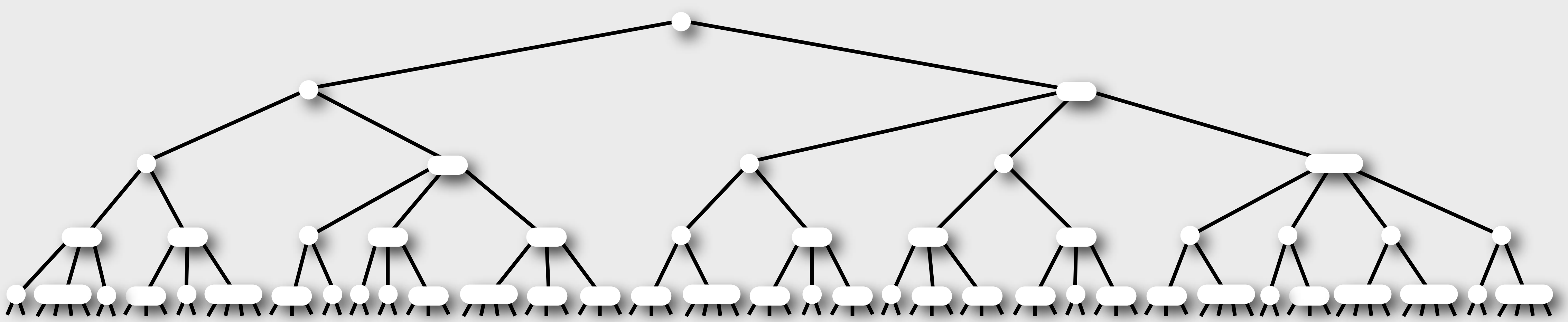
Introduction

2-3-4 Trees

Red-Black Trees

Left-Leaning RB Trees

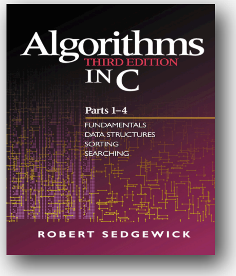
Deletion

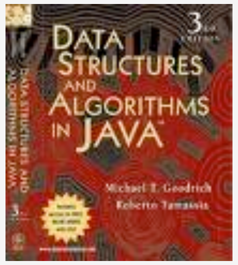


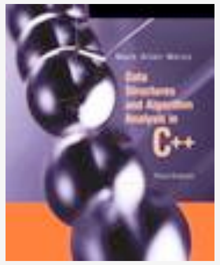
Red-black trees

are now found throughout our computational infrastructure

Textbooks on algorithms







. . .

Library search function in many programming environments

. . .

Popular culture (stay tuned)

Worth revisiting?

Red-black trees

are now found throughout our computational infrastructure

Typical:

> ya thanks,
> i got the idea
> but is there some other place on the web where only the algorithms
> used by STL is
> explained. (that is the underlying data structures etc.) without
> explicit reference to the code (as it is pretty confusing if I try to
> read through).
>
> thanks[/color]

The standard does not specify which algorithms the STL must use. Implementers are free to choose which ever algorithm or data structure that fulfils the functional and efficiency requirements of the standard.

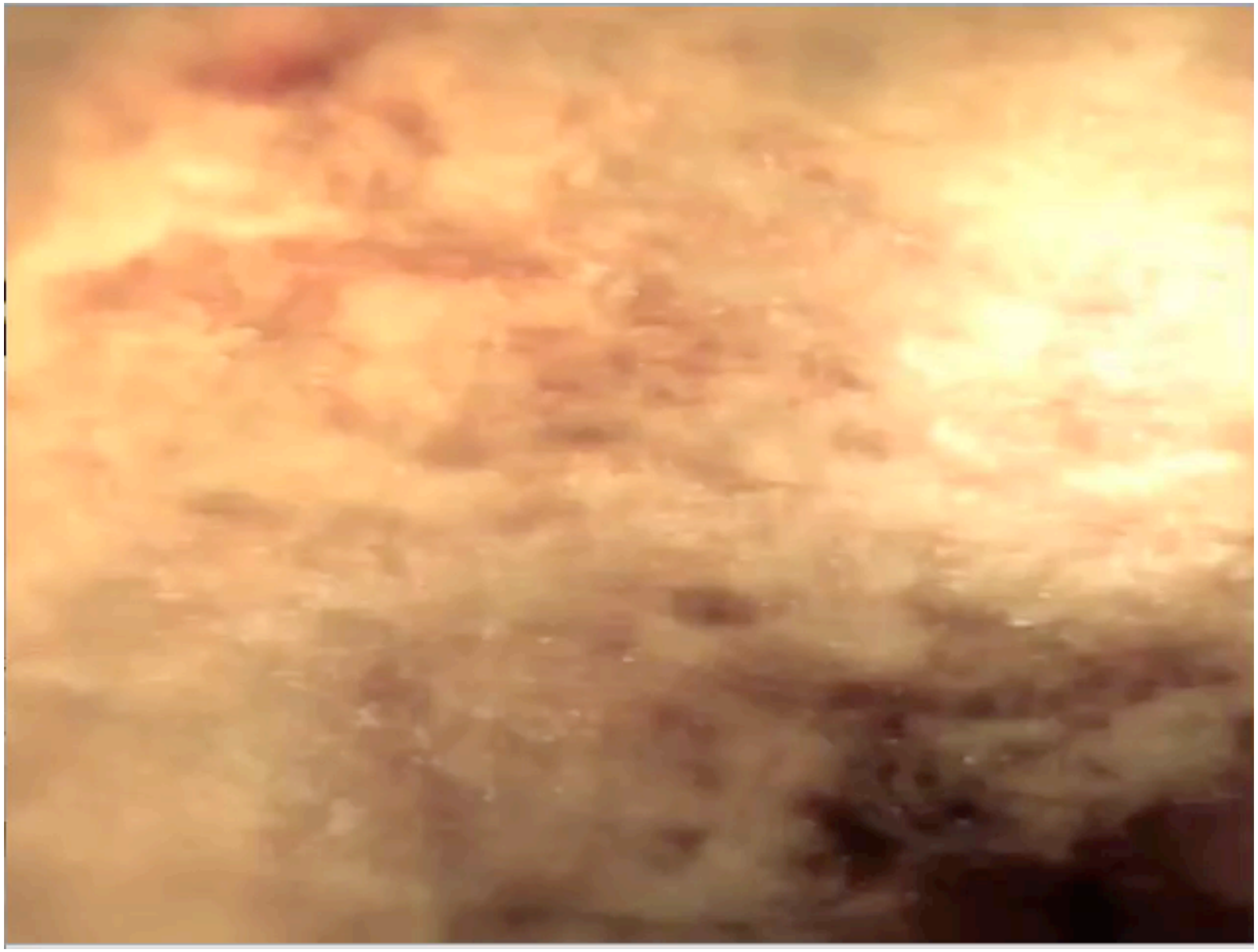
There are some common choices however. For instance every implementation of map, multimap, set and multiset that I have ever seen uses a structure called a red black tree. Typing 'red black tree algorithm' in google produces a number of likely looking links.

john

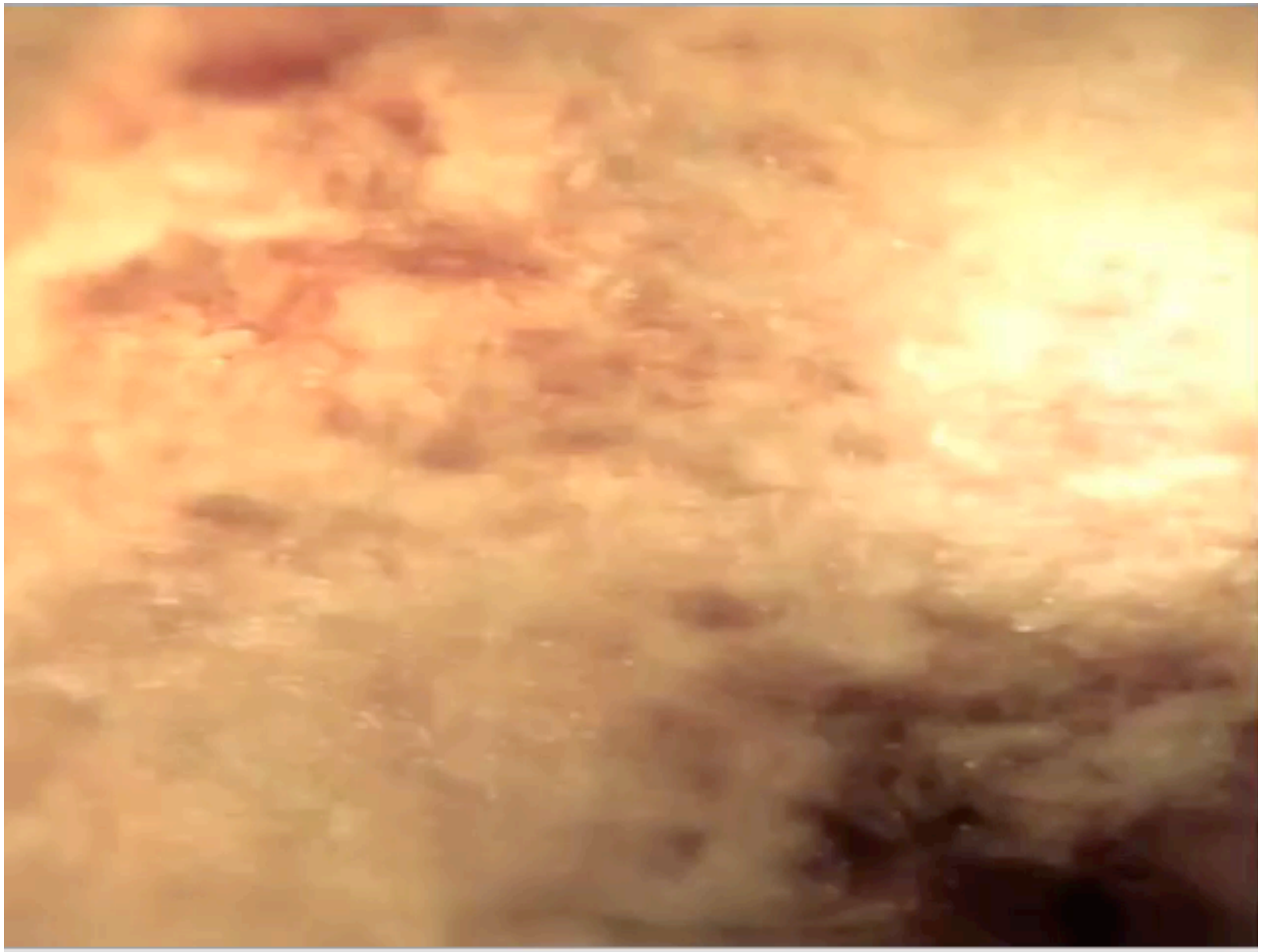
Digression:

Red-black trees are found in popular culture??

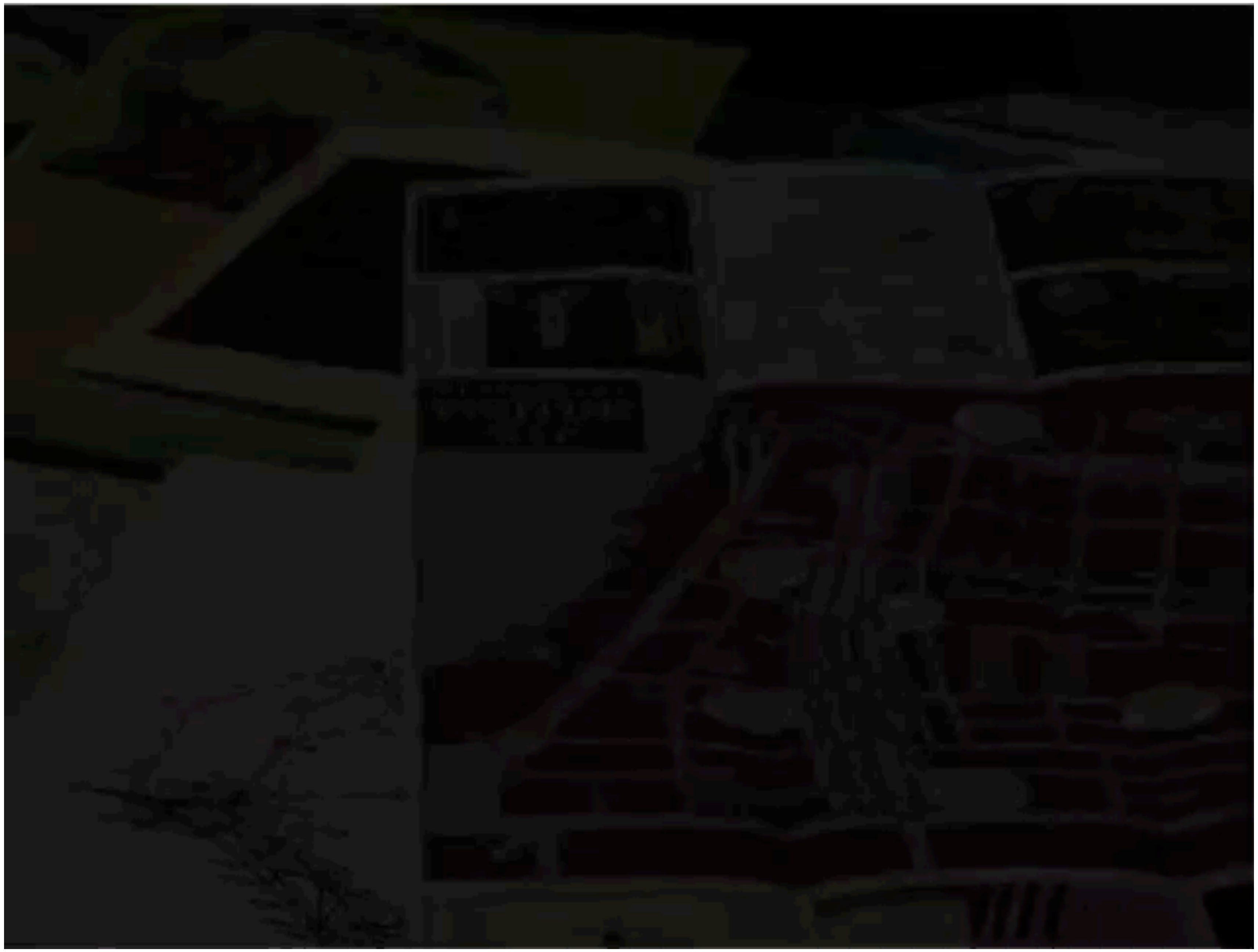
Mystery: black door?



Mystery: red door?



An explanation ?



Primary goals

Red-black trees (Guibas-Sedgewick, 1978)

- reduce code complexity
- minimize or eliminate space overhead
- unify balanced tree algorithms
- single top-down pass (for concurrent algorithms)
- find version amenable to average-case analysis

Current implementations

- maintenance
- migration
- space not so important (??)
- guaranteed performance
- support full suite of operations

Worth revisiting ?

Primary goals

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Current implementations

- maintenance
- migration
- space not so important (??)
- guaranteed performance
- support full suite of operations

Worth revisiting ? YES. Code complexity is out of hand.

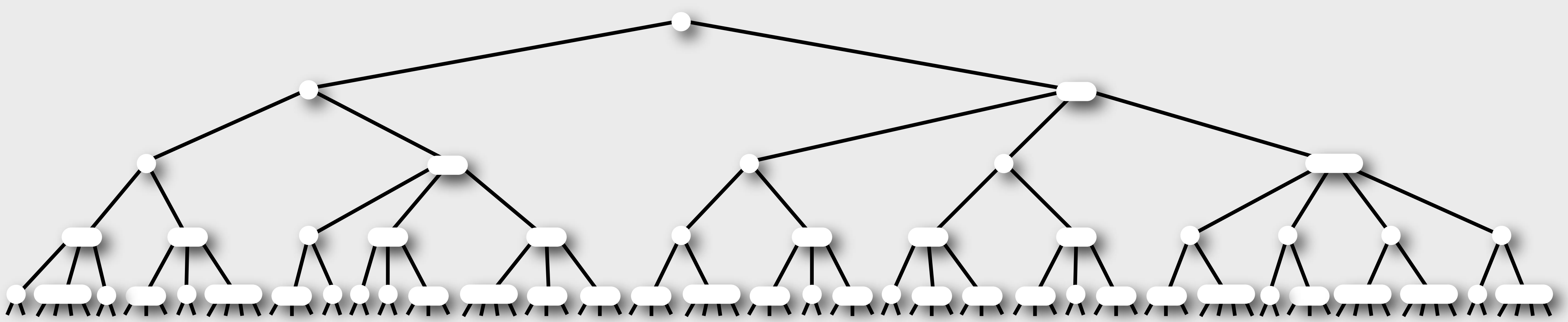
Introduction

2-3-4 Trees

LLRB Trees

Deletion

Analysis



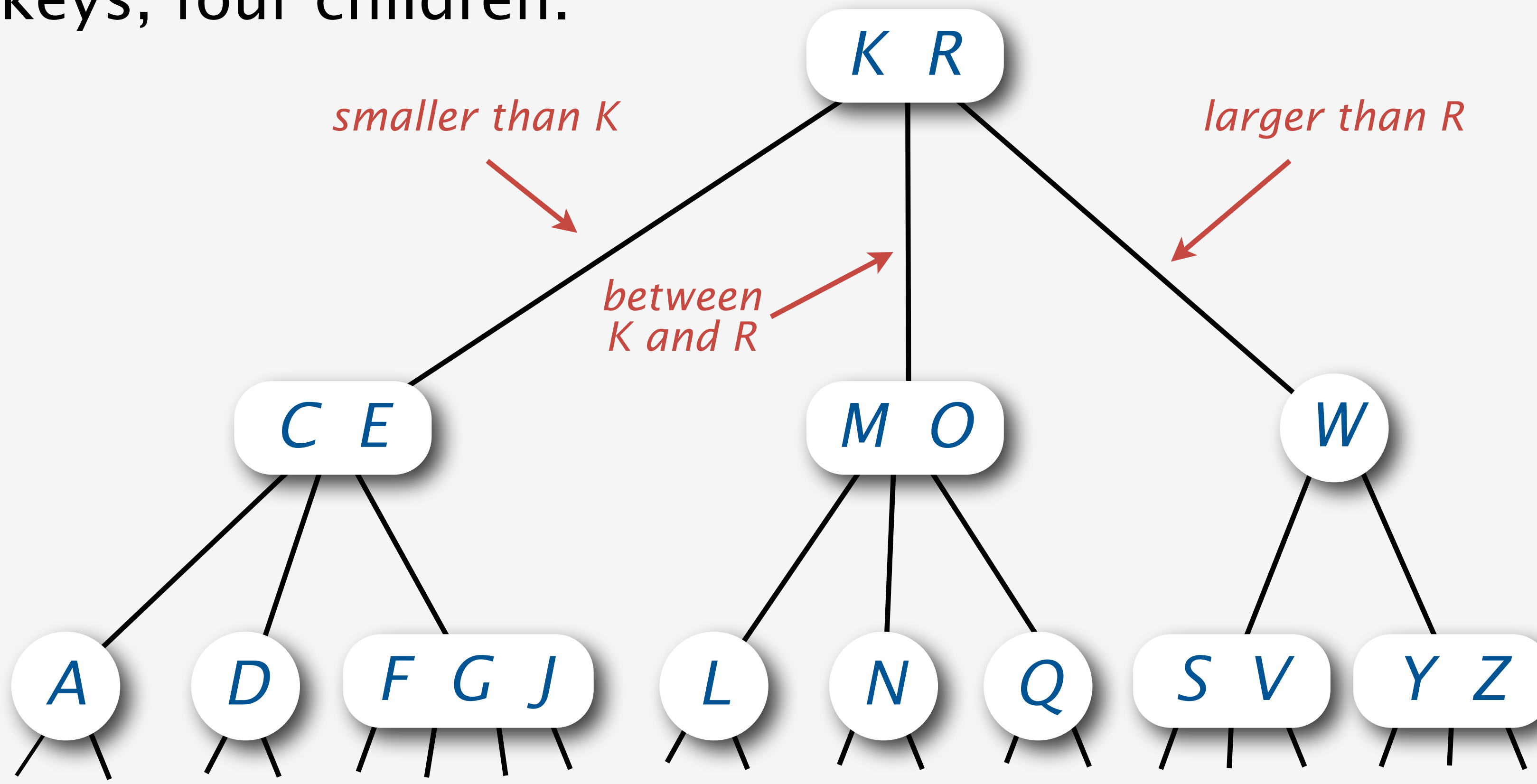
2-3-4 Tree

Generalize BST node to allow multiple keys.
Keep tree in perfect balance.

Perfect balance. Every path from root to leaf has same length.

Allow 1, 2, or 3 keys per node.

- 2-node: one key, two children.
- 3-node: two keys, three children.
- 4-node: three keys, four children.



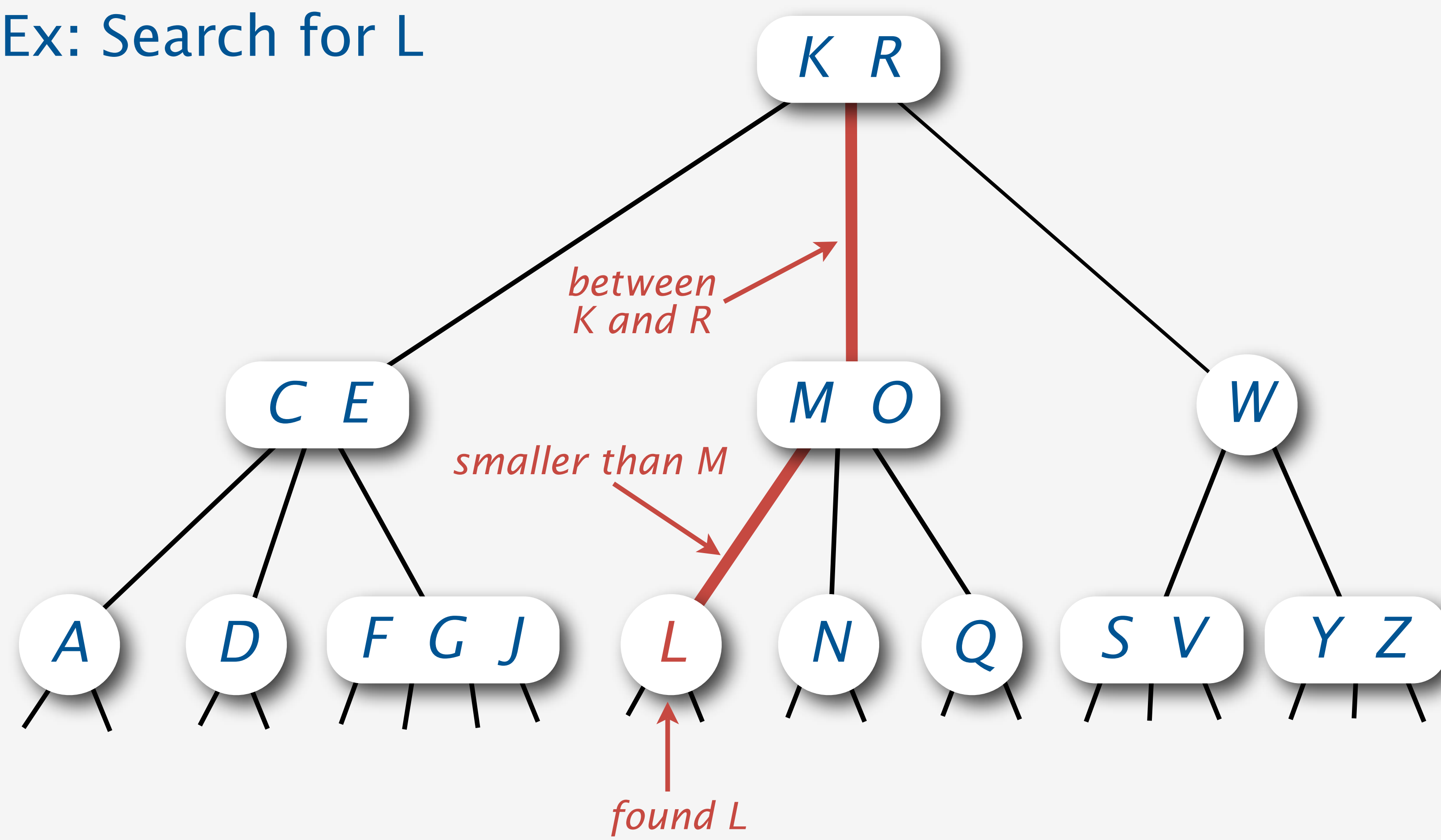
Search in a 2-3-4 Tree

Compare node keys against search key to guide search.

Search.

- Compare search key against keys in node.
- Find interval containing search key.
- Follow associated link (recursively).

Ex: Search for L



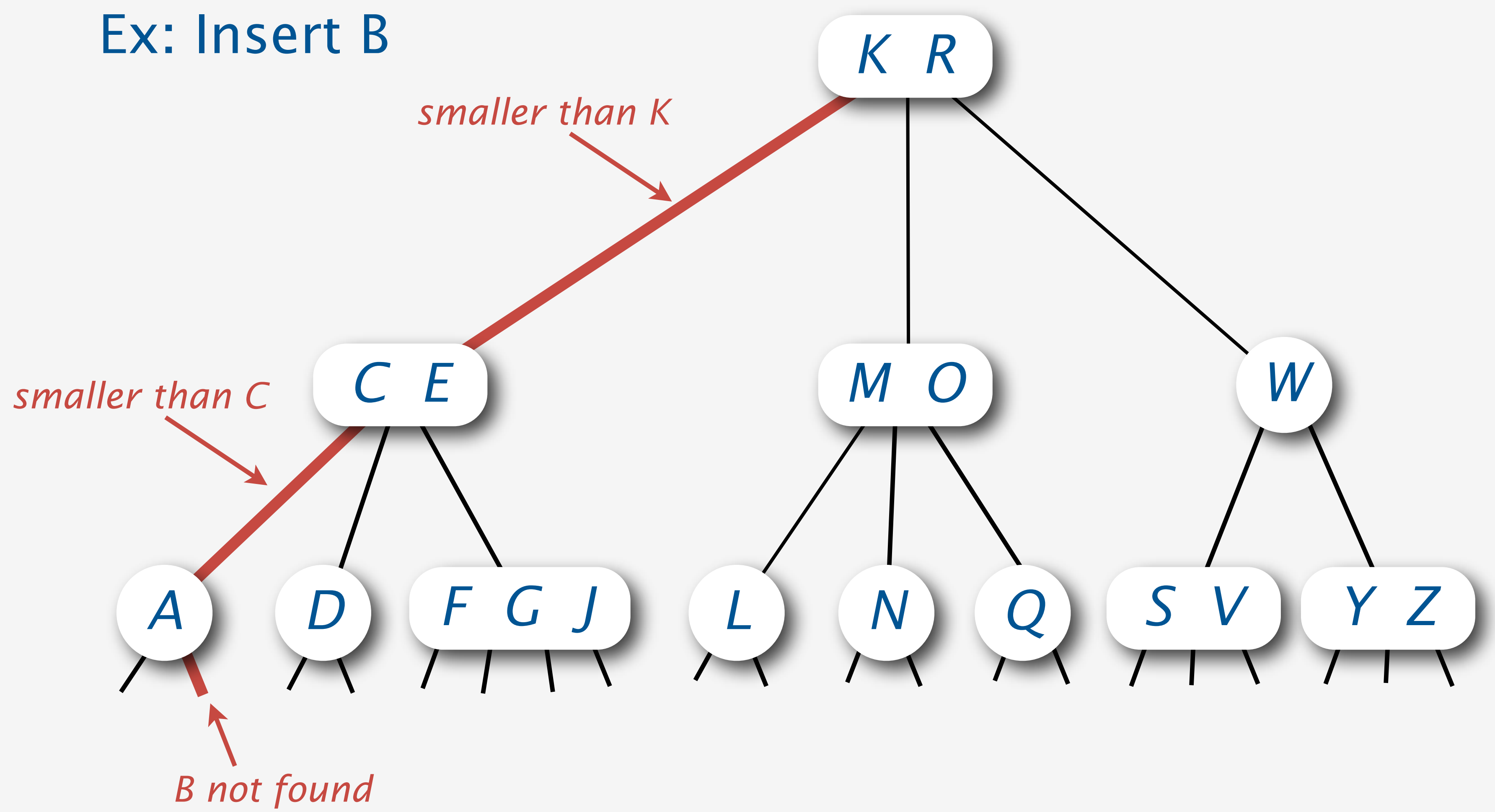
Insertion in a 2-3-4 Tree

Add new keys at the bottom of the tree.

Insert.

- Search to bottom for key.

Ex: Insert B



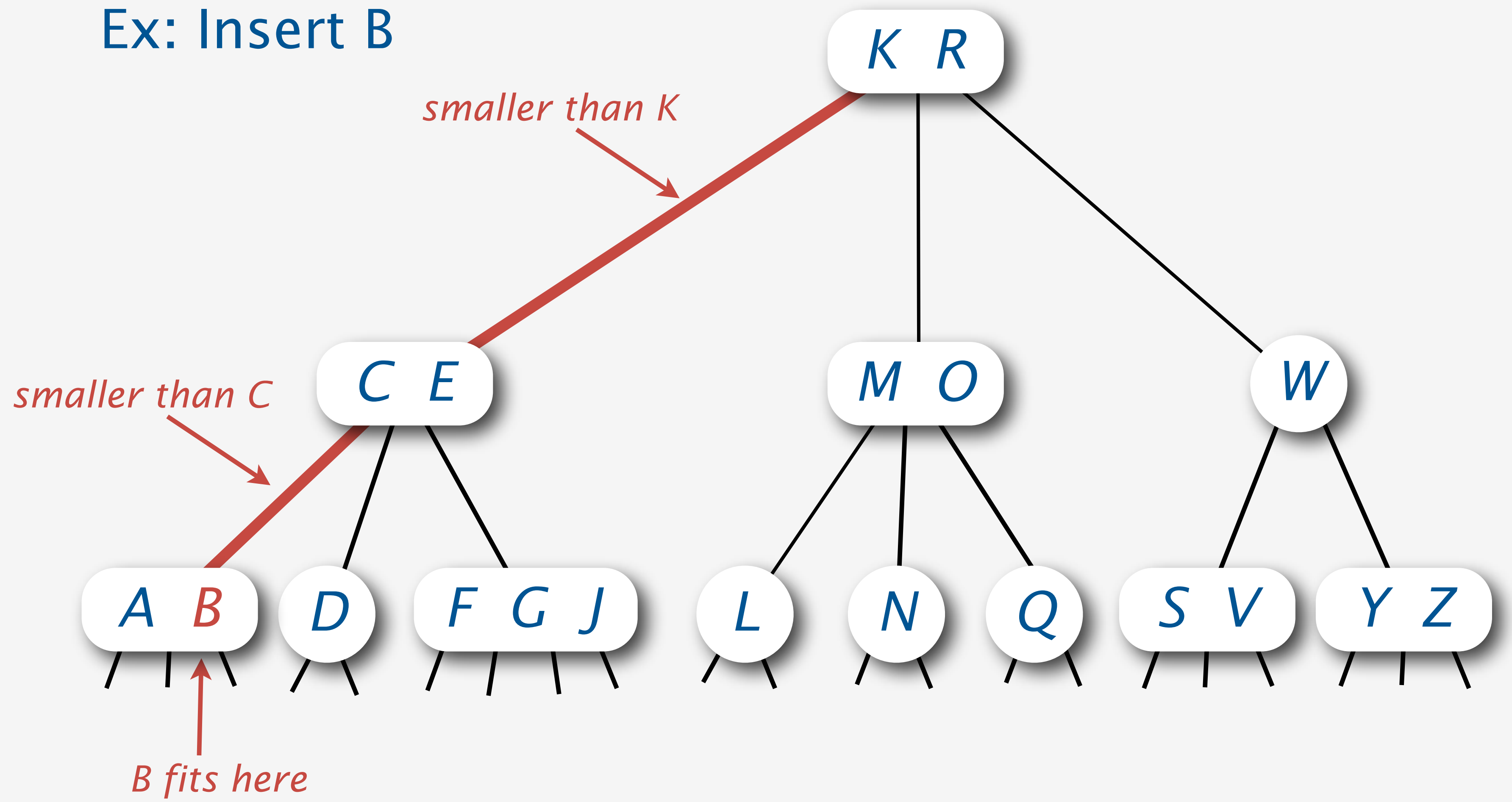
Insertion in a 2-3-4 Tree

Add new keys at the bottom of the tree.

Insert.

- Search to bottom for key.
- 2-node at bottom: convert to a 3-node.

Ex: Insert B



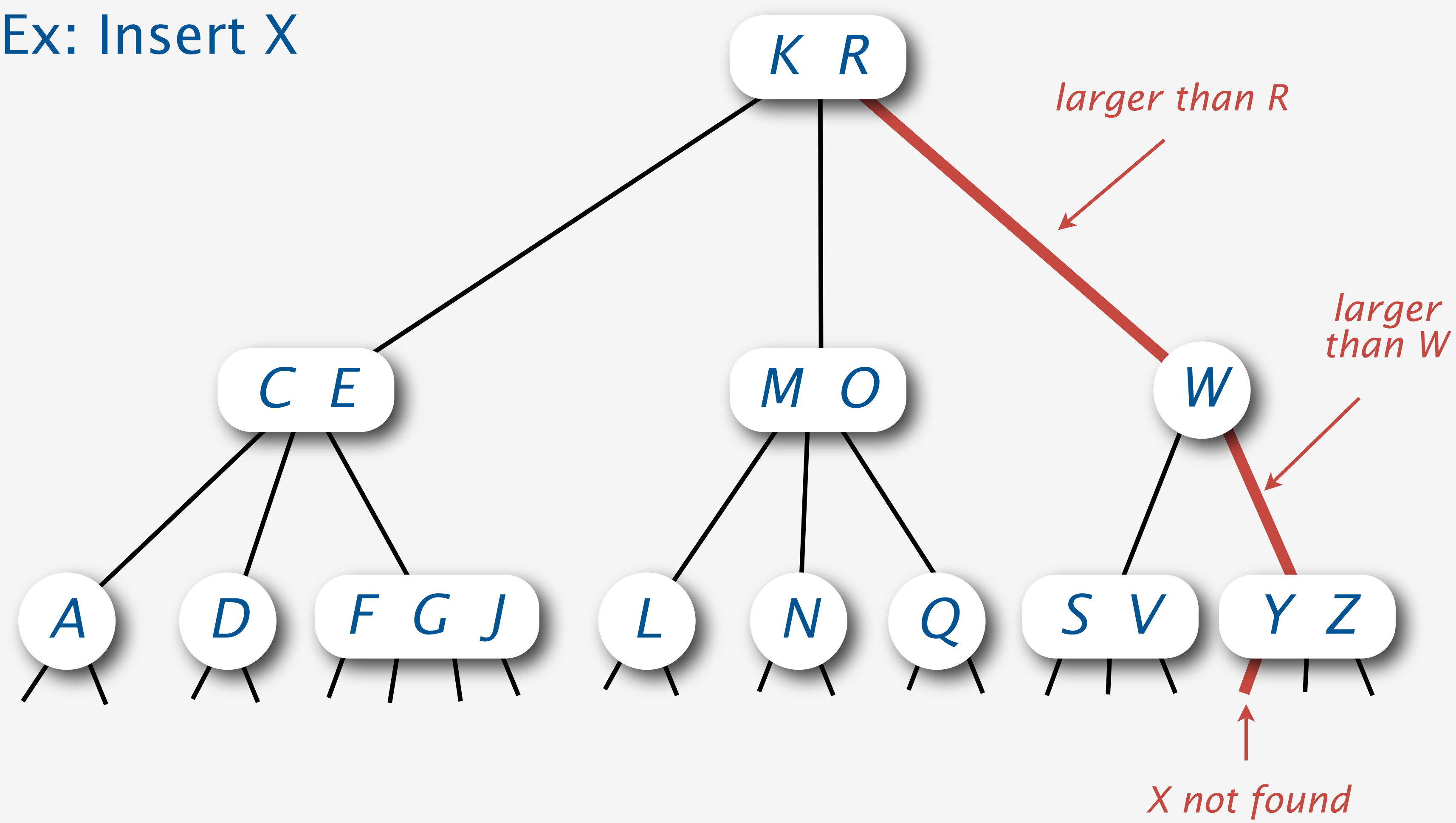
Insertion in a 2-3-4 Tree

Add new keys at the bottom of the tree.

Insert.

- Search to bottom for key.

Ex: Insert X



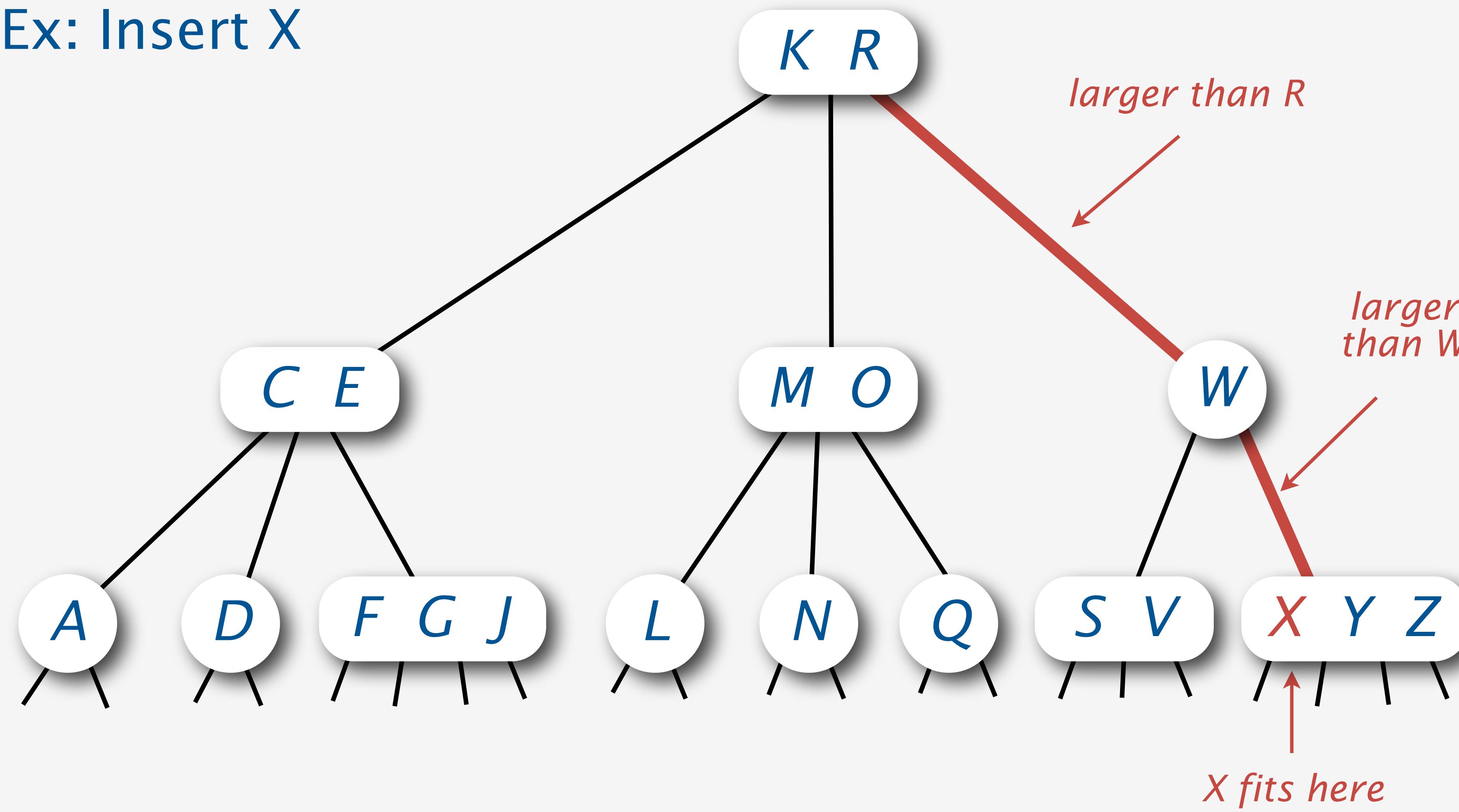
Insertion in a 2-3-4 Tree

Add new keys at the bottom of the tree.

Insert.

- Search to bottom for key.
- 3-node at bottom: convert to a 4-node.

Ex: Insert X



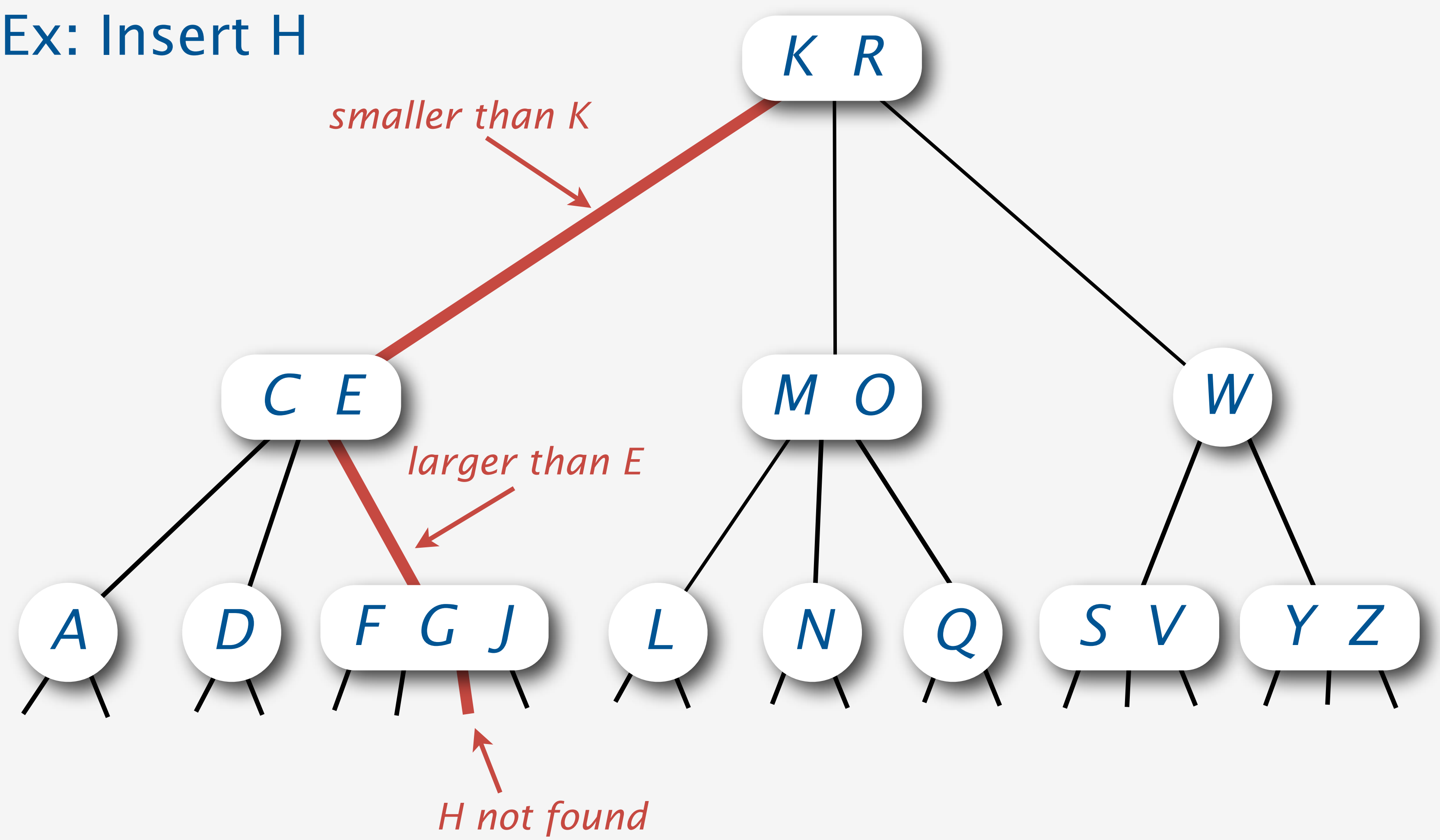
Insertion in a 2-3-4 Tree

Add new keys at the bottom of the tree.

Insert.

- Search to bottom for key.

Ex: Insert H



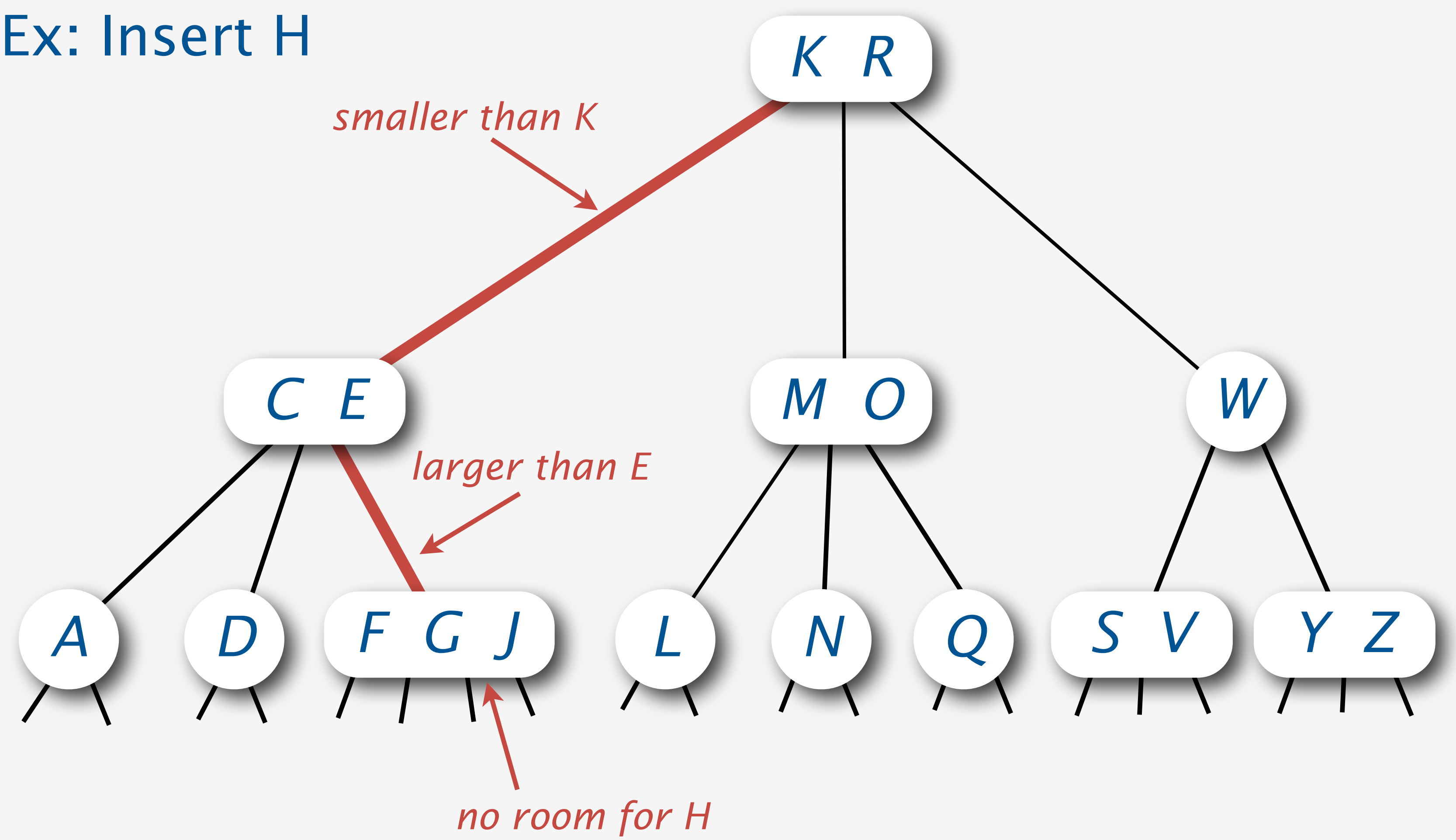
Insertion in a 2-3-4 Tree

Add new keys at the bottom of the tree.

Insert.

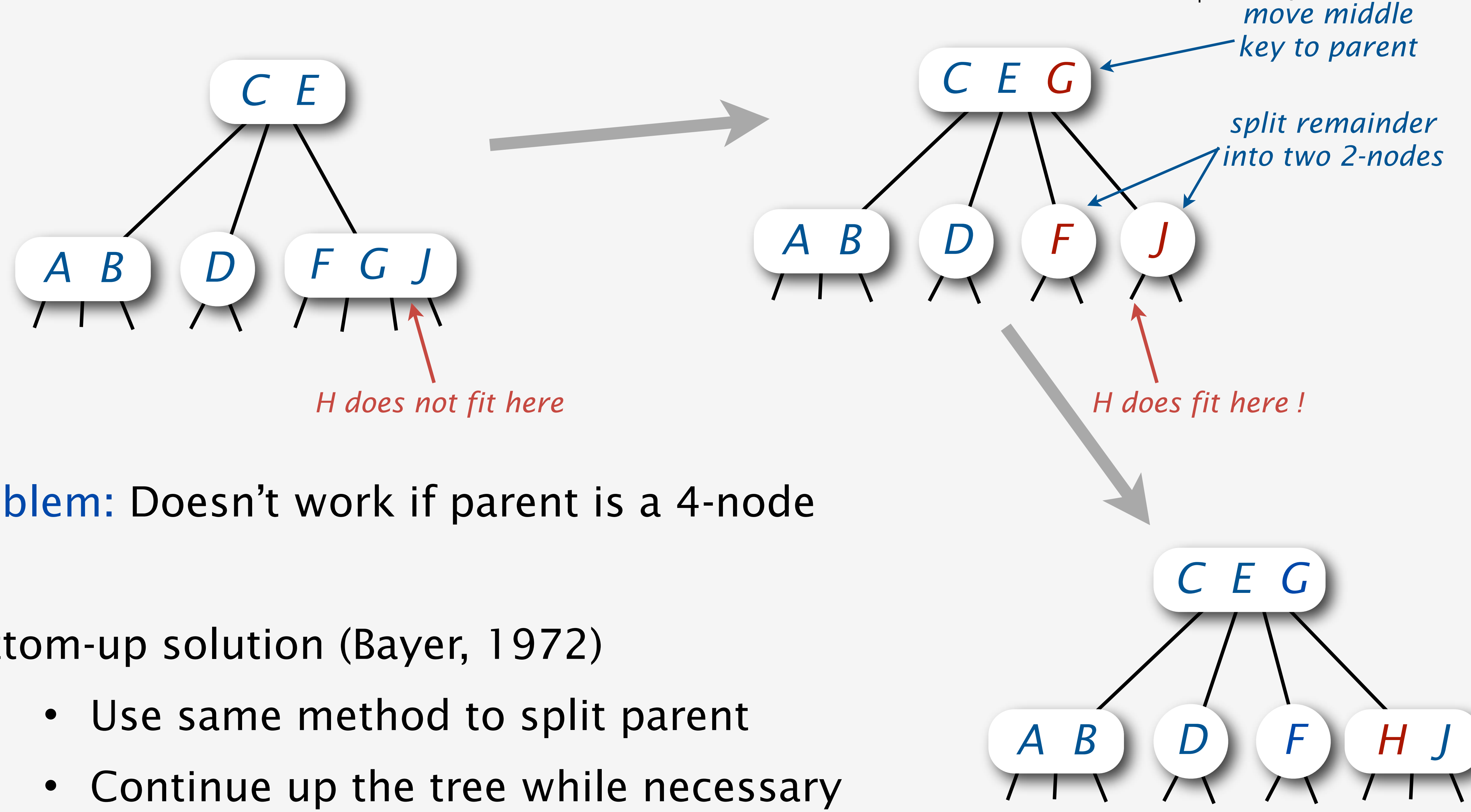
- Search to bottom for key.
- 2-node at bottom: convert to a 3-node.
- 3-node at bottom: convert to a 4-node.
- 4-node at bottom: no room for new key.

Ex: Insert H



Splitting 4-nodes in a 2-3-4 tree

is an effective way to make room for insertions



Problem: Doesn’t work if parent is a 4-node

Bottom-up solution (Bayer, 1972)

- Use same method to split parent
- Continue up the tree while necessary

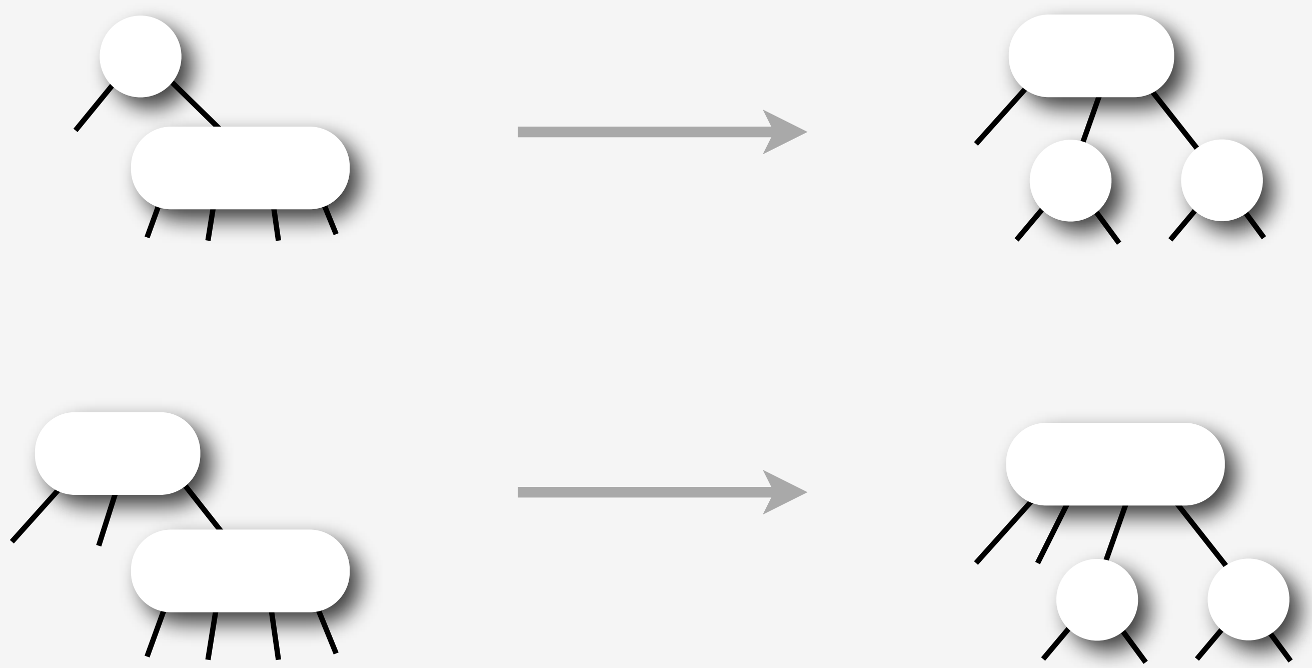
Top-down solution (Guibas-Sedgewick, 1978)

- Split 4-nodes on the way down
- Insert at bottom

Splitting 4-nodes on the way down

ensures that the "current" node is not a 4-node

Transformations to split 4-nodes:



local transformations that work anywhere in the tree

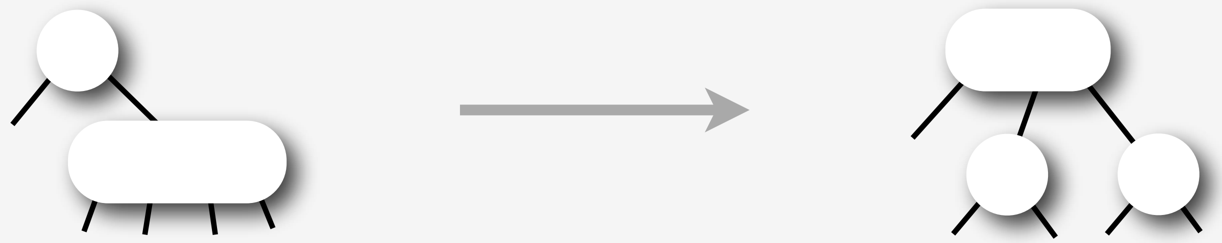
Invariant: "Current" node is not a 4-node

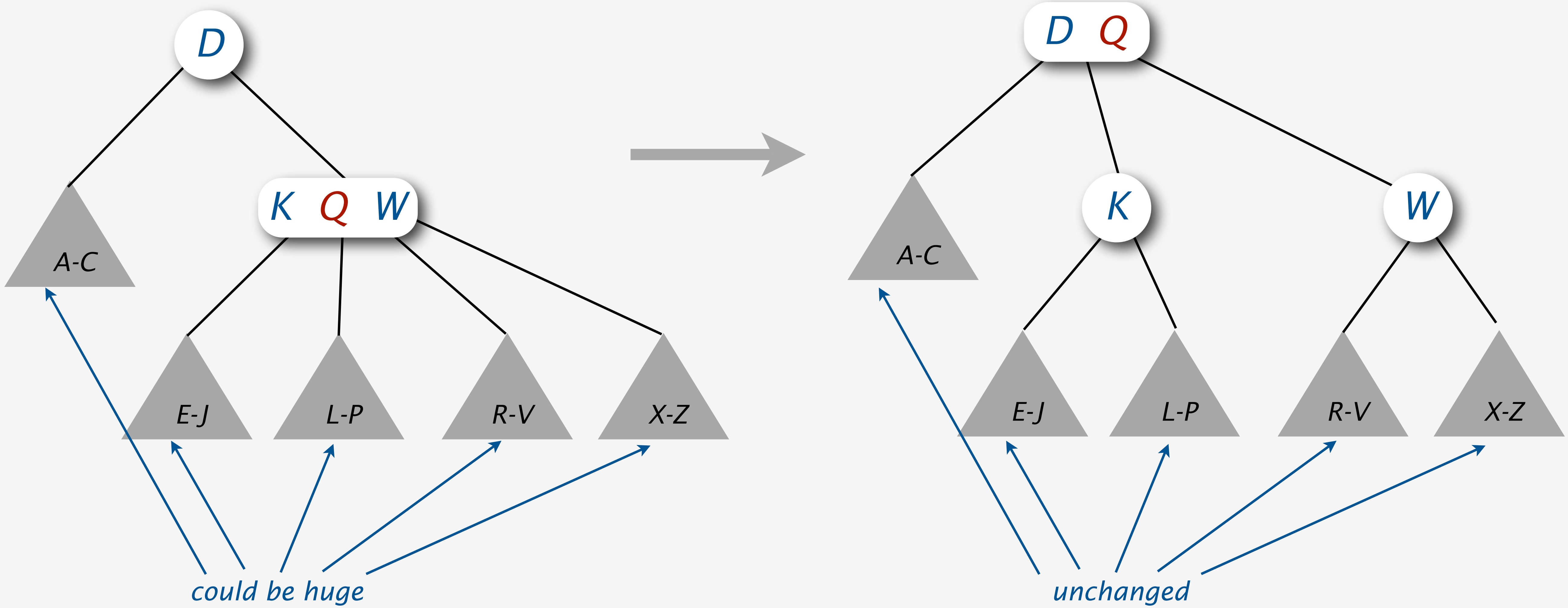
Consequences:

- 4-node below a 4-node case never happens
- Bottom node reached is always a 2-node or a 3-node

Splitting a 4-node below a 2-node

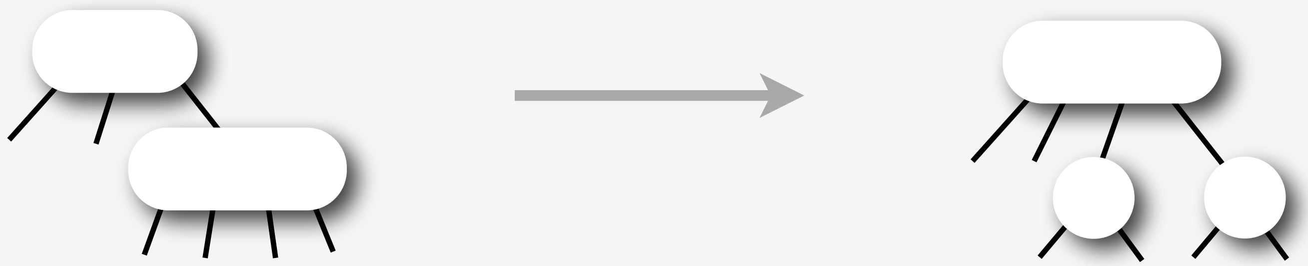
is a local transformation that works anywhere in the tree

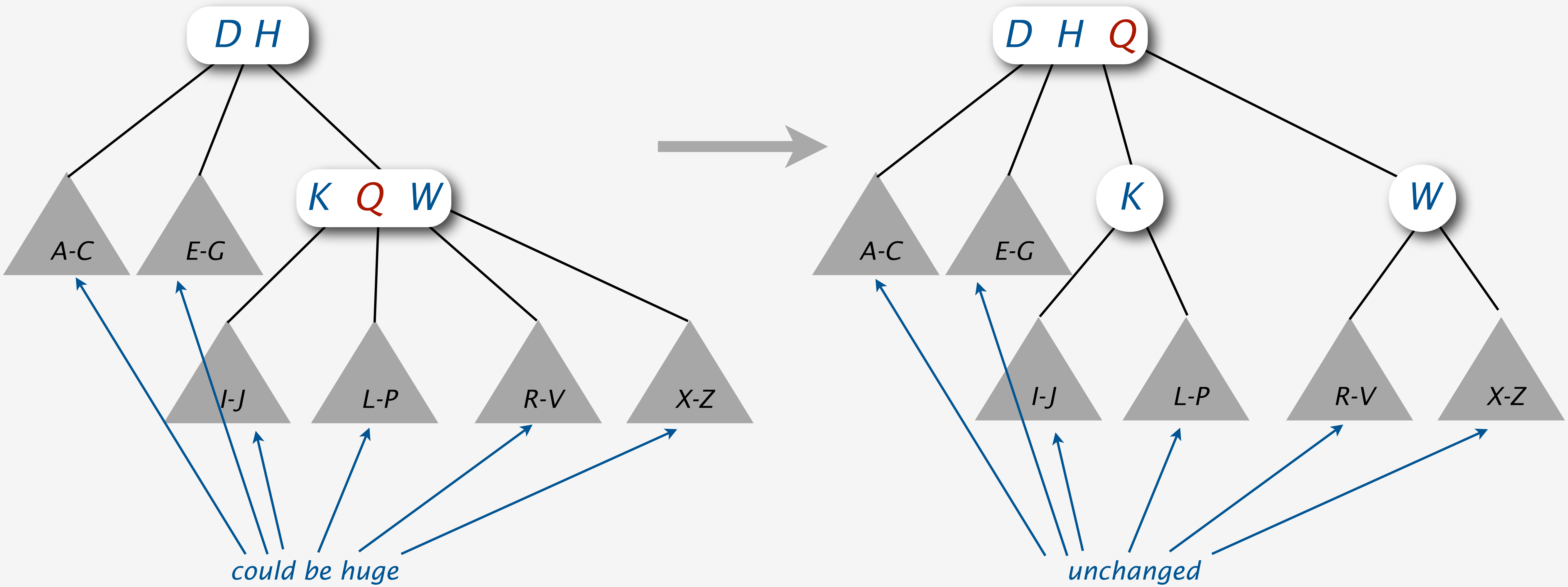




Splitting a 4-node below a 3-node

is a local transformation that works anywhere in the tree

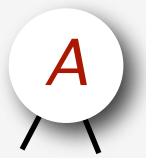




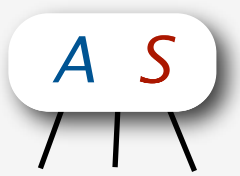
Growth of a 2-3-4 tree

happens upwards from the bottom

insert A

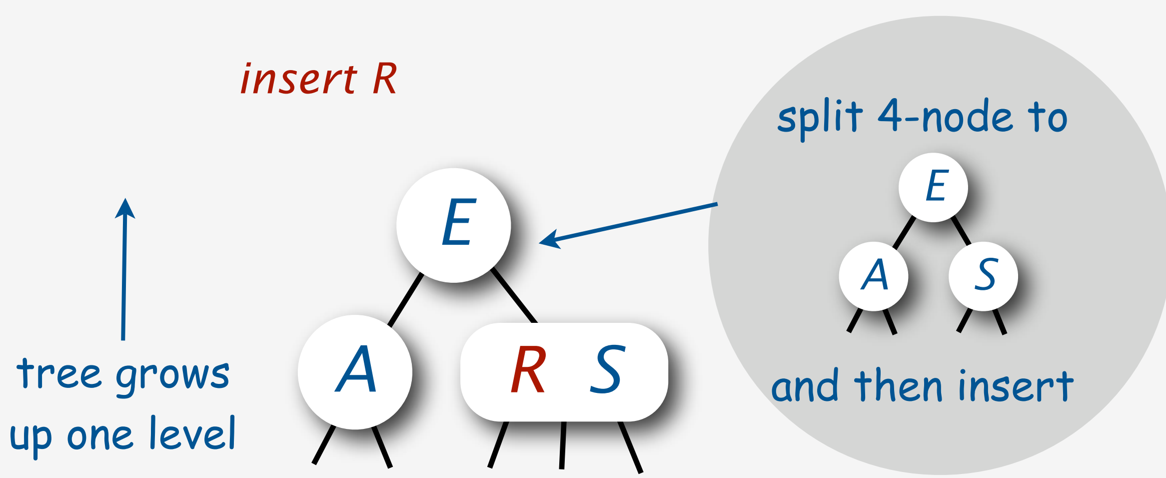


insert S

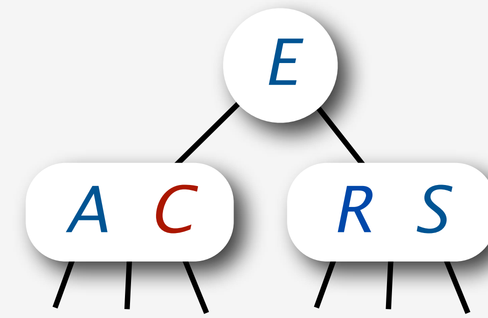


insert E

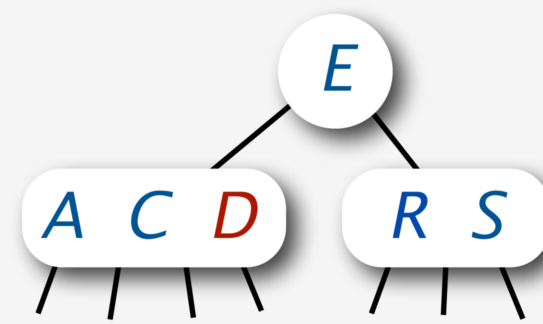
insert R



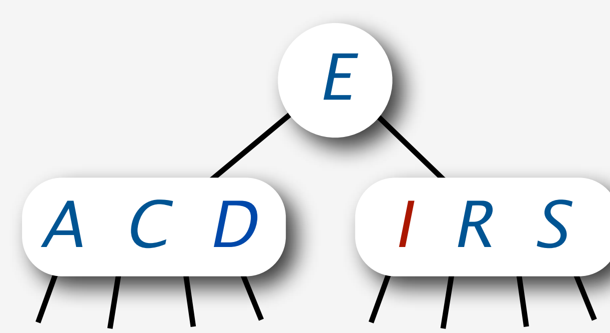
insert C



insert D

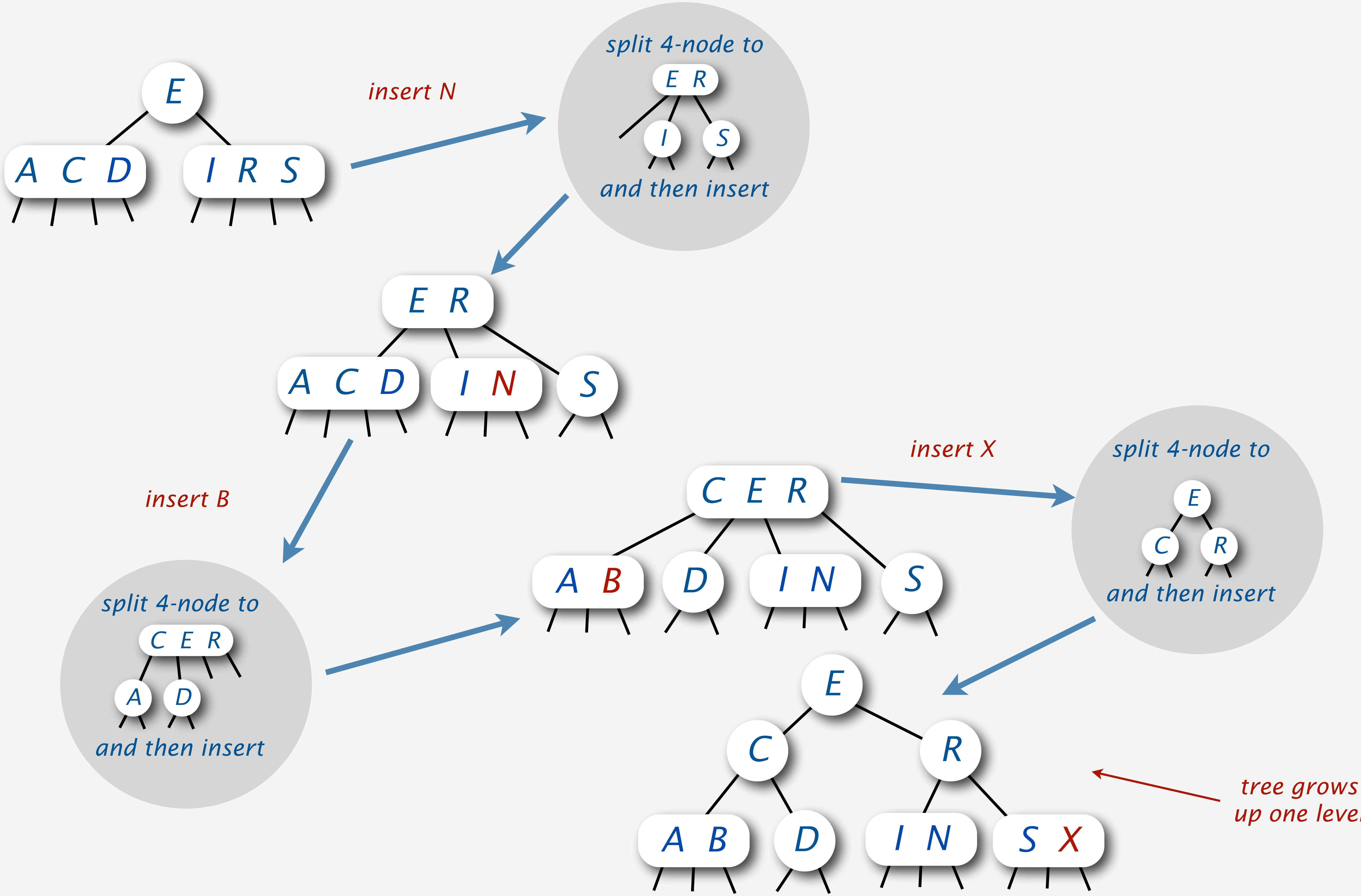


insert I



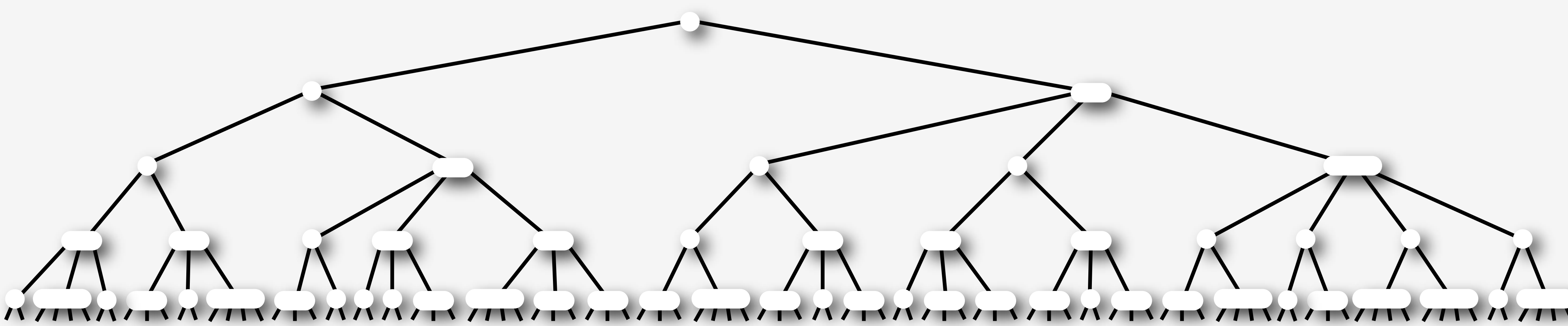
Growth of a 2-3-4 tree (continued)

happens upwards from the bottom



Balance in 2-3-4 trees

Key property: All paths from root to leaf are the same length



Tree height.

- Worst case: lg N [all 2-nodes]
- Best case: log4 N = 1/2 lg N [all 4-nodes]
- Between 10 and 20 for 1 million nodes.
- Between 15 and 30 for 1 billion nodes.

Guaranteed logarithmic performance for both search and insert.

Direct implementation of 2-3-4 trees

is complicated because of code complexity.

Maintaining multiple node types is cumbersome.

- Representation?
- Need multiple compares to move down in tree.
- Large number of cases for splitting.
- Need to convert 2-node to 3-node and 3-node to 4-node.

fantasy code

```
private void insert(Key key, Val val)
{
   Node x = root;
   while (x.getTheCorrectChild(key) != null)
   {
      x = x.getTheCorrectChild(key);
      if (x.is4Node()) x.split();
   }
   if      (x.is2Node()) x.make3Node(key, val);
   else if (x.is3Node()) x.make4Node(key, val);
   return x;
}
```

Bottom line: Could do it, but stay tuned for an easier way.

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2-3-4 Trees

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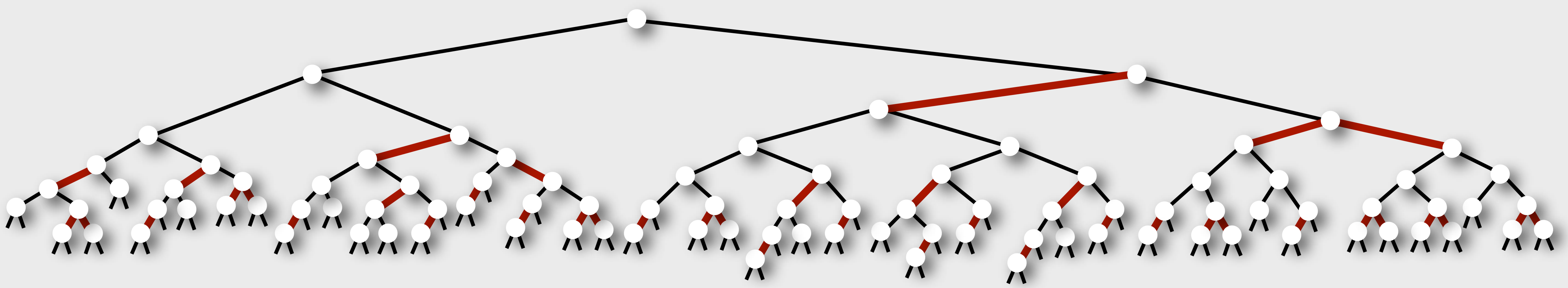
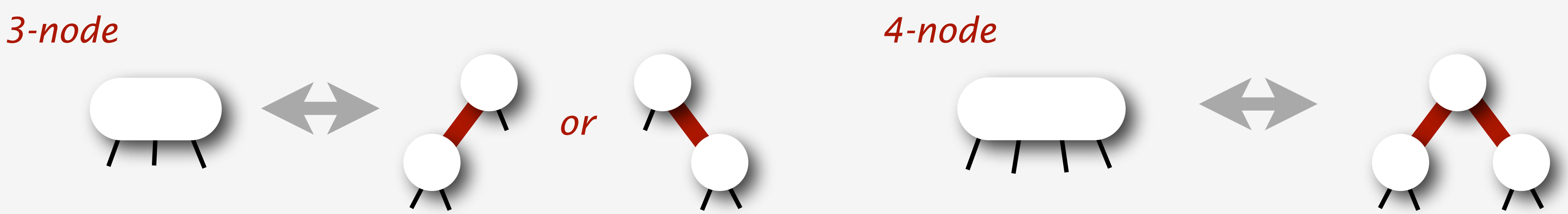


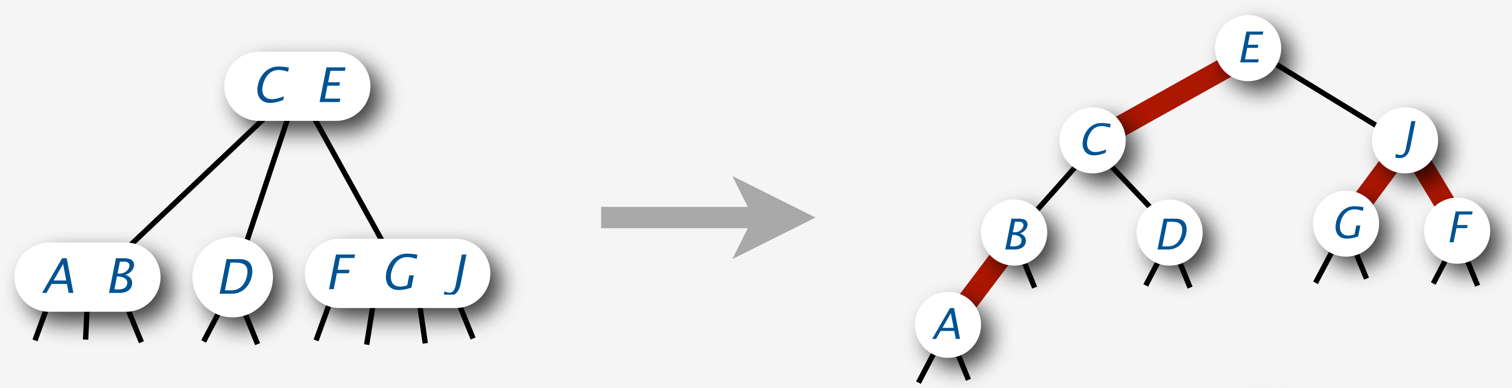
Red-black trees (Guibas-Sedgewick, 1978)

1. Represent 2-3-4 tree as a BST.
2. Use "internal" red edges for 3- and 4- nodes.

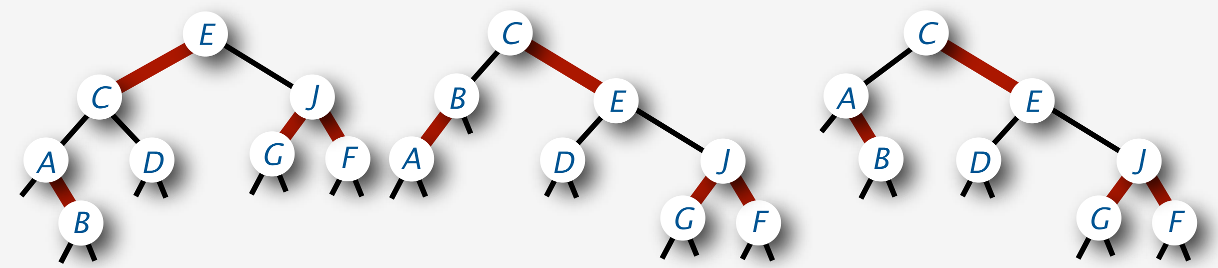


Key Properties

- elementary BST search works
- easy to maintain a correspondence with 2-3-4 trees (and several other types of balanced trees)



Note: correspondence is not 1-1.
(3-nodes can lean either way)

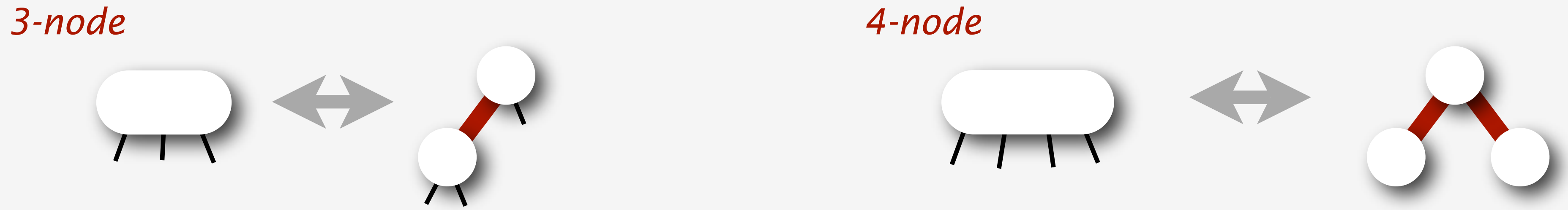


Many variants studied (details omitted.)

NEW VARIANT (this talk): Left-leaning red-black trees

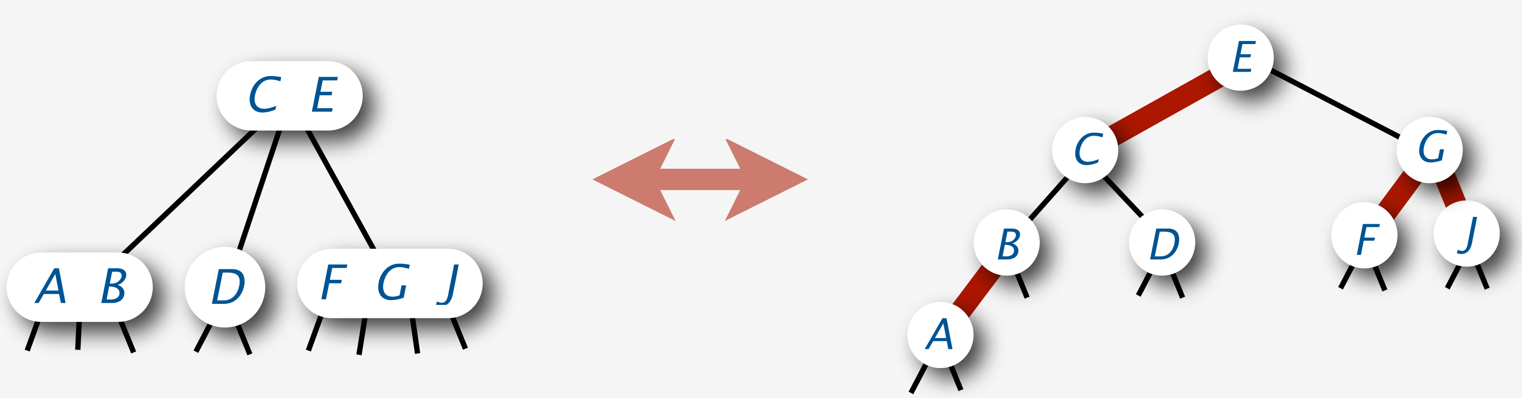
Left-leaning red-black trees

1. Represent 2-3-4 tree as a BST.
2. Use "internal" red edges for 3- and 4- nodes.
3. Require that 3-nodes be left-leaning.



Key Properties

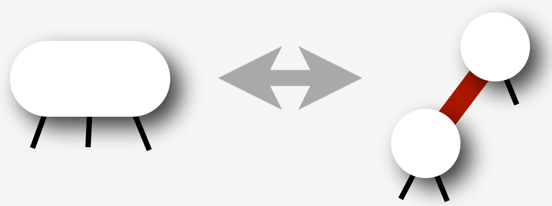
- elementary BST search works
- easy-to-maintain 1-1 correspondence with 2-3-4 trees
- trees therefore have perfect black-link balance



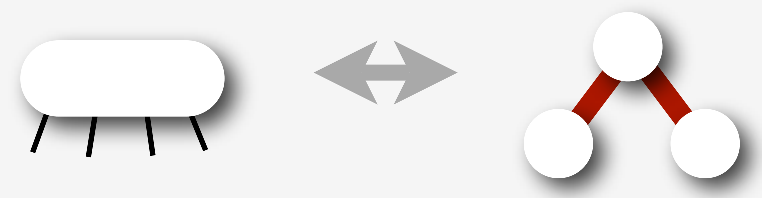
Left-leaning red-black trees

1. Represent 2-3-4 tree as a BST.
2. Use "internal" red edges for 3- and 4- nodes.
3. Require that 3-nodes be left-leaning.

3-node

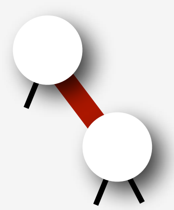


4-node



Disallowed

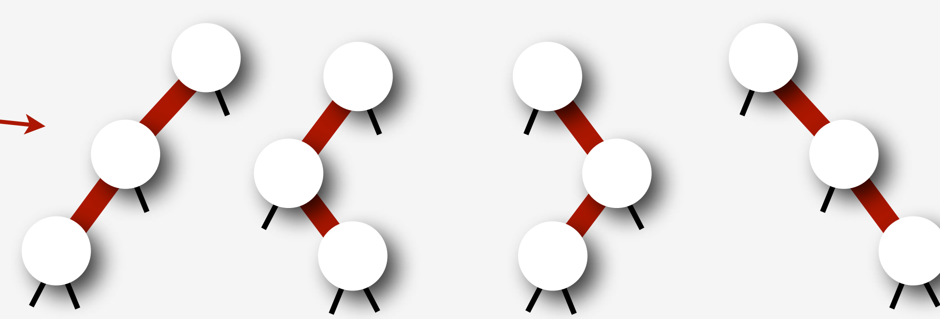
- right-leaning 3-node representation



standard red-black trees allow this one

- two reds in a row

original version of left-leaning trees used this 4-node representation



single-rotation trees allow all of these

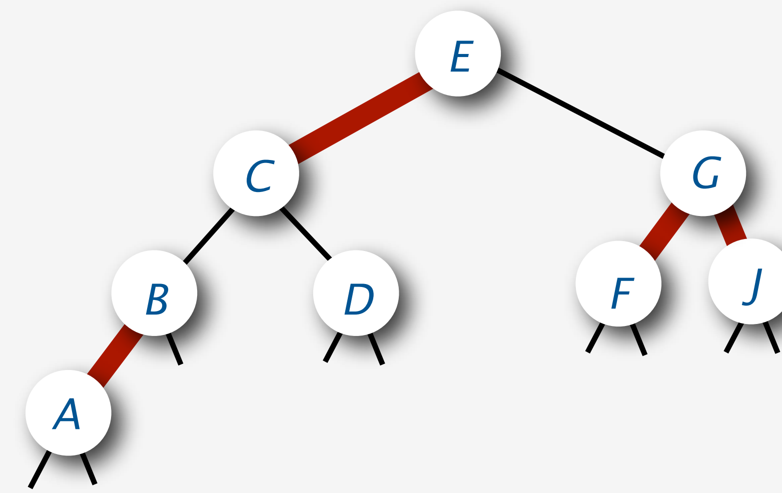
Java data structure for red-black trees

adds one bit for color to elementary BST data structure

```
public class BST<Key extends Comparable<Key>, Value>
{
    private static final boolean RED   = true;
    private static final boolean BLACK = false;
    private Node root;

    private class Node
    {
        Key key;
        Value val;
        Node left, right;
        boolean color;
        Node(Key key, Value val, boolean color)
        {
            this.key   = key;
            this.val = val;
            this.color = color;
        }
    }

   public Value get(Key key)
   // Search method.

   public void put(Key key, Value val)
   // Insert method.
}
```

constants

color of incoming link



helper method to test node color

```
private boolean isRed(Node x)
{
   if (x == null) return false;
   return (x.color == RED);
}
```

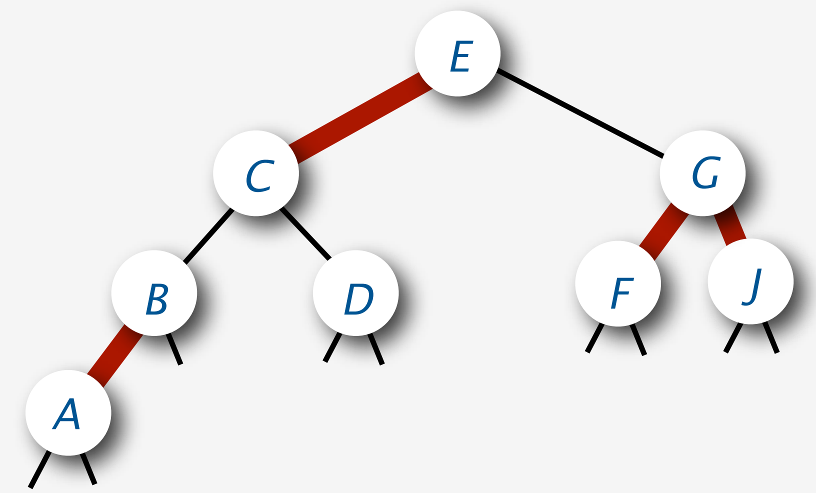
Search implementation for red-black trees

is the same as for elementary BSTs

(but typically runs faster because of better balance in the tree).

BST (and LLRB tree) search implementation

```
public Value get(Key key)
{
   Node x = root;
   while (x != null)
   {
      int cmp = key.compareTo(x.key);
      if (cmp == 0)      return x.val;
      else if (cmp < 0) x = x.left;
      else if (cmp > 0) x = x.right;
   }
   return null;
}
```



Important note: Other BST methods also work

- order statistics
- iteration

Ex: Find the minimum key

```
public Key min()
{
   Node x = root;
   while (x != null) x = x.left;
   if (x == null) return null;
   else           return x.key;
}
```

Insert implementation for LLRB trees

is best expressed in a recursive implementation

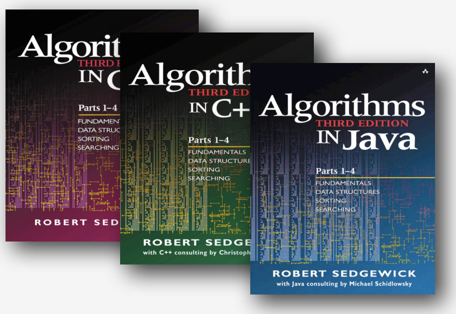
Recursive insert() *implementation for elementary BSTs*

```
private Node insert(Node h, Key key, Value val)
{
   if (h == null)
      return new Node(key, val);

   int cmp = key.compareTo(h.key);
   if (cmp == 0) h.val = val;
   else if (cmp < 0)
      h.left = insert(h.left, key, val);
   else
      h.right = insert(h.right, key, val);

   return h;
}
```

associative model (no duplicate keys)

Nonrecursive

Recursive



Note: effectively travels down the tree and then up the tree.

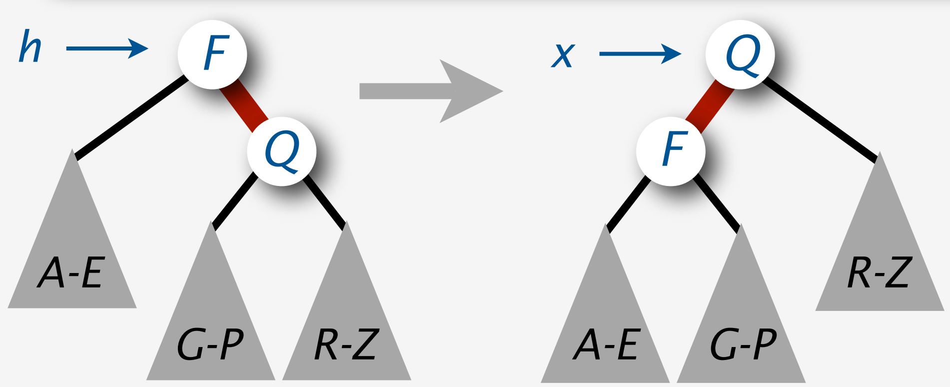
- simplifies correctness proof
- simplifies code for balanced BST implementations
- could remove recursion to get stack-based single-pass algorithm

Balanced tree code

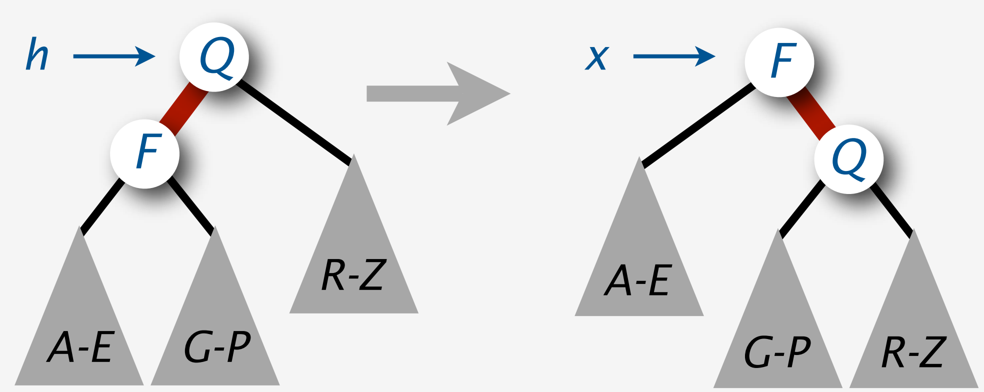
is based on local transformations known as rotations

In red-black trees, we only rotate red links
(to maintain perfect black-link balance)

```
private Node rotateLeft(Node h)
{
   Node x = h.right;
   h.right = x.left;
   x.left = h;
   x.color = x.left.color;
   x.left.color = RED;
   return x;
}
```



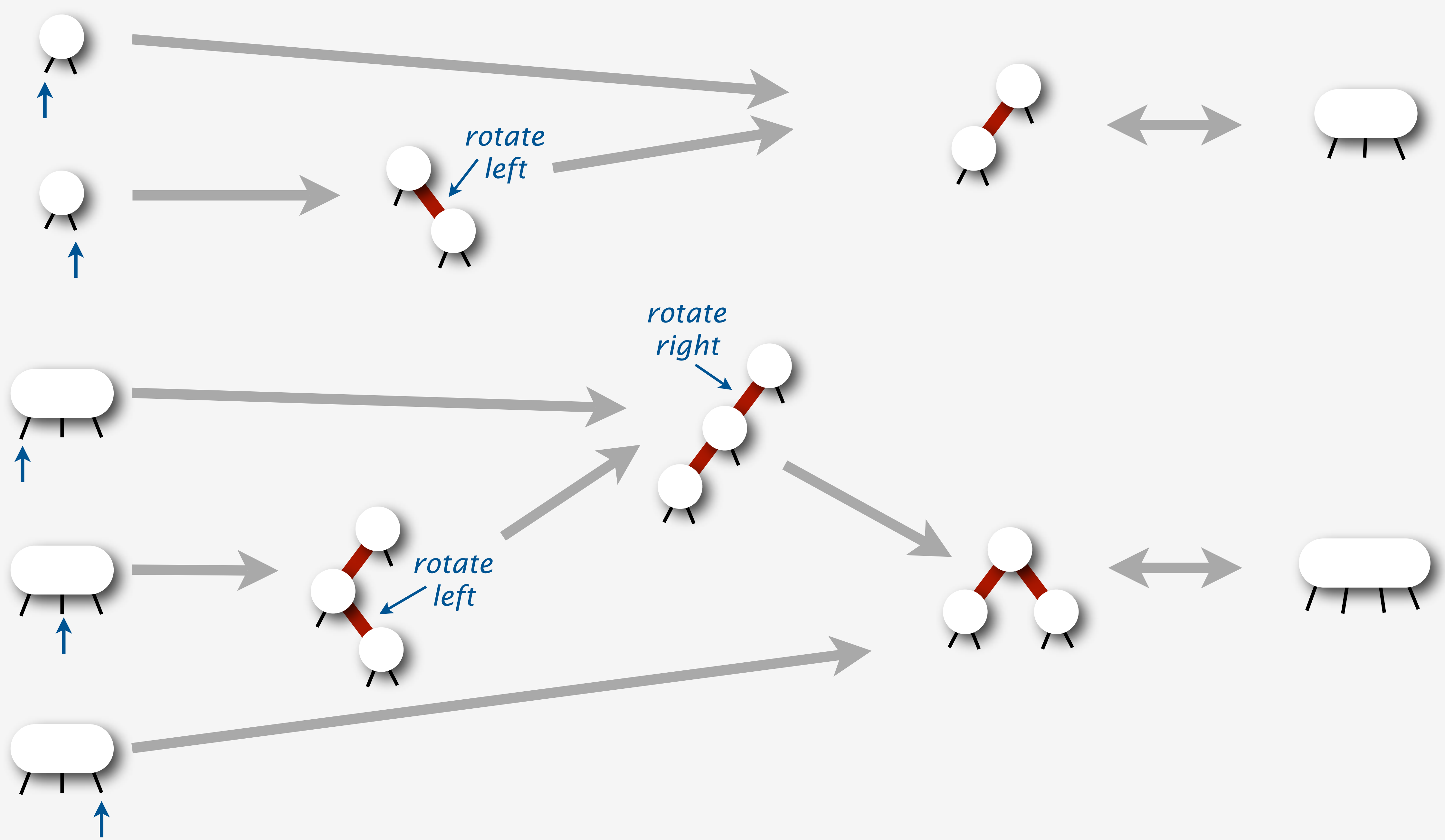

```
private Node rotateRight(Node h)
{
   Node x = h.left;
   h.left = x.right;
   x.right = h;
   x.color = x.right.color;
   x.right.color = RED;
   return x;
}
```



Insert a new node at the bottom in a LLRB tree

follows directly from 1-1 correspondence with 2-3-4 trees

1. Add new node as usual, with red link to glue it to node above
2. Rotate if necessary to get correct 3-node or 4-node representation

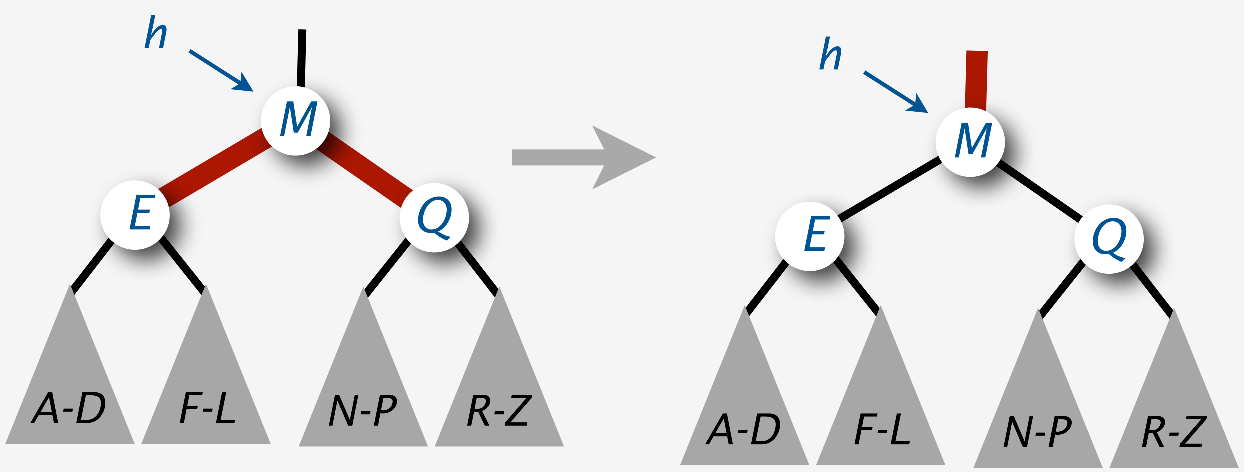


Splitting a 4-node

is accomplished with a color flip

Flip the colors of the three nodes

```
private Node colorFlip(Node h)
{
   x.color       = !x.color;
   x.left.color  = !x.left.color;
   x.right.color = !x.right.color;
   return x;
}
```



Key points:

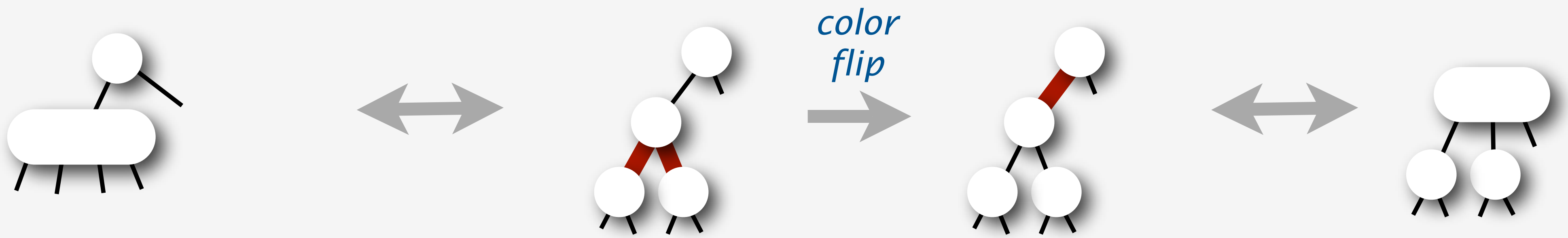
- preserves prefect black-lin balance
- passes a RED link up the tree
- reduces problem to inserting (that link) into parent

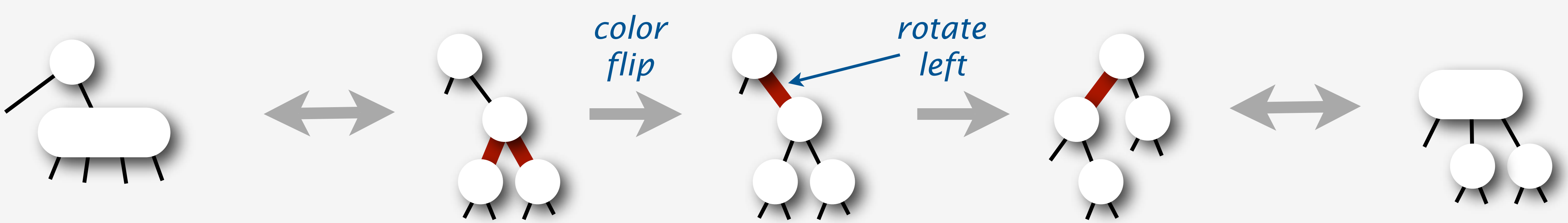
Splitting a 4-node in a LLRB tree

follows directly from 1-1 correspondence with 2-3-4 trees

1. Flip colors, which passes red link up one level
2. Rotate if necessary to get correct representation in parent (using precisely the same transformations as for insert at bottom)

Parent is a 2-node: two cases



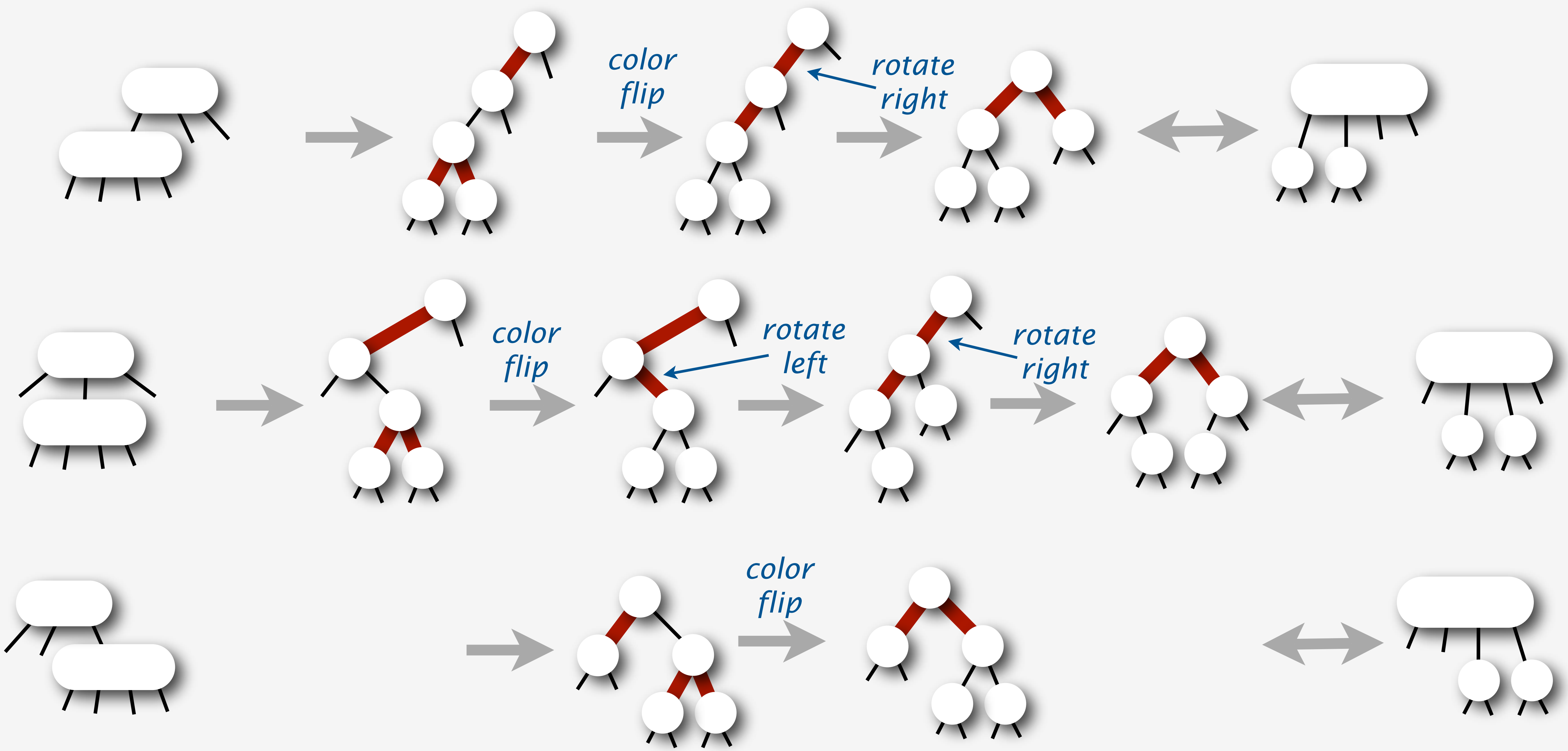


Splitting a 4-node in a LLRB tree

follows directly from 1-1 correspondence with 2-3-4 trees

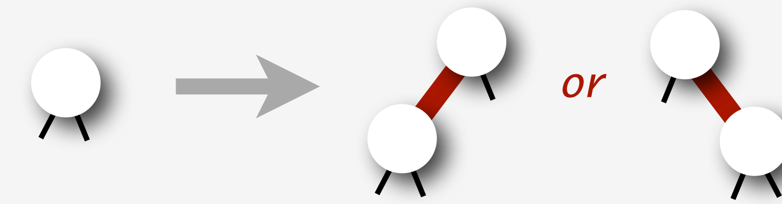
1. Flip colors, which passes red link up one level
2. Rotate if necessary to get correct representation in parent (using precisely the same transformations as for insert at bottom)

Parent is a 3-node: three cases



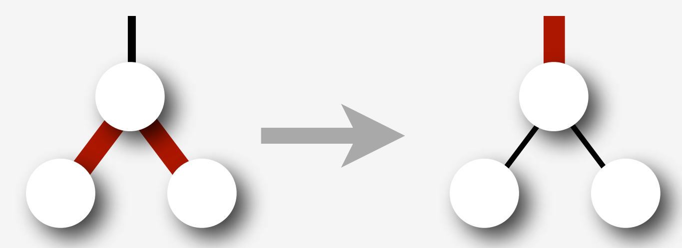
Inserting and splitting nodes in LLRB trees

are easier when rotates are done on the way up the tree.



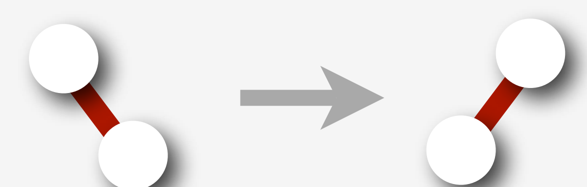
Search as usual

- if key found reset value, as usual
- if key not found insert new red node at the bottom
- might leave right-leaning red or two reds in a row higher up in the tree



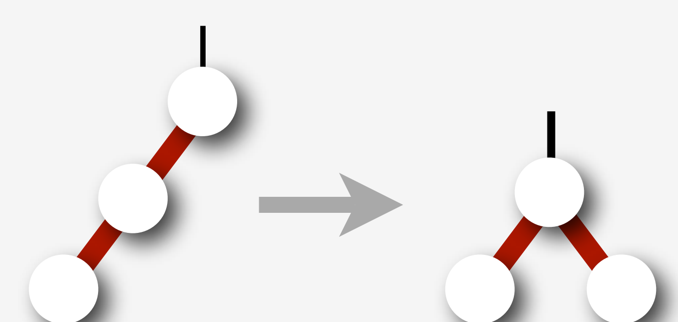
Split 4-nodes on the way down the tree.

- flip color
- might leave right-leaning red or two reds in a row higher up in the tree



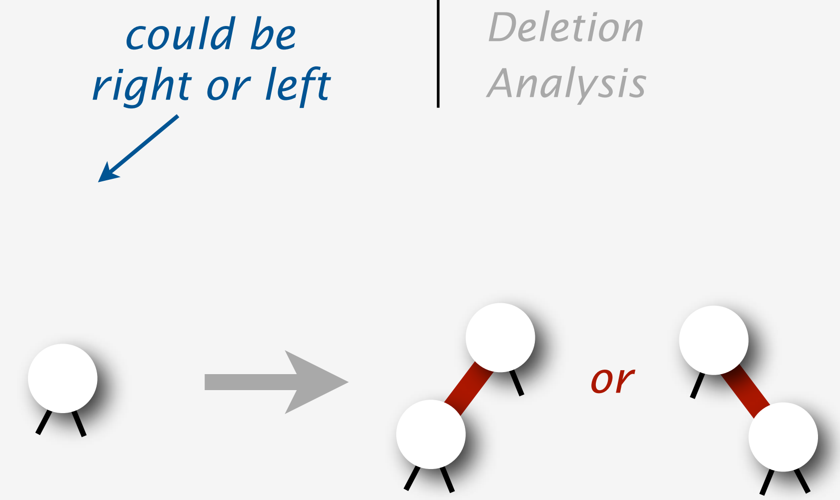
NEW TRICK: Do rotates on the way UP the tree.

- left-rotate any right-leaning link on search path
- right-rotate top link if two reds in a row found
- trivial with recursion (do it after recursive calls)
- no corrections needed elsewhere



Insert code for LLRB trees

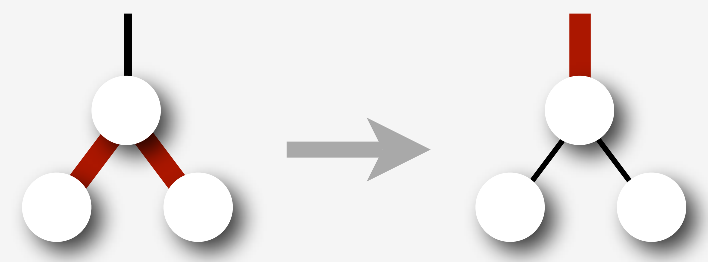
is based on four simple operations.



1. Insert a new node at the bottom.

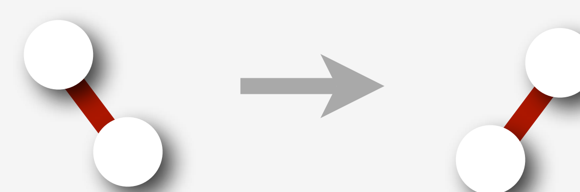
```
if (h == null)
      return new Node(key, value, RED);
```

2. Split a 4-node.



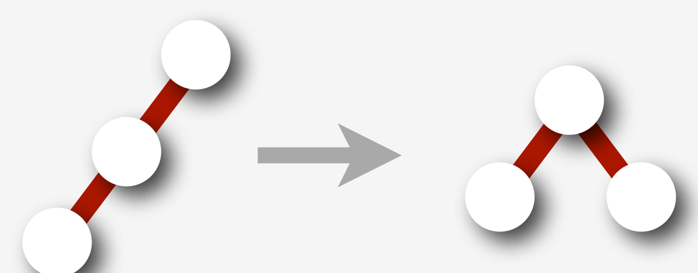
```
if (isRed(h.left) && isRed(h.right))
      colorFlip(h);
```

3. Enforce left-leaning condition.



```
if (isRed(h.right))
      h = rotateLeft(h);
```

4. Balance a 4-node.



```
if (isRed(h.left) && isRed(h.left.left))
      h = rotateRight(h);
```

Insert implementation for LLRB trees

is a few lines of code added to elementary BST insert

```
private Node insert(Node h, Key key, Value val)
{
   if (h == null)
      return new Node(key, val, RED);

   if (isRed(h.left) && isRed(h.right))
      colorFlip(h);

   int cmp = key.compareTo(h.key);
   if (cmp == 0) h.val = val;
   else if (cmp < 0)
      h.left = insert(h.left, key, val);
   else
      h.right = insert(h.right, key, val);

   if (isRed(h.right))
      h = rotateLeft(h);

   if (isRed(h.left) && isRed(h.left.left))
      h = rotateRight(h);

   return h;
}
```

insert at the bottom

split 4-nodes on the way down

standard BST insert code

fix right-leaning reds on the way up

fix two reds in a row on the way up

LLRB (top-down 2-3-4) insert movie

A surprise

Q. What happens if we move color flip to the end?

```
private Node insert(Node h, Key key, Value val)
{
   if (h == null)
      return new Node(key, val, RED);

   if (isRed(h.left) && isRed(h.right))
      colorFlip(h);

   int cmp = key.compareTo(h.key);
   if (cmp == 0) h.val = val;
   else if (cmp < 0)
      h.left = insert(h.left, key, val);
   else
      h.right = insert(h.right, key, val);

   if (isRed(h.right))
      h = rotateLeft(h);

   if (isRed(h.left) && isRed(h.left.left))
      h = rotateRight(h);

   return h;
}
```

A surprise

Q. What happens if we move color flip to the end?

```
private Node insert(Node h, Key key, Value val)
{
   if (h == null)
      return new Node(key, val, RED);

   int cmp = key.compareTo(h.key);
   if (cmp == 0) h.val = val;
   else if (cmp < 0)
      h.left = insert(h.left, key, val);
   else
      h.right = insert(h.right, key, val);

   if (isRed(h.right))
      h = rotateLeft(h);

   if (isRed(h.left) && isRed(h.left.left))
      h = rotateRight(h);

   if (isRed(h.left) && isRed(h.right))
      colorFlip(h);

   return h;
}
```

A surprise

Q. What happens if we move color flip to the end?

A. It becomes an implementation of 2-3 trees (!)

```
private Node insert(Node h, Key key, Value val)
{
   if (h == null)
      return new Node(key, val, RED);

   int cmp = key.compareTo(h.key);
   if (cmp == 0) h.val = val;
   else if (cmp < 0)
      h.left = insert(h.left, key, val);
   else
      h.right = insert(h.right, key, val);

   if (isRed(h.right))
      h = rotateLeft(h);

   if (isRed(h.left) && isRed(h.left.left))
      h = rotateRight(h);

   if (isRed(h.left) && isRed(h.right))
      colorFlip(h);

   return h;
}
```

Insert in 2-3 tree:

attach new node with red link

2-node → 3-node
3-node → 4-node

split 4-node

pass red link up to parent and repeat

no 4-nodes left!

Insert implementation for 2-3 trees (!)

is a few lines of code added to elementary BST insert

```
private Node insert(Node h, Key key, Value val)
{
   if (h == null)
      return new Node(key, val, RED);

   int cmp = key.compareTo(h.key);
   if (cmp == 0) h.val = val;
   else if (cmp < 0)
      h.left = insert(h.left, key, val);
   else
      h.right = insert(h.right, key, val);

   if (isRed(h.right))
      h = rotateLeft(h);

   if (isRed(h.left) && isRed(h.left.left))
      h = rotateRight(h);

   if (isRed(h.left) && isRed(h.right))
      colorFlip(h);

   return h;
}
```

insert at the bottom

standard BST insert code

fix right-leaning reds on the way up

fix two reds in a row on the way up

split 4-nodes on the way up

LLRB (bottom-up 2-3) insert movie

Why revisit red-black trees?

Which do you prefer?


```
private Node insert(Node x, Key key, Value val, boolean sw)
{
   if (x == null)
      return new Node(key, value, RED);
   int cmp = key.compareTo(x.key);

   if (isRed(x.left) && isRed(x.right))
   {
      x.color = RED;
      x.left.color  = BLACK;
      x.right.color = BLACK;
   }
   if (cmp == 0) x.val = val;
   else if (cmp < 0))
   {
     x.left = insert(x.left, key, val, false);
     if (isRed(x) && isRed(x.left) && sw)
        x = rotR(x);
     if (isRed(x.left) && isRed(x.left.left))
      {
         x = rotR(x);
         x.color = BLACK; x.right.color = RED;
      }
   }
   else // if (cmp > 0)
   {
      x.right = insert(x.right, key, val, true);
      if (isRed(h) && isRed(x.right) && !sw)
         x = rotL(x);
      if (isRed(h.right) && isRed(h.right.right))
      {
         x = rotL(x);
         x.color = BLACK; x.left.color = RED;
      }
   }
   return x;
}
```

very tricky

```
private Node insert(Node h, Key key, Value val)
{
   if (h == null)
      return new Node(key, val, RED);

   int cmp = key.compareTo(h.key);
   if (cmp == 0) h.val = val;
   else if (cmp < 0)
      h.left = insert(h.left, key, val);
   else
      h.right = insert(h.right, key, val);

   if (isRed(h.right))
      h = rotateLeft(h);
   if (isRed(h.left) && isRed(h.left.left))
      h = rotateRight(h);
   if (isRed(h.left) && isRed(h.right))
      colorFlip(h);

   return h;
}
```

straightforward

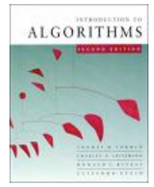
Left-Leaning
Red-Black Trees
Robert Sedgewick
Princeton University

Why revisit red-black trees?

Take your pick:

TreeMap.java

Adapted from CLR by experienced professional programmers (2004)



150

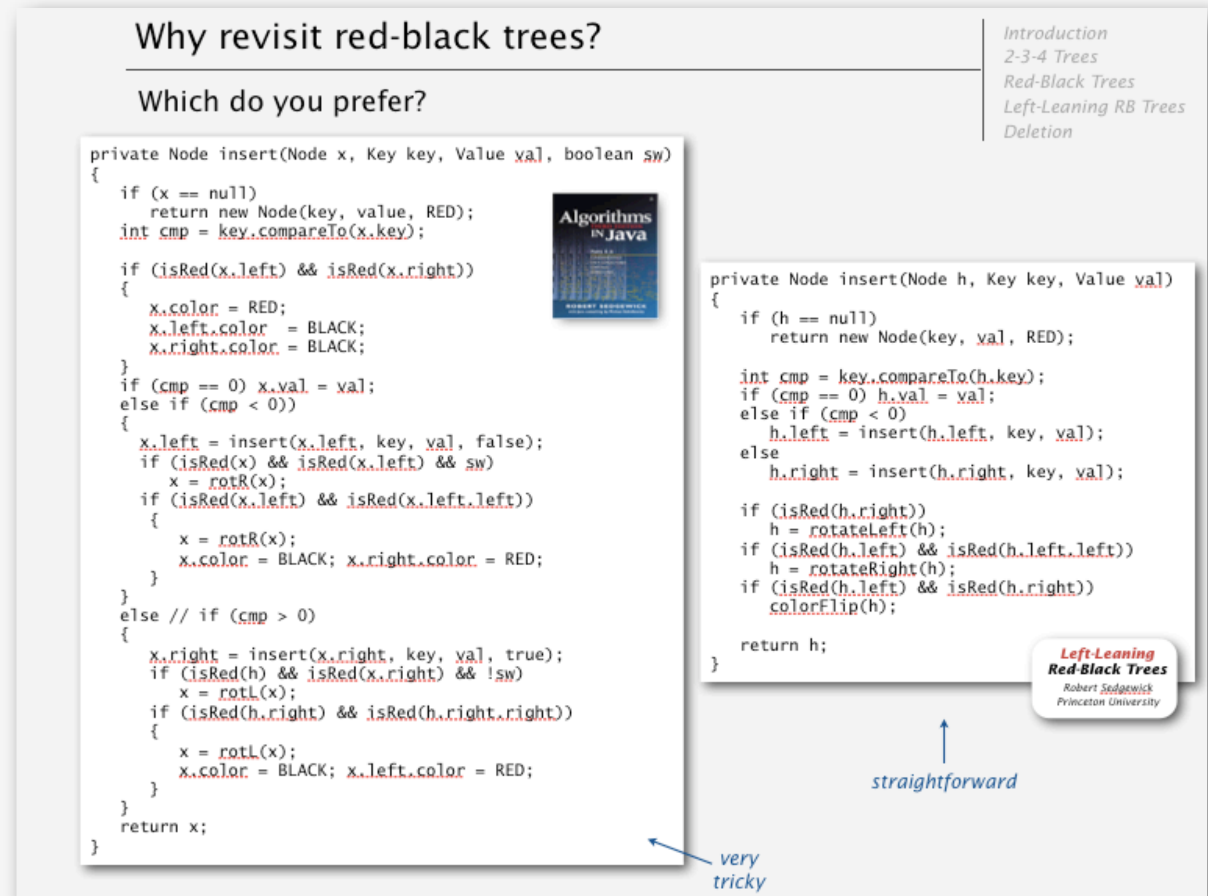
46

Left-Leaning Red-Black Trees
Robert Sedgewick
Princeton University

33

lines of code for insert (lower is better!)

wrong scale!



Why revisit red-black trees?

Introduction
2-3-4 Trees
Red-Black Trees
Left-Leaning RB Trees
Deletion

Which do you prefer?

```
private Node insert(Node x, Key key, Value val, boolean sw)
{
   if (x == null)
      return new Node(key, value, RED);
   int cmp = key.compareTo(x.key);

   if (isRed(x.left) && isRed(x.right))
   {
      x.color = RED;
      x.left.color  = BLACK;
      x.right.color = BLACK;
   }
   if (cmp == 0) x.val = val;
   else if (cmp < 0))
   {
     x.left = insert(x.left, key, val, false);
     if (isRed(x) && isRed(x.left) && sw)
        x = rotR(x);
     if (isRed(x.left) && isRed(x.left.left))
      {
         x = rotR(x);
         x.color = BLACK; x.right.color = RED;
      }
   }
   else // if (cmp > 0)
   {
      x.right = insert(x.right, key, val, true);
      if (isRed(h) && isRed(x.right) && !sw)
         x = rotL(x);
      if (isRed(h.right) && isRed(h.right.right))
      {
         x = rotL(x);
         x.color = BLACK; x.left.color = RED;
      }
   }
   return x;
}
```

very tricky

```
private Node insert(Node h, Key key, Value val)
{
   if (h == null)
      return new Node(key, val, RED);

   int cmp = key.compareTo(h.key);
   if (cmp == 0) h.val = val;
   else if (cmp < 0)
      h.left = insert(h.left, key, val);
   else
      h.right = insert(h.right, key, val);

   if (isRed(h.right))
      h = rotateLeft(h);
   if (isRed(h.left) && isRed(h.left.left))
      h = rotateRight(h);
   if (isRed(h.left) && isRed(h.right))
      colorFlip(h);

   return h;
}
```

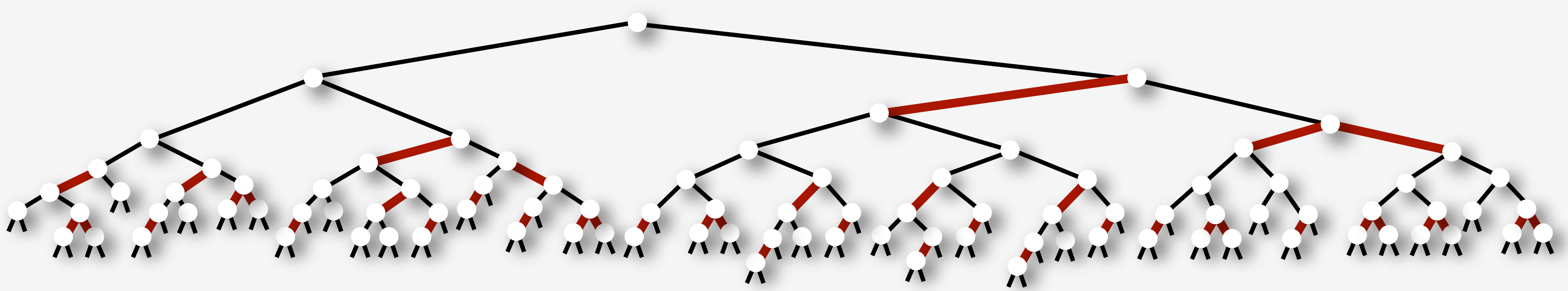
Left-Leaning Red-Black Trees
Robert Sedgewick
Princeton University

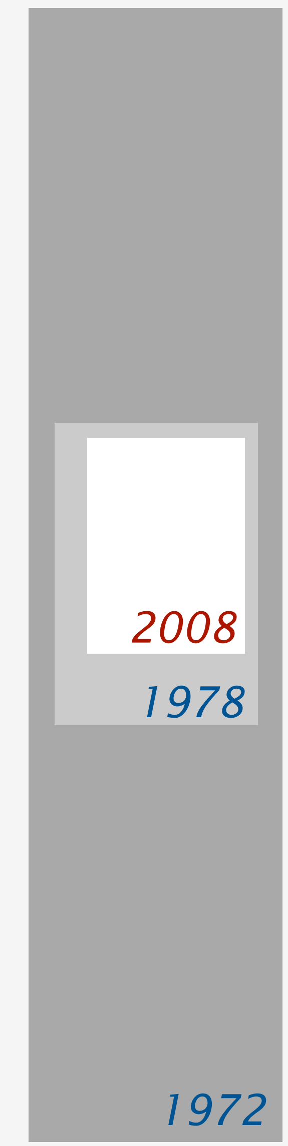
straightforward

Why revisit red-black trees?

LLRB implementation is far simpler than previous attempts.

- left-leaning restriction reduces number of cases
- recursion gives two (easy) chances to fix each node
- take your pick: top-down 2-3-4 or bottom-up 2-3





Improves widely used implementations

- AVL, 2-3, and 2-3-4 trees
- red-black trees

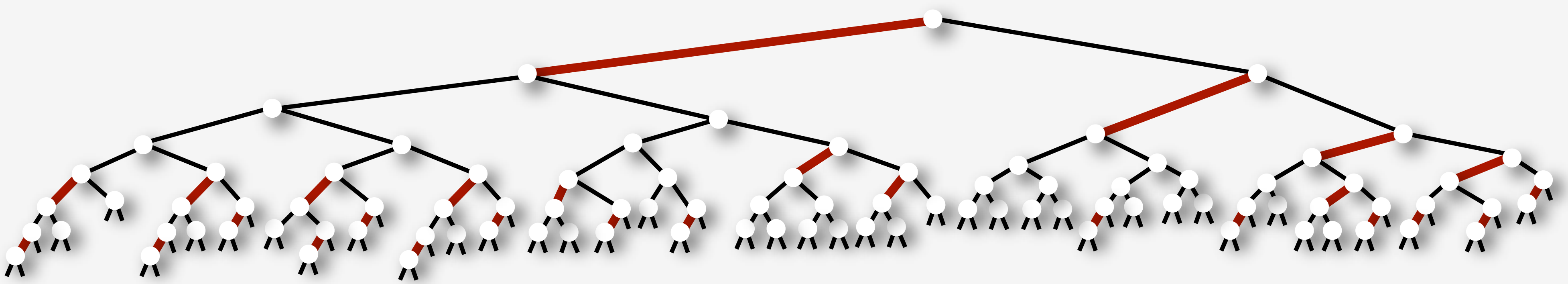
Same ideas simplify implementation of other operations

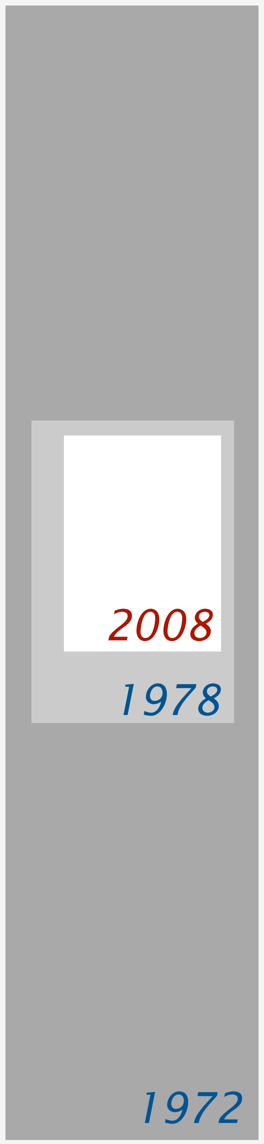
- delete min, max
- arbitrary delete

Why revisit red-black trees?

LLRB implementation is far simpler than previous attempts.

- left-leaning restriction reduces number of cases
- recursion gives two (easy) chances to fix each node
- take your pick: top-down 2-3-4 or bottom-up 2-3





Improves widely used implementations

- AVL, 2-3, and 2-3-4 trees
- red-black trees

Same ideas simplify implementation of other operations

- delete min, max
- arbitrary delete

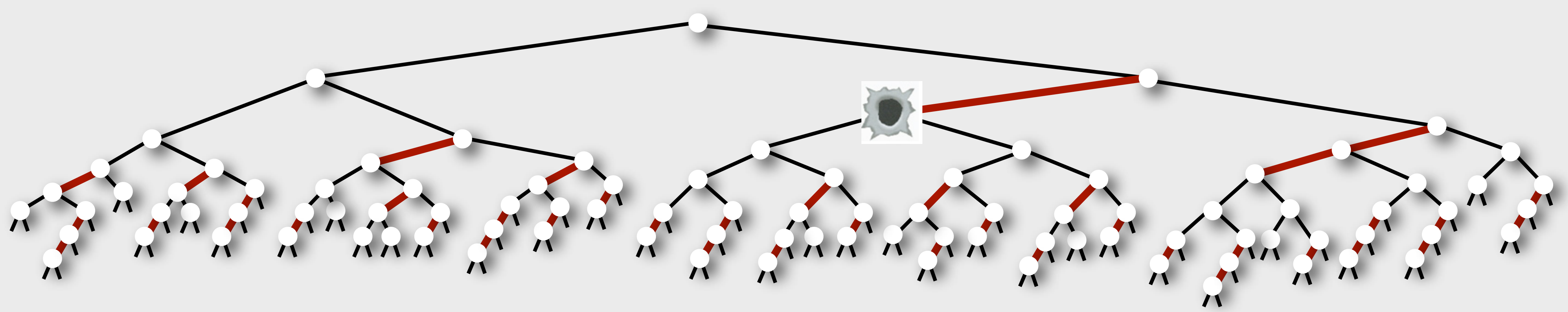
Introduction

2-3-4 Trees

LLRB Trees

Deletion

Analysis



Lessons learned from insert() implementation

also simplify delete() implementations

1. Color flips and rotations preserve perfect black-link balance.
2. Fix right-leaning reds and eliminate 4-nodes on the way up.

```
private Node fixUp(Node h)
{
   if (isRed(h.right))
      h = rotateLeft(h);
   if (isRed(h.left) && isRed(h.left.left))
      h = rotateRight(h);
   if (isRed(h.left) && isRed(h.right))
      colorFlip(h);
   return h;
}
```

rotate-left right-leaning reds

rotate-right red-red pairs

split 4-nodes

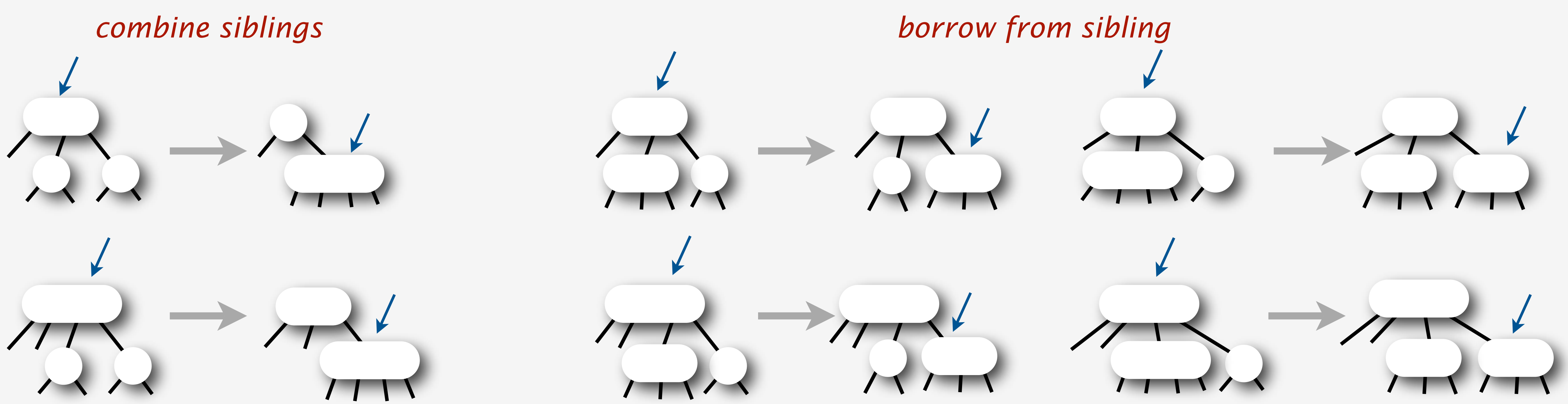
Delete strategy (works for 2-3 and 2-3-4 trees)

- invariant: current node is not a 2-node
- introduce 4-nodes if necessary
- remove key from bottom
- eliminate 4-nodes on the way up

Warmup 1: delete the maximum

1. Search down the right spine of the tree.
2. If search ends in a 3-node or 4-node: just remove it.

3. Removing a 2-node would destroy balance
   - transform tree on the way down the search path
   - Invariant: current node is not a 2-node

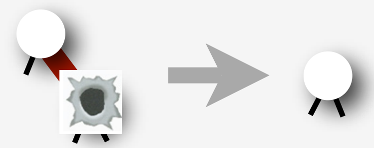


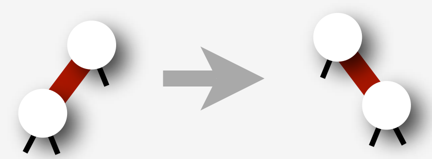
Note: LLRB representation reduces number of cases (as for insert)

Warmup 1: delete the maximum

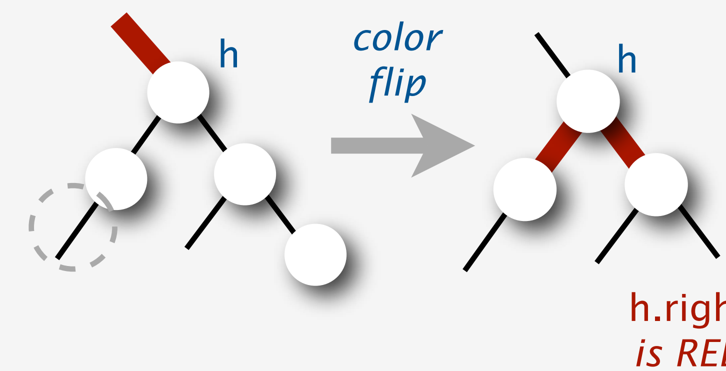
by carrying a red link down the right spine of the tree.

Invariant: either h or h.right is RED

Implication: deletion easy at bottom 

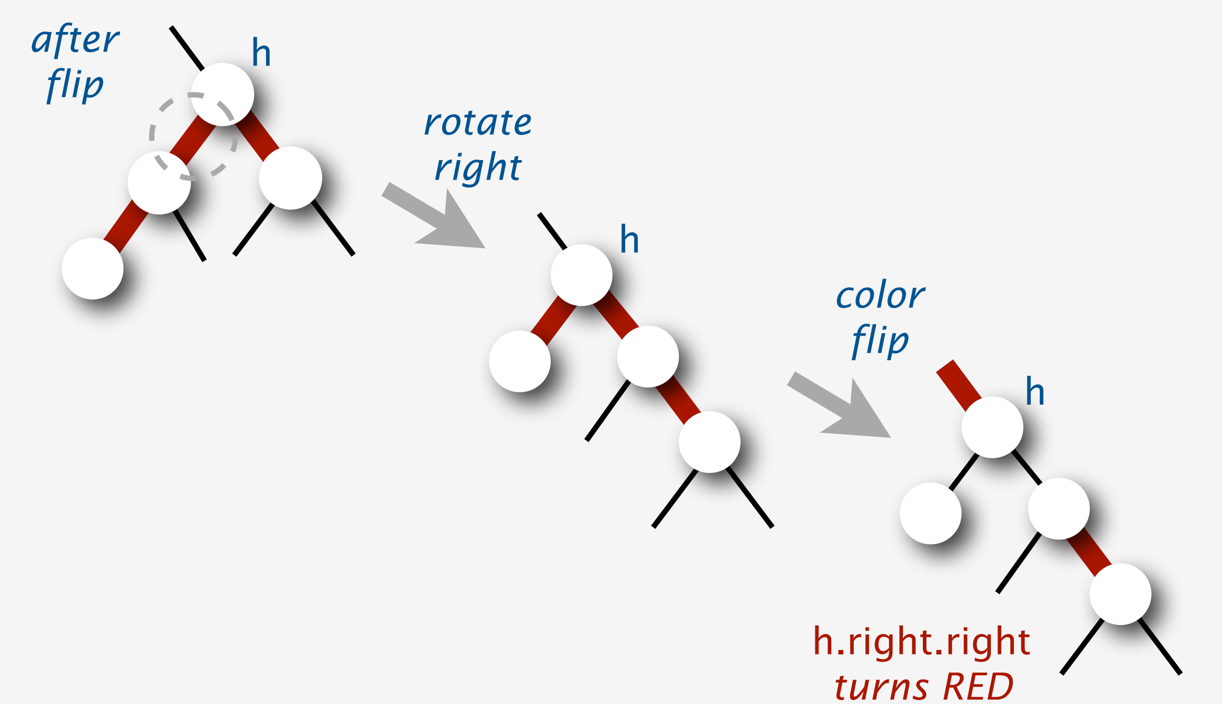
1. Rotate red links to the right 
2. Borrow from sibling if necessary
   - when h.right and h.right.left are both BLACK
   - Two cases, depending on color of h.left.left

Easy case: h.left.left *is BLACK*



Harder case: h.left.left *is RED*

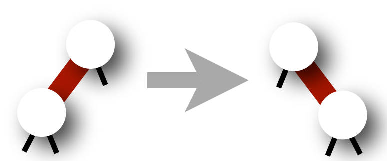
```
private Node moveRedRight(Node h)
{
   colorFlip(h);
   if (isRed(h.left.left))
   {
      h = rotateRight(h);
      colorFlip(h);
   }
   return h;
}
```



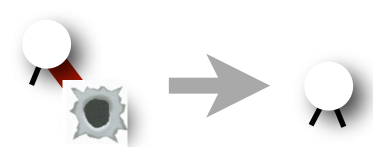
deleteMax() implementation for LLRB trees

is otherwise a few lines of code

```
public void deleteMax()
{
  root = deleteMax(root);
  root.color = BLACK;
}

private Node deleteMax(Node h)
{
   if (isRed(h.left))
      h = rotateRight(h);

   if (h.right == null)
      return null;

   if (!isRed(h.right) && !isRed(h.right.left))
      h = moveRedRight(h);

   h.left = deleteMax(h.left);

   return fixUp(h);
}
```



lean 3-nodes to the right



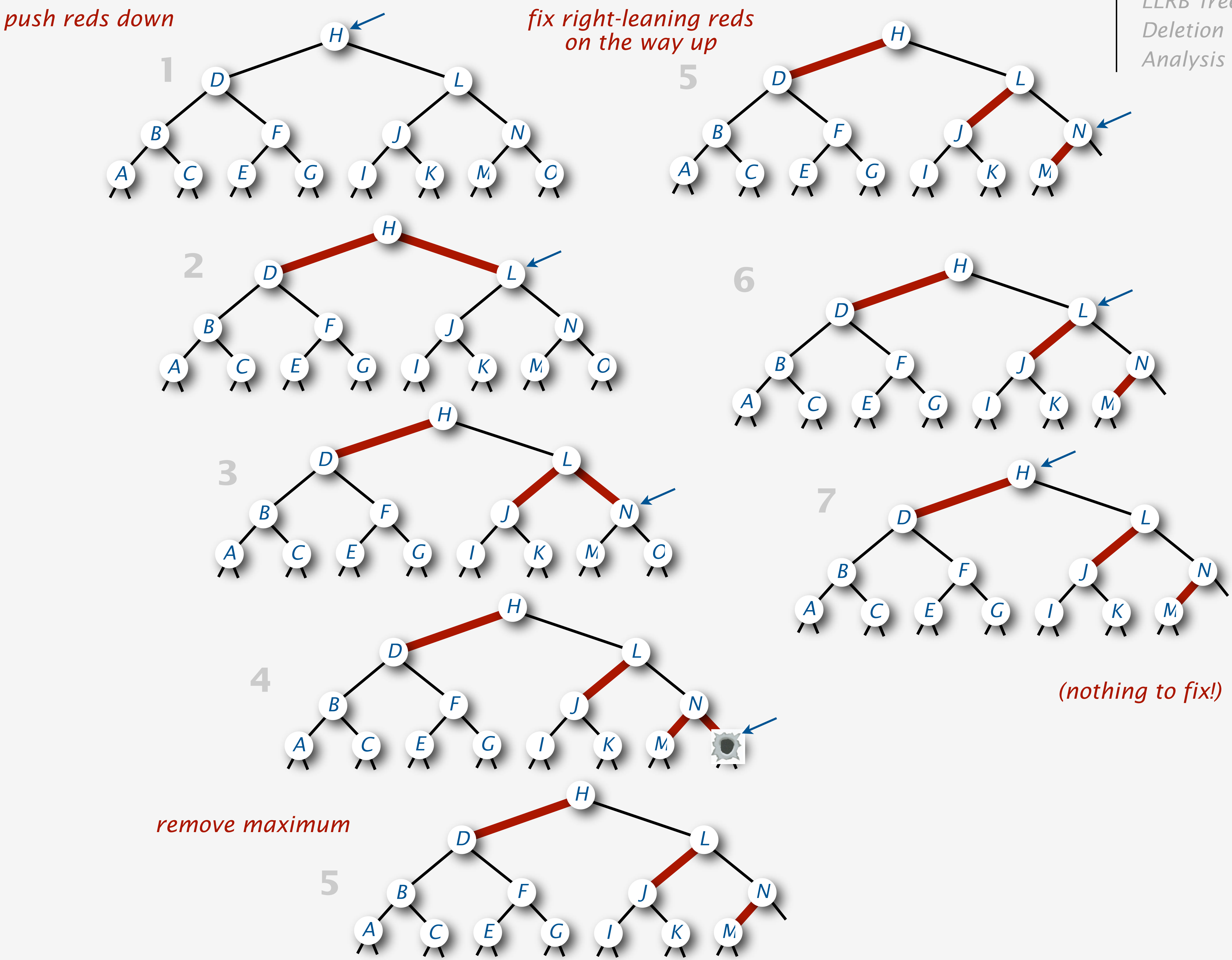
remove node on bottom level
(h *must be* RED *by invariant*)

borrow from sibling if necessary

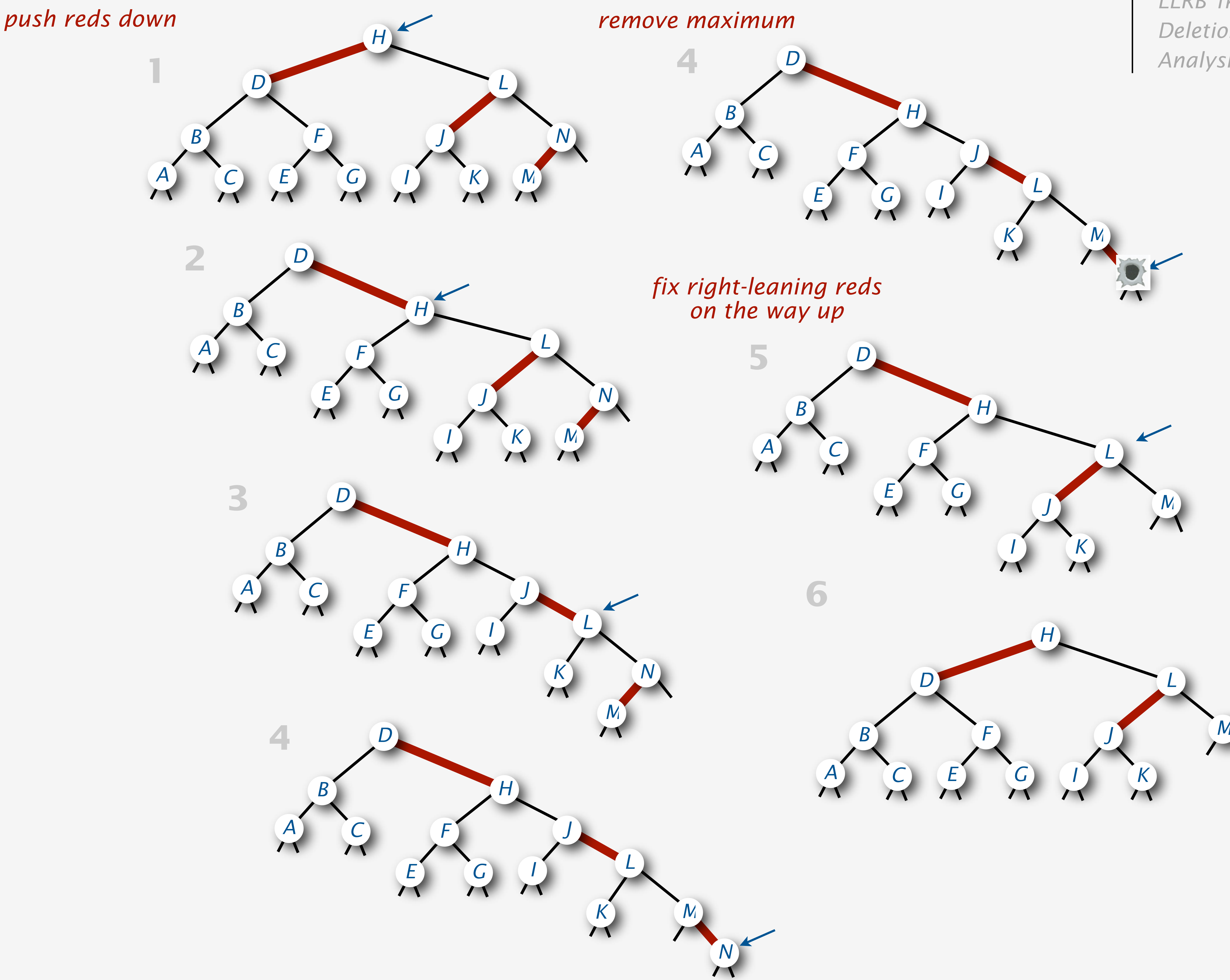
move down one level

fix right-leaning red links and eliminate 4-nodes on the way up

deleteMax() example 1



deleteMax() example 2



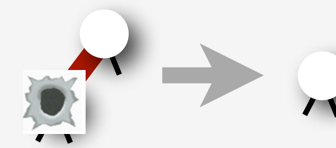
LLRB deleteMax() movie

Warmup 2: delete the minimum

is similar but slightly different (since trees lean left).

Invariant: either h or h.left is RED

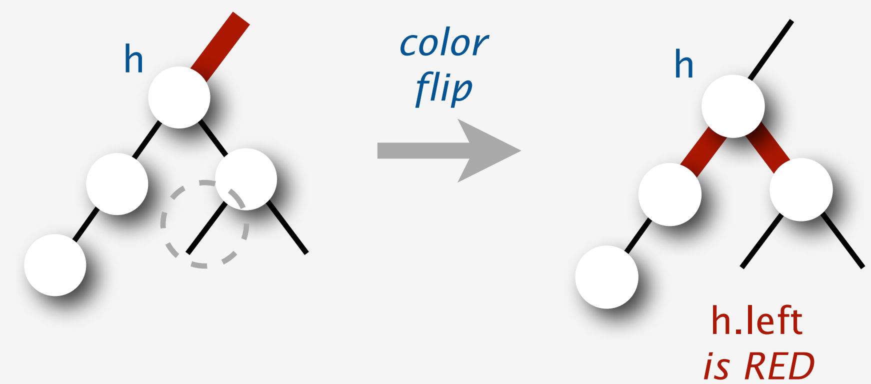
Implication: deletion easy at bottom



Borrow from sibling

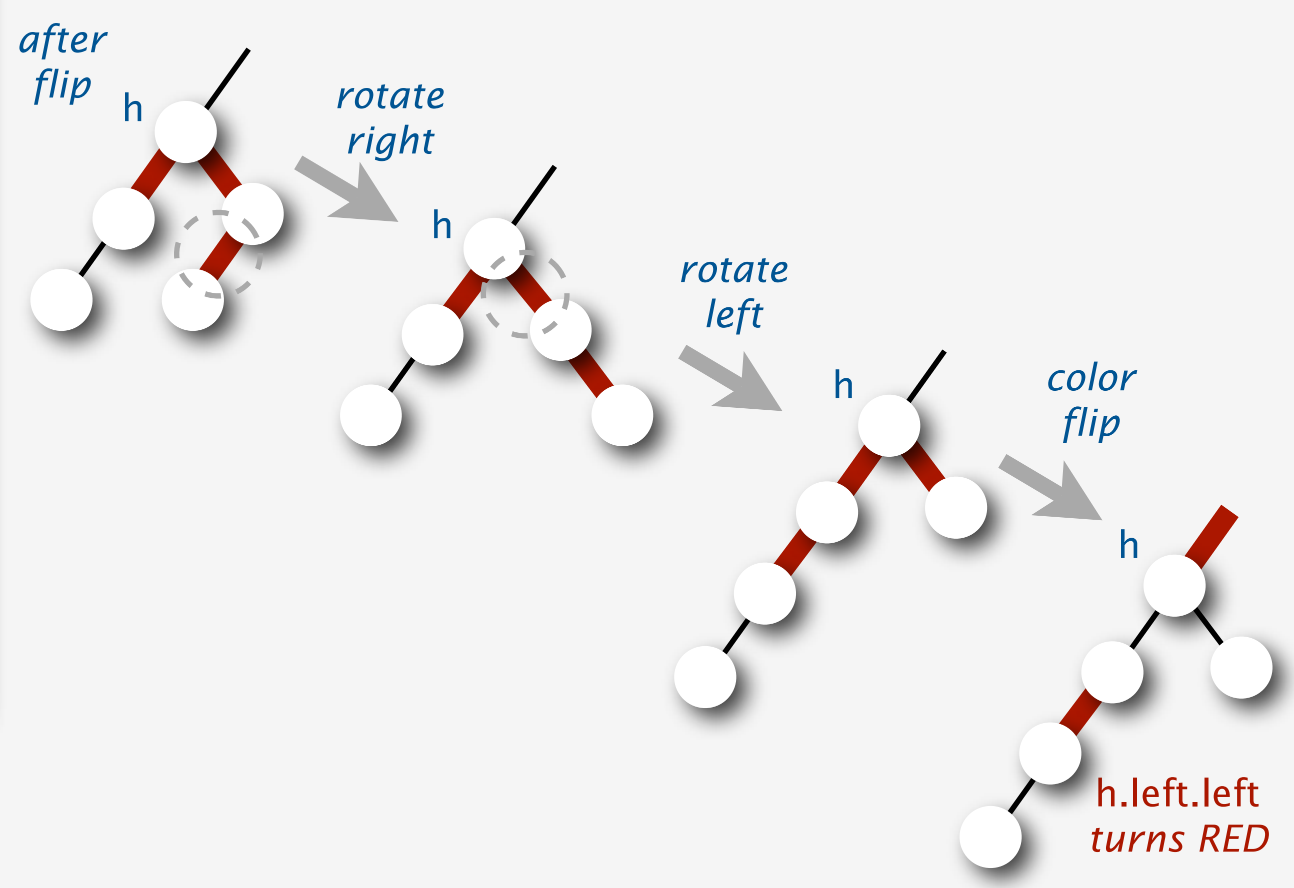
- if h.left and h.left.left are both BLACK
- two cases, depending on color of h.right.left

Easy case: h.right.left *is* BLACK




```
private Node moveRedLeft(Node h)
{
   colorFlip(h);
   if (isRed(h.right.left))
   {
      h.right = rotateRight(h.right);
      h = rotateLeft(h);
      colorFlip(h);
   }
   return h;
}
```

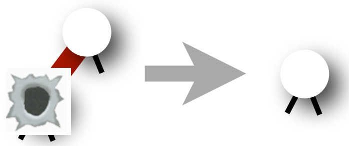
Harder case: h.right.left *is* RED



deleteMin() implementation for LLRB trees

is a few lines of code

```
public void deleteMin()
{
  root = deleteMin(root);
  root.color = BLACK;
}

private Node deleteMin(Node h)
{
   if (h.left == null)
      return null;

   if (!isRed(h.left) && !isRed(h.left.left))
      h = moveRedLeft(h);

   h.left = deleteMin(h.left);

   return fixUp(h);

}
```



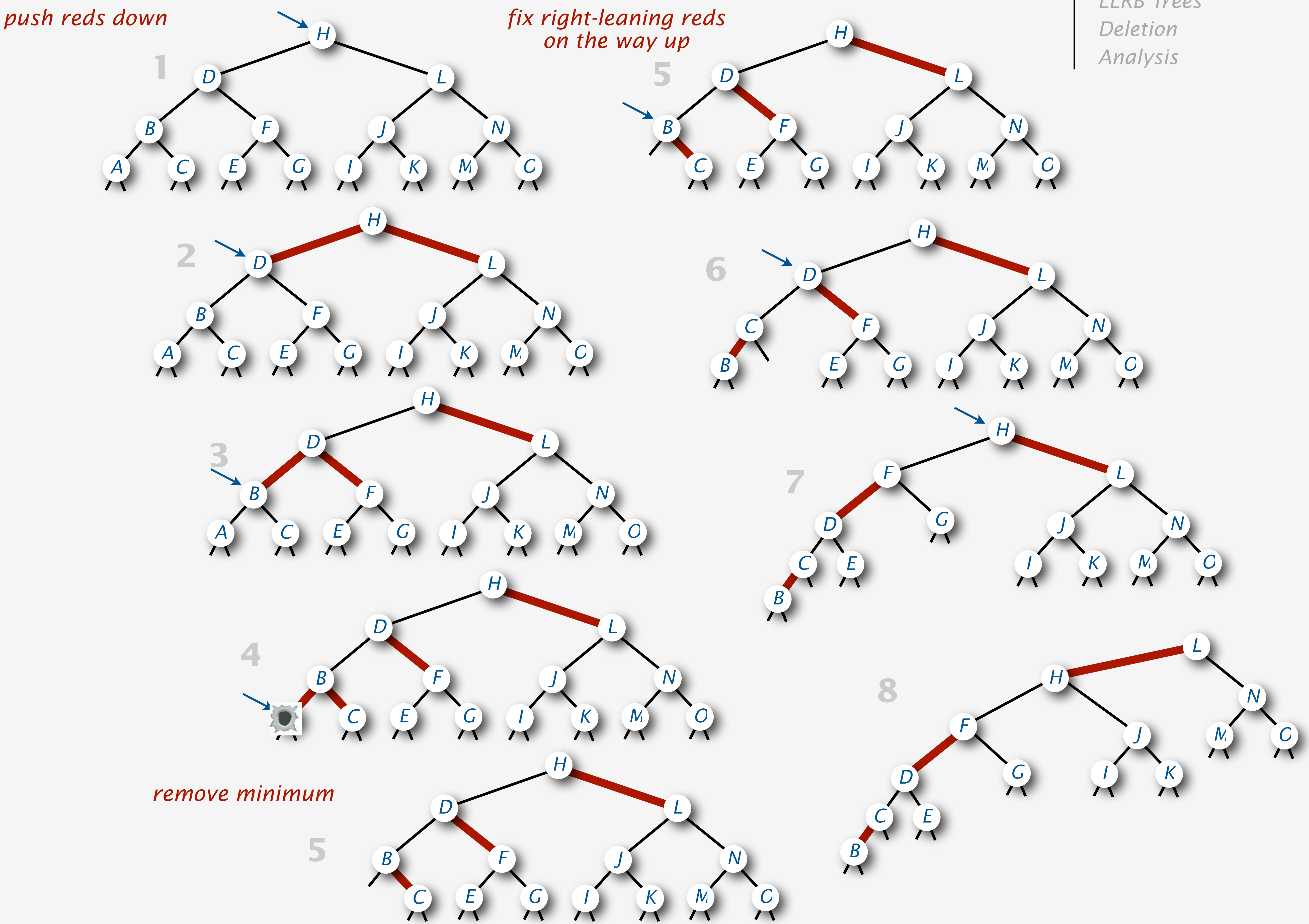
remove node on bottom level
(h *must be* RED *by invariant*)

push red link down if necessary

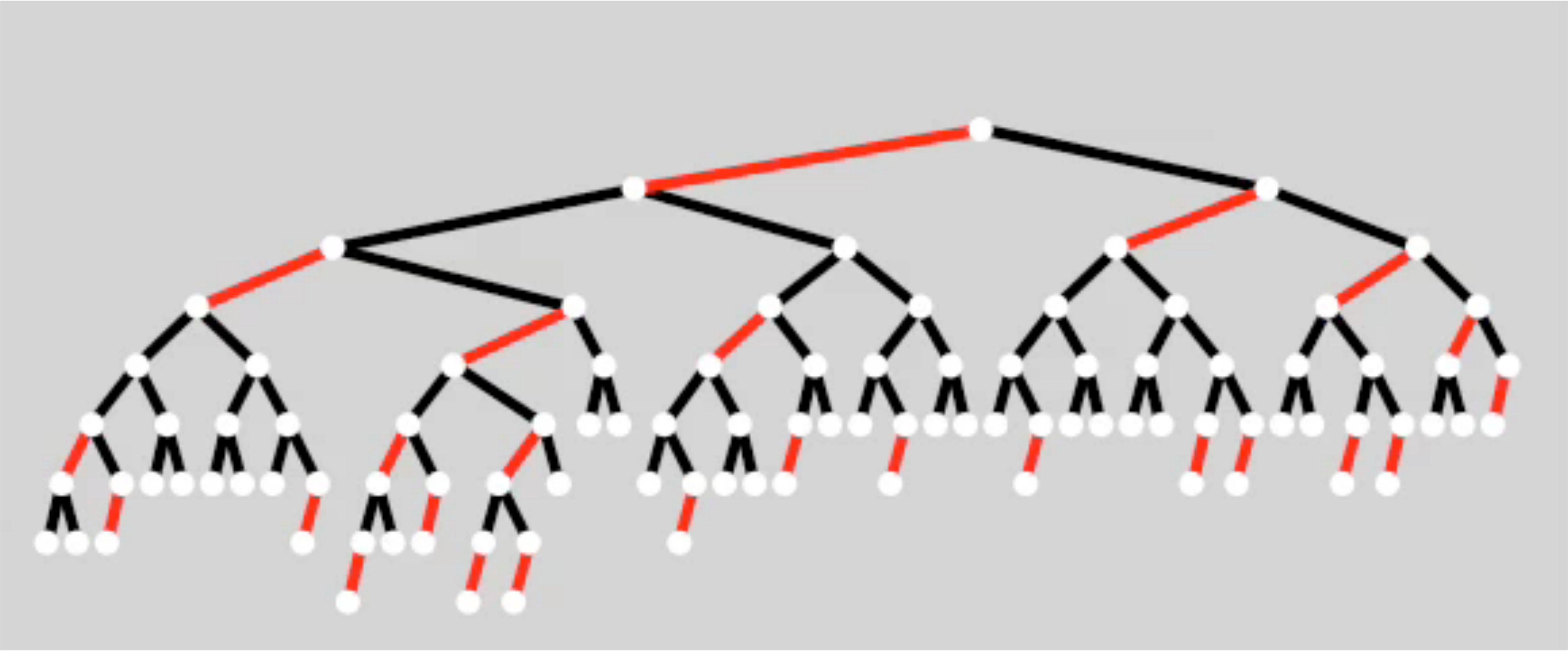
move down one level

fix right-leaning red links
and eliminate 4-nodes
on the way up

deleteMin() example



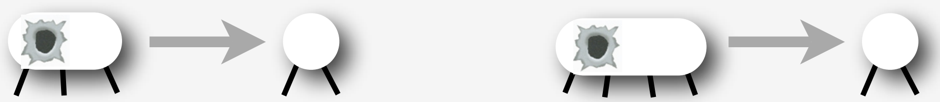
LLRB deleteMin() movie



Deleting an arbitrary node

involves the same general strategy.

1. Search down the left spine of the tree.
2. If search ends in a 3-node or 4-node: just remove it.



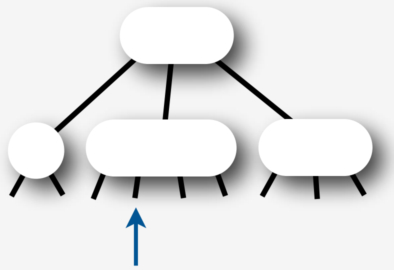
3. Removing a 2-node would destroy balance
   - transform tree on the way down the search path
   - Invariant: current node is not a 2-node

Difficulty:

- Far too many cases!
- LLRB representation dramatically reduces the number of cases.

Q: How many possible search paths in two levels ?

A: 9 * 6 + 27 * 9 + 81 * 12 = 1269 (! !)

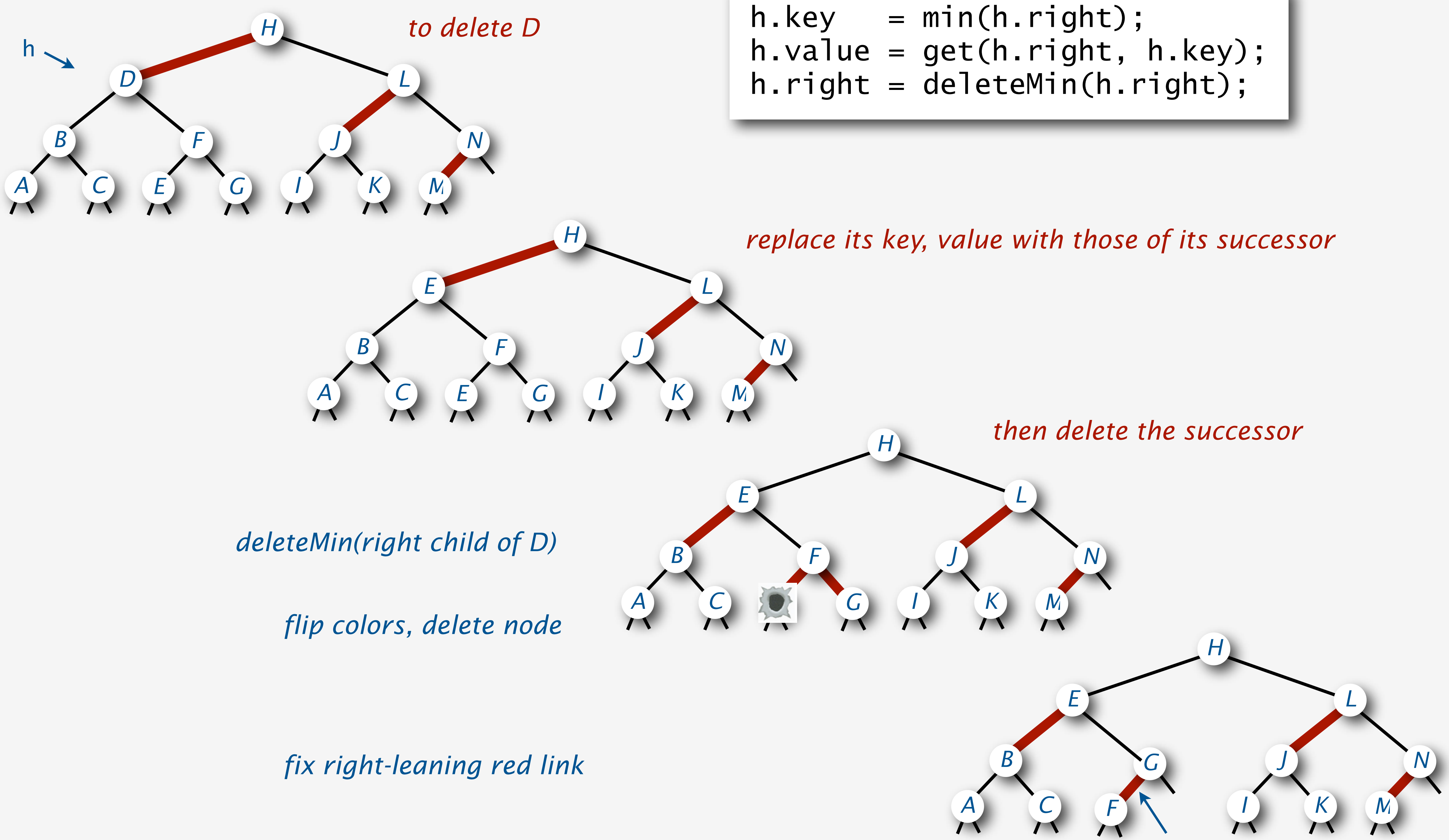


Deleting an arbitrary node

reduces to `deleteMin()`

A standard trick:

```
h.key   = min(h.right);
h.value = get(h.right, h.key);
h.right = deleteMin(h.right);
```

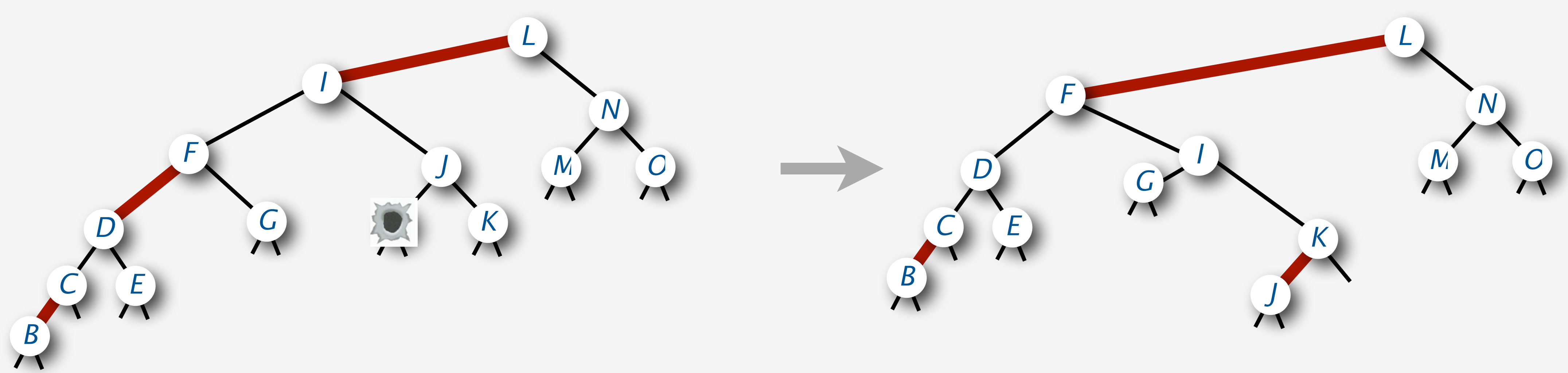


Deleting an arbitrary node at the bottom

can be implemented with the same helper methods used for `deleteMin()` and `deleteMax()`.

Invariant: h or one of its children is RED

- search path goes left: use `moveRedLeft()`.
- search path goes right: use `moveRedRight()`.
- delete node at bottom
- fix right-leaning reds on the way up



delete() implementation for LLRB trees


```
private Node delete(Node h, Key key)
{
   int cmp = key.compareTo(h.key);
   if (cmp < 0)
   {
      if (!isRed(h.left) && !isRed(h.left.left))
          h = moveRedLeft(h);
      h.left =  delete(h.left, key);
   }
   else
   {
      if (isRed(h.left)) h = leanRight(h);

      if (cmp == 0 && (h.right == null))
         return null;

      if (!isRed(h.right) && !isRed(h.right.left))
         h = moveRedRight(h);

      if (cmp == 0)
      {
         h.key = min(h.right);
         h.value = get(h.right, h.key);
         h.right = deleteMin(h.right);
      }
      else h.right = delete(h.right, key);
   }

   return fixUp(h);
}
```

LEFT

push red right if necessary

move down (left)

RIGHT or EQUAL

rotate to push red right

EQUAL (at bottom)
delete node

push red right if necessary

EQUAL (not at bottom)

replace current node with successor key, value

delete successor

move down (right)

fix right-leaning red links and eliminate 4-nodes on the way up

LLRB `delete()` movie

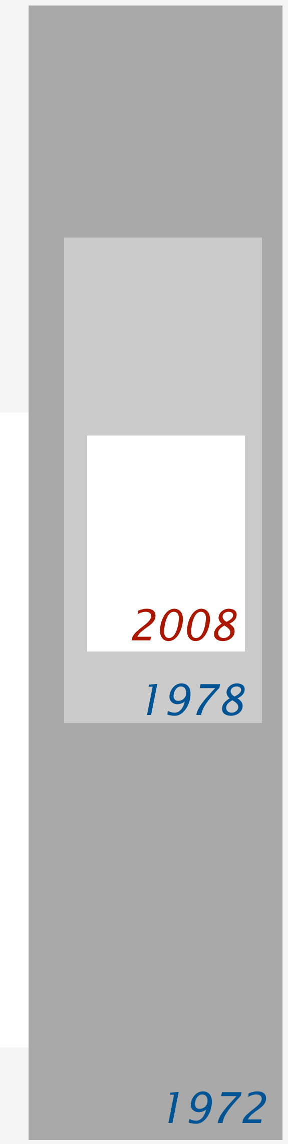
Alternatives

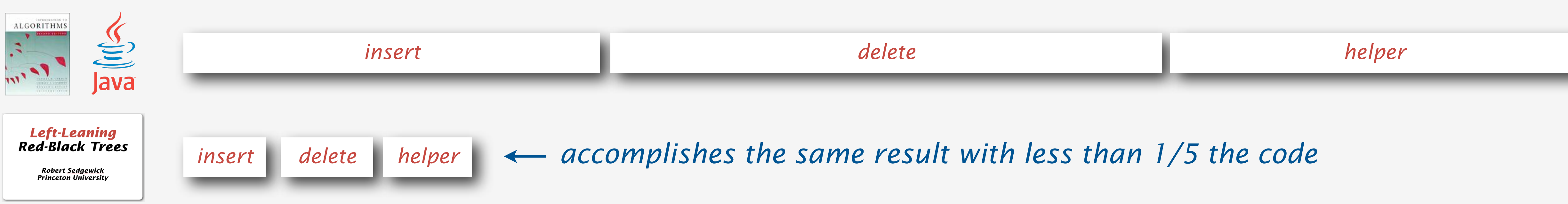
Red-black-tree implementations in widespread use:

- are based on pseudocode with “case bloat”
- use parent pointers (!)
- 400+ lines of code for core algorithms

Left-leaning red-black trees

- you just saw all the code
- single pass (remove recursion if concurrency matters)
- <80 lines of code for core algorithms
- less code implies faster insert, delete
- less code implies easier maintenance and migration





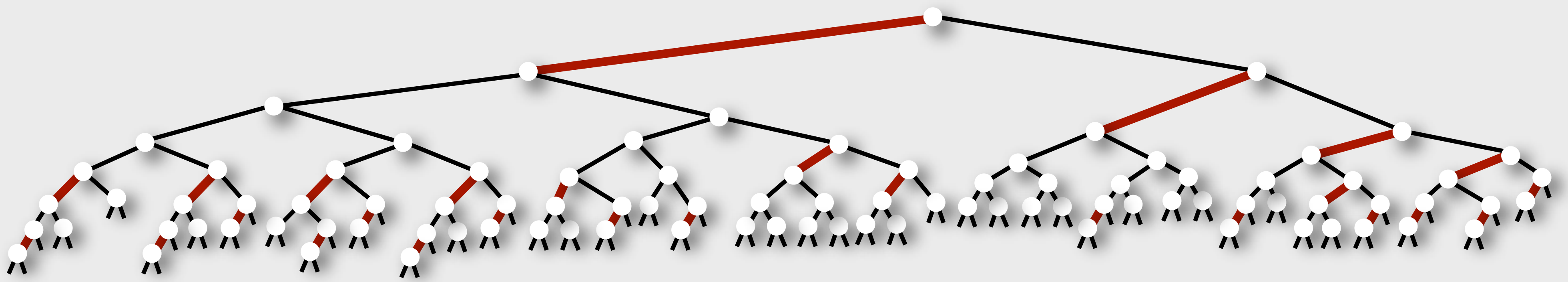
Introduction

2-3-4 Trees

LLRB Trees

Deletion

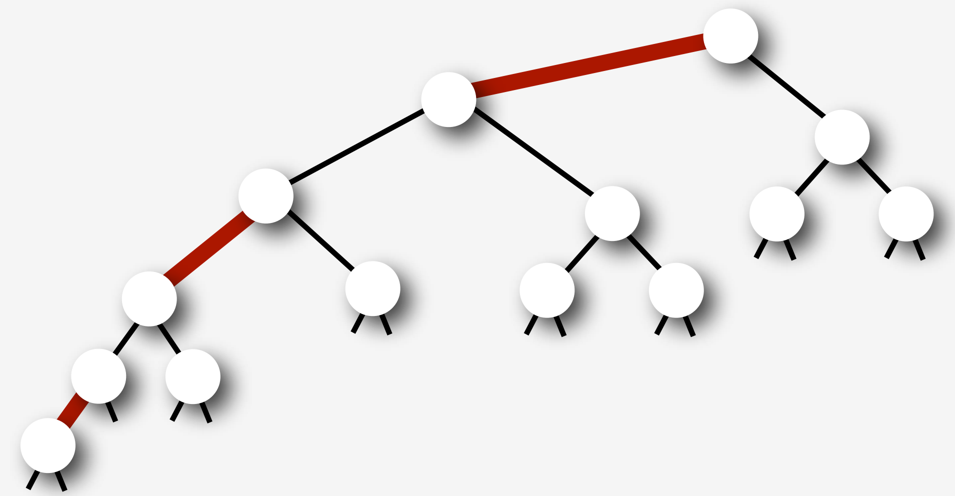
Analysis



Worst-case analysis

follows immediately from 2-3-4 tree correspondence

1. All trees have perfect black balance.
2. No two red links in a row on any path.



Shortest path: lg N (all black)

Longest path: 2 lg N (alternating red-black)

Theorem: *With red-black BSTs as the underlying data structure, we can implement an ordered symbol-table API that supports insert, delete, delete the minimum, delete the maximum, find the minimum, find the maximum, rank, select the kth largest, and range count in guaranteed logarithmic time.*

Red-black trees are the method of choice for many applications.

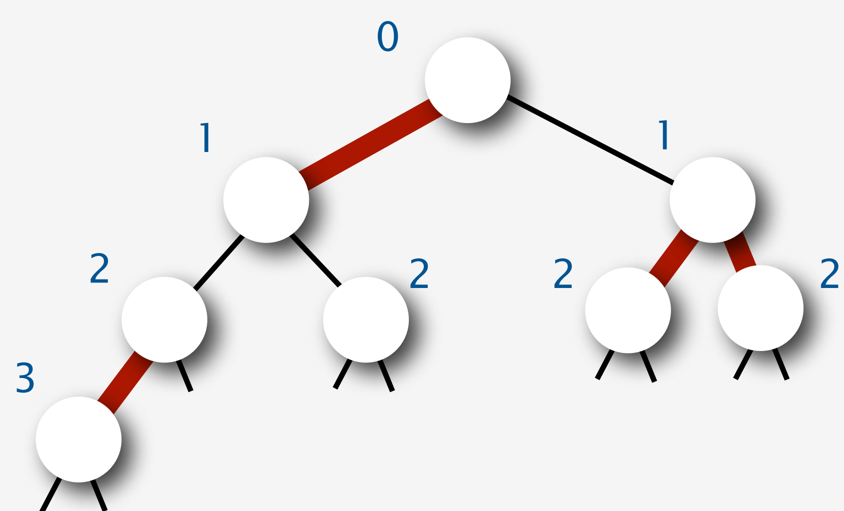
One remaining question

that is of interest in typical applications

The number of searches far exceeds the number of inserts.

Q. What is the cost of a typical search?

A. If each tree node is equally likely to be sought, compute the internal path length of the tree and divide by N.



N: 8

internal path length: 0 + 1 + 1 + 2 + 2 + 2 + 2 + 3 = 13

average search cost: 13/8 = 1.625

Q. What is the expected internal path length of a tree built with randomly ordered keys (average cost of a search)?

Average-case analysis of balanced trees

deserves another look!

Main questions:

Is average path length in tree built from random keys ~ c lg N ?
If so, is c = 1 ?

Average-case analysis of balanced trees

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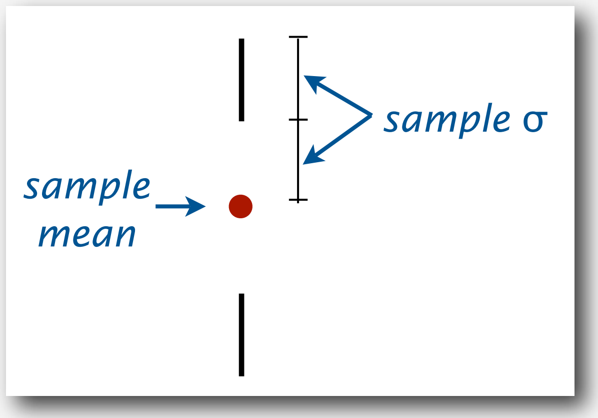
Experimental evidence

Tufte plot



Ex: Tufte plot of average path length in 2-3 trees

- N = 100, 200, . . . , 50,000
- 100 trees each size

Average-case analysis of balanced trees

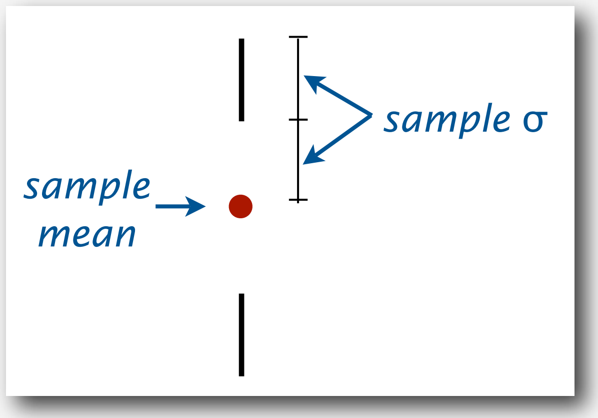
deserves another look!

Main questions:

Is average path length in tree built from random keys ~ c lg N ?
If so, is c = 1 ?

Experimental evidence strongly suggests YES!

Tufte plot

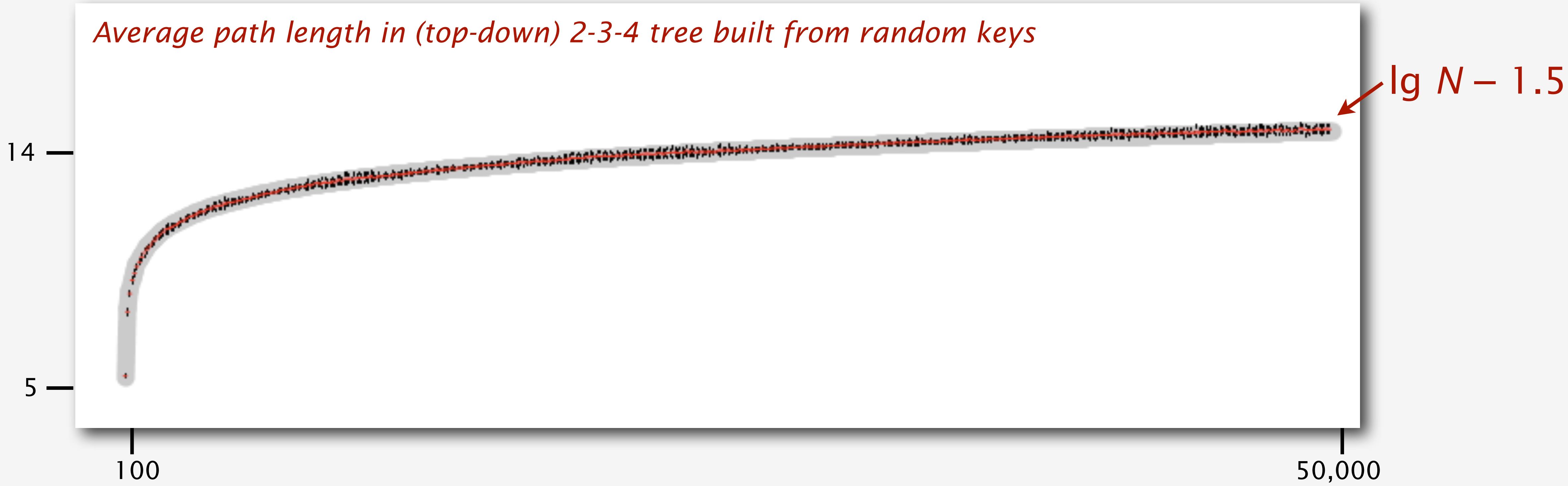


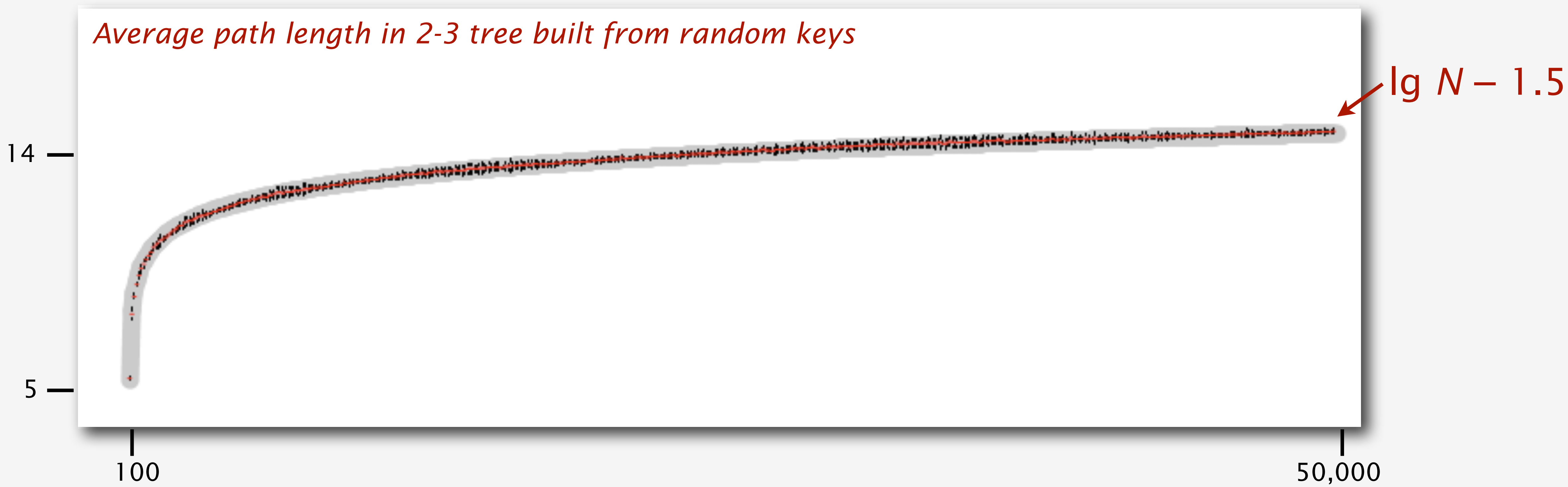
Ex: Tufte plot of average path length in 2-3 trees

- N = 100, 200, . . . , 50,000
- 100 trees each size

Experimental evidence

can suggest and confirm hypotheses





Average-case analysis of balanced trees

deserves another look!

Main questions:

Is average path length in tree built from random keys ~ c lg N ?
If so, is c = 1 ?

Some known facts:

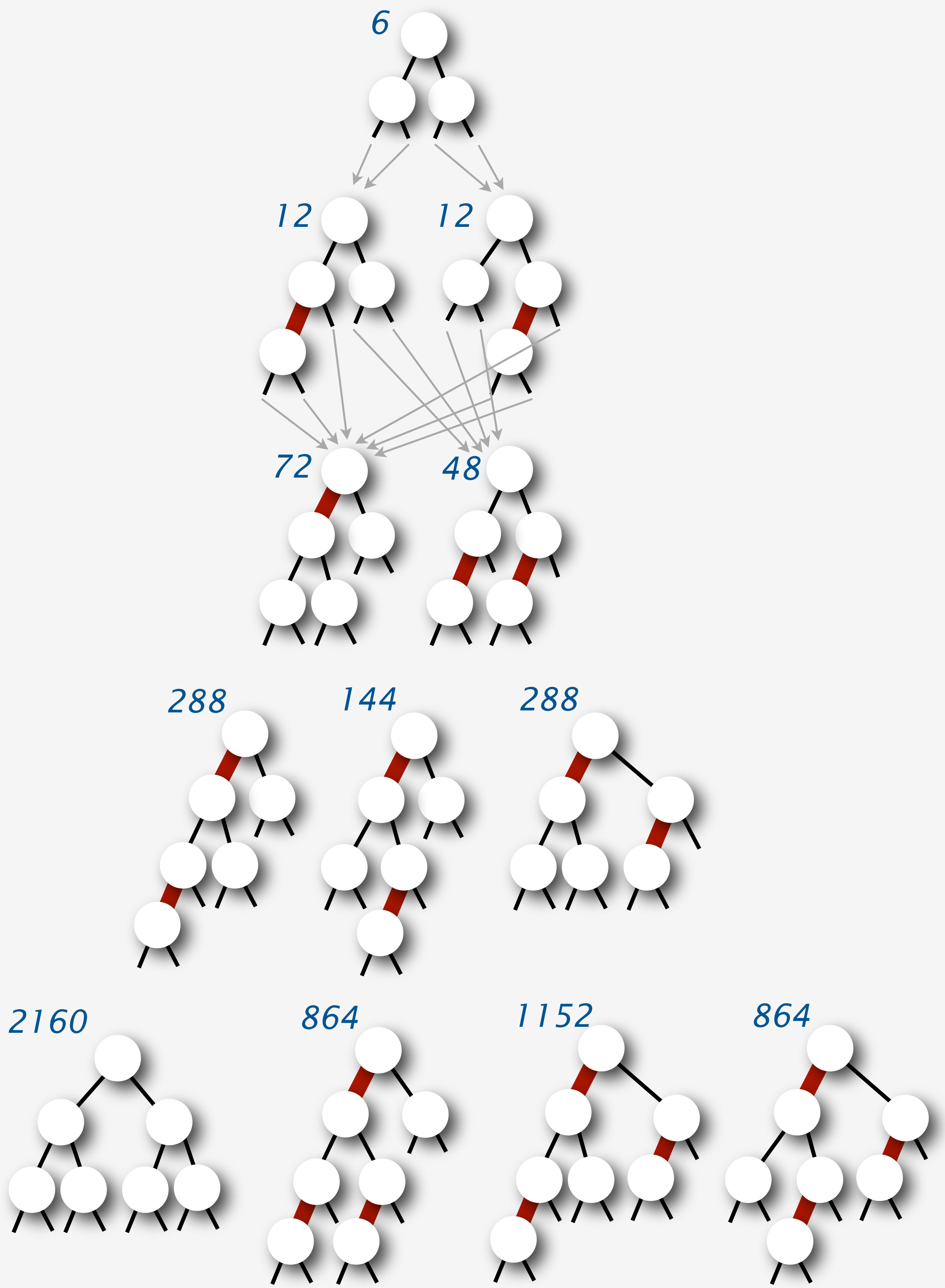
- worst case gives easy 2 lg N upper bound
- fringe analysis of gives upper bound of c_k lgN with $c_k > 1$
- analytic combinatorics gives path length in random trees

Are simpler implementations simpler to analyze?

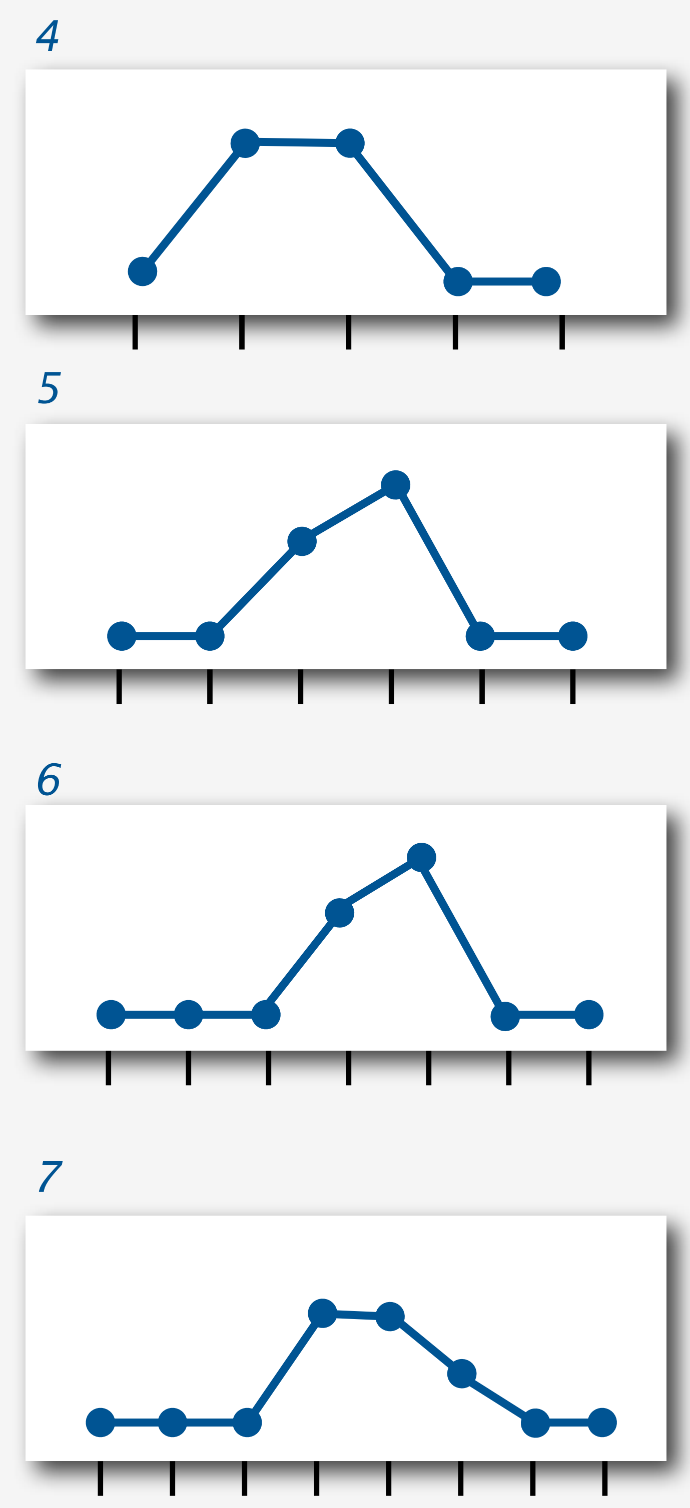
Is the better experimental evidence that is now available helpful?

A starting point: study balance at the root (left subtree size)

Left subtree size in left-leaning 2-3 trees

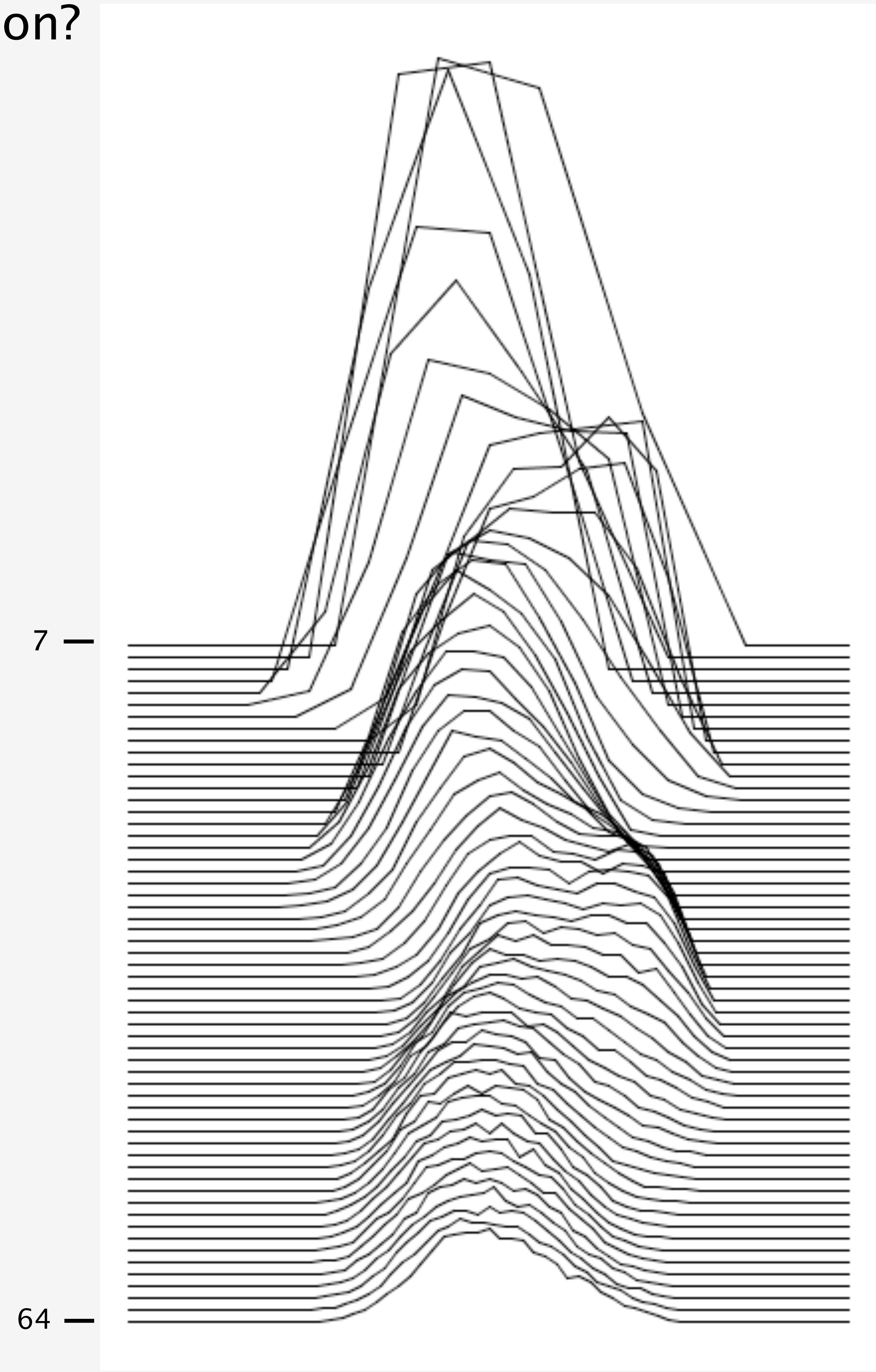


Exact distributions

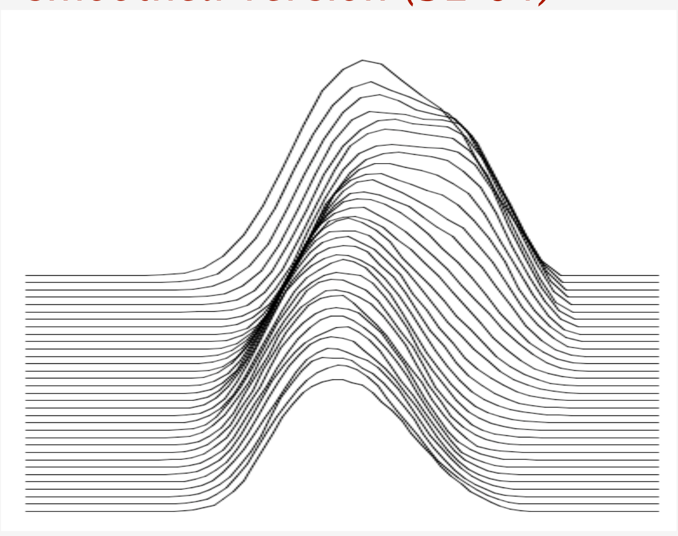


Left subtree size in left-leaning 2-3 trees

Limiting distribution?

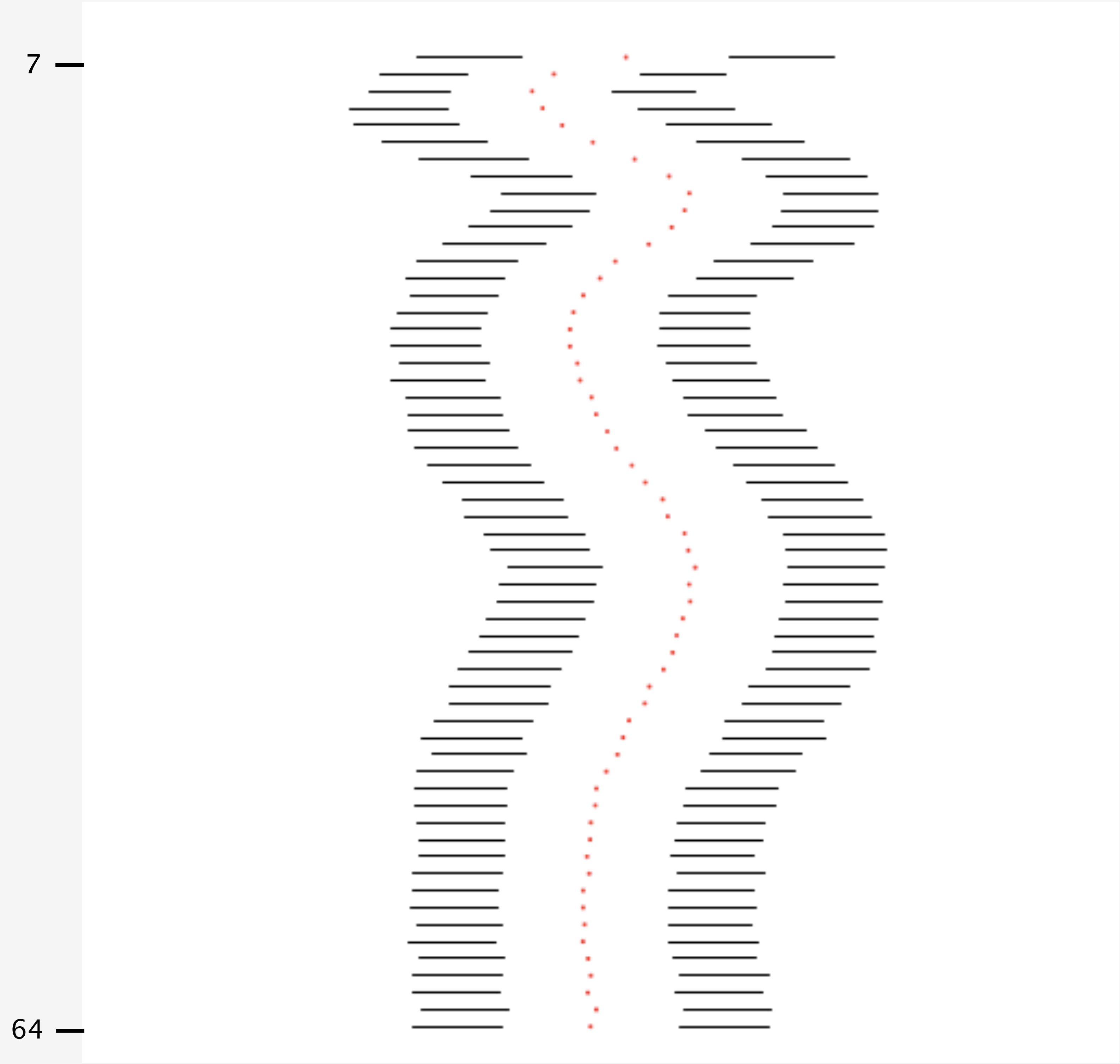


smoothed version (32-64)



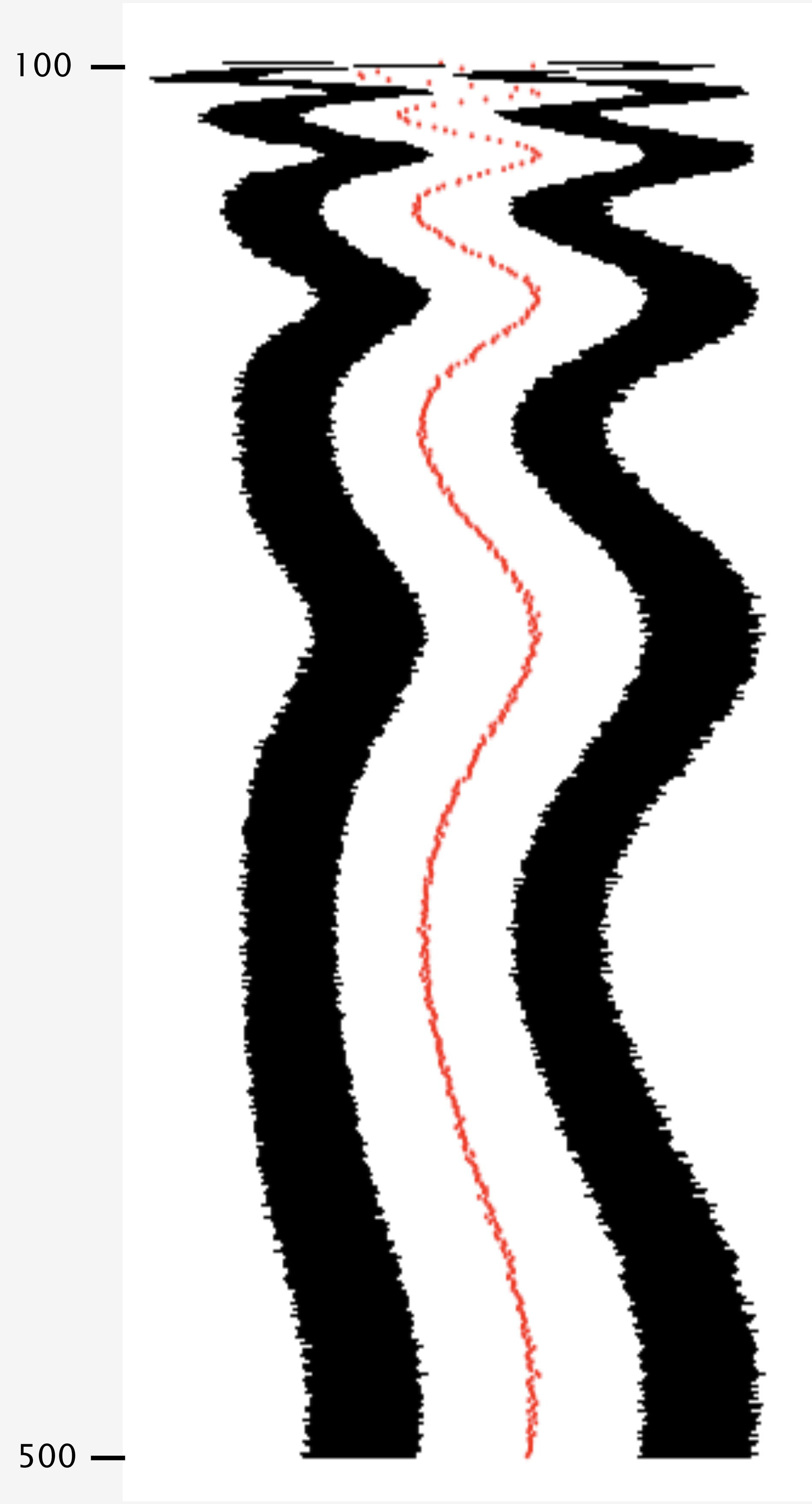
Left subtree size in left-leaning 2-3 trees

Tufte plot



Left subtree size in left-leaning 2-3 trees

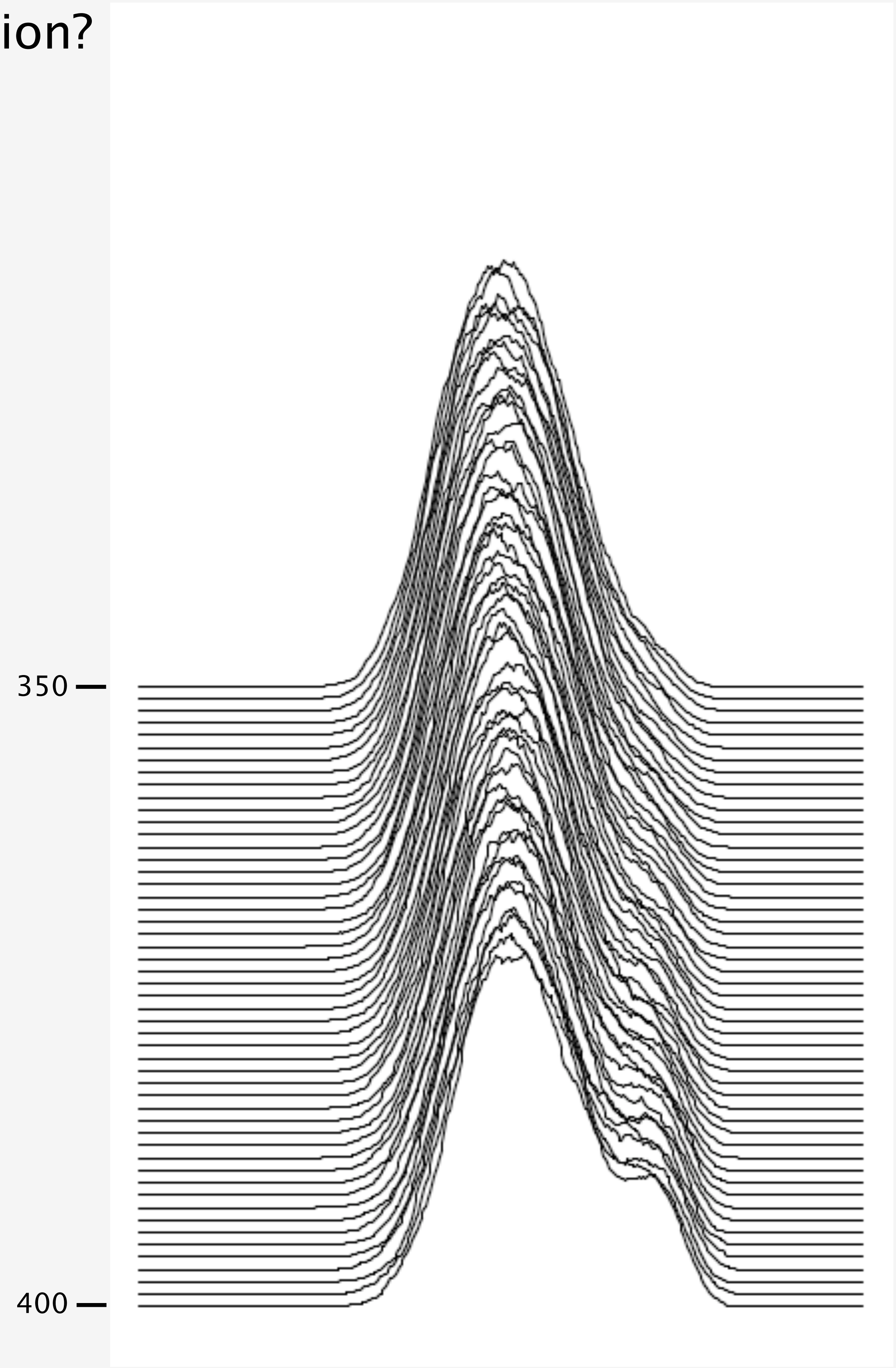
Tufte plot



view of highway for bus driver who has had one Caipirinha too many ?

Left subtree size in left-leaning 2-3 trees

Limiting distribution?

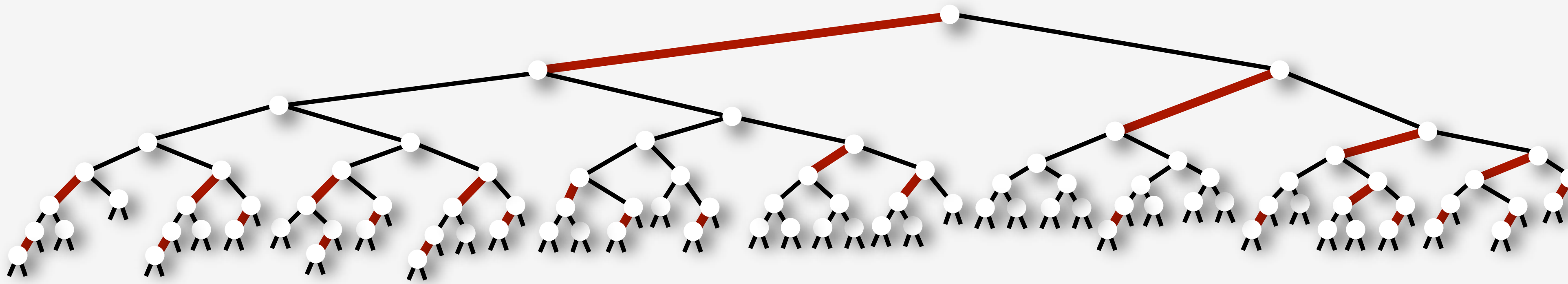


10,000 trees for each size

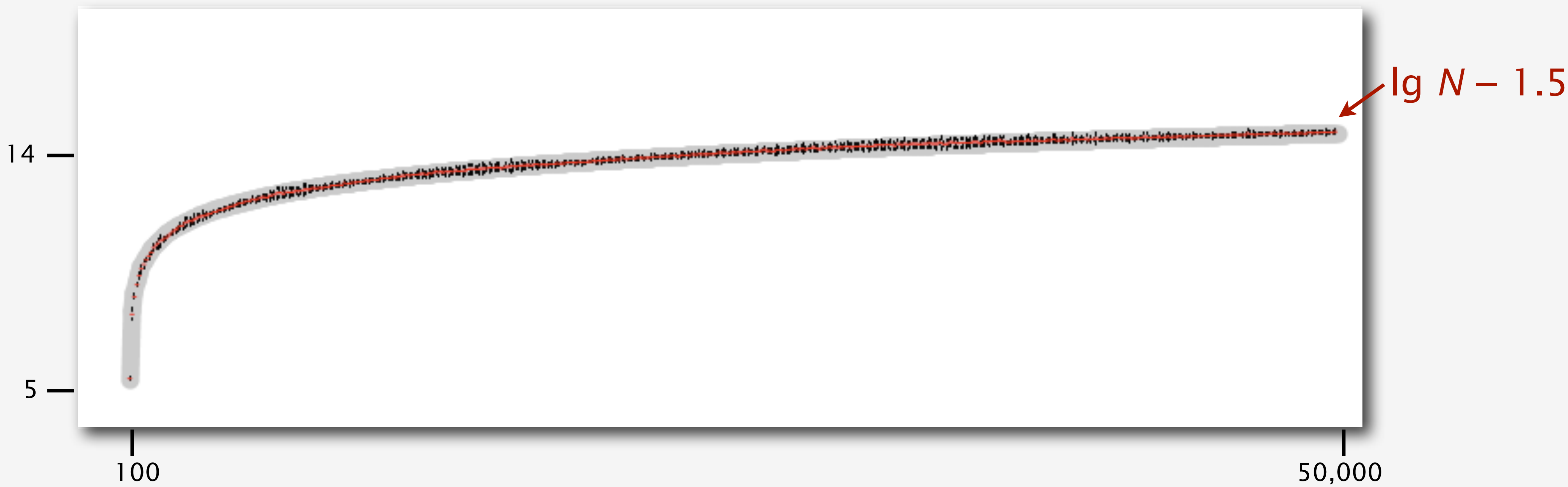
smooth factor 10

An exercise in the analysis of algorithms

Find a proof !



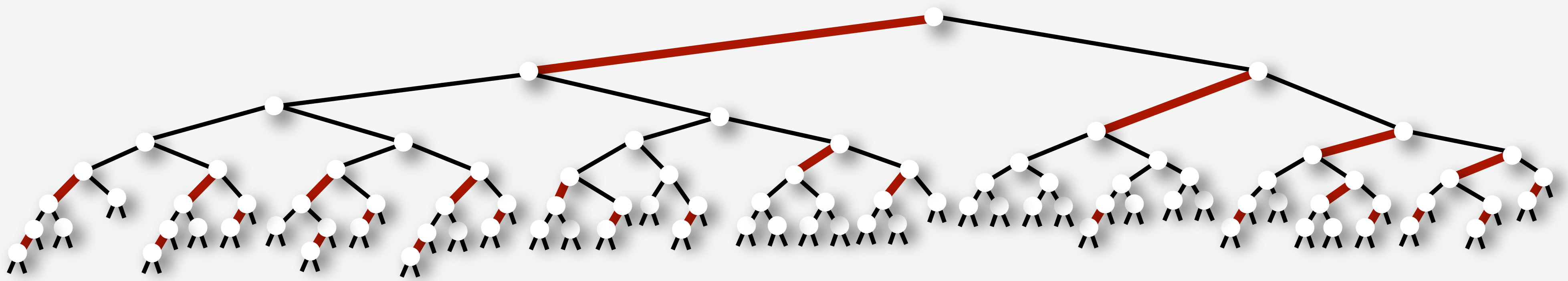
Average path length in 2-3 tree built from random keys



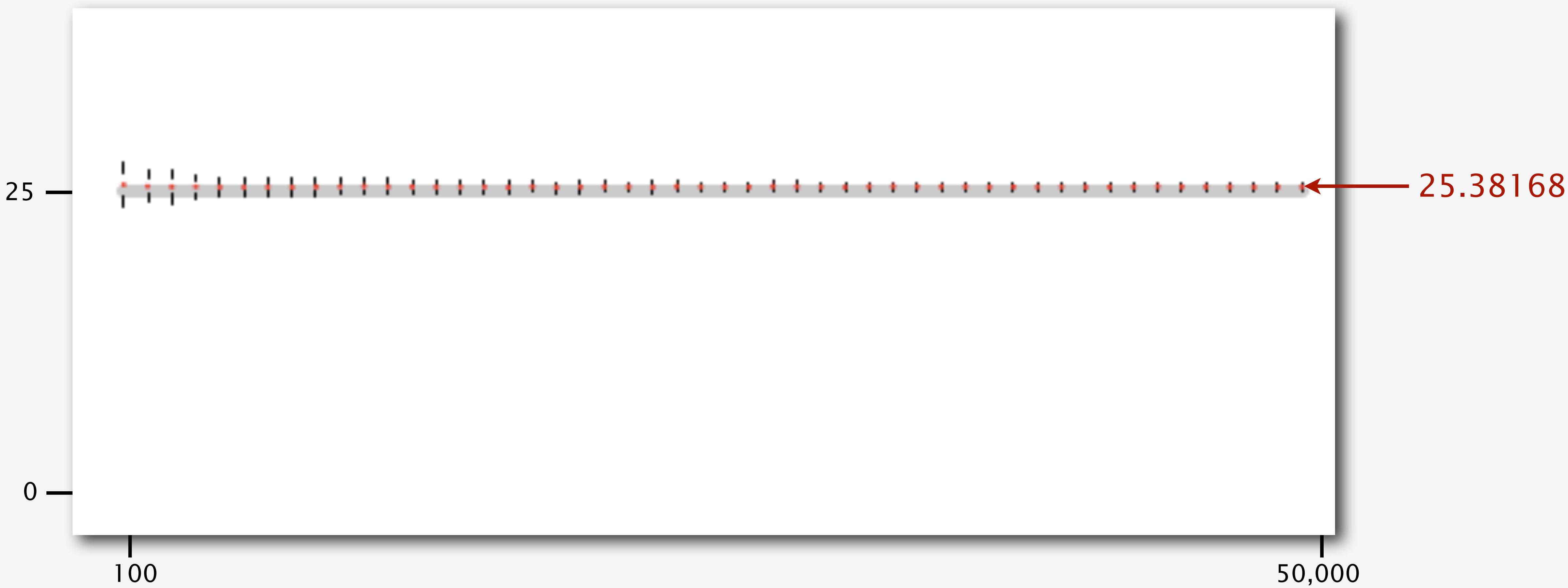
Addendum: Observations

Observation 1

The percentage of red nodes in a 2-3 tree is between 25 and 25.5%

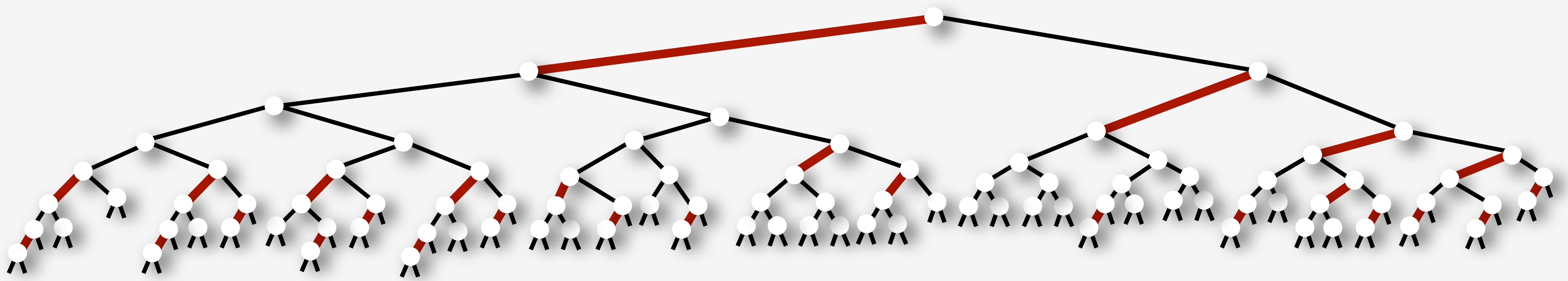


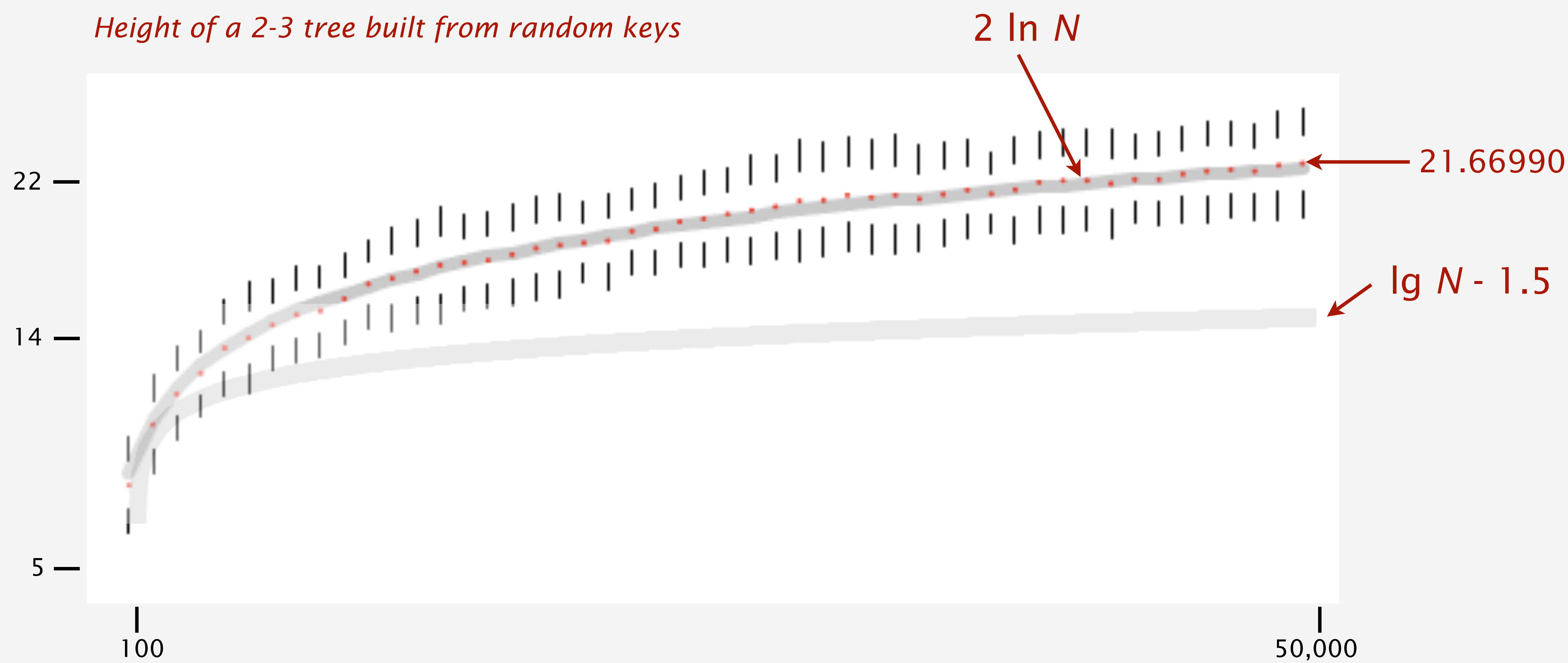
Percentage of red nodes in 2-3 tree built from random keys



Observation 2

The height of a 2-3 tree is ~2 ln N (!!!)

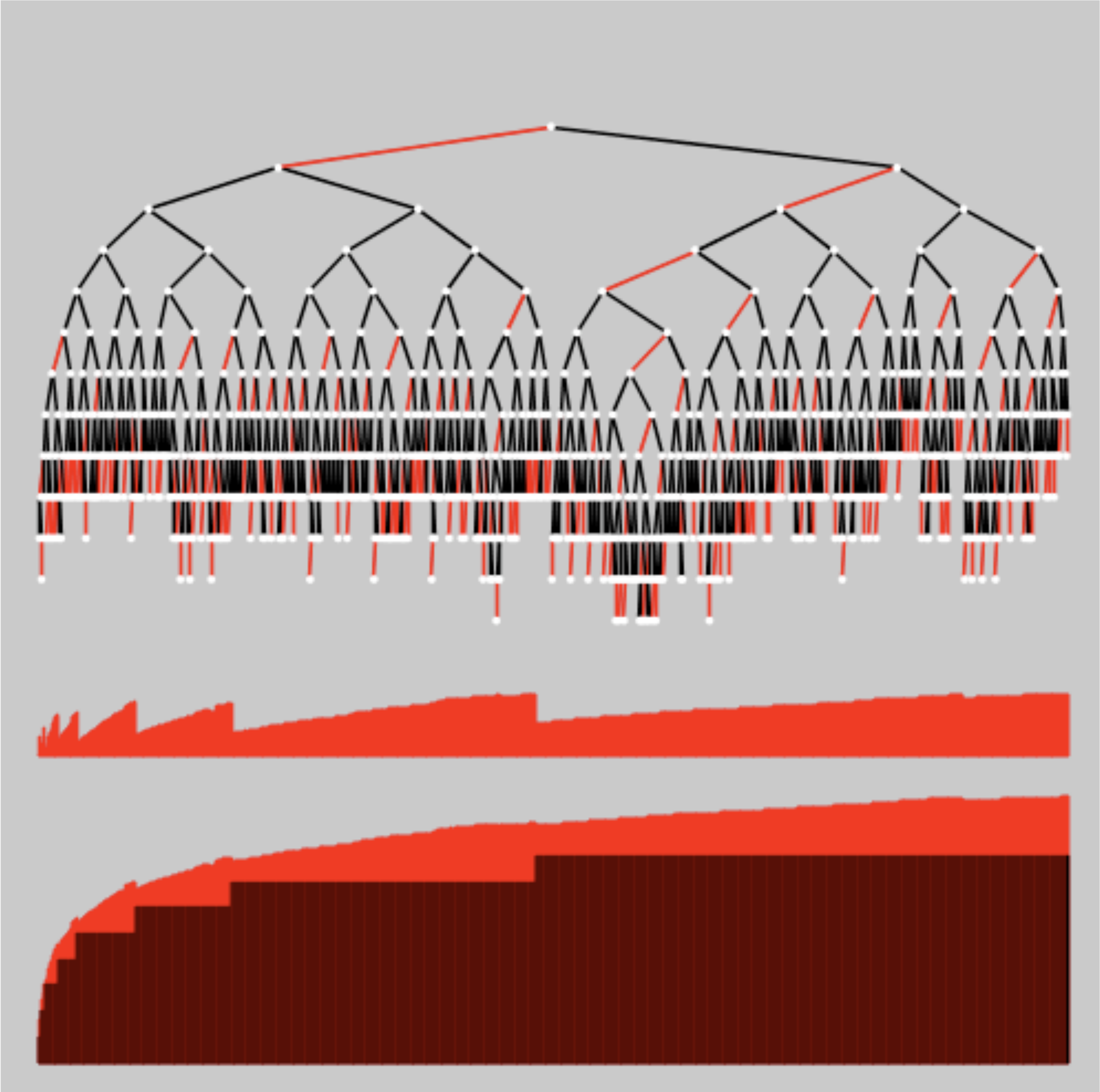




Very surprising because the average path length in an elementary BST is also ~2 ln *N ≈ 1.386* lg *N*

Observation 3

The percentage of red nodes on each path in a 2-3 tree rises to about 25%, then drops by 2 when the root splits



Observation 4

In aggregate, the observed number of red links per path log-alternates between periods of steady growth and not-so-steady decrease (because root-split times vary widely)