Chapter 1 STERN-GERLACH EXPERIMENTS

It was not a dark and stormy night when Otto Stern and Walther Gerlach performed their now famous experiment in 1922. The Stern-Gerlach experiment demonstrated that measurements on microscopic or quantum particles are not always as certain as we might expect. Quantum particles behave as mysteriously as Erwin's socks—sometimes forgetting what we have already measured. Erwin's adventure with the mystery socks is farfetched because you know that everyday objects do not behave like his socks. If you observe a sock to be black, it remains black no matter what other properties of the sock you observe. However, the Stern-Gerlach experiment goes against these ideas. Microscopic or quantum particles do not behave like the classical objects of your everyday experience. The act of observing a quantum particle affects its measurable properties in a way that is foreign to our classical experience.

In these first three chapters, we focus on the Stern-Gerlach experiment because it is a conceptually simple experiment that demonstrates many basic principles of quantum mechanics. We discuss a variety of experimental results and the quantum theory that has been developed to predict those results. The mathematical formalism of quantum mechanics is based upon six postulates that we will introduce as we develop the theoretical framework. (A complete list of these postulates is in Section 1.5.) We use the Stern-Gerlach experiment to learn about quantum mechanics theory for two primary reasons: (1) It demonstrates how quantum mechanics works in principle by illustrating the postulates of quantum mechanics, and (2) It demonstrates how quantum mechanics works in practice through the use of Dirac notation and matrix mechanics to solve problems. By using a simple example, we can focus on the principles and the new mathematics, rather than having the complexity of the physics obscure these new aspects.

1.1 Stern-Gerlach experiment

In 1922 Otto Stern and Walther Gerlach performed a seminal experiment in the history of quantum mechanics. In its simplest form, the experiment consists of an oven that produces a beam of neutral atoms, a region of space with an inhomogeneous magnetic field, and a detector for the atoms, as depicted in Fig. 1.1. Stern and Gerlach used a beam of silver atoms and found that the beam was split into two in its passage through the magnetic field. One beam was deflected upwards and one downwards in relation to the direction of the magnetic field gradient.

To understand why this result is so at odds with our classical expectations, we must first analyze the experiment classically. The results of the experiment suggest an interaction between a neutral particle and a magnetic field. We expect such an interaction if the particle possesses a magnetic moment $\boldsymbol{\mu}$. The potential energy of this interaction is $E = -\boldsymbol{\mu}\cdot\mathbf{B}$, which results in a force $\mathbf{F} = \nabla(\boldsymbol{\mu}\cdot\mathbf{B})$. In the Stern-Gerlach experiment, the magnetic field gradient is primarily in the z-direction, and

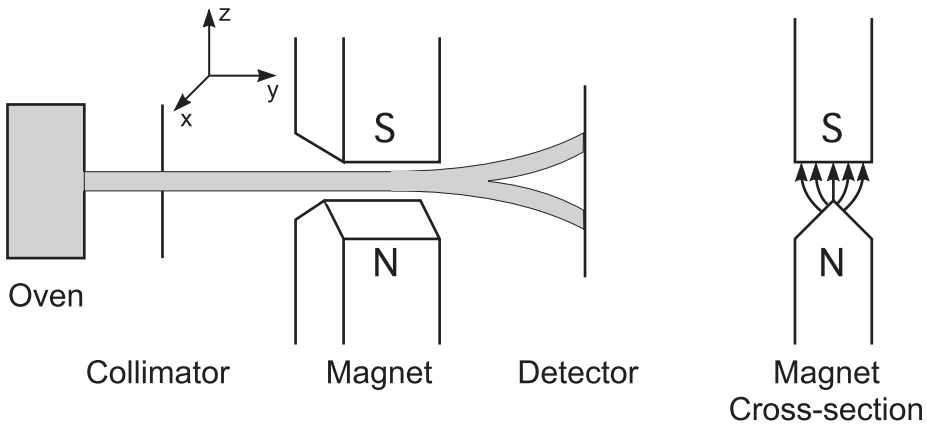

Figure 1.1 Stern-Gerlach experiment to measure the spin component of neutral particles along the z-axis. The magnetic cross-section at right shows the inhomogeneous field used in the experiment.

the resulting z-component of the force is

$$\begin{aligned} F_z &= \frac{\partial}{\partial z}(\boldsymbol{\mu}\bullet\mathbf{B}) \\ &\cong \mu_z \frac{\partial B_z}{\partial z} \end{aligned} \quad . \tag{1.1}$$

This force is perpendicular to the direction of motion and deflects the beam in proportion to the component of the magnetic moment in the direction of the magnetic field gradient.

Now consider how to understand the origin of the atom's magnetic moment from a classical viewpoint. The atom consists of charged particles, which, if in motion, can produce loops of current that give rise to magnetic moments. A loop of area A and current I produces a magnetic moment

$$\mu = IA \tag{1.2}$$

in MKS units. If this loop of current arises from a charge q traveling at speed v in a circle of radius r, then

$$\begin{aligned} \mu &= \frac{q}{2\pi r/v}\pi r^2 \\ &= \frac{qrv}{2} \\ &= \frac{q}{2m}L \end{aligned} \quad , \tag{1.3}$$

where $L = mrv$ is the orbital angular momentum of the particle. In the same way that the earth revolves around the sun and rotates around its own axis, we can also imagine a charged particle in an atom having **orbital angular momentum L** and a new property, the **intrinsic angular momentum**, which we label **S** and call **spin**. The intrinsic angular momentum also creates current loops, so we expect a similar relation between the magnetic moment $\boldsymbol{\mu}$ and **S**. The exact calculation involves an integral over the charge

distribution, which we will not do. We simply assume that we can relate the magnetic moment to the intrinsic angular momentum in the same fashion as Eq. (1.3), giving

$$\boldsymbol{\mu} = g\frac{q}{2m}\mathbf{S}\,, \tag{1.4}$$

where the dimensionless **gyroscopic ratio** g contains the details of that integral.

A silver atom has 47 electrons, 47 protons, and 60 or 62 neutrons (for the most common isotopes). The magnetic moments depend on the inverse of the particle mass, so we expect the heavy protons and neutrons ($\approx 2000\, m_e$) to have little effect on the magnetic moment of the atom and so we neglect them. From your study of the periodic table in chemistry, you recall that silver has an electronic configuration $1s^2 2s^2 2p^6 3s^2 3p^6 4s^2 3d^{10} 4p^6 4d^{10} 5s^1$, which means that there is only the lone 5*s* electron outside of the closed shells. The electrons in the closed shells can be represented by a spherically symmetric cloud with no orbital or intrinsic angular momentum (unfortunately we are injecting some quantum mechanical knowledge of atomic physics into this classical discussion). That leaves the lone 5*s* electron as a contributor to the magnetic moment of the atom as a whole. An electron in an *s* state has no orbital angular momentum, but it does have spin. Hence the magnetic moment of this electron, and therefore of the entire neutral silver atom, is

$$\boldsymbol{\mu} = -g\frac{e}{2m_e}\mathbf{S}\,, \tag{1.5}$$

where e is the magnitude of the electron charge. The classical force on the atom can now be written as

$$F_z \cong -g\frac{e}{2m_e}S_z\frac{\partial B_z}{\partial z}\,. \tag{1.6}$$

The deflection of the beam in the Stern-Gerlach experiment is thus a measure of the component (or projection) S_z of the spin along the z-axis, which is the orientation of the magnetic field gradient.

If we assume that the 5*s* electron of each atom has the same magnitude $|\mathbf{S}|$ of the intrinsic angular momentum or spin, then classically we would write the z-component as $S_z = |\mathbf{S}|\cos\theta$, where θ is the angle between the z-axis and the direction of the spin **S**. In the thermal environment of the oven, we expect a random distribution of spin directions and hence all possible angles θ. Thus we expect some continuous distribution (the details are not important) of spin components from $S_z = -|\mathbf{S}|$ to $S_z = +|\mathbf{S}|$, which would yield a continuous spread in deflections of the silver atomic beam. Rather, the experimental result that Stern and Gerlach observed was that there are only two deflections, indicating that there are only two possible values of the z-component of the electron spin. The magnitudes of these deflections are consistent with values of the spin component of

$$S_z = \pm\frac{\hbar}{2}\,, \tag{1.7}$$

where $\hbar$ is Planck's constant h divided by 2π and has the numerical value

$$\begin{aligned}\hbar &= 1.0546\times10^{-34}\,J\cdot\mathrm{s}\\ &= 6.5821\times10^{-16}\,\mathrm{eV}\cdot\mathrm{s}\end{aligned}. \tag{1.8}$$

This result of the Stern-Gerlach experiment is evidence of the **quantization** of the electron's spin angular momentum component along an axis. This quantization is at odds with our classical expectations for this measurement. The factor of 1/2 in Eq. (1.7) leads us to refer to this as a **spin 1/2** system.

In this example, we have chosen the z-axis along which to measure the spin component, but there is nothing special about this direction in space. We could have chosen any other axis and we would have obtained the same results.

Now that we know the fine details of the Stern-Gerlach experiment, we simplify the experiment for the rest of our discussions by focusing on the essential features. A simplified schematic representation of the experiment is shown in Fig. 1.2, which depicts an oven that produces the beam of atoms, a Stern-Gerlach device with two output ports for the two possible values of the spin component, and two counters to detect the atoms leaving the output ports of the Stern-Gerlach device. The Stern-Gerlach device is labeled with the axis along which the magnetic field is oriented. The up and down arrows indicate the two possible measurement results for the device; they correspond respectively to the results $S_z = \pm\hbar/2$ in the case where the field is oriented along the z-axis. There are only two possible results in this case, so they are generally referred to as **spin up** and **spin down**. The physical quantity that is measured, S_z in this case, is called an **observable**. In our detailed discussion of the experiment above, we chose the field gradient in such a manner that the spin up states were deflected upwards. In this new simplification, the deflection itself is not an important issue. We simply label the output port with the desired state and count the particles leaving that port. The Stern-Gerlach device sorts (or filters or selects or analyzes) the incoming particles into the two possible outputs $S_z = \pm\hbar/2$ in the same way that Erwin sorted his socks according to color or length. We follow convention and refer to a Stern-Gerlach device as an **analyzer**.

In Fig. 1.2, the input and output beams are labeled with a new symbol called a **ket**. We use the ket $|+\rangle$ as a mathematical representation of the quantum state of the atoms that exit the upper port corresponding to $S_z = +\hbar/2$. The lower output beam is labeled with the ket $|-\rangle$, which corresponds to $S_z = -\hbar/2$, and the input beam is labeled

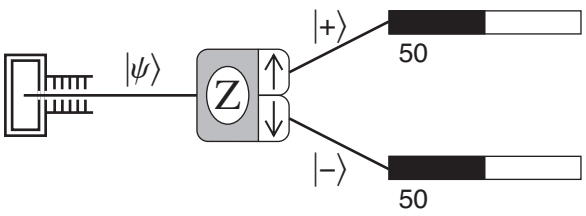

Figure 1.2 Simplified schematic of the Stern-Gerlach experiment, depicting a source of atoms, a Stern-Gerlach analyzer, and two counters.

with the more generic ket $|\psi\rangle$. The kets are simple labels for the quantum states. They are used in mathematical expressions and they represent all the information that we can know about the state. This ket notation was developed by Paul A. M. Dirac and is central to the approach to quantum mechanics that we take in this text. We will discuss the mathematics of these kets in full detail later. With regard to notation, you will find many different ways of writing the same ket. The symbol within the ket brackets is any simple label to distinguish the ket from other different kets. For example, the kets $|+\rangle$, $|+\hbar/2\rangle$, $|S_z = +\hbar/2\rangle$, $|+\hat{\mathbf{z}}\rangle$, and $|\uparrow\rangle$ are all equivalent ways of writing the same thing, which in this case signifies that we have measured the *z*-component of the spin and found it to be $+\hbar/2$ or spin up. Though we may label these kets in different ways, they all refer to the same physical state and so they all behave the same mathematically. The symbol $|\pm\rangle$ kets refers to both the $|+\rangle$ and $|-\rangle$ kets. The first postulate of quantum mechanics tells us that kets in general describe the quantum state mathematically and that they contain all the information that we can know about the state. We denote a general ket as $|\psi\rangle$.

Postulate 1

The state of a quantum mechanical system, including all the information you can know about it, is represented mathematically by a normalized ket $|\psi\rangle$.

We have chosen the particular simplified schematic representation of Stern-Gerlach experiments shown in Fig. 1.2 because it is the same representation used in the SPINS software program that you may use to simulate these experiments. The SPINS program allows you to perform all the experiments described in this text. This software is freely available, as detailed in section 1.8. In the SPINS program, the components are connected with simple lines to represent the paths the atoms take. The directions and magnitudes of deflections of the beams in the program are not relevant. That is, whether the spin up output beam is drawn as deflected upwards, or downwards, or not at all is not relevant. The labeling on the output port is enough to tell us what that state is. Thus the extra ket label $|+\rangle$ on the spin up output beam in Fig. 1.2 is redundant and will be dropped soon.

The SPINS program permits alignment of Stern-Gerlach analyzing devices along all three axes and also at any angle ϕ measured from the *x*-axis in the *x*-*y* plane. This would appear to be difficult, if not impossible, given that the atomic beam in Fig. 1.1 is directed along the *y*-axis, making it unclear how to align the magnet in the *y*-direction and measure a deflection. In our depiction and discussion of Stern-Gerlach experiments, we ignore this technical complication.

In the SPINS program, as in real Stern-Gerlach experiments, the numbers of atoms detected in particular states can be predicted by probability rules that we will discuss later. To simplify our schematic depictions of Stern-Gerlach experiments, the numbers shown for detected atoms are those obtained by using the calculated probabilities without any regard to possible statistical uncertainties. That is, if the theoretically predicted probabilities of two measurement possibilities are each 50%, then

our schematics will display equal numbers for those two possibilities, whereas in a real experiment, statistical uncertainties might yield a 55%/45% split in one experiment and a 47%/53% split in another, etc. The SPINS program simulations are designed to give statistical uncertainties and so you will need to perform enough experiments to convince yourself that you have a sufficiently good estimate of the probability (see SPINS Lab 1 for more information on statistics).

Now let's consider a series of simple Stern-Gerlach experiments with slight variations that help to illustrate the main features of quantum mechanics. We first describe the experiments and their results and draw some qualitative conclusions about the nature of quantum mechanics. Then we introduce the formal mathematics of the ket notation and show how it can be used to predict the results of each of the experiments.

1.1.1 Experiment 1

The first experiment is shown in Fig. 1.3 and consists of a source of atoms, two Stern-Gerlach analyzers both aligned along the z-axis, and counters for the output ports of the analyzers. The atomic beam coming into the 1st Stern-Gerlach analyzer is split into two beams at the output, just like the original experiment. Now instead of counting the atoms in the upper output beam, the spin component is measured again by directing those atoms into the 2nd Stern-Gerlach analyzer. The result of this experiment is that no atoms are ever detected coming out of the lower output port of the 2nd Stern-Gerlach analyzer. All atoms that are output from the upper port of the 1st analyzer also pass through the upper port of the 2nd analyzer. Thus we say that when the 1st Stern-Gerlach analyzer measures an atom to have a z-component of spin $S_z = +\hbar/2$, then the 2nd analyzer also measures $S_z = +\hbar/2$ for that atom. This result is not surprising, but sets the stages for results of experiments to follow.

Though both Stern-Gerlach analyzers in Experiment 1 are identical, they play different roles in this experiment. The 1st analyzer *prepares* the beam in a particular quantum state ($|+\rangle$) and the 2nd analyzer *measures* the resultant beam, so we often refer to the 1st analyzer as a **state preparation device**. By preparing the state with the 1st analyzer, the details of the source of atoms can be ignored. Thus our main focus in Experiment 1 is what happens at the 2nd analyzer, because we know that any atom entering the 2nd analyzer is represented by the $|+\rangle$ ket prepared by the 1st analyzer. All the experiments we will describe employ a 1st analyzer as a state preparation device,

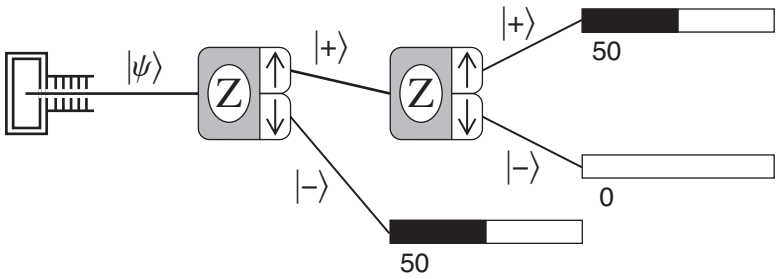

Figure 1.3 Experiment 1 measures the spin component along the z-axis twice in succession.

though the SPINS program has a feature where the state of the atoms coming from the oven is determined but unknown and the user can perform experiments to determine the unknown state using only one analyzer in the experiment.

1.1.2 Experiment 2

The second experiment is shown in Fig. 1.4 and is identical to Experiment 1 except that the 2nd Stern-Gerlach analyzer has been rotated by 90° to be aligned with the *x*-axis. Now the 2nd analyzer measures the spin component along the *x*-axis rather the *z*-axis. Atoms input to the 2nd analyzer are still represented by the ket $|+\rangle$ because the 1st analyzer is unchanged. The result of this experiment is that atoms appear at both possible output ports of the analyzer. Atoms leaving the upper port of the analyzer have been measured to have $S_x = +\hbar/2$ and atoms leaving the lower port have $S_x = -\hbar/2$. On average, each of these ports has 50% of the atoms that left the upper port of the 1st analyzer. As shown in Fig. 1.4, the output states of the 2nd analyzer have new labels $|+\rangle_x$ and $|-\rangle_x$, where the *x* subscript denotes that the spin component has been measured along the *x*-axis. We assume that if no subscript is present on the quantum ket (*e.g.*, $|+\rangle$), then the spin component is along the *z*-axis. This use of the *z*-axis as the default is a common convention throughout our work and also in much of physics.

A few items are noteworthy about this experiment. First, we notice that there are still only two possible outputs of the 2nd Stern-Gerlach analyzer. The fact that it is aligned along a different axis doesn't affect the fact that we only get two possible results for the case of a spin 1/2 particle. Second, it turns out that the results of this experiment would be unchanged if we used the lower port of the polarizer. That is, atoms entering the analyzer in state $|-\rangle$ would also result in half the atoms in each of the $|\pm\rangle_x$ output ports. Finally, we cannot predict which of the 2nd analyzer output ports any particular atom will come out. This can be demonstrated in actual experiments by recording the individual counts out of each port. The arrival sequences at any counter are completely random. We can only say that there is a 50% probability that an atom from the 2nd analyzer will exit the upper analyzer port and a 50% probability that it will exit the lower port. The random arrival of atoms at the detectors can be seen clearly in the SPINS program simulations.

This probabilistic nature is at the heart of quantum mechanics. One might be tempted to say that we just don't know enough about the system to predict which port the

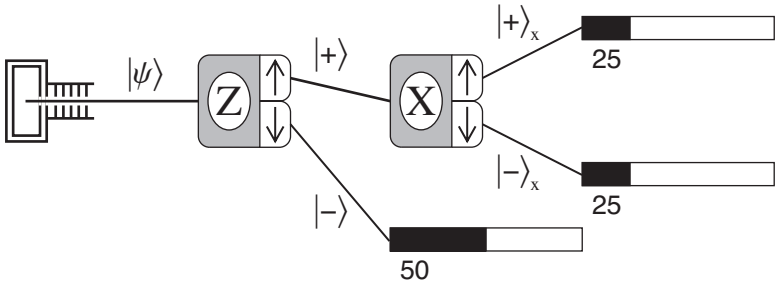

Figure 1.4 Experiment 2 measures the spin component along the *z*-axis and then along the *x*-axis.

atom will exit. That is to say, there may be some other variables, of which we are ignorant, that would allow us to predict the results. Such a viewpoint is known as a **local hidden variable theory**. John Bell proved that such theories are not compatible with the experimental results of quantum mechanics. The conclusion to draw from this is that even though quantum mechanics is a probabilistic theory, it is a complete description of reality. We will have more to say about this in Chap. 4.

Note that the 50% probability referred to above is the probability that an atom input to the 2nd analyzer exits one particular output port. It is not the probability for an atom to pass through the whole system of Stern-Gerlach analyzers. It turns out that the results of this experiment (the 50/50 split at the 2nd analyzer) are the same for any combination of two orthogonal axes of the 1st and 2nd analyzers.

1.1.3 Experiment 3

Experiment 3, shown in Fig. 1.5, extends Experiment 2 by adding a 3rd Stern-Gerlach analyzer aligned along the *z*-axis. Atoms entering the 3rd analyzer have been measured by the 1st Stern-Gerlach analyzer to have spin component up along the *z*-axis, and by the 2nd analyzer to have spin component up along the *x*-axis. The 3rd analyzer then measures how many atoms have spin component up or down along the *z*-axis. Classically, one would expect that the final measurement would yield the result spin up along the *z*-axis, because that was measured at the 1st analyzer. That is to say: classically the first 2 analyzers tell us that the atoms have $S_z = +\hbar/2$ and $S_x = +\hbar/2$, so the third measurement must yield $S_z = +\hbar/2$. But that doesn't happen, as Erwin learned with his quantum socks in the Prologue. The quantum mechanical result is that the atoms are split with 50% probability into each output port at the 3rd analyzer. Thus the last two analyzers behave like the two analyzers of Experiment 2 (except with the order reversed), and the fact that there was an initial measurement that yielded $S_z = +\hbar/2$ is somehow forgotten or erased.

This result demonstrates another key feature of quantum mechanics: a measurement disturbs the system. The 2nd analyzer has disturbed the system such that the spin component along the *z*-axis does not have a unique value, even though we measured it with the 1st analyzer. Erwin saw this when he sorted, or measured, his socks by color and then by length. When he looked, or measured, a third time, he found that the color he had measured originally was now random—the socks had forgotten about the first

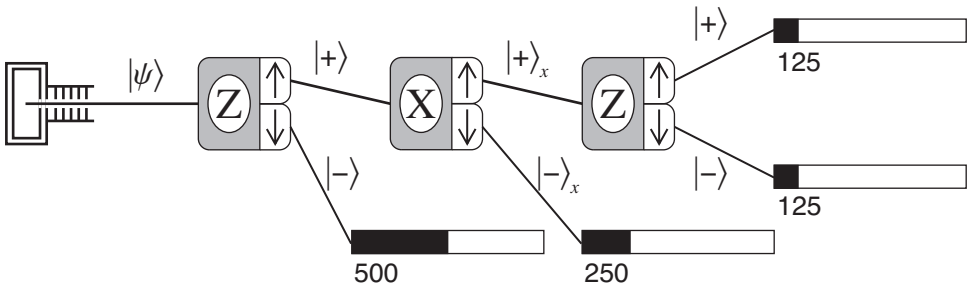

Figure 1.5 Experiment 3 measures the spin component three times in succession.

measurement. One might ask: Can I be more clever in designing the experiment such that I don't disturb the system? The short answer is no. There is a fundamental incompatibility in trying to measure the spin component of the atom along two different directions. So we say that S_x and S_z are **incompatible observables**. We cannot know the measured values of both simultaneously. The state of the system can be represented by the ket $|+\rangle = |S_z = +\hbar/2\rangle$ or by the ket $|+\rangle_x = |S_x = +\hbar/2\rangle$, but it cannot be represented by a ket $|S_z = +\hbar/2, S_x = +\hbar/2\rangle$ that specifies values of both components. Having said this, it should be said that not all pairs of quantum mechanical observables are incompatible. It is possible to do some experiments without disturbing some of the other aspects of the system. And we will see in Chap. 2.4 that whether two observables are compatible or not is very important in how we analyze a quantum mechanical system.

Not being able to measure both the S_z and S_x spin components is clearly distinct from the classical case where we can measure all three components of the spin vector, which tells us which direction the spin is pointing. In quantum mechanics, the incompatibility of the spin components means that we cannot know which direction the spin is pointing. So when we say "the spin is up," we really mean only that the spin component along that one axis is up (vs. down). The quantum mechanical spin vector cannot be said to be pointing in any given direction. As is often the case, we must check our classical intuition at the door of quantum mechanics.

1.1.4 Experiment 4

Experiment 4 is depicted in Fig. 1.6 and is a slight variation on Experiment 3. Before we get into the details, note a few changes in the schematic drawings. As promised, we have dropped the ket labels on the beams because they are redundant. We have deleted the counters on all but the last analyzer and instead simply block the unwanted beams and give the average number of atoms passing from one analyzer to the next. Note also that in Experiment 4c two output beams are combined as input to the following analyzer. This is simple in principle and in the SPINS program, but can be difficult in practice. The recombination of the beams must be done properly so as to avoid "disturbing" the beams. If you care to read more about this problem, see Feynman's Lectures on Physics, volume 3. We will have more to say about the "disturbance" later in Chap. 2.2. For now we simply assume that the beams can be recombined in the proper manner.

Experiment 4a is identical to Experiment 3. In Experiment 4b, the upper beam of the 2nd analyzer is blocked and the lower beam is sent to the 3rd analyzer. In Experiment 4c, both beams are combined with our new method and sent to the 3rd analyzer. It should be clear from our previous experiments that Experiment 4b has the same results as Experiment 4a. We now ask about the results of Experiment 4c. If we were to use classical probability analysis, then Experiment 4a would indicate that the probability for an atom leaving the 1st analyzer to take the upper path through the 2nd analyzer and then exit through the upper port of the 3rd analyzer is 25%, where we are now referring to the total probability for those two steps. Likewise, Experiment 4b would indicate that the probability total to take the lower path through the 2nd analyzer and exit through the

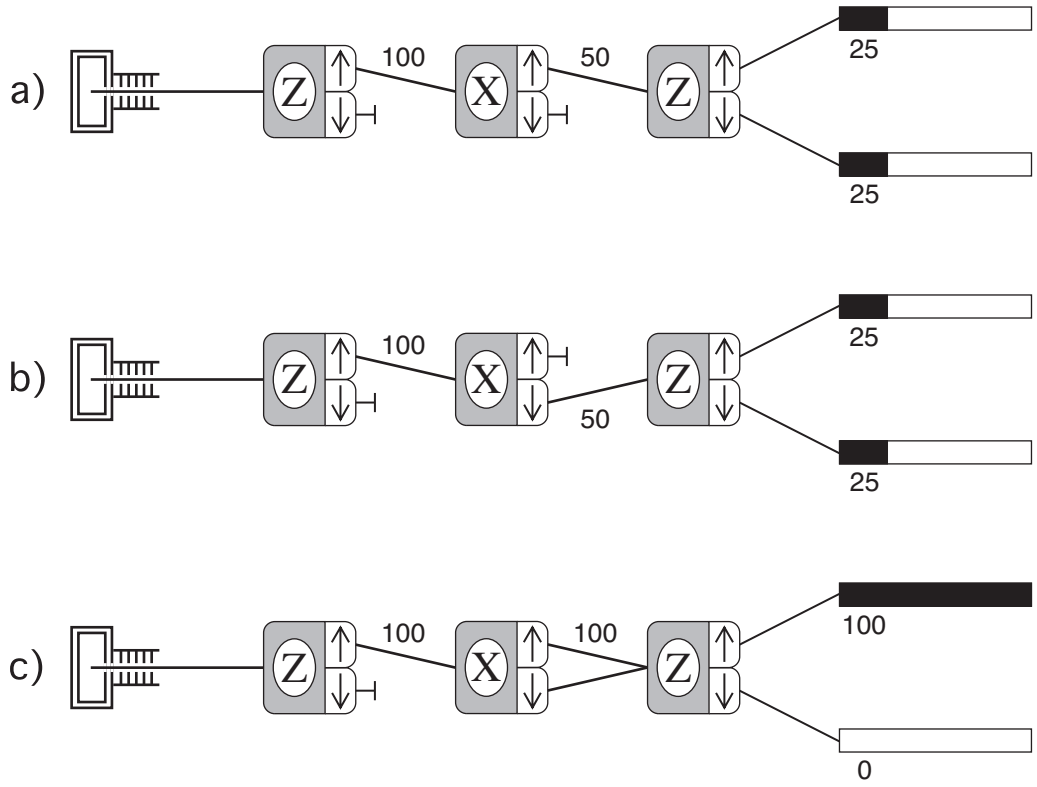

Figure 1.6 Experiment 4 measures the spin component three times in succession and uses (a and b) one or (c) two beams from the 2nd analyzer.

upper port of the 3rd analyzer is also 25%. Hence the total probability to exit from the upper port of the 3rd analyzer when both paths are available, which is Experiment 4c, would be 50%, and likewise for the exit from the lower port.

However, the quantum mechanical result in Experiment 4c is that all the atoms exit the upper port of the 3rd analyzer and none exits the lower port. The atoms now appear to "remember" that they were initially measured to have spin up along the *z*-axis. By combining the two beams from the 2nd analyzer, we have avoided the quantum mechanical disturbance that was evident in Experiments 3, 4a, and 4b. The result is now the same as Experiment 1, which means it is as if the 2nd analyzer is not there.

To see how odd this is, look carefully at what happens at the lower port of the 3rd analyzer. In this discussion, we refer to percentages of atoms leaving the 1st analyzer, because that analyzer is the same in all three experiments. In Experiments 4a and 4b, 50% of the atoms are blocked after the middle analyzer and 25% of the atoms exit the lower port of the 3rd analyzer. In Experiment 4c, 100% of the atoms pass from the 2nd analyzer to the 3rd analyzer, yet fewer atoms come out of the lower port. In fact, no atoms make it through the lower port! So we have a situation where allowing more ways or paths to reach a counter results in fewer counts. Classical probability theory cannot explain this aspect of quantum mechanics. It is as if you opened a second window in a room to get more sunlight and the room went dark!

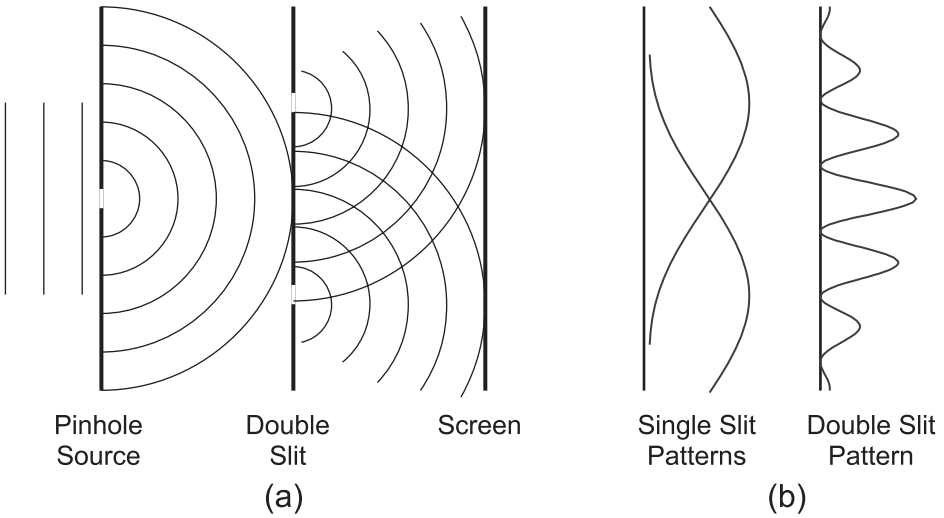

Figure 1.7 **(a)** Young's double slit interference experiment and (b) resultant intensity patterns observed on the screen, demonstrating single-slit diffraction and double-slit interference.

However, you may already know of a way to explain this effect. Imagine a procedure whereby combining two effects leads to cancellation rather than enhancement. The concept of wave interference, especially in optics, comes to mind. In the Young's double-slit experiment, light waves pass through two narrow slits and create an interference pattern on a distant screen, as shown in Fig. 1.7. Either slit by itself produces a nearly uniform illumination of the screen, but the two slits combined produce bright and dark interference fringes, as shown in Fig. 1.7(b). We explain this by adding together the electric field vectors of the light from the two slits, then squaring the resultant vector to find the light intensity. We say that we add the amplitudes and then square the total amplitude to find the resultant intensity. See Section 6.6 or an optics textbook for more details about this experiment.

We follow a similar prescription in quantum mechanics. We add together amplitudes and then take the square to find the resultant probability, which opens the door to interference effects. Before we discuss quantum mechanical interference, we must explain what we mean by an amplitude in quantum mechanics and how we calculate it.

1.2 Quantum State Vectors

Postulate 1 of quantum mechanics stipulates that kets are to be used for a mathematical description of a quantum mechanical system. These kets are abstract entities that obey many of the rules you know about ordinary spatial vectors. Hence they are called **quantum state vectors**. As we will show in Example 1.3, these vectors must employ complex numbers in order to properly describe quantum mechanical systems. Quantum state vectors are part of a vector space whose dimensionality is determined by the physics of the system at hand. In the Stern-Gerlach example, the two possible results for a spin component measurement dictate that the vector space has only two dimensions.

That makes this problem mathematically as simple as it can be, which is why we have chosen to study it. Because the quantum state vectors are abstract, it is hard to say much about what they are, other than how they behave mathematically and how they lead to physical predictions.

In the two-dimensional vector space of a spin 1/2 system, the two kets $|\pm\rangle$ form a basis, just like the **unit vectors** $\hat{\mathbf{i}}, \hat{\mathbf{j}}$, and $\hat{\mathbf{k}}$ form a basis for describing vectors in three-dimensional space. However, the analogy we want to make with these spatial vectors is only mathematical, not physical. The spatial unit vectors have three important mathematical properties that are characteristic of a basis: the basis vectors $\hat{\mathbf{i}}, \hat{\mathbf{j}}$, and $\hat{\mathbf{k}}$ are **normalized, orthogonal**, and **complete**. Spatial vectors are normalized if their magnitudes are unity and they are orthogonal if they are geometrically perpendicular to each other. The basis is complete if any general vector in the space can be written as a linear superposition of the basis vectors. These properties of spatial basis vectors can be summarized as follows:

$$\begin{aligned} &\hat{\mathbf{i}}\bullet\hat{\mathbf{i}} = \hat{\mathbf{j}}\bullet\hat{\mathbf{j}} = \hat{\mathbf{k}}\bullet\hat{\mathbf{k}} = 1 && \text{normalization} \\ &\hat{\mathbf{i}}\bullet\hat{\mathbf{j}} = \hat{\mathbf{i}}\bullet\hat{\mathbf{k}} = \hat{\mathbf{j}}\bullet\hat{\mathbf{k}} = 0 && \text{orthogonality ,} \\ &\mathbf{A} = a_x\hat{\mathbf{i}} + a_y\hat{\mathbf{j}} + a_z\hat{\mathbf{k}} && \text{completeness} \end{aligned} \tag{1.9}$$

where $\mathbf{A}$ is a general vector. Note that the **dot product**, also called the **scalar product**, is central to the description of these properties.

Continuing the mathematical analogy between spatial vectors and abstract vectors, we require that these same properties (at least conceptually) apply to quantum mechanical basis vectors. For the S_z measurement, there are only two possible results, corresponding to the states $|+\rangle$ and $|-\rangle$, so these two states comprise a complete set of basis vectors. This basis is known as the $\boldsymbol{S_z}$ **basis**. We focus on this basis for now and refer to other possible basis sets later. The completeness of the basis kets $|\pm\rangle$ implies that a general quantum state vector $|\psi\rangle$ is a linear combination of the two basis kets:

$$|\psi\rangle = a|+\rangle + b|-\rangle, \tag{1.10}$$

where a and b are complex scalar numbers multiplying each ket. This addition of two kets yields another ket in the same abstract space. The complex scalar can appear either before or after the ket without affecting the mathematical properties of the ket (*i.e.*, $a|+\rangle = |+\rangle a$). It is customary to use the Greek letter ψ (psi) for a general quantum state. You may have seen $\psi(x)$ used before as a quantum mechanical wave function. However, the state vector or ket $|\psi\rangle$ is not a wave function. Kets do not have any spatial dependence as wave functions do. We will study wave functions in Chap. 5.

To discuss orthogonality and normalization (known together as **orthonormality**) we must first define scalar products as they apply to these new kets. As we said above, the machinery of quantum mechanics requires the use of complex numbers. You may have seen other fields of physics use complex numbers. For example, sinusoidal oscillations can be described using the complex exponential $e^{i\omega t}$ rather than $\cos(\omega t)$. However, in such cases, the complex numbers are not required, but are rather a

convenience to make the mathematics easier. When using complex notation to describe classical vectors like electric and magnetic fields, the definition of the dot product is generalized slightly such that one of the vectors is complex conjugated. A similar approach is taken in quantum mechanics. The analog to the complex conjugated vector of classical physics is called a **bra** in the Dirac notation of quantum mechanics. Thus corresponding to a general ket $|\psi\rangle$, there is a bra, or bra vector, which is written as $\langle\psi|$. If a general ket $|\psi\rangle$ is specified as $|\psi\rangle = a|+\rangle + b|-\rangle$, then the corresponding bra $\langle\psi|$ is defined as

$$\langle\psi| = a^*\langle+| + b^*\langle-|, \tag{1.11}$$

where the basis bras $\langle+|$ and $\langle-|$ correspond to the basis kets $|+\rangle$ and $|-\rangle$, respectively, and the coefficients *a* and *b* have been complex conjugated.

The scalar product in quantum mechanics is defined as the product of a bra and a ket taken in the proper order—bra first, then ket second:

$$\big(\langle bra|\big)\big(|ket\rangle\big) \tag{1.12}$$

When the bra and ket are combined together in this manner we get a bracket (*bra ket*)—*a little physics humor*—that is written in shorthand as

$$\langle bra|ket\rangle \tag{1.13}$$

Thus, given the basis kets $|+\rangle$ and $|-\rangle$, one inner product, for example, is written as

$$\big(\langle+|\big)\big(|-\rangle\big) = \langle+|-\rangle \tag{1.14}$$

and so on. Note that we have eliminated the extra vertical bar in the middle. The scalar product in quantum mechanics is generally referred to as an **inner product** or a **projection**.

So, how do we calculate the inner product $\langle+|+\rangle$? We do it the same way we calculate the dot product $\hat{\mathbf{i}}\bullet\hat{\mathbf{i}}$. We define it to be unity because we like basis vectors to be unit vectors. There is a little more to it than that because in quantum mechanics (as we will see shortly) using normalized basis vectors is more rooted in physics than in our personal preferences for mathematical cleanliness. But for all practical purposes, if someone presents a set of basis vectors to you, you can probably assume that they are normalized. So the normalization of the spin-½ basis vectors is expressed in this new notation as $\langle+|+\rangle = 1$ and $\langle-|-\rangle = 1$.

Now, what about orthogonality? The spatial unit vectors $\hat{\mathbf{i}}$, $\hat{\mathbf{j}}$, and $\hat{\mathbf{k}}$ used for spatial vectors are orthogonal to each other because they are at 90° with respect to each other. That orthogonality is expressed mathematically in the dot products $\hat{\mathbf{i}}\bullet\hat{\mathbf{j}} = \hat{\mathbf{i}}\bullet\hat{\mathbf{k}} = \hat{\mathbf{j}}\bullet\hat{\mathbf{k}} = 0$. For the spin basis kets $|+\rangle$ and $|-\rangle$, there is no spatial geometry involved. Rather, the spin basis kets $|+\rangle$ and $|-\rangle$ are orthogonal in the mathematical sense, which we express with the inner product as $\langle+|-\rangle = 0$. Again, we do not prove to you that these basis vectors are orthogonal, but we assume that a well-behaved basis set obeys orthogonality. Though there is no geometry in this property for quantum

mechanical basis vectors, the fundamental idea of orthogonality is the same, so we use the same language—if a general vector "points" in the direction of a basis vector, then there is no component in the "direction" of the other unit vectors.

In summary, the properties of normalization, orthogonality, and completeness can be expressed in the case of a two-state spin-½ quantum system as:

$$\boxed{\begin{array}{ll} \left.\begin{array}{l}\langle +|+\rangle = 1 \\ \langle -|-\rangle = 1\end{array}\right\} & \text{normalization} \\ \left.\begin{array}{l}\langle +|-\rangle = 0 \\ \langle -|+\rangle = 0\end{array}\right\} & \text{orthogonality} \\ |\psi\rangle = a|+\rangle + b|-\rangle & \text{completeness} \end{array}} \quad . \tag{1.15}$$

Note that a product of kets (*e.g.*, $|+\rangle|+\rangle$) or a similar product of bras (*e.g.*, $\langle +|\langle +|$) is meaningless in this new notation, while a product of a ket and a bra in the "wrong" order (*e.g.*, $|+\rangle\langle +|$) has a meaning that we will define in Chap. 2.2.3. Equations (1.15) are sufficient to define how the basis kets behave mathematically. Note that the inner product is defined using a bra and a ket, though it is common to refer to the inner product of two kets, where it is understood that one is converted to a bra first. The order does matter, as we will see shortly.

Using this new notation, we can learn a little more about general quantum states and derive some expressions that will be useful later. Consider the general state vector $|\psi\rangle = a|+\rangle + b|-\rangle$. Take the inner product of this ket with the bra $\langle +|$ and obtain

$$\begin{aligned} \langle +|\psi\rangle &= \langle +| \ \left(a|+\rangle \ + \ b|-\rangle\right) \\ &= \langle +|a|+\rangle \ + \ \langle +|b|-\rangle \\ &= a\langle +|+\rangle \ + \ b\langle +|-\rangle \\ &= a \end{aligned} \quad , \tag{1.16}$$

using the properties that inner products are distributive and that scalars can be moved freely through bras or kets. Likewise, you can show that $\langle -|\psi\rangle = b$. Hence the coefficients multiplying the basis kets are simply the inner products or projections of the general state $|\psi\rangle$ along each basis ket, albeit in an abstract complex vector space, rather than the concrete three-dimensional space of normal vectors. Using these results, we rewrite the general state as

$$\begin{aligned} |\psi\rangle &= a|+\rangle + b|-\rangle \\ &= |+\rangle a + |-\rangle b \\ &= |+\rangle\left\{\langle +|\psi\rangle\right\} + |-\rangle\left\{\langle -|\psi\rangle\right\} \end{aligned} \quad , \tag{1.17}$$

where the rearrangement of the 2nd equation again uses the property that scalars (*e.g.*, $a = \langle +|\psi\rangle$) can be moved through bras or kets.

For a general state vector $|\psi\rangle = a|+\rangle + b|-\rangle$ we defined the corresponding bra to be $\langle\psi| = a^*\langle+| + b^*\langle-|$. Thus, the inner product of the state $|\psi\rangle$ with the basis ket $|+\rangle$ taken in the reverse order compared to Eq. (1.16) yields

$$\begin{aligned}\langle\psi|+\rangle &= \langle+|a^*|+\rangle \;+\; \langle-|b^*|+\rangle \\ &= a^*\langle+|+\rangle \;+\; b^*\langle-|+\rangle \;. \\ &= a^*\end{aligned} \tag{1.18}$$

Thus we see that an inner product with the states reversed results in a complex conjugation of the inner product:

$$\langle+|\psi\rangle = \langle\psi|+\rangle^* . \tag{1.19}$$

This important property holds for any inner product. For example, the inner product of two general states is

$$\boxed{\langle\phi|\psi\rangle = \langle\psi|\phi\rangle^*}. \tag{1.20}$$

Now we come to a new mathematical aspect of quantum vectors that differs from the use of vectors in classical mechanics. The rules of quantum mechanics (postulate 1) require that all state vectors describing a quantum system be normalized, not just the basis kets. This is clearly different from ordinary spatial vectors, where the length or magnitude of a vector means something and only the unit vectors $\hat{\mathbf{i}}, \hat{\mathbf{j}}$, and $\hat{\mathbf{k}}$ are normalized to unity. This new rule means that in the quantum mechanical state space only the direction—in an abstract sense—is important. If we apply this normalization requirement to a general state $|\psi\rangle$, then we obtain

$$\begin{aligned}&\langle\psi|\psi\rangle = \left\{a^*\langle+| + b^*\langle-|\right\}\left\{a|+\rangle + b|-\rangle\right\} = 1 \\ &\Rightarrow a^*a\langle+|+\rangle + a^*b\langle+|-\rangle + b^*a\langle-|+\rangle + b^*b\langle-|-\rangle = 1 \\ &\Rightarrow a^*a + b^*b = 1 \\ &\Rightarrow |a|^2 + |b|^2 = 1\end{aligned}\;, \tag{1.21}$$

or using the expressions for the coefficients obtained above,

$$\left|\langle+|\psi\rangle\right|^2 + \left|\langle-|\psi\rangle\right|^2 = 1 . \tag{1.22}$$

Example 1.1

Normalize the vector $|\psi\rangle = C\left(1|+\rangle + 2i|-\rangle\right)$. The complex constant C is often referred to as the **normalization constant**.

To normalize $|\psi\rangle$, we set the inner product of the vector with itself equal to unity and then solve for C—note the requisite complex conjugations

$$\begin{aligned} 1 &= \langle\psi|\psi\rangle \\ &= C^*\{1\langle+|-2i\langle-|\}C\{1|+\rangle+2i|-\rangle\} \\ &= C^*C\{1\langle+|+\rangle-2i\langle+|-\rangle+2i\langle-|+\rangle+4\langle-|-\rangle\}, \\ &= 5|C|^2 \\ \Rightarrow |C| &= \frac{1}{\sqrt{5}} \end{aligned} \tag{1.23}$$

The overall phase of the normalization constant is not physically meaningful (HW), so we follow the standard convention and choose it to be real and positive. This yields $C = 1/\sqrt{5}$. The normalized quantum state vector is then

$$|\psi\rangle = \frac{1}{\sqrt{5}}\left(1|+\rangle+2i|-\rangle\right), \tag{1.24}$$

Now comes the crucial element of quantum mechanics. We postulate that each term in the sum of Eq. (1.22) is equal to the **probability** that the quantum state described by the ket $|\psi\rangle$ is measured to be in the corresponding basis state. Thus

$$\mathcal{P}_{S_z=+\hbar/2} = \left|\langle+|\psi\rangle\right|^2 \tag{1.25}$$

is the probability that the state $|\psi\rangle$ is found to be in the state $|+\rangle$ when a measurement of S_z is made, meaning that the result $S_z = +\hbar/2$ is obtained. Likewise,

$$\mathcal{P}_{S_z=-\hbar/2} = \left|\langle-|\psi\rangle\right|^2 \tag{1.26}$$

is the probability that the measurement yields the result $S_z = -\hbar/2$. The subscript on the probability indicates the measured value. For the spin component measurements, we will usually abbreviate this to, for example, $\mathcal{P}_+$ for a $S_z = +\hbar/2$ result or $\mathcal{P}_{-y}$ for a $S_y = -\hbar/2$ measurement.

We now have a prescription for predicting the outcomes of the experiments we have been discussing. For example, the experiment shown in Fig. 1.8 has the state $|\psi\rangle = |+\rangle$ prepared by the 1st Stern-Gerlach device and then input to the 2nd Stern-Gerlach device aligned along the *z*-axis. Therefore the probabilities of measuring the input state $|\psi\rangle = |+\rangle$ to have the two output values are as shown. Because the spin-1/2 system has only two possible measurement results, these two probabilities must sum to unity—there

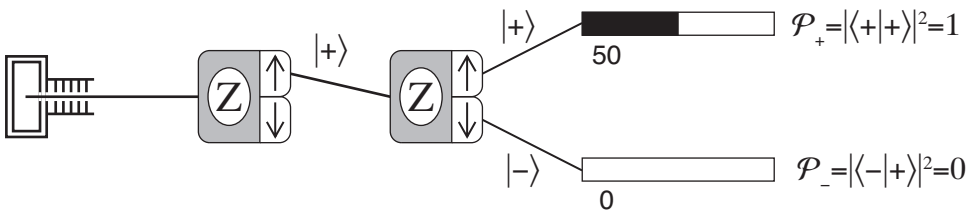

Figure 1.8 Probabilities of spin component measurements.

is a 100% probability of recording some value in the experiment. This basic rule of probabilities is why the rules of quantum mechanics require that all state vectors be properly normalized before they are used in any calculation of probabilities. The experimental predictions shown in Fig. 1.8 are an example of the 4th postulate of quantum mechanics, which is presented below.

Postulate 4 (Spin-1/2 system)

The probability of obtaining the value $\pm\hbar/2$ in a measurement of the observable S_z on a system in the state $|\psi\rangle$ is

$$\mathcal{P}_{\pm} = \left|\langle \pm|\psi\rangle\right|^2,$$

where $|\pm\rangle$ is the basis ket of S_z corresponding to the result $\pm\hbar/2$.

This is labeled as the 4th postulate because we have written this postulate using the language of the spin-1/2 system, while the general statement of the 4th postulate presented in section 1.5 requires the 2nd and 3rd postulates of section 2.1. A general spin component measurement is shown in Fig. 1.9, along with a histogram that compactly summarizes the measurement results.

Because the quantum mechanical probability is found by squaring an inner product, we refer to an inner product, $\langle +|\psi\rangle$ for example, as a **probability amplitude** or sometimes just an **amplitude**; much like a classical wave intensity is found by squaring the wave amplitude. Note that the convention is to put the input or initial state on the right and the output or final state on the left: $\langle out|in\rangle$, so one would read from right to left in describing a problem. Because the probability involves the complex square of the amplitude, and $\langle out|in\rangle = \langle in|out\rangle^*$, this convention is not critical for calculating probabilities. Nonetheless, it is the accepted practice and is important in situations where several amplitudes are combined.

Armed with these new quantum mechanical rules and tools, let's continue to analyze the experiments discussed earlier. Using the experimental results and the new rules we have introduced, we can learn more about the mathematical behavior of the kets and the relationships among them. We will focus on the first two experiments for now and return to the others in the next chapter.

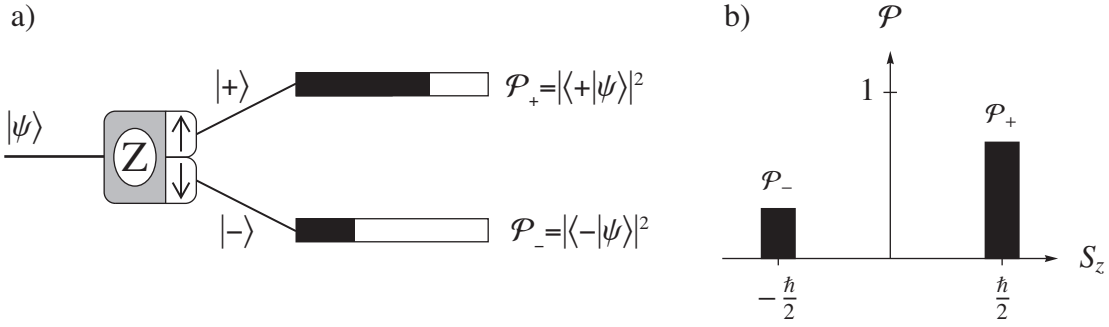

Figure 1.9 (a) Spin component measurement for a general input state and (b) histogram of measurement results.

1.2.1 Analysis of Experiment 1

In Experiment 1, the 1st Stern-Gerlach analyzer prepared the system in the $|+\rangle$ state and the 2nd analyzer later measured this state to be in the $|+\rangle$ state and not in the $|-\rangle$ state. The results of the experiment are summarized in the histogram in Fig. 1.10. We can use the 4th postulate to predict the results of this experiment. We take the inner product of the input state $|+\rangle$ with each of the possible output basis states $|+\rangle$ and $|-\rangle$. Because we know that the basis states are normalized and orthogonal, we calculate the probabilities to be

$$\begin{aligned} \mathcal{P}_+ &= |\langle + | + \rangle|^2 = 1 \\ \mathcal{P}_- &= |\langle - | + \rangle|^2 = 0 \end{aligned} . \tag{1.27}$$

These predictions agree exactly with the histogram of experimental results shown in Fig. 1.10. A $|+\rangle$ state is always measured to have $S_z = +\hbar/2$.

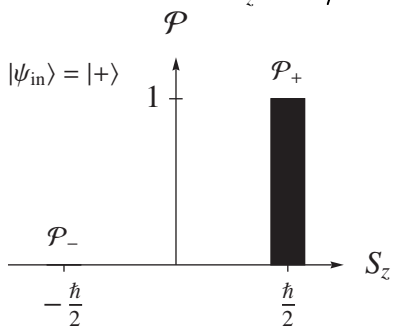

Figure 1.10 Histogram of S_z spin component measurements for experiment 1 with $|\psi_{in}\rangle = |+\rangle$.

1.2.2 Analysis of Experiment 2

In Experiment 2, the 1st Stern-Gerlach analyzer prepared the system in the $|+\rangle$ state and the 2nd analyzer performed a measurement of the spin component along the *x*-axis, finding 50% probabilities for each of the two possible states $|+\rangle_x$ and $|-\rangle_x$, as shown in the histogram in Fig. 1.11(a). For this experiment, we cannot predict the results of the measurements, because we do not yet have enough information about how the states $|+\rangle_x$ and $|-\rangle_x$ behave mathematically. Rather, we will use the results of the experiment to determine these states. Recalling that the experimental results would be the same if the 1st analyzer prepared the system to be in the $|-\rangle$ state (see Fig. 1.11(b)), we have four results for the two experiments:

$$\begin{aligned} \mathcal{P}_{1,+x} &= |{}_x\langle + | + \rangle|^2 = \tfrac{1}{2} \\ \mathcal{P}_{1,-x} &= |{}_x\langle - | + \rangle|^2 = \tfrac{1}{2} \\ \mathcal{P}_{2,+x} &= |{}_x\langle + | - \rangle|^2 = \tfrac{1}{2} \\ \mathcal{P}_{2,-x} &= |{}_x\langle - | - \rangle|^2 = \tfrac{1}{2} \end{aligned} \tag{1.28}$$

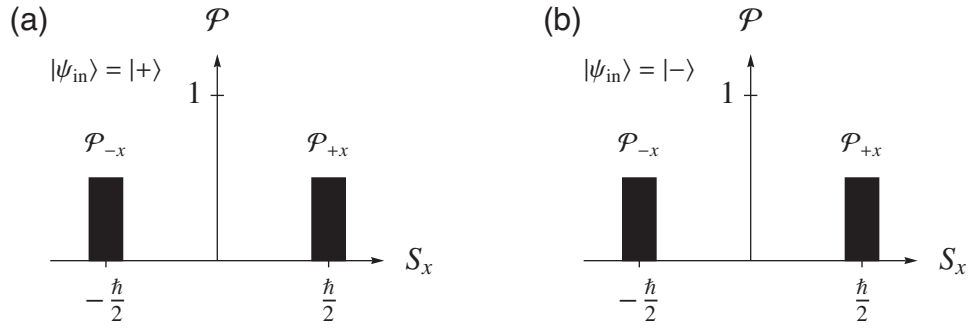

Figure 1.11 Histograms of S_x spin component measurements for experiment 2 for different input states (a) $|\psi_{in}\rangle = |+\rangle$ and (b) $|\psi_{in}\rangle = |-\rangle$.

Because the kets $|+\rangle$ and $|-\rangle$ form a complete basis, the kets describing the S_x measurement, $|+\rangle_x$ and $|-\rangle_x$, can be written in terms of them. We do not yet know the specific coefficients of the $|\pm\rangle_x$ states, so we use general expressions

$$\begin{aligned} |+\rangle_x &= a|+\rangle + b|-\rangle \\ |-\rangle_x &= c|+\rangle + d|-\rangle \end{aligned}, \tag{1.29}$$

and now our task is to use the results of Experiment 2 to determine the coefficients a, b, c, and d. The first measured probability in Eqn. (1.28) is

$$\mathcal{P}_{1,+x} = \left|{}_x\langle +|+\rangle\right|^2 = \tfrac{1}{2} \tag{1.30}$$

Using the general expression for $|+\rangle_x$ in Eqn. (1.29), we calculate the probability that the $|+\rangle$ input state is measured to be in the $|+\rangle_x$ output state, *i.e.*, to have $S_x = +\hbar/2$:

$$\begin{aligned} \mathcal{P}_{1,+x} &= \left|{}_x\langle +|+\rangle\right|^2 \\ &= \left|\left\{a^*\langle +| + b^*\langle -|\right\}|+\rangle\right|^2 \\ &= \left|a^*\right|^2 = |a|^2 \end{aligned} \tag{1.31}$$

where we convert the $|+\rangle_x$ ket to a bra ${}_x\langle +|$ in order to calculate the inner product. Equating the experimental result in Eqn. (1.30) and the prediction in Eqn. (1.31), we find

$$|a|^2 = \tfrac{1}{2} \tag{1.32}$$

Similarly, one can calculate the other three probabilities to arrive at $|b|^2 = |c|^2 = |d|^2 = \frac{1}{2}$. (HW) Because each coefficient is complex, each has an amplitude and phase. However, the overall phase of a quantum state vector is not physically meaningful (see problem 1.3). Only the relative phase between different components of the state vector is physically measurable. Hence, we are free to choose *one* coefficient of each vector to be real and positive without any loss of generality. This allows us to write the desired states as

$$\begin{aligned} |+\rangle_x &= \tfrac{1}{\sqrt{2}}\left[|+\rangle + e^{i\alpha}|-\rangle\right] \\ |-\rangle_x &= \tfrac{1}{\sqrt{2}}\left[|+\rangle + e^{i\beta}|-\rangle\right] \end{aligned}, \tag{1.33}$$

where α and β are relative phases that we have yet to determine. Note that these states are already normalized because we used all of the experimental results, which reflect the fact that the probability for all possible results of an experiment must sum to unity.

We have used all the experimental results from Experiment 2, but the $|\pm\rangle_x$ kets are still not determined. We need some more information. If we perform Experiment 1 with both analyzers aligned along the *x*-axis, the results will be as you expect—all $|+\rangle_x$ states from the 1st analyzer will be measured to have $S_x = +\hbar/2$ at the 2nd analyzer, *i.e.*, all atoms exit in the $|+\rangle_x$ state and none in the $|-\rangle_x$. The probability calculations for this experiment are

$$\begin{aligned} \mathcal{P}_{+x} &= \left|{}_x\langle +|+\rangle_x\right|^2 = 1 \\ \mathcal{P}_{-x} &= \left|{}_x\langle -|+\rangle_x\right|^2 = 0 \end{aligned}, \tag{1.34}$$

which tell us mathematically that the $|\pm\rangle_x$ states are orthonormal to each other, just like the $|\pm\rangle$ states. This also implies that the $|\pm\rangle_x$ kets form a basis, the S_x basis, which you might expect because they correspond to the distinct results of a different spin component measurement. The general expressions we used for the $|\pm\rangle_x$ kets are already normalized, but are not yet orthogonal. That is the new piece of information we need. The orthogonality condition leads to

$$\begin{aligned} &{}_x\langle -|+\rangle_x = 0 \\ &\tfrac{1}{\sqrt{2}}\left[\langle +| + e^{-i\beta}\langle -|\right]\tfrac{1}{\sqrt{2}}\left[|+\rangle + e^{i\alpha}|-\rangle\right] = 0 \\ &\tfrac{1}{2}\left[1 + e^{i(\alpha-\beta)}\right] = 0 \\ &e^{i(\alpha-\beta)} = -1 \\ &e^{i\alpha} = -e^{i\beta} \end{aligned} \quad . \tag{1.35}$$

where the complex conjugation of the second coefficient of the ${}_x\langle -|$ bra should be noted.

We now have an equation relating the remaining coefficients α and β, but need some more information to determine their values. Unfortunately, there is no more information to be obtained, so we are free to choose the value of the phase α. This freedom comes from the fact that we have required only that the *x*-axis be perpendicular to the *z*-axis, which limits the *x*-axis only to a plane rather than to a unique direction. We follow convention here and choose the phase $\alpha = 0$. Thus we can express the S_x basis kets in terms of the S_z basis kets as

$$\begin{aligned} |+\rangle_x &= \tfrac{1}{\sqrt{2}}\left[|+\rangle + |-\rangle\right] \\ |-\rangle_x &= \tfrac{1}{\sqrt{2}}\left[|+\rangle - |-\rangle\right] \end{aligned}. \tag{1.36}$$

We generally use the S_z basis as the preferred basis for writing general states, but we could use any basis we choose. If we were to use the S_x basis, then we could write the $|\pm\rangle$ kets as general states in terms of the $|\pm\rangle_x$ kets. This can be done by solving Eqs. (1.36) for the $|\pm\rangle$ kets, yielding

$$\begin{aligned} |+\rangle &= \tfrac{1}{\sqrt{2}}\left[|+\rangle_x + |-\rangle_x\right] \\ |-\rangle &= \tfrac{1}{\sqrt{2}}\left[|+\rangle_x - |-\rangle_x\right] \end{aligned}. \tag{1.37}$$

With respect to the measurements performed in Experiment 2, Eqn. (1.37) tells us that the $|+\rangle$ state is a combination of the states $|+\rangle_x$ and $|-\rangle_x$. The coefficients tell us that there is a 50% probability for measuring the spin component to be up along the *x*-axis, and likewise for the down possibility, which is in agreement with the histogram of measurements shown in Fig. 1.11(a). We must now take a moment to describe carefully what a combinations of states, such as in Eqns. (1.36) and (1.37), is and what it is not.

1.2.3 Superposition states

A general spin-1/2 state vector $|\psi\rangle$ can be expressed as a combination of the basis kets $|+\rangle$ and $|-\rangle$

$$|\psi\rangle = a|+\rangle + b|-\rangle. \tag{1.38}$$

We refer to such a combination of states as a **superposition state**. To understand the importance of a quantum mechanical superposition state, consider the particular state

$$|\psi\rangle = \tfrac{1}{\sqrt{2}}\left(|+\rangle + |-\rangle\right). \tag{1.39}$$

and measurements on this state, as shown in Fig. 1.12(a). Note that the state $|\psi\rangle$ is none other than the state $|+\rangle_x$ that we found in Eqn. (1.36), so we already know what the measurement results are. If we measure the spin component along the *x*-axis for this state, then we record the result $S_x = +\hbar/2$ with 100% probability (Experiment 1 with both analyzers along the *x*-axis). If we measure the spin component along the orthogonal *z*-axis, then we record the two results $S_z = \pm\hbar/2$ with 50% probability each (Experiment 2 with the 1st and 2nd analyzers along the *x*– and *z*-axes, respectively). Based upon this second set of results, one might be tempted to consider the state $|\psi\rangle$ as describing a beam that contains a mixture of atoms with 50% of the atoms in the $|+\rangle$ state and 50% in the $|-\rangle$ state. Such a state is called a **mixed state**, and is very different from a superposition state.

To clarify the difference between a mixed state and a superposition state, let's carefully examine the results of experiments on the proposed mixed state beam, as shown in Fig. 1.12(b). If we measure the spin component along the *z*-axis, then each atom in the $|+\rangle$ state yields the result $S_z = +\hbar/2$ with 100% certainty and each atom in the $|-\rangle$ state yields the result $S_z = -\hbar/2$ with 100% certainty. The net result is that 50% of the atoms yield $S_z = +\hbar/2$ and 50% yield $S_z = -\hbar/2$. This is exactly the same result as that obtained with all atoms in the $|+\rangle_x$ state, as seen in Fig. 1.12(a). If we instead measure the spin component along the *x*-axis, then each atom in the $|+\rangle$ state yields the two

results $S_x = \pm\hbar/2$ with 50% probability each (Experiment 2 with the 1st and 2nd analyzers along the *z*- and *x*-axes, respectively). The atoms in the $|-\rangle$ state yield the same results. The net result is that 50% of the atoms yield $S_x = +\hbar/2$ and 50% yield $S_x = -\hbar/2$. This is in stark contrast to the results of Experiment 1, which tells us that once we have prepared the state to be $|+\rangle_x$, then subsequent measurements yield $S_x = +\hbar/2$ with certainty, as seen in Fig. 1.12(a).

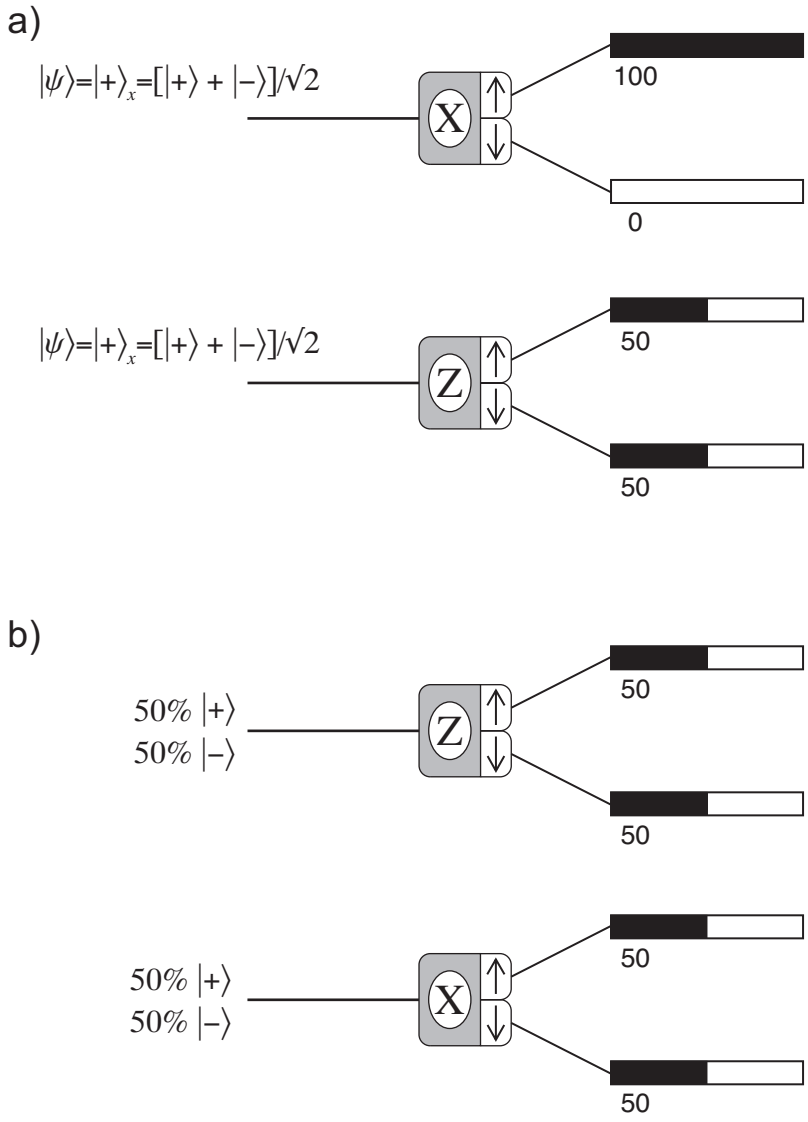

Figure 1.12 (a) Superposition state measurements and (b) mixed state measurements.

Hence we must conclude that the system described by the $|\psi\rangle = |+\rangle_x$ state is not a mixed state with some atoms in the $|+\rangle$ state and some in the $|-\rangle$ state. Rather, each atom in the $|+\rangle_x$ beam is in a state that itself is a superposition of the $|+\rangle$ and $|-\rangle$ states. A superposition state is often called a **coherent superposition** because the relative phase of the two terms is important. For example, if the input beam were in the $|-\rangle_x$ state, then

there would be a relative minus sign between the two coefficients, which would result in an $S_x = -\hbar/2$ measurement but would not affect the S_z measurement.

We will not have any further need to speak of mixed states, so any combination of states we use is a superposition state. Note that we cannot even write down a ket describing a mixed state. So, if someone gives you a quantum state written as a ket, then it must be a superposition state and not a mixed state. The random option in the SPINS program produces a mixed state, while the unknown states are all superposition states.

Example 1.2

Consider the input state

$$|\psi_{in}\rangle = 3|+\rangle + 4|-\rangle . \tag{1.40}$$

Normalize this state vector and find the probabilities of measuring the spin component along the *z*-axis to be $S_z = \pm\hbar/2$.

To normalize this state, introduce an overall complex multiplicative factor and solve for this factor by imposing the normalization condition:

$$\begin{aligned}
&|\psi_{in}\rangle = C\left[3|+\rangle + 4|-\rangle\right] \\
&\langle\psi_{in}|\psi_{in}\rangle = 1 \\
&\left\{C^*\left[3\langle+| + 4\langle-|\right]\right\}\left\{C\left[3|+\rangle + 4|-\rangle\right]\right\} = 1 \\
&C^*C\left[9\langle+|+\rangle + 12\langle+|-\rangle + 12\langle-|+\rangle + 16\langle-|-\rangle\right] = 1 , \\
&C^*C\left[25\right] = 1 \\
&|C|^2 = \frac{1}{25}
\end{aligned} \tag{1.41}$$

Because an overall phase is physically meaningless, we choose *C* to be real and positive: $C = 1/5$. Hence the normalized input state is

$$|\psi_{in}\rangle = \tfrac{3}{5}|+\rangle + \tfrac{4}{5}|-\rangle . \tag{1.42}$$

The probability of measuring $S_z = +\hbar/2$ is

$$\begin{aligned}
\mathcal{P}_+ &= \left|\langle+|\psi_{in}\rangle\right|^2 \\
&= \left|\left\langle+\left[\tfrac{3}{5}|+\rangle + \tfrac{4}{5}|-\rangle\right]\right|\right|^2 \\
&= \left|\tfrac{3}{5}\langle+|+\rangle + \tfrac{4}{5}\langle+|-\rangle\right|^2 \\
&= \left|\tfrac{3}{5}\right|^2 = \tfrac{9}{25}
\end{aligned} \tag{1.43}$$

The probability of measuring $S_z = -\hbar/2$ is

$$\begin{aligned}\mathcal{P}_- &= \left|\langle -|\psi_{in}\rangle\right|^2 \\ &= \left|\left\langle -\left[\tfrac{3}{5}|+\rangle + \tfrac{4}{5}|-\rangle\right]\right\|^2 \\ &= \left|\tfrac{3}{5}\langle -|+\rangle + \tfrac{4}{5}\langle -|-\rangle\right|^2 \\ &= \left|\tfrac{4}{5}\right|^2 = \tfrac{16}{25}\end{aligned} \tag{1.44}$$

Note that the two probabilities add to unity, which indicates that we normalized the input state properly. A histogram of the predicted measurement results is shown in Fig. 1.13.

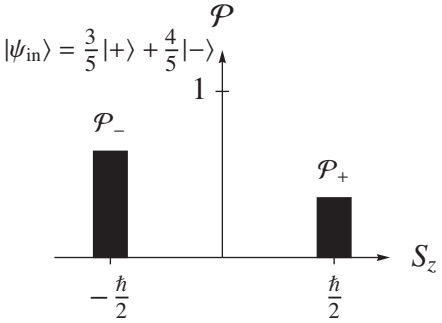

Figure 1.13 Histogram of S_z spin component measurements.

1.3 Matrix notation

Up to this point, we have defined kets mathematically in terms of their inner products with other kets. Thus in the general case we write a ket as

$$|\psi\rangle = \langle +|\psi\rangle\,|+\rangle \;+\; \langle -|\psi\rangle\,|-\rangle\,, \tag{1.45}$$

or in a specific case, we write

$$\begin{aligned}|+\rangle_x &= \langle +|+\rangle_x\,|+\rangle \;+\; \langle -|+\rangle_x\,|-\rangle \\ &= \tfrac{1}{\sqrt{2}}|+\rangle + \tfrac{1}{\sqrt{2}}|-\rangle\end{aligned}\,. \tag{1.46}$$

In both of these cases, we have chosen to write the kets in terms of the $|+\rangle$ and $|-\rangle$ basis kets. If we agree on that choice of basis as a convention, then the two coefficients $\langle +|+\rangle_x$ and $\langle -|+\rangle_x$ uniquely specify the quantum state, and we can simplify the notation by using those just numbers. Thus, we represent a ket as a **column vector** containing the two coefficients that multiply each basis ket. For example, we represent $|+\rangle_x$ as

$$|+\rangle_x \doteq \begin{pmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \end{pmatrix}, \tag{1.47}$$

where we have used the new symbol $\doteq$ to signify "is represented by", and it is understood that we are using the $|+\rangle$ and $|-\rangle$ basis or the S_z basis. We cannot say that the ket *equals* the column vector, because the ket is an abstract vector in the state space

and the column vector is just two complex numbers. If we were to choose a different basis for representing the vector, then the complex coefficients would be different even though the vector is unchanged. We need to have a convention for the ordering of the amplitudes in the column vector. The standard convention is to put the spin up amplitude first (at the top). Thus the representation of the $|-\rangle_x$ state (Eqn. (1.37)) is

$$|-\rangle_x \doteq \begin{pmatrix} 1/\sqrt{2} \\ -1/\sqrt{2} \end{pmatrix} \begin{matrix} \leftarrow |+\rangle \\ \leftarrow |-\rangle \end{matrix} . \tag{1.48}$$

where we have explicitly labeled the rows according to their corresponding basis kets. Using this convention, it should be clear that the basis kets themselves are written as

$$\begin{aligned} |+\rangle &\doteq \begin{pmatrix} 1 \\ 0 \end{pmatrix} \\ |-\rangle &\doteq \begin{pmatrix} 0 \\ 1 \end{pmatrix} \end{aligned} . \tag{1.49}$$

This demonstrates the important feature that *basis kets are unit vectors when written in their own basis*.

This new way of expressing a ket simply as the collection of coefficients that multiply the basis kets is referred to as a **representation**. Because we have assumed the S_z kets as the basis kets, this is called the S_z representation. It is always true that basis kets have the simple form shown in Eq. (1.49) when written in their own representation. A general ket $|\psi\rangle$ is written as

$$|\psi\rangle \doteq \begin{pmatrix} \langle +|\psi\rangle \\ \langle -|\psi\rangle \end{pmatrix} . \tag{1.50}$$

This use of matrix notation simplifies the mathematics of bras and kets. The advantage is not so evident for the simple 2-dimensional state space of spin-1/2 systems, but it is very evident for larger dimensional problems. This notation is indispensable when using computers to calculate quantum mechanical results. For example, the SPINS program employs matrix calculations coded in the Java computer language to simulate the Stern-Gerlach experiments using the same probability rules you are learning here.

We saw earlier (Eqn. (1.11)) that the coefficients of a bra are the complex conjugates of the coefficients of the corresponding ket. We also know that an inner product of a bra and a ket yields a single complex number. In order for the matrix rules of multiplication to be used, a bra must be represented by a **row vector**, with the entries being the coefficients ordered in the same sense as for the ket. For example, if we use the general ket

$$|\psi\rangle = a|+\rangle + b|-\rangle , \tag{1.51}$$

which is represented as

$$|\psi\rangle \doteq \begin{pmatrix} a \\ b \end{pmatrix}, \tag{1.52}$$

then the corresponding bra

$$\langle\psi| = a^*\langle +| + b^*\langle -| \tag{1.53}$$

is represented by a row vector as

$$\langle\psi| \doteq \begin{pmatrix} a^* & b^* \end{pmatrix}. \tag{1.54}$$

The rules of matrix algebra can then be applied to find an inner product. For example,

$$\begin{aligned} \langle\psi|\psi\rangle &= \begin{pmatrix} a^* & b^* \end{pmatrix}\begin{pmatrix} a \\ b \end{pmatrix}. \\ &= |a|^2 + |b|^2 \end{aligned} \tag{1.55}$$

So a bra is represented by a row vector that is the complex conjugate and transpose of the column vector representing the corresponding ket.

Example 1.3:

To get some practice using this new matrix notation, and to learn some more about the spin-1/2 system, use the results of Experiment 2 to determine the S_y basis kets using the matrix approach instead of the Dirac bra-ket approach.

Consider Experiment 2 in the case where the 2nd Stern-Gerlach analyzer is aligned along the *y*-axis. We said before that the results are the same as in the case shown in Fig. 1.4. Thus we have

$$\begin{aligned} \mathcal{P}_{1,+y} &= \left|{}_y\langle +|+\rangle\right|^2 = \tfrac{1}{2} \\ \mathcal{P}_{1,-y} &= \left|{}_y\langle -|+\rangle\right|^2 = \tfrac{1}{2} \\ \mathcal{P}_{2,+y} &= \left|{}_y\langle +|-\rangle\right|^2 = \tfrac{1}{2} \\ \mathcal{P}_{2,-y} &= \left|{}_y\langle -|-\rangle\right|^2 = \tfrac{1}{2} \end{aligned} \quad . \tag{1.56}$$

as depicted in the histograms of Fig. 1.14.

These results allows us to determine the kets $|\pm\rangle_y$ corresponding to the spin component up and down along the *y*-axis. The argument and calculation proceeds exactly as it did earlier for the $|\pm\rangle_x$ states up until the point (Eqn. (1.35)) where we arbitrarily choose the phase α to be zero. Having done that for the $|\pm\rangle_x$ states, we are no longer free to make that same choice for the $|\pm\rangle_y$ states. Thus we use Eqn. (1.35) to write the $|\pm\rangle_y$ states as

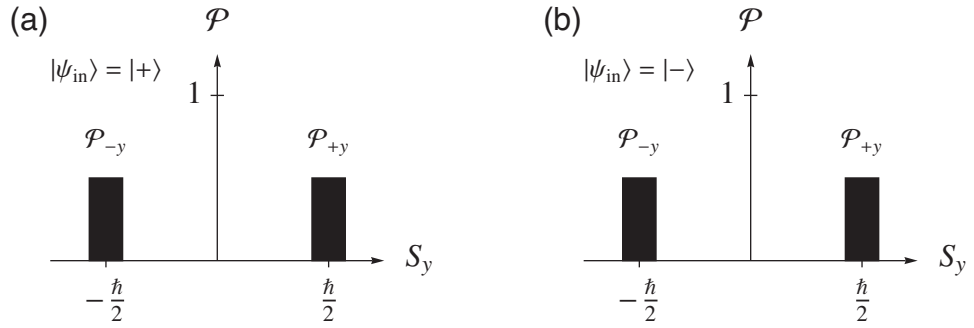

Figure 1.14 Histograms of S_y spin component measurements for input states (a) $|\psi_{in}\rangle = |+\rangle$ and (b) $|\psi_{in}\rangle = |-\rangle$.

$$\begin{aligned} |+\rangle_y &= \tfrac{1}{\sqrt{2}}\left[|+\rangle + e^{i\alpha}|-\rangle\right] \doteq \tfrac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ e^{i\alpha} \end{pmatrix} \\ |-\rangle_y &= \tfrac{1}{\sqrt{2}}\left[|+\rangle - e^{i\alpha}|-\rangle\right] \doteq \tfrac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ -e^{i\alpha} \end{pmatrix} \end{aligned} \tag{1.57}$$

To determine the phase α, we use some more information at our disposal. Experiment 2 could be performed with the 1st Stern-Gerlach analyzer aligned along the *x*-axis and the 2nd analyzer along the *y*-axis. Again the results would be identical (50% at each output port), yielding

$$\mathcal{P}_{+y} = \left|{}_y\langle +|+\rangle_x\right|^2 = \tfrac{1}{2} \tag{1.58}$$

as one of the measured quantities. Now use matrix algebra to calculate this:

$$\begin{aligned} {}_y\langle +|+\rangle_x &= \tfrac{1}{\sqrt{2}} \begin{pmatrix} 1 & e^{-i\alpha} \end{pmatrix} \tfrac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 1 \end{pmatrix} \\ &= \tfrac{1}{2}\left(1 + e^{-i\alpha}\right) \\ \left|{}_y\langle +|+\rangle_x\right|^2 &= \tfrac{1}{2}\left(1 + e^{-i\alpha}\right)\tfrac{1}{2}\left(1 + e^{i\alpha}\right) \\ &= \tfrac{1}{4}\left(1 + e^{i\alpha} + e^{-i\alpha} + 1\right) \\ &= \tfrac{1}{2}\left(1 + \cos\alpha\right) = \tfrac{1}{2} \end{aligned} \tag{1.59}$$

This result requires that $\cos\alpha = 0$, or that $\alpha = \pm\pi/2$. The two choices for the phase correspond to the two possibilities for the direction of the *y*-axis relative to the already determined *x*- and *z*-axes. The choice $\alpha = +\pi/2$ can be shown to correspond to a right-handed coordinate system, which is the standard convention, so we choose that phase. We thus represent the $|\pm\rangle_y$ kets as

$$\begin{aligned} |+\rangle_y &\doteq \tfrac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ i \end{pmatrix} \\ |-\rangle_y &\doteq \tfrac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ -i \end{pmatrix} \end{aligned} \tag{1.60}$$

Note that the imaginary components of these kets are required. They are not merely a mathematical convenience as one sees in classical mechanics. In general, quantum mechanical state vectors have complex coefficients. But this does not mean that the results of physical measurements are complex. On the contrary, we always calculate a measurement probability using a complex square, so all quantum mechanics predictions of probabilities are real.

1.4 General Quantum Systems

The machinery we have developed for spin-1/2 systems can be generalized to other quantum systems. For example, if an observable A yields quantized measurement results a_n for some finite range of n, then we generalize the schematic depiction of a Stern-Gerlach measurement to a measurement of the observable A, as shown in Fig. 1.15. The observable A labels the measurement device and the possible results a_1, a_2, a_3, *etc.* label the output ports. The basis kets corresponding to the results a_n are then $|a_n\rangle$. The mathematical rules about kets in this general case are

$$\begin{aligned} \langle a_i | a_j \rangle &= \delta_{ij} && \text{orthonormality} \\ |\psi\rangle &= \sum_i \langle a_i | \psi \rangle \, |a_i\rangle && \text{completeness} \end{aligned} \tag{1.61}$$

where we use the **Kronecker delta**

$$\delta_{ij} = \begin{cases} 0 & i \neq j \\ 1 & i = j \end{cases} \tag{1.62}$$

to express the orthonormality condition compactly. In this case, the generalization of postulate 4 says that the probability of a measurement of one of the possible results a_n is

$$\mathcal{P}_{a_n} = \left| \langle a_n | \psi_{in} \rangle \right|^2 \tag{1.63}$$

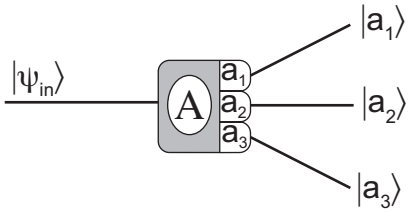

Figure 1.15 Generic depiction of the quantum mechanical measurement of observable A.

Example 1.4.

Imagine a quantum system with an observable A that has three possible measurement results: a_1, a_2, and a_3. The three kets $|a_1\rangle$, $|a_2\rangle$, and $|a_3\rangle$ corresponding to these possible results form a complete orthonormal basis. The system is prepared in the state

$$|\psi\rangle = 2|a_1\rangle - 3|a_2\rangle + 4i|a_3\rangle \tag{1.64}$$

Calculate the probabilities of all possible measurement results of the observable A.

The state vector in Eqn. (1.64) is not normalized, so we must normalize it before calculating probabilities. Introducing a complex normalization constant C, we find

$$\begin{aligned}
1 &= \langle\psi|\psi\rangle \\
&= C^*\left(2\langle a_1| - 3\langle a_2| - 4i\langle a_3|\right)C\left(2|a_1\rangle - 3|a_2\rangle + 4i|a_3\rangle\right) \\
&= |C|^2\left\{4\langle a_1|a_1\rangle - 6\langle a_1|a_2\rangle + 8i\langle a_1|a_3\rangle\right. \\
&\qquad -6\langle a_2|a_1\rangle + 9\langle a_2|a_2\rangle - 12i\langle a_2|a_3\rangle \\
&\qquad \left. -8i\langle a_3|a_1\rangle + 12i\langle a_3|a_2\rangle + 16\langle a_3|a_3\rangle\right\} \\
&= |C|^2\{4+9+16\} = |C|^2\,29 \\
&\Rightarrow C = \frac{1}{\sqrt{29}}
\end{aligned} \tag{1.65}$$

The normalized state is

$$|\psi\rangle = \frac{1}{\sqrt{29}}\left(2|a_1\rangle - 3|a_2\rangle + 4i|a_3\rangle\right) \tag{1.66}$$

The probabilities of measuring the results a_1, a_2, and a_3 are

$$\begin{aligned}
\mathcal{P}_{a_1} &= \left|\langle a_1|\psi\rangle\right|^2 \\
&= \left|\langle a_1|\frac{1}{\sqrt{29}}\left\{2|a_1\rangle - 3|a_2\rangle + 4i|a_3\rangle\right\}\right|^2 \\
&= \frac{1}{29}\left|2\langle a_1|a_1\rangle - 3\langle a_1|a_2\rangle + 4i\langle a_1|a_3\rangle\right|^2 = \frac{4}{29}
\end{aligned} \tag{1.67}$$

$$\mathcal{P}_{a_2} = \left|\langle a_2|\psi\rangle\right|^2 = \left|\langle a_2|\frac{1}{\sqrt{29}}\left\{2|a_1\rangle - 3|a_2\rangle + 4i|a_3\rangle\right\}\right|^2 = \frac{9}{29}$$

$$\mathcal{P}_{a_3} = \left|\langle a_3|\psi\rangle\right|^2 = \left|\langle a_3|\frac{1}{\sqrt{29}}\left\{2|a_1\rangle - 3|a_2\rangle + 4i|a_3\rangle\right\}\right|^2 = \frac{16}{29}$$

A schematic of this experiment is shown in Fig. 1.16(a) and a histogram of the predicted probabilities is shown in Fig. 1.16(b).

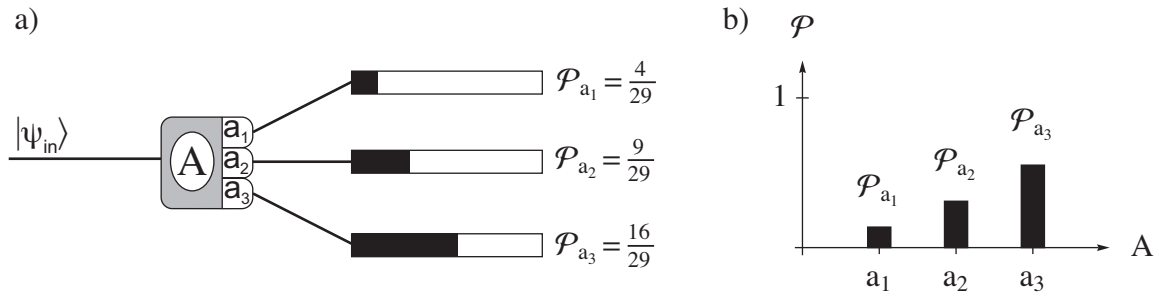

Figure 1.16 (a) Schematic diagram of the measurement of observable *A* and (b) histogram of the predicted measurement probabilities.

1.5 Postulates

We have introduced two of the postulates of quantum mechanics in this chapter. The postulates of quantum mechanics dictate how to treat a quantum mechanical system mathematically and how to interpret the mathematics to learn about the physical system in question. These postulates cannot be proven, but they have been successfully tested by many experiments, and so we accept them as an accurate way to describe quantum mechanical systems. New results could force us to reevaluate these postulates at some later time. All six postulates are listed below to give you an idea where we are headed and a framework into which you can place the new concepts as we confront them.

Postulates of Quantum Mechanics

1. The state of a quantum mechanical system, including all the information you can know about it, is represented mathematically by a normalized ket $|\psi\rangle$.
2. A physical observable is represented mathematically by an operator *A* that acts on kets.
3. The only possible result of a measurement of an observable is one of the eigenvalues a_n of the corresponding operator *A*.
4. The probability of obtaining the eigenvalue a_n in a measurement of the observable *A* on the system in the state $|\psi\rangle$ is

$$\mathcal{P}_{a_n} = \left|\langle a_n|\psi\rangle\right|^2,$$

where $|a_n\rangle$ is the normalized eigenvector of *A* corresponding to the eigenvalue a_n.
5. After a measurement of *A* that yields the result a_n, the quantum system is in a new state that is the normalized projection of the original system ket onto the ket (or kets) corresponding to the result of the measurement:

$$|\psi'\rangle = \frac{P_n|\psi\rangle}{\sqrt{\langle\psi|P_n|\psi\rangle}}.$$

6. The time evolution of a quantum system is determined by the Hamiltonian or total energy operator $H(t)$ through the Schrödinger equation

$$i\hbar\frac{d}{dt}|\psi(t)\rangle = H(t)|\psi(t)\rangle .$$

As you read these postulates for the first time, you will undoubtedly encounter new terms and concepts. Rather than explain them all here, the plan of this text is to continue to explain them through their manifestation in the Stern-Gerlach spin-1/2 experiment. We have chosen this example because it is inherently quantum mechanical and forces us to break away from reliance on classical intuition or concepts. Moreover, this simple example is a paradigm for many other quantum mechanical systems. By studying it in detail, we can appreciate much of the richness of quantum mechanics.

1.6 Summary

Through the Stern-Gerlach experiment we have learned several key concepts about quantum mechanics in this chapter.

- Quantum mechanics is probabilistic.
  We cannot predict the results of experiments precisely. We can predict only the probability that a certain result is obtained in a measurement.
- Spin measurements are quantized.
  The possible results of a spin component measurement are quantized. Only these discrete values are measured.
- Quantum measurements disturb the system.
  Measuring one physical observable can "destroy" information about other observables.

We have learned how to describe the state of a quantum mechanical system mathematically using a ket, which represents all the information we can know about that state. The kets $|+\rangle$ and $|-\rangle$ result when the spin component S_z along the z-axis is measured to be up or down, respectively. These kets form an orthonormal basis, which we denote by the inner products

$$\begin{aligned} \langle +|+\rangle &= 1 \\ \langle -|-\rangle &= 1 \\ \langle +|-\rangle &= 0 \end{aligned} \qquad (1.68)$$

The basis is also complete, which means that it can be used to express all possible kets as superposition states

$$|\psi\rangle = a|+\rangle + b|-\rangle . \qquad (1.69)$$

For spin component measurements, the kets corresponding to spin up or down along the three Cartesian axes are

$$\begin{array}{lll} |+\rangle & |+\rangle_x = \frac{1}{\sqrt{2}}\left[|+\rangle + |-\rangle\right] & |+\rangle_y = \frac{1}{\sqrt{2}}\left[|+\rangle + i|-\rangle\right] \\ |-\rangle & |-\rangle_x = \frac{1}{\sqrt{2}}\left[|+\rangle - |-\rangle\right] & |-\rangle_y = \frac{1}{\sqrt{2}}\left[|+\rangle - i|-\rangle\right] \end{array} \tag{1.70}$$

We also found it useful to introduce a matrix notation for calculations. In this matrix language the kets in Eqn. (1.70) are represented by

$$\begin{array}{lll} |+\rangle \doteq \begin{pmatrix} 1 \\ 0 \end{pmatrix} & |+\rangle_x \doteq \begin{pmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \end{pmatrix} & |+\rangle_y \doteq \begin{pmatrix} 1/\sqrt{2} \\ i/\sqrt{2} \end{pmatrix} \\ |-\rangle \doteq \begin{pmatrix} 0 \\ 1 \end{pmatrix} & |-\rangle_x \doteq \begin{pmatrix} 1/\sqrt{2} \\ -1/\sqrt{2} \end{pmatrix} & |-\rangle_y \doteq \begin{pmatrix} 1/\sqrt{2} \\ -i/\sqrt{2} \end{pmatrix} \end{array} \tag{1.71}$$

The most important tool we have learned so far is the probability postulate (postulate 4). To calculate the probability that a measurement on an input state $|\psi_{in}\rangle$ will yield a particular result, for example $S_z = \hbar/2$, we complex square the inner product of the input state with the ket corresponding to the measured result, $|+\rangle$ in this case:

$$\mathcal{P}_+ = \left|\langle + | \psi_{in} \rangle\right|^2 \tag{1.72}$$

This is generalized to other systems where a measurement yields a particular result a_n corresponding to the ket $|a_n\rangle$ as:

$$\mathcal{P}_{a_n} = \left|\langle a_n | \psi_{in} \rangle\right|^2 \tag{1.73}$$

1.7 Problems

1.1 Consider the following state vectors:

$$|\psi_1\rangle = 3|+\rangle + 4|-\rangle$$
$$|\psi_2\rangle = |+\rangle + 2i|-\rangle$$
$$|\psi_3\rangle = 3|+\rangle - e^{i\pi/3}|-\rangle$$

a) Normalize each state vector.

b) For each state vector, calculate the probability that the spin component is up or down along each of the three Cartesian axes. Use bra-ket notation for the entire calculation.

c) Write each normalized state in matrix notation.

d) Repeat part (b) using matrix notation for the entire calculation.

1.2 Consider the three quantum states:

$$|\psi_1\rangle = \frac{1}{\sqrt{3}}|+\rangle + i\frac{\sqrt{2}}{\sqrt{3}}|-\rangle$$
$$|\psi_2\rangle = \frac{1}{\sqrt{5}}|+\rangle - \frac{2}{\sqrt{5}}|-\rangle$$
$$|\psi_3\rangle = \frac{1}{\sqrt{2}}|+\rangle + \frac{e^{i\pi/4}}{\sqrt{2}}|-\rangle$$

Use bra-ket notation (not matrix notation) to solve the following problems. Note that $\langle+|+\rangle = 1$, $\langle-|-\rangle = 1$, and $\langle+|-\rangle = 0$.

a) For each of the $|\psi_i\rangle$ above, find the normalized vector $|\phi_i\rangle$ that is orthogonal to it.

b) Calculate the inner products $\langle\psi_i|\psi_j\rangle$ for i and $j = 1, 2, 3$.

1.3 Show that a change in the overall phase of a quantum state vector does not change the probability of obtaining a particular result in a measurement. To do this, consider how the probability is affected by changing the state $|\psi\rangle$ to the state $e^{i\delta}|\psi\rangle$.

1.4 Show by explicit bra-ket calculations using the states in Eqn. (1.29) that the four experimental results in Eqn. (1.28) lead to the results $|b|^2 = |c|^2 = |d|^2 = \frac{1}{2}$.

1.5 A beam of spin-½ particles is prepared in the state

$$|\psi\rangle = \frac{2}{\sqrt{13}}|+\rangle + i\frac{3}{\sqrt{13}}|-\rangle$$

a) What are the possible results of a measurement of the spin component S_z, and with what probabilities would they occur?

b) What are the possible results of a measurement of the spin component S_x, and with what probabilities would they occur?

c) Plot histograms of the predicted measurement results from parts (a) and (b).

1.6 A beam of spin-½ particles is prepared in the state

$$|\psi\rangle = \frac{2}{\sqrt{13}}|+\rangle_x + i\frac{3}{\sqrt{13}}|-\rangle_x$$

a) What are the possible results of a measurement of the spin component S_z, and with what probabilities would they occur?

b) What are the possible results of a measurement of the spin component S_x, and with what probabilities would they occur?

c) Plot histograms of the predicted measurement results from parts (a) and (b).

1.7 A classical coin is thrown in the air and lands on the ground, where a measurement is made of its state.

a) What are the possible results of this measurement?

b) What are the predicted probabilities for these possible outcomes?

c) Plot a histogram of the predicted measurement results.

1.8 A classical cubical die is thrown onto a table and comes to rest, where a measurement is made of its state.

a) What are the possible results of this measurement?

b) What are the predicted probabilities for these possible outcomes?

c) Plot a histogram of the predicted measurement results.

1.9 A pair of dice (classical cubes) are thrown onto a table and come to rest, where a measurement is made of the state of the system (*i.e.*, the sum of the two die).

a) What are the possible results of this measurement?

b) What are the predicted probabilities for these possible outcomes?

c) Plot a histogram of the predicted measurement results.

1.10 Consider the three quantum states:

$$|\psi_1\rangle = \frac{4}{5}|+\rangle + i\frac{3}{5}|-\rangle$$

$$|\psi_2\rangle = \frac{4}{5}|+\rangle - i\frac{3}{5}|-\rangle$$

$$|\psi_3\rangle = -\frac{4}{5}|+\rangle + i\frac{3}{5}|-\rangle$$

a) For each of the $|\psi_i\rangle$ above, calculate the probabilities of spin component measurements along the *x*-, *y*-, and *z*-axes.

b) Use your results from (a) to comment on the importance of the overall phase and of the relative phases of the quantum state vector.

1.11 A spin-½ particle is prepared in the state

$$|\psi\rangle = \frac{3}{\sqrt{34}}|+\rangle + i\frac{5}{\sqrt{34}}|-\rangle$$

a) What are the possible results of a measurement of the spin component S_z, and with what probabilities would they occur?

b) Suppose that the S_z measurement on the particle yields the result $S_z = -\hbar/2$. Subsequent to that result a second measurement is performed to measure the spin component S_x. What are the possible results of that measurement, and with what probabilities would they occur?

c) Draw a schematic diagram depicting the successive measurements in parts (a) and (b).

1.12 Consider a quantum system with an observable A that has three possible measurement results: a_1, a_2, and a_3. Write down the orthogonality, normalization, and completeness relations for the three kets comprising the basis corresponding to the possible results of the A measurement.

1.13 Consider a quantum system with an observable A that has three possible measurement results: a_1, a_2, and a_3.

a) Write down the three kets $|a_1\rangle$, $|a_2\rangle$, and $|a_3\rangle$ corresponding to these possible results using matrix notation.

b) The system is prepared in the state

$$|\psi\rangle = 1|a_1\rangle - 2|a_2\rangle + 5|a_3\rangle$$

Write this state in matrix notation and calculate the probabilities of all possible measurement results of the observable A. Plot a histogram of the predicted measurement results.

c) In a different experiment, the system is prepared in the state

$$|\psi\rangle = 2|a_1\rangle + 3i|a_2\rangle$$

Write this state in matrix notation and calculate the probabilities of all possible measurement results of the observable A. Plot a histogram of the predicted measurement results.

1.14 Consider a quantum system in which the energy E is measured and there are four possible measurement results: 2 eV, 4 eV, 7 eV, and 9 eV.

a) The system is prepared in the state

$$|\psi\rangle = \frac{1}{\sqrt{39}}\left\{3|2\ eV\rangle - i|4\ eV\rangle + 2e^{i\pi/7}|7\ eV\rangle + 5|9\ eV\rangle\right\}$$

Calculate the probabilities of all possible measurement results of the energy E. Plot a histogram of the predicted measurement results.

1.15 Consider a quantum system described by a basis $|a_1\rangle$, $|a_2\rangle$, and $|a_3\rangle$. The system is initially in a state

$$|\psi_i\rangle = \frac{i}{\sqrt{3}}|a_1\rangle + \sqrt{\frac{2}{3}}|a_2\rangle .$$

Find the probability that the system is measured to be in the final state

$$|\psi_f\rangle = \frac{1+i}{\sqrt{3}}|a_1\rangle + \frac{1}{\sqrt{6}}|a_2\rangle + \frac{1}{\sqrt{6}}|a_3\rangle .$$

1.16 The spin components of a beam of atoms prepared in the state $|\psi_{in}\rangle$ are measured and the following experimental probabilities are obtained:

$$\mathcal{P}_+ = \tfrac{1}{2} \quad \mathcal{P}_{+x} = \tfrac{3}{4} \quad \mathcal{P}_{+y} = 0.067$$
$$\mathcal{P}_- = \tfrac{1}{2} \quad \mathcal{P}_{-x} = \tfrac{1}{4} \quad \mathcal{P}_{-y} = 0.933$$

From the experimental data, determine the input state.

1.17 In part (1) of SPINS Lab #2, you measured the probabilities of all the possible spin components for each of the unknown initial states $|\psi_i\rangle$ (i = 1, 2, 3, 4). Using your data from that lab, find the unknown states $|\psi_1\rangle$, $|\psi_2\rangle$, $|\psi_3\rangle$ and $|\psi_4\rangle$. Express each of the unknown states as a linear superposition of the S_z basis states $|+\rangle$ and $|-\rangle$. For each state, use your result to calculate the theoretical values of the probabilities for each component measurement and compare these theoretical predictions with your experimental results.

1.8 Resources

1.8.1 Activities

SPINS: A software program to simulate Stern-Gerlach spin experiments. The Java software runs on all platforms and can be downloaded in two forms:

Open Source Physics framework

www.physics.oregonstate.edu/~mcintyre/ph425/spins/index_SPINS_OSP.html

or

Standalone Java

www.physics.oregonstate.edu/~mcintyre/ph425/spins

SPINS Lab 1: An introduction to successive Stern-Gerlach spin 1/2 measurements. The randomness of measurements is demonstrated and students use statistical analysis to deduce probabilities from measurements.

www.physics.oregonstate.edu/portfolioswiki/doku.php?id=activities:main&file=spin1

SPINS Lab 2: Students deduce unknown quantum state vectors from measurements of spin projections. (part 3 requires material from Chap. 2 to do the calculations)

www.physics.oregonstate.edu/portfolioswiki/doku.php?id=activities:main&file=spin2

A different simulation of the Stern-Gerlach experiment (somewhat Flashier):

phet.colorado.edu/simulations/sims.php?sim=SternGerlach_Experiment

1.8.2 Further reading

The history of the Stern-Gerlach experiment and how a bad cigar helped are chronicled in a Physics Today article:

B. Friedrich and D. Herschbach, "Stern and Gerlach: How a Bad Cigar Helped Reorient Atomic Physics," *Phys. Today* **56**, 53-59 (2003). http://dx.doi.org/10.1063/1.1650229

Nature has published a supplement on the milestones in spin physics. An extensive timeline of historical events, review articles, and links to original articles are included.

Nature Phys. 4, S1-S43 (2008). www.nature.com/milestones/spin

The SPINS lab software is described in this pedagogical article:

D. V. Schroeder and T. A. Moore, "A computer-simulated Stern-Gerlach laboratory," *Am. J. Phys.* **61**, 798-805 (1993).

Some other textbooks that take a spins-first approach or have an extensive treatment of Stern-Gerlach experiments:

R. P. Feynman, R. B. Leighton, and M. Sands, *The Feynman Lectures on Physics, Volume 3, Quantum Mechanics*, Addison-Wesley Publishing Company, Inc., Reading, Massachusetts, 1965.

J. J. Sakurai, *Modern Quantum Mechanics*, Addison-Wesley Publishing Company, Inc., Redwood City, California, 1985.

J. S. Townsend, *A Modern Approach to Quantum Mechanics*, McGraw Hill, Inc., New York, 1992.

C. Cohen-Tannoudji, B. Diu, and F. Laloë, *Quantum Mechanics*, John Wiley & Sons, New York, 1977.