

The beauty, mystery, and utility of prime numbers

Tom Marley

University of Nebraska-Lincoln

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By a *factor* of a number we mean a whole number which *evenly* divides the number; i.e., divides with no remainder.

- 4 is a factor of 20 because 20 divided by 4 is 5 with no remainder.
- 3 is **not** a factor of 20 because 20 divided by 3 is 6 with remainder 2.

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- The factors of 24 are 1, 2, 3, 4, 6, 8, 12, and 24.

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- The factors of 24 are 1, 2, 3, 4, 6, 8, 12, and 24.
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The first few prime numbers are:

2, 3, 5, 7, 11, 13, 17, 19, 23, ...

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This fact is known as *The Fundamental Theorem of Arithmetic*



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This is a method of logic whereby one assumes a statement is false and shows this leads to an 'absurdity'.

Euclid's proof

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Question: What is the remainder when you divide N by one of the primes?

Answer: **One!**

This means that N is not divisible by any prime! This is our 'abusurdity'.

A simple primality test

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So, to see if a number between 2 and 100 is prime, we just have to check if it is divisible by 2, 3, 5, or 7.

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We can make an algorithm out of this, which is called the

Sieve of Eratosthenes.

Sieve of Eratosthenes

	2	3	4	5	6	7	8	9	10
11	12	13	14	15	16	17	18	19	20
21	22	23	24	25	26	27	28	29	30
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Prime: 2

All multiples of 2 crossed out.

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Primes: 2, 3

All multiples of 2 and 3 crossed out.

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Primes: 2, 3, 5

All multiples of 2, 3, and 5 crossed out.

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Primes: 2, 3, 5, 7 and all other uncrossed numbers

All multiples of 2, 3, 5, and 7 crossed out.

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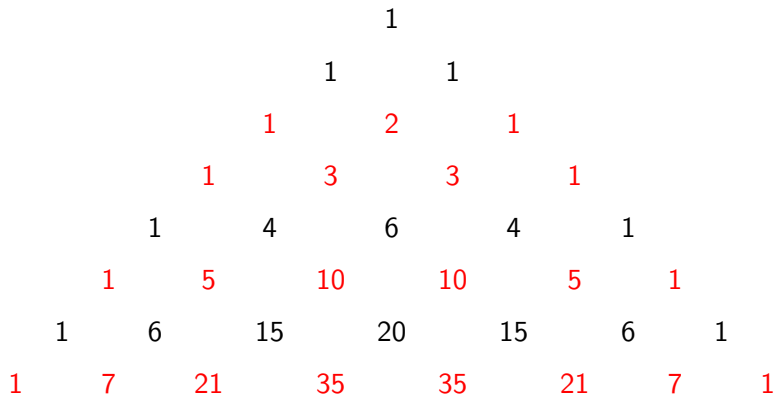
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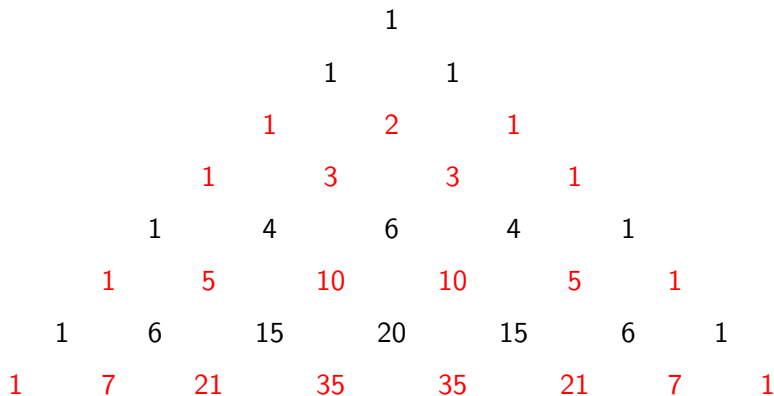
This function grows very fast with n . Consequently, it would take thousands of years for even the world's fastest supercomputers to check if a 400-digit number is prime using the Sieve.



Pascal's Triangle



Pascal's Triangle



Fact: If n is prime then n divides all the middle terms in it's row.

Fermat's Theorem

For example,

$$(a + b)^5 = a^5 + 5a^4b + 10a^3b^2 + 10a^2b^3 + 5ab^4 + b^5.$$

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In general, if p is prime then $(a + b)^p - a^p - b^p$ is divisible by p for all numbers a and b .

It's a very short argument from there to...

Fermat's Theorem: If p is a prime number then p divides $a^p - a$ for all numbers a .

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Question: Does Fermat's Theorem only work for prime numbers?

That is, suppose n is a number and $a^n - a$ is divisible by n for all numbers a . Must n be prime?

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The good news is: Carmichael numbers are quite rare relative to prime numbers!

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Continue the loop until, with increasing probability, you conclude that N is either prime or (if you are really unlucky) a Carmichael number.

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However, the function

$$\frac{(3.16)^n}{(1.15)^n}$$

is **not** bounded by a polynomial function of n .

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Their algorithm, now known as the AKS primality test, determines with certainty whether an n -digit number. The number of divisions needed in their algorithm is bounded by a polynomial function in n of degree 12.

This has now been improved to a polynomial of degree 6.

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It forms the basis for *public key cryptography*, which was discovered by three mathematicians at M.I.T.: [Ron Rivest](#), [Adi Shamir](#), and [Leonard Adleman](#) in 1977. It is now known as the [RSA cryptosystem](#).

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Bob then sends r to me using any public channel he wishes (e.g., the internet).

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Why is this secure? Because there is no known way to find d without first knowing p and q . So the security depends on the factorization problem being "hard".

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It is unknown if any other Fermat primes exist.

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(Mersenne was a French monk who lived in the 17th century.)

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It is unknown if there are infinitely many Mersenne primes.



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- The divisors of 28 are 1, 2, 4, 7, and 14. Since $1 + 2 + 4 + 7 + 14 = 28$, 28 is a perfect number.

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- The divisors of 6 are 1, 2, and 3. Since $1 + 2 + 3 = 6$, 6 is a perfect number.
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It can be shown (with a little arithmetic) that if N is a Mersenne prime, then $\frac{N(N+1)}{2}$ is a perfect number.

Moreover, every even perfect number has this form. So there are exactly as many even perfect numbers as there are Mersenne primes!

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- 29, 31
- 41, 43
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The case $N = 2$ remains elusive....waiting for YOU to solve it.

Thank you!