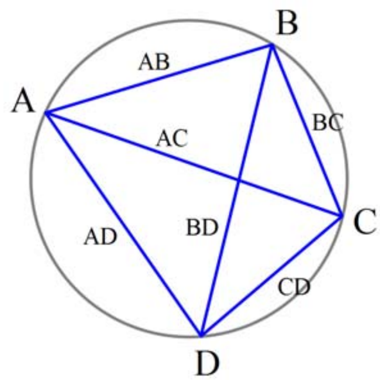
Ptolemy's Theorem

Ptolemy's Theorem is a relation in Euclidean geometry between the four sides and two diagonals of a **cyclic quadrilateral** (i.e., a quadrilateral whose vertices lie on a common circle). The theorem is named after the Greek astronomer and mathematician Ptolemy (Claudius Ptolemaeus).

If the quadrilateral is given with its four vertices *A*, *B*, *C*, and *D* in order, then the theorem states that:

$$|AC| \cdot |BD| = |AB| \cdot |CD| + |BC| \cdot |AD|$$

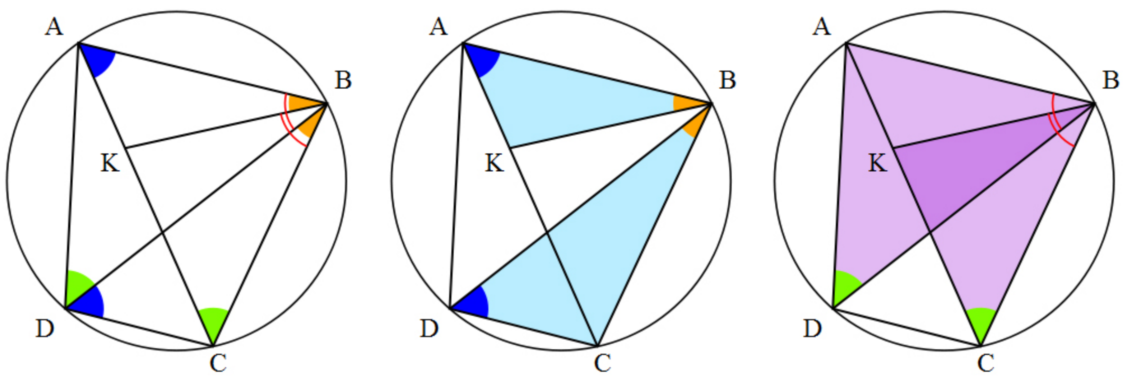


This relation may be verbally expressed as follows:

> *If a quadrilateral is inscribed in a circle then the sum of the products of its two pairs of opposite sides is equal to the product of its diagonals.*

Proof

Geometric proof of Ptolemy's Theorem



1. Let ABCD be a cyclic quadrilateral.
2. Note that on the chord BC, the inscribed angles ∠BAC = ∠BDC, and on AB, ∠ADB = ∠ACB.
3. Construct K on AC such that ∠ABK = ∠CBD;

   [Note that: ∠ABK + ∠CBK = ∠ABC = ∠CBD + ∠ABD ⇒ ∠CBK = ∠ABD.]

4. Now, by common angles △ABK is similar to △DBC, and likewise △KBC ~ △ABD.

5. Thus, $\frac{|AK|}{|AB|} = \frac{|DC|}{|DB|}$ and $\frac{|KC|}{|BC|} = \frac{|AD|}{|BD|}$ due to the similarities noted above:

[△ABK ~ △DBC and △KBC ~ △ABD]

1. So |AK|·|DB| = |AB|·|DC|, and |KC|·|BD| = |BC|·|AD|;
2. Adding, |AK|·|DB|+|KC|·|BD| = |AB|·|DC| + |BC|·|AD|;
3. Equivalently, (|AK| + |KC|)·| BD| = |AB|·|CD| + |BC|·|AD|;
4. But |AK|+|KC| = |AC|, so
5. |AC|·|BD| = |AB|·|CD| + |BC|·|DA|; Q.E.D.

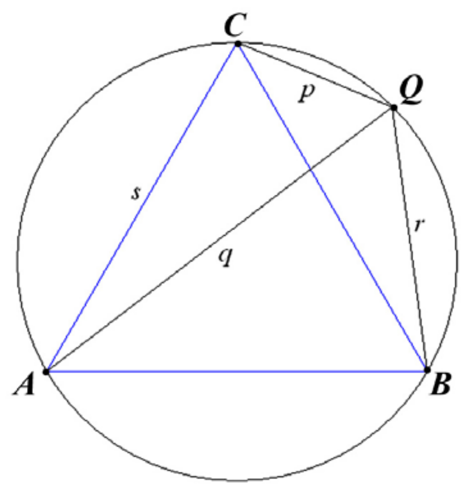
Some Corollaries to Ptolemy's Theorem

CORROLARY 1:

Given an equilateral triangle △ABC inscribed in a circle and a point Q on the circle.
Then the distance from point Q to the most distant vertex of the triangle is the sum of the distances from the point to the two nearer vertices.

In the figure, it follows that

$$q = p + r$$



CORROLARY 2:

In any regular pentagon the ratio of the length of a diagonal to the length of a side is the golden ratio, φ .

In the figure, it follows that

$$\varphi = \frac{b}{a}$$