DERIVED SMOOTH MANIFOLDS

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Abstract

We define a simplicial category called the category of derived manifolds. It contains the category of smooth manifolds as a full discrete subcategory, and it is closed under taking arbitrary intersections in a manifold. A derived manifold is a space together with a sheaf of local C^∞-rings that is obtained by patching together homotopy zero sets of smooth functions on Euclidean spaces. We show that derived manifolds come equipped with a stable normal bundle and can be imbedded into Euclidean space. We define a cohomology theory called derived cobordism, and use a Pontrjagin-Thom argument to show that the derived cobordism theory is isomorphic to the classical cobordism theory. This allows us to define fundamental classes in cobordism for all derived manifolds. In particular, the intersection $A \cap B$ of submanifolds $A, B \subset X$ exists on the categorical level in our theory, and a cup product formula $[A] \smile [B] = [A \cap B]$ holds, even if the submanifolds are not transverse. One can thus consider the theory of derived manifolds as a categorification *of intersection theory.*

Contents

1. Introduction

Let Ω denote the unoriented cobordism ring (although what we say applies to other cobordism theories as well, e.g., oriented cobordism). The fundamental class of a

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compact manifold X is an element $[X] \in \Omega$. By the Pontrjagin-Thom construction, such an element is classified by a homotopy class of maps from a large enough sphere S^n to a large enough Thom space MO. One can always choose a map $f\colon S^n \to MO$ which represents this homotopy class, is smooth (away from the base point), and which meets the zero section $B \subset MO$ transversely. The pullback $f^{-1}(B)$ is a compact manifold which is cobordant to X, so we have an equality $[X] = [f^{-1}(B)]$ of elements in Ω.

This construction provides a correspondence which is homotopical in nature: one begins with a homotopy class of maps and receives a cobordism class. However, it is close to existing on the nose, in that a dense subset of all representing maps $f\colon S^n \to MO$ is transverse to B and yields an imbedded manifold rather than merely its image in Ω. If transversality were not an issue, Pontrjagin and Thom would indeed provide a correspondence between smooth maps $S^n \to MO$ and their zero sets.

The purpose of this article is to introduce the category of derived manifolds wherein nontransverse intersections make sense. In this setting, $f^{-1}(B)$ is a derived manifold which is derived cobordant to X, regardless of one's choice of smooth map f, and in terms of fundamental cobordism classes we have $[f^{-1}(B)] = [X]$. Our hope is that by using derived manifolds, researchers can avoid having to make annoying transversality arguments. This could be of use in string topology or Floer homology, for example.

As an example, let us provide a short story that can only take place in the derived setting. Consider the case of a smooth degree d hypersurface $X \subset \mathbb{C}P^3$ in complex projective space. One can express the K-theory fundamental class of X as

$$[X] = \binom{d}{1}[\mathbb{C}P^2] - \binom{d}{2}[\mathbb{C}P^1] + \binom{d}{3}[\mathbb{C}P^0]. \tag{1.0.1}$$

It may be difficult to see exactly where this formula comes from; let us give the derived perspective, which should make the formula more clear.

The union Y of d distinct hyperplanes in $\mathbb{C}P^3$ is not smooth, but it does exist as an object in the category of derived manifolds. Moreover, as the zero set of a section of the line bundle $\mathcal{O}(d)$, one has that Y is a degree d derived submanifold of $\mathbb{C}P^3$ which is derived cobordant to X. As such, the fundamental class of Y is equal to that of X (i.e., $[Y] = [X]$).

The point is that the above formula (1.0.1) takes on added significance as the fundamental class of Y, because it has the form of an inclusion-exclusion formula. One could say that the K-theory fundamental class of Y is obtained by adding the fundamental classes of d hyperplanes, subtracting off the fundamental classes of the overcounted $\binom{d}{2}$ lines of two-fold intersection, and adding back the fundamental classes of the missed $\binom{d}{3}$ points of three-fold intersection. Hopefully, this convinces the reader that derived manifolds may be of use.

To construct the virtual fundamental class on an arbitrary intersection of compact submanifolds, we follow an idea of Kontsevich (see [15, Section 1.4]), explained to us by Jacob Lurie. Basically, we take our given intersection $\mathcal{X} = A \cap B$, realize it as the zero set of a section of a vector bundle, and then deform that section until it is transverse to zero. The result is a derived cobordism between $\mathcal{X}$ and a smooth manifold.

While dispensing with the transversality requirement for intersecting manifolds is appealing, it does come with a cost, namely that defining the category of derived manifolds is a bit technical. However, anyone familiar with homotopy sheaves should not be too surprised by our construction. One starts with Lawvere's algebraic theory of C^∞-rings, which are rings whose elements are closed under composition with smooth functions. Simplicial (lax) C^∞-rings are the appropriate homotopy-theoretic analogue and as such are objects in a simplicial model category. We then form the category of local C^∞-ringed spaces, wherein an object is a topological space together with a homotopy sheaf of simplicial C^∞-rings whose stalks are local rings. Euclidean space, with its (discrete) C^∞-ring of smooth real-valued functions is such an object, and the zero set of finitely many smooth functions on Euclidean space is called an affine derived manifold. A derived manifold is a local C^∞-ringed space which is obtained by patching together affine derived manifolds (see Definition 6.15).

Notation 1.1
Let **sSets** denote the monoidal category of simplicial sets. A simplicial category $\mathcal{C}$ is a category enriched over **sSets**; we denote the mapping space functor $\mathrm{Map}_{\mathcal{C}}(-,-)$. If all of the mapping spaces in $\mathcal{C}$ are fibrant (i.e., Kan complexes), we call $\mathcal{C}$ *fibrant* as a simplicial category; in the following discussion we consider only this case. By a *map* between objects X and Y in $\mathcal{C}$, we mean a zero-simplex in $\mathrm{Map}_{\mathcal{C}}(X, Y)$.

An object $X \in \mathcal{C}$ is called *homotopy initial* if for every $Y \in \mathcal{C}$, the mapping space $\mathrm{Map}(X, Y)$ is contractible. Similarly, X is called *homotopy terminal* if for every $Y \in \mathcal{C}$, the mapping space $\mathrm{Map}(Y, X)$ is contractible. We say that a vertex in a contractible space is *homotopy unique*. We sometimes abuse notation and refer to a contractible space as though it were just a single point, saying something like "*the* natural map $Y \to X$."

Let $\mathcal{C}$ be a simplicial category. The homotopy pullback of a diagram $A \xrightarrow{f} B \xleftarrow{g} C$ is a diagram

$$\begin{array}{ccc} A \times_B C & \xrightarrow{f'} & C \\ & \diamond & \\ {\scriptstyle g'}\downarrow & & \downarrow{\scriptstyle g} \\ A & \xrightarrow[f]{} & B \end{array} \tag{1.1.1}$$

By this we mean an object $A \times_B C$ equipped with maps g', h', and f' to A, B, and C, respectively, and further equipped with homotopies between gf' and h and between fg' and h. Finally, we require that $A \times_B C$ is homotopy terminal in the category of such objects. More succinctly, Diagram (1.1.1) expresses that for any object $X \in \mathcal{C}$, the natural map

$$\mathrm{Map}(X, A \times_B C) \to \mathrm{Map}(X, A) \times^h_{\mathrm{Map}(X,B)} \mathrm{Map}(X, C)$$

is a weak equivalence in the usual model category of simplicial sets (see [11, Definition 7.8.10]), where by $\times^h$ we mean the homotopy pullback in **sSets**. The diamond in the upper left corner of the square in Diagram (1.1.1) serves to remind the reader that object in the upper left corner is a homotopy pullback, and that the diagram does not commute on the nose but up to chosen homotopies. We can define homotopy pushouts similarly.

Two objects X and Y in $\mathcal{C}$ are said to be *equivalent* if there exist maps $f : X \to Y$ and $g : Y \to X$ such that $g \circ f$ and $f \circ g$ are homotopic to the identity maps on X and Y. By [20, Proposition 1.2.4.1], this is equivalent to the assertion that the map $\mathrm{Map}(Z, X) \to \mathrm{Map}(Z, Y)$ is a weak equivalence for all $Z \in \mathcal{C}$.

If $\mathcal{C}$ is a discrete simplicial category (i.e., a category in the usual sense), then the homotopy pullback of a diagram in $\mathcal{C}$ is the pullback of that diagram. The pullback of a diagram $A \xrightarrow{f} B \xleftarrow{g} C$ is a commutative diagram

$$\begin{array}{ccc} A \times_B C & \xrightarrow{f'} & C \\ & \ulcorner & \\ {\scriptstyle g'}\downarrow & & \downarrow{\scriptstyle g} \\ A & \xrightarrow[f]{} & B \end{array}$$

The symbol in the upper left corner serves to remind the reader that the object in the upper left corner is a pullback in the usual sense. Two objects are equivalent if and only if they are isomorphic.

Remark 1.2

If $\mathcal{C}$ is a simplicial model category (see, e.g., [11] for an introduction to this subject), then the full subcategory of cofibrant-fibrant objects is a simplicial category in which all mapping spaces are fibrant. Moreover, we can replace any diagram with a diagram of the same shape in which all objects are cofibrant-fibrant. Our definitions of homotopy pullback, homotopy pushout, and equivalence match the model category terminology. In keeping with this, if we are in the setting of model categories, the result of a construction (such as taking a homotopy limit) is always assumed to be cofibrant and fibrant.

Whenever we speak of pullbacks in a simplicial category, we are always referring to homotopy pullbacks unless otherwise specified. Similarly, whenever we speak of terminal (resp., initial) objects, we are always referring to homotopy terminal (resp., homotopy initial) objects. Finally, we sometimes use the word *category* to mean *simplicial category*.

Given a functor $F\colon \mathcal{C} \to \mathcal{D}$, we say that an object $D \in \mathcal{D}$ is in the *essential image* of F if there is an object $C \in \mathcal{C}$ such that $F(C)$ is equivalent to D.

We denote the category of smooth manifolds by **Man**; whenever we refer to a manifold, it is always assumed smooth. It is discrete as a simplicial category. In other words, we *do not* include any kind of homotopy information in **Man**.

We now recall a few well-known facts and definitions about **Man**. Every manifold A has a tangent bundle $T_A \to A$ which is a vector bundle whose rank is equal to the dimension of A. A morphism of smooth manifolds $f\colon A \to B$ induces a morphism $T_f\colon T_A \to f^*T_B$ of vector bundles on A, from the tangent bundle of A to the pullback of the tangent bundle on B. We say that f is an *immersion* if T_f is injective and we say that f is a *submersion* if T_f is surjective. A pair of maps $f\colon A \to B$ and $g\colon C \to B$ are called *transverse* if the induced map $f \amalg g\colon A \amalg C \to B$ is a submersion. If f and g are transverse, then their fiber product (over B) exists in **Man**.

1.1. Results

In this article, we hope to convince the reader that we have a reasonable category in which to do intersection theory on smooth manifolds. The following definition expresses what we mean by *reasonable*. Definition 1.7 expresses what we mean by *doing intersection theory* on such a category. The main result of the article, Theorem 1.8, is that there is a simplicial category which satisfies Definition 1.7.

Definition 1.3

A simplicial category $\mathcal{C}$ is called *geometric* if it satisfies the following axioms:

(1) *Fibrant.* For any two objects $\mathcal{X}, \mathcal{Y} \in \mathcal{C}$, the mapping space $\mathrm{Map}_{\mathcal{C}}(\mathcal{X}, \mathcal{Y})$ is a fibrant simplicial set.

(2) *Smooth manifolds.* There exists a fully faithful functor $\mathbf{i}\colon \mathbf{Man} \to \mathcal{C}$. We say that $M \in \mathcal{C}$ is a manifold if it is in the essential image of $\mathbf{i}$.

(3) *Manifold limits.* The functor $\mathbf{i}$ commutes with transverse intersections. That is, if $A \to M$ and $B \to M$ are transverse, then a homotopy limit $\mathbf{i}(A) \times_{\mathbf{i}(M)} \mathbf{i}(B)$ exists in $\mathcal{C}$, and the natural map

$$\mathbf{i}(A \times_M B) \to \mathbf{i}(A) \times_{\mathbf{i}(M)} \mathbf{i}(B)$$

is an equivalence in $\mathcal{C}$.
Furthermore, the functor $\mathbf{i}$ preserves the terminal object (i.e., the object $\mathbf{i}(\mathbb{R}^0)$ is homotopy terminal in $\mathcal{C}$).

(4) *Underlying spaces.* Let $\mathbf{CG}$ denote the discrete simplicial category of compactly generated Hausdorff spaces. There exists an *underlying space* functor $\mathbf{U}\colon \mathcal{C} \to \mathbf{CG}$, such that the diagram

$$\begin{array}{ccc} \mathbf{Man} & \xrightarrow{\mathbf{i}} & \mathcal{C} \\ \downarrow & \swarrow_{\mathbf{U}} & \\ \mathbf{CG} & & \end{array}$$

commutes, where the vertical arrow is the usual underlying space functor on smooth manifolds. Furthermore, the functor $\mathbf{U}$ commutes with finite limits when they exist.

Remark 1.4
When we speak of an object (resp., a morphism or a set of morphisms) in $\mathcal{C}$ having some topological property (e.g., Hausdorff or compact object, proper morphism, open cover, and so on), we mean that the underlying object (resp., the underlying morphism or set of morphisms) in $\mathbf{CG}$ has that property.

Since any discrete simplicial category has fibrant mapping spaces, it is clear that $\mathbf{Man}$ and $\mathbf{CG}$ are geometric.

If $\mathcal{C}$ is a geometric category and if $M \in \mathbf{Man}$ is a manifold, then we generally abuse notation and write M in place of $\mathbf{i}(M)$, as though M were itself an object of $\mathcal{C}$.

Remark 1.5
Note that in Axiom (3) of Definition 1.3, we are not requiring that $\mathbf{i}$ commute with all limits which exist in $\mathbf{Man}$, only those which we have deemed appropriate. For example, if one has a line L and a parabola P which meet tangentially in $\mathbb{R}^2$, their fiber product $L \times_{\mathbb{R}^2} P$ *does* exist in the category of manifolds (it is a point). However,

limits like these are precisely the kind we wish to avoid! We are searching for a category in which intersections are in some sense stable under perturbations (see Definition 1.7, Condition (4)), and thus we should not ask **i** to preserve intersections which are not stable in this sense.

Remark 1.6

In all of the axioms of Definition 1.3, we are working with simplicial categories, so when we speak of pullbacks and pushouts, we mean homotopy pullbacks and homotopy pushouts. Axiom (4) requires special comment however. We take **CG**, the category of compactly generated Hausdorff spaces, as a *discrete* simplicial category, so homotopy pullbacks and pushouts are just pullbacks and pushouts in the usual sense. The underlying space functor **U** takes finite homotopy pullbacks in $\mathcal{C}$ to pullbacks in **CG**.

Again, our goal is to find a category in which intersections of arbitrary submanifolds exist at the categorical level and descend correctly to the level of cobordism groups. We make this precise in the following definition.

Definition 1.7

We say that a simplicial category $\mathcal{C}$ *has the general cup product formula in cobordism* if the following conditions hold.

(1) *Geometric.* The simplicial category $\mathcal{C}$ is geometric in the sense of Definition 1.3.

(2) *Intersections.* If M is a manifold and if A and B are submanifolds (possibly not transverse), then there exists a homotopy pullback $A \times_M B$ in $\mathcal{C}$, which we denote by $A \cap B$ and which we call the *derived intersection* of A and B in M.

(3) *Derived cobordism.* There exists an equivalence relation on the compact objects of $\mathcal{C}$ called derived cobordism, which extends the cobordism relation on manifolds. That is, for any manifold T, there is a ring $\Omega^{\mathrm{der}}(T)$ called *the derived cobordism ring over* T, and the functor $\mathbf{i}\colon \mathbf{Man} \to \mathcal{C}$ induces a homomorphism of cobordism rings over T,

$$\mathbf{i}_*\colon \Omega(T) \to \Omega^{\mathrm{der}}(T).$$

We further impose the condition that $\mathbf{i}_*$ be an injection.

(4) *Cup product formula.* If A and B are compact submanifolds of a manifold M with derived intersection $A \cap B := A \times_M B$, then the cup product formula

$$[A] \smile [B] = [A \cap B] \tag{1.7.1}$$

holds, where $[-]$ is the functor taking a compact derived submanifold of M to its image in the derived cobordism ring $\Omega^{\mathrm{der}}(M)$, and where $\smile$ denotes the multiplication operation in that ring (i.e., the cup product).

Without the requirement (Condition (3)) that $\mathbf{i}_*\colon \Omega(T) \to \Omega^{\mathrm{der}}(T)$ be an injection, the general cup product formula could be trivially attained. For example, one could extend **Man** by including nontransverse intersections which were given no more structure than their underlying space, and the derived cobordism relation could be chosen to be maximal (i.e., one equivalence class); then the cup product formula would trivially hold.

In fact, when we eventually prove that there is a category which has the general cup product formula, we find that $\mathbf{i}_*$ is not just an injection but an isomorphism (see Theorem 2.6). We did not include that as an axiom here, however, because it does not seem to us to be an inherently necessary aspect of a good intersection theory.

The category of smooth manifolds does not have the general cup product formula because it does not satisfy Condition (2). Indeed, suppose that A and B are submanifolds of M. If A and B meet transversely, then their intersection naturally has the structure of a manifold, and the cup product formula (1.7.1) holds. If they do not, then one of two things can happen: either their intersection cannot be given the structure of a manifold (so in the classical setting, equation (1.7.1) does not have meaning), or their intersection can be given the structure of a smooth manifold, but it *does not* satisfy equation (1.7.1).

Therefore, we said that a category which satisfies the conditions of Definition 1.7 satisfies the *general* cup product formula because Condition (4) holds even for nontransverse intersections. Of course, to accomplish this, one needs to find a more refined notion of intersection; that is, find a setting in which homotopy limits have the desired properties.

Suppose that a simplicial category $\mathcal{C}$ has the general cup product formula in cobordism. It follows that any cohomology theory E which has fundamental classes for compact manifolds (i.e., for which there exists a map $MO \to E$) also has fundamental classes for compact objects of $\mathcal{C}$, and that these satisfy the cup product formula as well. Returning to our previous example, the union of d hyperplanes in complex projective space is a derived manifold which is derived cobordant to a smooth degree d hypersurface (see Example 2.7). Thus, these two subspaces have the same K-theory fundamental classes, which justifies equation 1.0.1.

Our main result is that the conditions of Definition 1.7 can be satisfied.

THEOREM 1.8
There exists a simplicial category **dMan**, *called* the category of derived manifolds, *which has the general cup product formula in cobordism, in the sense of Definition 1.7.*

The category **dMan** is defined in Definition 6.15, and the above theorem is proved as Theorem 9.6 (see also Definition 2.1 for a list of axioms satisfied by **dMan**).

Remark 1.9

We do not offer a uniqueness counterpart to Theorem 1.8. We purposely left Definition 1.7 loose, because we could not convince the reader that any more structure on a category $\mathcal{C}$ was necessary in order to say that it *has the general cup formula*. For example, we could have required that the morphism $\mathbf{i}_*$ be an isomorphism instead of just an injection (this is indeed the case for **dMan**, see Corollary 3.13); however, doing so would be hard to justify as being necessary. Because of the generality of Definition 1.7, we are precluded from offering a uniqueness result here.

Finally, the following proposition justifies the need for simplicial categories in this article.

PROPOSITION 1.10

If $\mathcal{C}$ is a discrete simplicial category (i.e., a category in the usual sense), then $\mathcal{C}$ cannot have the general cup product formula in cobordism.

Proof

We assume that Conditions (1), (2), and (3) hold, and we show that Condition (4) cannot.

Since $\mathcal{C}$ is geometric, the object $\mathbb{R}^0$ (technically $\mathbf{i}(\mathbb{R}^0)$) is homotopy terminal in $\mathcal{C}$. Since $\mathcal{C}$ is discrete, $\mathbb{R}^0$ is terminal in $\mathcal{C}$, all equivalences in $\mathcal{C}$ are isomorphisms, and all homotopy pullbacks in $\mathcal{C}$ are categorical pullbacks. Let $0\colon \mathbb{R}^0 \to \mathbb{R}$ be the origin, and let X be defined as the pullback in the diagram

$$\begin{array}{ccc} X & \longrightarrow & \mathbb{R}^0 \\ \big\downarrow & \lrcorner & \big\downarrow{\scriptstyle 0} \\ \mathbb{R}^0 & \underset{0}{\longrightarrow} & \mathbb{R} \end{array}$$

A morphism from an arbitrary object Y to X consists of two morphisms $Y \to \mathbb{R}^0$ which agree under composition with 0. For any Y, there is exactly one such morphism $Y \to X$ because $\mathbb{R}^0$ is terminal in $\mathcal{C}$. That is, X represents the same functor as $\mathbb{R}^0$ does, so $X \cong \mathbb{R}^0$ (i.e., $\mathbb{R}^0 \cap \mathbb{R}^0 = \mathbb{R}^0$). This equation forces Condition (4) to fail.

Indeed, to see that the cup product formula

$$[\mathbb{R}^0] \smile [\mathbb{R}^0] \stackrel{?}{=} [\mathbb{R}^0 \cap \mathbb{R}^0]$$

does not hold in $\Omega(\mathbb{R})$, note that the left-hand side is homogeneous of degree two, whereas the right-hand side is homogeneous of degree one in the cohomology ring. □

1.2. Structure of the article

We have decided to present this article in a hierarchical fashion. In the introduction, we presented the goal: to find a geometric category that has the general cup product formula in cobordism (see Definition 1.7).

In Section 2, we present a set of axioms that suffice to achieve this goal. In other words, any category that satisfies the axioms of Definition 2.1 is said to be "good for doing intersection theory on manifolds," and we prove in Theorem 2.6 that such a category has the general cup product formula. Of course, we could have chosen our axioms in a trivial way, but this would not have given a useful layer of abstraction. Instead, we chose axioms that resemble axioms satisfied by smooth manifolds. These axioms imply the general cup product formula, but are not implied by it.

In Sections 5 – 9, we give an explicit formulation of a category that is good for doing intersection theory. This category can be succinctly described as *the category of homotopical C^∞-schemes of finite type*. To make this precise and prove that it satisfies the axioms of Definition 2.1 takes five sections. We lay out our methodology for this undertaking in Section 4.

Finally, in Section 10, we discuss related constructions. First we see the way that derived manifolds are related to Jacob Lurie's "structured spaces" (see [21]). Then we discuss manifolds with singularities, Chen spaces, diffeological spaces, and synthetic C^∞-spaces, all of which are generalizations of the category **Man** of smooth manifolds. In this section, we show how the theory of derived manifolds fits into the existing literature.

2. The axioms

Theorem 1.8 makes clear our objectives: to find a simplicial category in which the general cup product formula holds. In this section, specifically in Definition 2.1, we provide a set of axioms which

- naturally extend corresponding properties of smooth manifolds, and
- together imply Theorem 1.8.

This is an attempt to give the reader the ability to work with derived manifolds (at least at a surface level) without fully understanding their technical underpinnings.

In the following section, Section 3, we prove Theorem 1.8 from the axioms presented in Definition 2.1. Then in Section 4, we give an outline of the internal structure of a simplicial category which satisfies the axioms in Definition 2.1. Finally, in the remaining sections we fulfill the outline given in Section 4, proving our main result in Section 9.

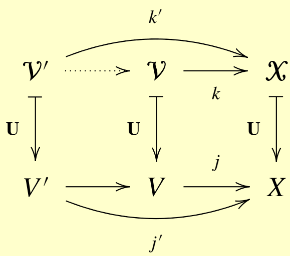
Definition 2.1
A simplicial category **dM** is called *good for doing intersection theory on manifolds* if it satisfies the following axioms:

(1) *Geometric.* The simplicial category **dM** is geometric in the sense of Definition 1.3. That is, roughly, **dM** has fibrant mapping spaces, contains the category **Man** of smooth manifolds, has reasonable limits, and has underlying Hausdorff spaces.

(2) *Open subobjects.*

Definition 2.2
Suppose that $\mathcal{X} \in \mathbf{dM}$ is an object with underlying space $X = \mathbf{U}(\mathcal{X})$, and suppose that $j: V \hookrightarrow X$ is a open inclusion of topological spaces. We say that a map $k: \mathcal{V} \to \mathcal{X}$ in **dM** is an *open subobject over* j if it is Cartesian over j; that is,

- $\mathbf{U}(\mathcal{V}) = V$,
- $\mathbf{U}(k) = j$, and
- if $k': \mathcal{V}' \to \mathcal{X}$ is a map with $\mathbf{U}(\mathcal{V}') = V'$ and $\mathbf{U}(k') = j'$, such that j' factors through j, then k' factors homotopy-uniquely through k; that is, the space of dotted arrows making the diagram



commute is contractible.

For any $\mathcal{X} \in \mathbf{dM}$ and for any open inclusion j as above, there exists an open subobject k over j. Moreover, if a map $f: Z \to X$ of topological spaces underlies a map $g: \mathcal{Z} \to \mathcal{X}$ in **dM**, then for any open neighborhood U of $f(Z)$, the map g factors through the open subobject $\mathcal{U}$ over U.

(3) *Unions.*

(a) Suppose that $\mathcal{X}$ and $\mathcal{Y}$ are objects of **dM** with underlying spaces X and Y, and suppose that $i: \mathcal{U} \to \mathcal{X}$ and $j: \mathcal{V} \to \mathcal{Y}$ are open subobjects with underlying spaces U and V. If $a: \mathcal{U} \to \mathcal{V}$ is an equivalence, and if the union of X and Y along $U \cong V$ is Hausdorff (so $X \cup Y$ exists in **CG**), then the union $\mathcal{X} \cup \mathcal{Y}$ (i.e., the colimit of $j \circ a$ and i) exists in **dM**, and one has as expected $\mathbf{U}(\mathcal{X} \cup \mathcal{Y}) = X \cup Y$.

(b) If $f\colon \mathcal{X} \to \mathcal{Z}$ and $g\colon \mathcal{Y} \to \mathcal{Z}$ are morphisms whose restrictions to $\mathcal{U}$ agree, then there is a morphism $\mathcal{X} \cup \mathcal{Y} \to \mathcal{Z}$ which restricts to f and g, respectively, on $\mathcal{X}$ and $\mathcal{Y}$.

(4) *Finite limits.* Given objects $\mathcal{X}, \mathcal{Y} \in \mathbf{dM}$, a smooth manifold M, and maps $f\colon \mathcal{X} \to M$ and $g\colon \mathcal{Y} \to M$, there exists an object $\mathcal{Z} \in \mathbf{dM}$ and a homotopy pullback diagram

$$\begin{array}{ccc} \mathcal{Z} & \longrightarrow & \mathcal{Y} \\ {\scriptstyle i}\downarrow & \diamond & \downarrow{\scriptstyle g} \\ \mathcal{X} & \xrightarrow[f]{} & M \end{array} \tag{2.2.1}$$

We denote $\mathcal{Z}$ by $\mathcal{X} \times_M \mathcal{Y}$. If $\mathcal{Y} = \mathbb{R}^0$, $M = \mathbb{R}^k$, and if $g\colon \mathbb{R}^0 \to \mathbb{R}^k$ is the origin, then we denote $\mathcal{Z}$ by $\mathcal{X}_{f=0}$, and we call i the canonical inclusion of the zero set of f into $\mathcal{X}$.

(5) *Local models.* We say that an object $\mathcal{U} \in \mathbf{dM}$ is a *local model* if there exists a smooth function $f\colon \mathbb{R}^n \to \mathbb{R}^k$ such that $\mathcal{U} \cong \mathbb{R}^n_{f=0}$. The *virtual dimension* of $\mathcal{U}$ is $n-k$. For any object $\mathcal{X} \in \mathbf{dM}$, the underlying space $X = \mathbf{U}(\mathcal{X})$ can be covered by open subsets in such a way that the corresponding open subobjects of $\mathcal{X}$ are all local models. More generally, any open cover of $\mathbf{U}(\mathcal{X})$ by open sets can be refined to an open cover whose corresponding open subobjects are local models.

(6) *Local extensions for imbeddings.*

Definition 2.3
For any map $f\colon \mathcal{Y} \to \mathbb{R}^n$ in $\mathbf{dM}$, the canonical inclusion of the zero set $\mathcal{Y}_{f=0} \to \mathcal{Y}$ is called a *model imbedding*. A map $g\colon \mathcal{X} \to \mathcal{Y}$ is called an *imbedding* if there is a cover of $\mathcal{Y}$ by open subobjects $\mathcal{Y}_i$ such that, if we set $\mathcal{X}_i = g^{-1}(\mathcal{Y}_i)$, the induced maps $g|_{\mathcal{X}_i}\colon \mathcal{X}_i \to \mathcal{Y}_i$ are model imbeddings. Such open subobjects $\mathcal{Y}_i \subset \mathcal{Y}$ are called *trivializing neighborhoods* of the imbedding.

Let $g\colon \mathcal{X} \to \mathcal{Y}$ be an imbedding, and let $h\colon \mathcal{X} \to \mathbb{R}$ be a map in $\mathbf{dM}$. Then there exists a dotted arrow such that the diagram

$$\begin{array}{ccc} \mathcal{X} & \xrightarrow{h} & \mathbb{R} \\ {\scriptstyle g}\downarrow & \nearrow & \\ \mathcal{Y} & & \end{array}$$

commutes up to homotopy.

(7) *Normal bundle.* Let M be a smooth manifold, and let $\mathcal{X} \in \mathbf{dM}$ with underlying space $X = \mathbf{U}(\mathcal{X})$. If $g\colon \mathcal{X} \to M$ is an imbedding, then there exists an open neighborhood $U \subset M$ of $\mathcal{X}$, a real vector bundle $E \to U$, and a section $s\colon U \to E$ such that

$$\begin{array}{ccc} \mathcal{X} & \xrightarrow{g} & U \\ {\scriptstyle g}\downarrow & \diamond & \downarrow{\scriptstyle z} \\ U & \xrightarrow[s]{} & E \end{array}$$

is a homotopy pullback diagram, where $z\colon U \to E$ is the zero section of the vector bundle. Let $g = \mathbf{U}(g)$ also denote the underlying map $X \to U$; then the pullback bundle $g^*(E)$ on X is unique up to isomorphism. We call $g^*(E)$ the *normal bundle of* X *in* M and we call s a *defining section*.

Objects in $\mathbf{dM}$ are called *derived manifolds (of type* $\mathbf{dM}$*)* and morphisms in $\mathbf{dM}$ are called *morphisms of derived manifolds (of type* $\mathbf{dM}$*)*.

Remark 2.4
We defined the virtual dimension of a local model $\mathcal{U} = \mathbb{R}^n_{f=0}$ in Axiom (5) in Definition 2.1. We often drop the word *virtual* and refer to the virtual dimension of $\mathcal{U}$ simply as *the dimension* of $\mathcal{U}$.

We eventually define the virtual dimension of an arbitrary derived manifold as the Euler characteristic of its cotangent complex (see Definition 7.5). For now, the reader only needs to know that if $\mathcal{Z}$, $\mathcal{X}$, $\mathcal{Y}$, and M are as in Diagram (2.2.1), and if these objects have constant dimension z, x, y, and m, respectively, then $z + m = x + y$, as expected.

Let us briefly explain the definition of imbedding (see Definition 2.3) given in Axiom (6) in Definition 2.1. If we add the word *transverse* in the appropriate places, we are left with the usual definition of imbedding for smooth manifolds. This is proved in Proposition 2.5.

Recall that a map of manifolds is called a (smooth) imbedding if the induced map on the total spaces of their respective tangent bundles is an injection. Say that a map of manifolds $X \to U$ is *the inclusion of a level manifold* if there exists a smooth function $f\colon U \to \mathbb{R}^n$, transverse to $0\colon \mathbb{R}^0 \to \mathbb{R}^n$, such that $X \cong U_{f=0}$ over U.

PROPOSITION 2.5
Let X and Y be smooth manifolds, and let $g\colon X \to Y$ be a smooth map. Then g is an imbedding if and only if there is a cover of Y by open submanifolds U^i such that, if

we set $X_i = g^{-1}(U^i)$, each of the induced maps $g|_{X_i}: X_i \to U^i$ is the inclusion of a level manifold.

Sketch of proof
We may assume that X is connected. If g is a smooth imbedding of codimension d, let U be a tubular neighborhood, take the $U^i \subset U$ to be open subsets that trivialize the normal bundle of X, and take the $f^i : U^i \to \mathbb{R}^k$ to be identity on the fibers. The zero sets of the f^i are open subsets of X, namely $U^i_{f^i=0} \cong X_i$. Since they are smooth of the correct codimension, the f^i are transverse to zero.

For the converse, note that the property of being an imbedding is local on the target, so we may assume that X is the preimage of the origin under a map $f: U \to \mathbb{R}^k$ that is transverse to zero, where $U \subset Y$ is some open subset. The induced map $X \to U$ is clearly injective on tangent bundles. □

We now present a refinement of Theorem 1.8, which we prove as Corollary 3.13 in the following section. Recall that $\mathbf{i}: \mathbf{Man} \to \mathbf{dMan}$ denotes the inclusion guaranteed by Axiom (1) of Definition 2.1.

THEOREM 2.6
If **dM** *is good for doing intersection theory on manifolds, then* **dM** *has the general cup product formula in cobordism, in the sense of Definition 1.7. Moreover, for any manifold T, the functor*

$$\mathbf{i}_*: \Omega(T) \to \Omega^{\mathrm{der}}(T)$$

is an isomorphism between the classical cobordism ring and the derived cobordism ring (over T).

Example 2.7
Let **dM** denote a simplicial category which is good for doing intersection theory. By the unproven theorem (2.6), we have a cobordism theory Ω^{der} and a nice cup product formula for doing intersection theory. We now give several examples which illustrate various types of intersections. The last few examples are cautionary.

Transverse planes. Consider a k-plane K and an ℓ-plane L inside of projective space $\mathbb{P}^n$. If K and L meet transversely, then they do so in a $(k+\ell-n)$-plane, which we denote A. In **dM** there is an equivalence $A \cong K \times_{\mathbb{P}^n} L$. Of course, this descends to an equality $[A] = [K] \smile [L]$ both in the cobordism ring $\Omega(\mathbb{P}^n)$ and in the derived cobordism ring $\Omega^{\mathrm{der}}(\mathbb{P}^n)$.

Nontransverse planes. Suppose now that K and L are as above but do not meet transversely. Their topological intersection A' has the structure of a smooth manifold but of *the wrong dimension* (i.e., $\dim(A) > k + \ell - n$). Moreover, the formula $[A'] =^? [K] \smile [L]$ does *not* hold in $\Omega(\mathbb{P}^n)$.

However, the intersection of K and L as a derived manifold is different from A'; let us denote it A. Although the underlying spaces $\mathbf{U}(A') = \mathbf{U}(A)$ are the same, the virtual dimension of A is $k + \ell - n$ as expected. Moreover, the formula $[A] = [K] \smile [L]$ holds in the derived cobordism ring $\Omega^{\mathrm{der}}(\mathbb{P}^n)$. (The formula does not makes sense in $\Omega(\mathbb{P}^n)$ because A is not a smooth manifold.)

Fiber products and zero sets. Let $M \to P \leftarrow N$ be a diagram of smooth manifolds with dimensions m, p, and n. The fiber product $\mathcal{X}$ of this diagram exists in **dM**, and the (virtual) dimension of $\mathcal{X}$ is $m + n - p$.

For example, if $f_1, \ldots f_k$ is a finite set of smooth functions $M \to \mathbb{R}$ on a manifold M, then their zero set is a derived manifold $\mathcal{X}$, even if the $f_1, \ldots, f_k$ are not transverse. To see this, let $f = (f_1, \ldots, f_k)\colon M \to \mathbb{R}^k$ and realize $\mathcal{X}$ as the fiber product in the diagram

$$\begin{array}{ccc} \mathcal{X} & \longrightarrow & \mathbb{R}^0 \\ \big\downarrow & \diamond & \big\downarrow {\scriptstyle 0} \\ M & \underset{f}{\longrightarrow} & \mathbb{R}^k \end{array}$$

The dimension of $\mathcal{X}$ is $m - k$, where m is the dimension of M.

For example, let T denote the two-dimensional torus, and let $f\colon T \to \mathbb{R}$ denote a Morse function. If $p \in \mathbb{R}$ is a critical value of index one, then the pullback $f^{-1}(p)$ is a *figure eight* (as a topological space). It comes with the structure of a derived manifold of dimension one. It is derived cobordant both to a pair of disjoint circles and to a single circle; however, it is not isomorphic as a derived manifold to either of these because its underlying topological space is different.

Euler classes. Let M denote a compact smooth manifold and let $p\colon E \to M$ denote a smooth vector bundle; consider M as a submanifold of E by way of the zero section $z\colon M \to E$. The Euler class $e(p)$ is often thought of as the cohomology class represented by the intersection of M with itself inside E. However, classically one must always be careful to perturb $M \subset E$ before taking this self intersection. In the theory of derived manifolds, it is not necessary to choose a perturbation. The fiber product $M \times_E M$ exists as a compact derived submanifold of M, and one has

$$e(p) = [M \times_E M].$$

Vector bundles. More generally, let M denote a smooth manifold, let $p\colon E \to M$ denote a smooth vector bundle, and let $z\colon M \to E$ denote the zero section. Given an arbitrary section $s\colon M \to E$, the zero set $Z(s) := z(M) \cap s(M)$ of s is a derived submanifold of M. If s is transverse to z inside E, then $Z(s)$ is a submanifold of M, and its manifold structure coincides with its derived manifold structure. Otherwise, $Z(s)$ is a derived manifold that is not equivalent to any smooth manifold.

Changing s by a linear automorphism of E (over M) does not change the homotopy type of the derived manifold $Z(s)$. Arbitrary changes of section do change the homotopy type: if s and t are any two sections of E, then $Z(s)$ is not generally equivalent to $Z(t)$ as a derived manifold. However, these two derived manifolds are *derived cobordant*. The derived cobordism can be given by a straight-line homotopy in E.

Failure of Nullstellensatz. Suppose that X and Y are varieties (resp., manifolds), and suppose that $X \to Y$ is a closed imbedding. Let $I(X)$ denote the ideal of functions on Y which vanish on X; given an ideal J, let $Z(J)$ denote the zero set of J in Y. A classical version of the Nullstellensatz states that if k is an algebraically closed field, then I induces a bijection between the Zariski closed subsets of affine space $\mathbb{A}^n = \operatorname{Spec} k[x_1, \ldots, x_n]$ and the radical ideals of the ring $k[x_1, \ldots, x_n]$. A corollary is that for any closed subset $X \subset Y$, one has $X = Z(I(X))$; that is, X is the zero set of the ideal of functions which vanish on X.

This radically fails in the derived setting. The simultaneous zero set of n functions $Y \to \mathbb{R}$ always has codimension n in Y. For example, the x-axis in $\mathbb{R}^2$ is the zero set of a single function $y\colon \mathbb{R}^2 \to \mathbb{R}$, as a derived manifold.

However, y is not the only function that vanishes on the x-axis; for example, so do the functions $2y, 3y, y^2$, and 0. If we find the simultaneous zero set of all five functions, the resulting derived manifold has codimension five inside of $\mathbb{R}^2$. Its underlying topological space is indeed the x-axis, but its structure sheaf is very different from that of $\mathbb{R}$.

Thus, if we take a closed subvariety of Y and quotient the coordinate ring of Y by the infinitely many functions which vanish on it, then the result has infinite codimension. The formula $X = Z(I(X))$ fails in the derived setting (i.e., both for derived manifolds and for derived schemes).

We note one upshot of this. Given a closed submanifold $N \subset M$, one cannot identify N with the zero set of the ideal sheaf of functions on M which vanish on N. Given an arbitrary closed subset X of a manifold, one may wish to find an appropriate derived manifold structure on X; this cannot be done in a canonical way as it can in classical algebraic geometry, unless X is a local complete intersection (see the "vector bundles" example above).

Unions not included. Note that the union of manifolds along closed subobjects does *not* generally come equipped with the structure of a derived manifold. The reason we include this cautionary note is that, in the introduction, we spoke of the union Y of d hyperplanes in $\mathbb{C}P^n$. However, we were secretly regarding Y as the zero set of a single section of the bundle $\mathcal{O}(d)$ on $\mathbb{C}P^n$, and not as a union. We referred to it as a union only so as to aid the reader's imagination of Y (see the "vector bundles" example above for more information on Y).

Twisted cubic. We include one more cautionary example. Let C denote the twisted cubic in $\mathbb{P}^3$; that is, the image of the map $[t^3, t^2u, tu^2, u^3]\colon \mathbb{P}^1 \to \mathbb{P}^3$. Scheme-theoretically, the curve C cannot be defined as the zero set of two homogeneous polynomials on $\mathbb{P}^3$, but it can be defined as the zero set of three homogeneous polynomials on $\mathbb{P}^3$ (or more precisely, three sections of the line bundle $\mathcal{O}(2)$ on $\mathbb{P}^3$); namely C is the zero set of the polynomials $f_1 = xz - y^2$, $f_2 = yw - z^2$, $f_3 = xw - yz$. This might lead one to conclude that C is $3 - 3 = 0$ dimensional as a derived manifold (see Definition 2.2(5)).

It is true that the zero set of f_1, f_2, f_3 is a zero-dimensional derived manifold, however this zero set is probably not what one means by C. Instead, one should think of C as locally the zero set of two functions. That is, C is the zero set of a certain section of a rank 2 vector bundle on $\mathbb{P}^3$. As such, C is a one-dimensional derived manifold.

The reason for the discrepancy is this. The scheme-theoretic intersection does not take into account the dependency relations among f_1, f_2, f_3. While these three functions are globally independent, they are locally dependent everywhere. That is, for every point $p \in C$, there is an open neighborhood on which two of these three polynomials generate the third (ideal-theoretically). This is not an issue scheme-theoretically, but it is an issue in the derived setting.

3. Main results

In this section, we prove Theorem 2.6, which says that any simplicial category that satisfies the axioms presented in the last section (see Definition 2.1) has the general cup product formula (see Definition 1.7).

Before we do so, we prove an imbedding theorem (see Proposition 3.3) for compact derived manifolds, which says that any compact derived manifold can be imbedded into a large enough Euclidean space. The proof very closely mimics the corresponding proof for compact smooth manifolds, except we do not have to worry about the rank of Jacobians.

Fix a category **dM** which is good for doing intersection theory on manifolds, in the sense of Definition 2.1. In this section, we refer to objects in **dM** as derived manifolds, and we refer to morphisms in **dM** as morphisms of derived manifolds. When we speak of an *axiom*, we are referring to the axioms of Definition 2.1.

Before proving Proposition 3.3, let us state a few lemmas.

LEMMA 3.1
Let $f\colon \mathcal{X} \to \mathcal{Y}$, and let $g\colon \mathcal{Y}' \to \mathcal{Y}$ be morphisms of derived manifolds such that g is an open subobject. Then there exists an open subobject $\mathcal{X}' \to \mathcal{X}$ and a homotopy pullback diagram

$$\begin{array}{ccc} \mathcal{X}' & \longrightarrow & \mathcal{Y}' \\ \downarrow & \diamond & \downarrow{\scriptstyle g} \\ \mathcal{X} & \xrightarrow[f]{} & \mathcal{Y} \end{array}$$

Sketch of proof
Apply $\mathbf{U}$ and let X' denote the preimage of $\mathbf{U}(\mathcal{Y}')$ in $\mathbf{U}(\mathcal{X})$. The corresponding open subobject $\mathcal{X}' \to \mathcal{X}$, guaranteed by Axiom (2), satisfies the universal property of the homotopy fiber product. □

We state one more lemma about imbeddings.

LEMMA 3.2
(1) *The pullback of an imbedding is an imbedding.*
(2) *If M is a manifold, if $\mathcal{X}$ is a derived manifold, and if $f\colon \mathcal{X} \to M$ is a morphism, then the graph $\Gamma(f)\colon \mathcal{X} \to \mathcal{X} \times M$ is an imbedding.*

Proof
The first result is obvious by Definition 2.3, Lemma 3.1, and the basic properties of pullbacks. For the second result, we may assume that $\mathcal{X}$ is an affine derived manifold and that $M = \mathbb{R}^p$. We have a homotopy pullback diagram

$$\begin{array}{ccc} \mathcal{X} & \longrightarrow & \mathbb{R}^0 \\ \downarrow & \diamond & \downarrow \\ \mathbb{R}^n & \longrightarrow & \mathbb{R}^k \end{array}$$

and by Axiom (6), the map $f\colon \mathcal{X} \to \mathbb{R}^p$ is homotopic to some composite $\mathcal{X} \to \mathbb{R}^n \xrightarrow{f'} \mathbb{R}^p$.

By Proposition 2.5, the graph $\Gamma(f')\colon \mathbb{R}^n \to \mathbb{R}^n \times \mathbb{R}^p$ is an imbedding, and since the diagram

$$\begin{array}{ccc} \mathcal{X} & \xrightarrow{\Gamma(f)} & \mathcal{X} \times \mathbb{R}^p \\ \downarrow & \diamond & \downarrow \\ \mathbb{R}^n & \xrightarrow[\Gamma(f')]{} & \mathbb{R}^n \times \mathbb{R}^p \end{array}$$

is a homotopy pullback, it follows from the first result that the top map is also an imbedding, as desired. □

The next theorem says that any compact derived manifold can be imbedded into Euclidean space. This result is proved for smooth manifolds in [6, Theorem II.10.7], and we simply adapt that proof to the derived setting.

PROPOSITION 3.3

Let **dM** *be good for intersection theory, and let* $\mathcal{X} \in$ **dM** *be an object whose underlying space* $\mathbf{U}(\mathcal{X})$ *is compact. Then* $\mathcal{X}$ *can be imbedded into some Euclidean space* $\mathbb{R}^N$.

Proof

By Axiom (5), we can cover $\mathcal{X}$ by local models $\mathcal{V}_i$; let $V_i = \mathbf{U}(\mathcal{V}_i)$ denote the underlying space. For each $\mathcal{V}_i$, there is a homotopy pullback diagram

$$\begin{array}{ccc} \mathcal{V}_i & \longrightarrow & \mathbb{R}^0 \\ {\scriptstyle x^i}\downarrow & \diamond & \downarrow{\scriptstyle 0} \\ \mathbb{R}^{m_i} & \xrightarrow[f^i]{} & \mathbb{R}^{n_i} \end{array}$$

in **dM**.

For each i, let $\beta^i : \mathbb{R}^{m_i} \to \mathbb{R}$ be a smooth function that is one on some open disk D^{m_i} (centered at the origin), zero outside of some bigger open disk, and nonnegative everywhere. Define $z^i = \beta^i \circ x^i : \mathcal{V}_i \to \mathbb{R}$. Then z^i is identically one on an open subset $V_i' \subset V_i$ (the preimage of D^{m_i}) and is identically zero outside some closed neighborhood of V_i' in V_i. Let $\mathcal{V}_i'$ denote the open subobject of $\mathcal{V}_i$ over $V_i' \subset V_i$. Define functions $y^i : \mathcal{V}_i \to \mathbb{R}^{m_i}$ by multiplication: $y^i = z^i x^i$.

By construction, we have

$$y^i|_{\mathcal{V}_i'} = x^i|_{\mathcal{V}_i'},$$

and each y^i is constantly equal to zero outside of a closed neighborhood of V_i' in V_i. We can thus extend the z^i and the y^i to all of $\mathcal{X}$ by making them zero outside of V_i (see Axiom (3)(b)). By Axiom (5) and by the compactness of $X = \mathbf{U}(\mathcal{X})$, we can choose a finite number of indices i so that the $\mathcal{V}_i'$ cover all of $\mathcal{X}$, say for indices $1 \le i \le k$. For each i, we have an all-Cartesian diagram

$$\begin{array}{ccccc} \mathcal{V}_i' & \longrightarrow & \mathcal{V}_i & \longrightarrow & \mathbb{R}^0 \\ {\scriptstyle y^i}\downarrow & \diamond & \downarrow{\scriptstyle x^i} & \diamond & \downarrow{\scriptstyle 0} \\ D^{m_i} & \longrightarrow & \mathbb{R}^{m_i} & \xrightarrow[f^i]{} & \mathbb{R}^{n_i} \end{array} \tag{3.3.1}$$

by Lemma 3.1.

Let $N = \sum_{i=1}^k m_i$. Let $b^1\colon \mathbb{R}^N \to \mathbb{R}^{m_1}$ denote the first m_1 coordinate projections, let $b^2\colon \mathbb{R}^N \to \mathbb{R}^{m_2}$ denote the next m_2 coordinate projections, and so on for each $1 \le i \le k$. The sequence $W = (y^1, \dots, y^k)$ gives a map

$$W\colon \mathcal{X} \to \mathbb{R}^N$$

such that for each $1 \le i \le k$ one has $b^i \circ W = y^i$.

We show that W is an imbedding in the sense of Definition 2.3; that is, that the restriction of W to each $\mathcal{V}_i'$ comes as the inclusion of the zero set of smooth functions c^i on a certain open subset of $\mathbb{R}^N$. The work has already been done; we just need to tease out what we already have. The c^i should act like f^i on the relevant coordinates for $\mathcal{V}_i'$, and should act like coordinate projections everywhere else. With that in mind, we define for each $1 \le i \le k$ the function

$$c^i = (b^1, \dots, b^{i-1}, f^i, b^{i+1}, \dots, b^k)\colon \mathbb{R}^N \to \mathbb{R}^{N-m_i+n_i}.$$

We construct the following diagram:

$$\begin{array}{ccccc} \mathcal{V}_i' & \longrightarrow & \mathcal{V}_i & \longrightarrow & \mathbb{R}^0 \\ {\scriptstyle y^i}\downarrow & \diamond & \downarrow{\scriptstyle x^i} & \diamond & \downarrow{\scriptstyle 0} \\ D^{m_i} & \longrightarrow & \mathbb{R}^{m_i} & \xrightarrow{f^i} & \mathbb{R}^{n_i} \\ \downarrow & \diamond & \downarrow & \diamond & \downarrow \\ D^{m_i} \times \mathbb{R}^{N-m_i} & \longrightarrow & \mathbb{R}^N & \xrightarrow{c^i} & \mathbb{R}^{(N-m_i)+n_i} \end{array}$$

The lower right-hand vertical map is a coordinate imbedding. The lower right square and the lower left square are pullbacks in the category of manifolds, so they are homotopy pullback in **dM** by Axiom (1). The upper squares are pullbacks (see (3.3.1)). Therefore, the diagram is (homotopy) all Cartesian. The vertical composite $\mathcal{V}_i' \to D^{m_i} \times \mathbb{R}^{N-m_i}$ is the restriction of W to $\mathcal{V}_i'$, and it is also the zero set of the horizontal composite $D^{m_i} \times \mathbb{R}^{N-m_i} \to \mathbb{R}^{(N-m_i)+n_i}$. Since $D^{m_i} \times \mathbb{R}^{N-m_i} \to \mathbb{R}^N$ is the inclusion of an open subset, we have shown that W is an imbedding in the sense of Definition 2.3. □

The following relative version of Proposition 3.3 is proven in almost exactly the same way. Recall that a map f of topological spaces is said to be *proper* if the inverse image under f of any compact subspace is compact. A morphism of derived manifolds is said to be proper if the underlying morphism of topological spaces is proper.

COROLLARY 3.4
Let **dM** *be good for intersection theory, let $\mathcal{X} \in$* **dM** *be a derived manifold, and let $M \in$* **Man** *be a manifold. Suppose that $f: \mathcal{X} \to M$ is proper, that $A \subset M$ is a compact subset, and that $A \subset f(\mathbf{U}(\mathcal{X}))$. Let A' denote the interior of A, and let $\mathcal{X}' = f^{-1}(A')$. There exists an imbedding $W: \mathcal{X}' \to \mathbb{R}^N \times A'$ such that the diagram*

$$\begin{array}{ccc}
\mathcal{X}' & \longrightarrow & \mathcal{X} \\
{\scriptstyle W}\downarrow & & \\
\mathbb{R}^N \times A' & & \downarrow{\scriptstyle f} \\
{\scriptstyle \pi_2}\downarrow & & \\
A' & \longrightarrow & M
\end{array}$$

commutes, and such that the composite $\pi_2 \circ W: \mathcal{X}' \to A'$ is proper.

Sketch of proof
Copy the proof of Proposition 3.3 verbatim until it requires the compactness of $X = \mathbf{U}(\mathcal{X})$, which does not hold in our case. Instead, use the compactness of $f^{-1}(A)$ and choose finitely many indices $1 \le i \le k$ such that the union $\cup_i \mathcal{V}_i'$ contains $f^{-1}(A)$. Continue to copy the proof verbatim, except replace instances of $\mathcal{X}$ with $\cup_i \mathcal{V}_i'$. In this way, we prove that one has an imbedding of the derived manifold

$$W' : \cup_i \mathcal{V}_i' \to \mathbb{R}^N.$$

Let $f': \mathcal{X}' \to A'$ be the pullback of f, and note that $\mathcal{X}' = f^{-1}(A')$ is contained in $\cup_i \mathcal{V}_i$. The restriction $W'|_{\mathcal{X}'}: \mathcal{X}' \to \mathbb{R}^N$ and the map f' together induce a map

$W\colon \mathcal{X}' \to \mathbb{R}^N \times A'$, which is an imbedding by Lemma 3.2. By construction, $f' = \pi_2 \circ W$ is the pullback of f, so it is proper. □

3.1. Derived cobordism

Fix a simplicial category **dM** that is good for doing intersection theory on manifolds in the sense of Definition 2.1. One generally defines cobordism using the idea of manifolds with boundary. Under the above definitions, manifolds with boundary are *not* derived manifolds.

One can define derived manifolds with boundary in a way that emulates Definition 2.1; that is, one could give axioms for a simplicial category with local models that look like generalizations of manifolds with boundary, and so on. One could then prove that the local definition was equivalent to a global one; that is, one could prove that derived manifolds with boundary can be imbedded into Euclidean half-space. All of this was the approach of the author's dissertation (see [27]). For this article, in the interest of space, we dispense with all that and define cobordism using the idea of proper maps to $\mathbb{R}$.

To orient the reader to Definition 6, we reformulate the usual cobordism relation for manifolds using this approach. Two compact smooth manifolds Z_0 and Z_1 are cobordant if there exists a manifold (without boundary) X and a proper map $f\colon X \to \mathbb{R}$ such that

- for the points $i = 0, 1 \in \mathbb{R}$, the map f is i-collared in the sense of Definition 3.5, and
- for $i = 0, 1$, there is an isomorphism $Z_i \cong f^{-1}(i)$, also written $Z_i \cong X_{f=i}$.

For the definition of derived cobordism, we simply drop the requirement that zero and one be regular values of f.

Definition 3.5

Let $\mathcal{X}$ denote a derived manifold, and let $f\colon \mathcal{X} \to \mathbb{R}$ define a morphism. Given a point $i \in \mathbb{R}$, we say that f is *i-collared* if there exists an $\epsilon > 0$ and an equivalence

$$\mathcal{X}_{|f-i|<\epsilon} \simeq \mathcal{X}_{f=i} \times (-\epsilon, \epsilon), \qquad (3.5.1)$$

where $\mathcal{X}_{|f-i|<\epsilon}$ denotes the open subobject of $\mathcal{X}$ over $(i-\epsilon, i+\epsilon) \subset \mathbb{R}$ (see Lemma 3.1).

Definition 3.6

Let **dM** be good for intersection theory. Compact derived manifolds $\mathcal{Z}_0$ and $\mathcal{Z}_1$ are said to be *derived cobordant* if there exists a derived manifold $\mathcal{X}$ and a proper map $f\colon \mathcal{X} \to \mathbb{R}$ such that for $i = 0, 1$, there is an equivalence $\mathcal{Z}_i \simeq \mathcal{X}_{f=i}$. The map

$f: \mathcal{X} \to \mathbb{R}$ is called *a derived cobordism between* $\mathcal{Z}_0$ *and* $\mathcal{Z}_1$. We refer to $\mathcal{Z}_0 \amalg \mathcal{Z}_1$ as the *boundary* of the cobordism.

If T is a CW complex, then a *derived cobordism over* T is a pair (a, f), where $f: \mathcal{X} \to \mathbb{R}$ is a derived cobordism and where $a: \mathbf{U}\mathcal{X} \to T$ is a continuous map to T.

If T is a manifold, we denote the ring whose elements are derived cobordism classes over T (with sum given by disjoint union and product given by fiber product over T) by $\Omega^{\text{der}}(T)$. We use Ω^{der} to denote $\Omega^{\text{der}}(\mathbb{R}^0)$.

Remark 3.7

In Definition 3.6, we speak of derived cobordism classes, even though we have not yet shown that derived cobordism is an equivalence relation on compact derived manifolds. We prove this fact in Proposition 3.10.

Remark 3.8

We did not define oriented derived manifolds, so all of our cobordism rings $\Omega(T)$ and derived cobordism rings $\Omega^{\text{der}}(T)$ should be taken to be unoriented.

However, the oriented case is no harder; we must simply define oriented derived manifolds and oriented derived cobordisms. To do so, imbed a compact derived manifold $\mathcal{X}$ into some $\mathbb{R}^n$ (which is possible by Proposition 3.3), and consider the normal bundle guaranteed by Axiom (7). We define an orientation on $\mathcal{X}$ to be an orientation on some such normal bundle. One orients derived cobordisms similarly.

All results which we prove in the unoriented case also work in the oriented case.

Remark 3.9

One show that if U is an open neighborhood of the closed interval $[0, 1] \subset \mathbb{R}$, then any proper map $f: \mathcal{X} \to U$ can be extended to a proper map $g: \mathcal{X} \to \mathbb{R}$ with an isomorphism $f^{-1}([0, 1]) \cong g^{-1}([0, 1])$ over $[0, 1]$. So, one can consider such an f to be a derived cobordism between $f^{-1}(0)$ and $f^{-1}(1)$.

PROPOSITION 3.10

Let **dM** *be good for intersection theory. Derived cobordism is an equivalence relation on compact objects in* **dM**.

Proof

Derived cobordism is clearly symmetric. To see that it is reflexive, let $\mathcal{X}$ be a compact derived manifold and consider the projection map $\mathcal{X} \times \mathbb{R} \to \mathbb{R}$. It is a derived cobordism between $\mathcal{X}$ and $\mathcal{X}$.

Finally, we must show that derived cobordism is transitive. Suppose that $f: \mathcal{X} \to \mathbb{R}$ is a derived cobordism between $\mathcal{Z}_0$ and $\mathcal{Z}_1$, and suppose that $g: \mathcal{Y} \to \mathbb{R}$ is a derived cobordism between $\mathcal{Z}_1$ and $\mathcal{Z}_2$. By Axiom (3), we can glue open subobjects $\mathcal{X}_{f<1+\epsilon} \subset \mathcal{X}$ and $\mathcal{Y}_{g>-\epsilon} \subset \mathcal{Y}$ together along the common open subset $\mathcal{Z}_1 \times (-\epsilon, \epsilon)$ to obtain a derived manifold $\mathcal{W}$ together with a proper map $h\colon \mathcal{W} \to \mathbb{R}$ such that for $i = 0, 2$, we have $\mathcal{W}_{h=i} = \mathcal{Z}_i$. The result follows. □

LEMMA 3.11
Let **dM** *be good for intersection theory. The functor* $\mathbf{i}\colon \mathbf{Man} \to \mathbf{dM}$ *induces a homomorphism on cobordism rings*

$$\mathbf{i}_*\colon \Omega \to \Omega^{\mathrm{der}}.$$

Proof
It suffices to show that manifolds which are cobordant are derived cobordant. If Z_0 and Z_1 are manifolds which are cobordant, then there exists a compact manifold X with boundary $Z_0 \amalg Z_1$. It is well known that one can imbed X into $\mathbb{R}^n$ in such a way that for $i = 0, 1$ we have $Z_i \cong X_{x_n=i}$, where x_n denotes the last coordinate of $\mathbb{R}^n$. Because each Z_i has a collar neighborhood, we can assume that $X_{|x_n-i|<\epsilon} \cong X_{x_n=i} \times (-\epsilon, \epsilon)$ for some $\epsilon > 0$. The preimage $\widetilde{X} \subset X$ of $(-\epsilon, 1+\epsilon)$ is a manifold (without boundary), and hence a derived manifold (under **i**). By Remark 3.9, $x_n\colon \widetilde{X} \to \mathbb{R}$ is a derived cobordism between Z_0 and Z_1. □

THEOREM 3.12
Let **dM** *be good for intersection theory. The functor*

$$\mathbf{i}_*\colon \Omega \to \Omega^{\mathrm{der}}$$

is an isomorphism.

Proof
We first show that $\mathbf{i}_*$ is surjective, that is, that every compact derived manifold $\mathcal{Z}$ is derived cobordant to a smooth manifold. Let $W\colon \mathcal{Z} \to \mathbb{R}^N$ denote a closed imbedding, which exists by Proposition 3.3. Let $U \subset \mathbb{R}^N$, $E \to U$, and $s, z\colon U \to E$ be the open neighborhood, vector bundle, zero section, and defining section from Axiom (7), so that the diagram

$$\begin{array}{ccc} \mathcal{Z} & \xrightarrow{g} & U \\ {\scriptstyle g}\downarrow & \diamond & \downarrow{\scriptstyle z} \\ U & \xrightarrow[s]{} & E \end{array}$$

is a homotopy pullback. Note that U and E are smooth manifolds and that the image $z(U) \subset E$ is a closed subset.

Since $\mathcal{Z}$ is compact, we can choose a compact subset $U' \subset U$ whose interior contains $\mathcal{Z}$. Let $A \subset U$ be the compliment of the interior of U'. Then $s(A) \cap z(U) = \emptyset$, so in particular they are transverse as closed submanifolds of E.

By [28, Appendix 2, p. 24], there exists a regular homotopy $H : [0, 1] \times U \to E$ such that

- $H_0 = s : U \to E$;
- $H_1 = t$, where $t : U \to E$ is transverse to the closed subset $z(U) \subset E$; and
- for all $0 \leq i \leq 1$, we have $H_i|_A = s|_A$.

For any $\epsilon > 0$ we can extend H to a homotopy $H : (-\epsilon, 1 + \epsilon) \times U \to E$, with the same three bulleted properties, by making it constant on $(-\epsilon, 0]$ and $[1, 1 + \epsilon)$.

Consider the all-Cartesian diagram

$$\begin{array}{ccccc}
\mathcal{Z}_i & \longrightarrow & \mathcal{P} & \longrightarrow & U \\
\downarrow & \diamond & \downarrow{\scriptstyle a} & \diamond & \downarrow{\scriptstyle z} \\
\mathbb{R}^0 \times U & \xrightarrow{i \times \mathrm{id}_U} & (-\epsilon, 1+\epsilon) \times U & \xrightarrow{H} & E \\
{\scriptstyle \pi}\downarrow & \diamond & \downarrow{\scriptstyle \pi} & & \\
\mathbb{R}^0 & \xrightarrow[i]{} & \mathbb{R} & &
\end{array}$$

for $i = 0, 1$. Notice that $\mathcal{Z}_0 = \mathcal{Z}$ and that $\mathcal{Z}_1$ is a smooth manifold. It suffices to show that the map $\pi \circ a : \mathcal{P} \to \mathbb{R}$ is proper, that is, that for each $0 \leq i \leq 1$, the intersection of $H_i(U)$ and $z(U)$ is compact. Since $H_i(A) \cap z(U) = \emptyset$, one has

$$H_i(U) \cap z(U) = H_i(U') \cap z(U),$$

and the right-hand side is compact. Thus H is a derived cobordism between $\mathcal{Z}$ and a smooth manifold.

To prove that $\mathbf{i}_*$ is injective, we must show that if smooth manifolds M_0, M_1 are derived cobordant, then they are smoothly cobordant. Let $f : \mathcal{X} \to \mathbb{R}$ denote a derived cobordism between M_0 and M_1. Let A' denote the open interval $(-1, 2) \subset \mathbb{R}$, and let $\mathcal{X}' = f^{-1}(A')$. By Corollary 3.4, we can find an imbedding $W : \mathcal{X}' \to \mathbb{R}^N \times A'$ such that

$$f' := f|_{\mathcal{X}'} = \pi_2 \circ W$$

is proper. Note that for $i = 0, 1$ we still have $\mathcal{X}'_{f'=i} = M_i$. By Remark 3.9, f' induces a derived cobordism between M_0 and M_1 with the added benefit of factoring through an imbedding $\mathcal{X}' \to \mathbb{R}^N$ into Euclidean space.

Again we use Axiom (7) to find a vector bundle and section s on an open subset of $\mathbb{R}^N \times \mathbb{R}$ whose zero set is $\mathcal{X}'$. We apply [28, Appendix 2, p. 24] to find a regular homotopy between s and a section t which is transverse to zero, all the while keeping the closed submanifold $M_0 \amalg M_1$ fixed. The zero set of t is a smooth cobordism between M_0 and M_1. □

COROLLARY 3.13

If **dM** *is good for doing intersection theory on manifolds, then* **dM** *has the general cup product formula in cobordism, in the sense of Definitions 1.3 and 1.7. Moreover, for any manifold T, the functor*

$$\mathbf{i}_*\colon \Omega(T) \to \Omega^{\mathrm{der}}(T)$$

is an isomorphism between the classical cobordism ring and the derived cobordism ring (over T).

Proof

Suppose that **dM** is good for doing intersection theory on manifolds. Since T is assumed to be a CW complex, if $Z \subset X$ is a closed subset of a metrizable topological space, then any map $f\colon Z \to T$ extends to a map $f'\colon U \to T$, where $U \subset X$ is an open neighborhood of Z. One can now modify the proof of Theorem 3.12 so that all constructions are suitably *over* T, which implies that i_* is an isomorphism. Let us quickly explain how to make this modification.

We begin with a compact derived manifold $\mathcal{Z} = (Z, \mathcal{O}_Z)$ and a map of topological spaces $\sigma\colon Z \to T$. Since Z is Hausdorff and paracompact, it is metrizable. It follows that the map σ extends to a map $\sigma'\colon U \to T$, where $U \subset \mathbb{R}^N$ is an open neighborhood of T and $\sigma'|_Z = \sigma$. By intersecting if necessary, we can take this U to be the open neighborhood given in the proof of Theorem 3.12. The vector bundle p is canonically defined over T (via σ'), as are the sections s, z. One can continue in this way and see that the proof extends without any additional work to the relative setting (over T).

Now that we have proved the second assertion, that i_* is an isomorphism, we go back and show the first, which is that **dM** has the general cup product formula in cobordism. The first three conditions of Definition 1.7 follow from Axiom (1), Axiom (4), and Theorem 3.12.

To prove the last condition, let M be a manifold and suppose that $j\colon A \to M$ and $k\colon B \to M$ are compact submanifolds. There is a map $H\colon A \times \mathbb{R} \to M$ such that $H_0 = j$ and such that $H_1\colon A \to M$ is transverse to k. For $i = 0, 1$, let $\mathcal{X}_i$ denote the intersection of (the images of) H_i and k. Note that $\mathcal{X}_0 = A \times_M B$ is a compact derived manifold which we also denote $A \cap B$. Further, $\mathcal{X}_1$ is a compact smooth manifold, often called the *transverse intersection of A and B* (because A is *made transverse to* B), and $\mathcal{X}_0$ and $\mathcal{X}_1$ are derived cobordant over M.

It is well known that $[\mathcal{X}_1] = [A] \smile [B]$ as elements of $\Omega(M)$. Since $\Omega(M) \cong \Omega^{\mathrm{der}}(M)$, and since $[\mathcal{X}_0] = [\mathcal{X}_1]$ as elements of $\Omega^{\mathrm{der}}(M)$, the formula

$$[A] \smile [B] = [A \cap B]$$

holds. □

4. Layout for the construction of dMan

In the next few sections we construct a simplicial category **dMan** which is good for doing intersection theory on manifolds, in the sense of Definition 2.1. An object of **dMan** is called a *derived manifold*, and a morphism in **dMan** is called a *morphism of derived manifolds*.

A derived manifold could be called a *homotopical C^∞-scheme of finite type*. In the coming sections, we build up to a precise definition. In this section, we give a brief outline of the construction.

Let us first recall the process by which one defines a scheme in algebraic geometry. One begins with the category of commutative rings, which can be defined as the category of algebras of a certain algebraic theory (see [5, Example 3.3.5.a]). One then defines a ringed space to be a space X together with a sheaf of rings $\mathcal{O}_X$ on X. A local ringed space is a ringed space in which all stalks are local rings. One must then functorially assign to each commutative ring A a local ringed space Spec A, called its prime spectrum. Once that is done, a scheme is then defined as a local ringed space which can be covered by open subobjects, each of which is isomorphic to the prime spectrum of a ring. Note that the purpose of defining local ringed spaces is that morphisms of schemes have to be morphisms of local ringed spaces, not just morphisms of ringed spaces.

If one is interested only in schemes of finite type (over $\mathbb{Z}$), one does not need to define prime spectra for all rings. One could get away with just defining the category of affine spaces $\mathbb{A}^n = \operatorname{Spec} \mathbb{Z}[x_1, \ldots, x_n]$ as local ringed spaces. One then uses as local models the fiber products of affine spaces, as taken in the category of local ringed spaces.

We recall quickly the notion of an algebraic theory (see [16]). A model for an algebraic theory is a category $\mathbb{T}$ with objects $\{T^i \mid i \in \mathbb{N}\}$ such that T^i is the i-fold product of T^1. An algebraic theory is a product preserving functor from $\mathbb{T}$ to **Sets**. A simplicial theory is a product preserving functor from $\mathbb{T}$ to **sSets**. The category of rings is the category of algebraic theories on the category with objects $\mathbb{A}^n_{\mathbb{Z}}$ and morphisms given by polynomial maps between affine spaces.

The basic outline of our construction of derived manifolds follows the above construction fairly closely, but with the following differences.

- Rings are not sufficient as our basic objects. We need a smooth version of the theory, whose algebras are called C^∞-rings.
- Everything must be done homotopically. Our basic objects are in fact lax simplicial C^∞-rings, which means that the defining functor is not required to be *product preserving on the nose* but instead weakly product preserving.
- Furthermore, our sheaves are homotopy sheaves, which means that they satisfy a homotopical version of descent.
- Our affine spaces are quite familiar, they are simply the Euclidean spaces $\mathbb{R}^n$ (as smooth manifolds). A morphism of affine spaces is a smooth function $\mathbb{R}^n \to \mathbb{R}^m$.
- We use as our local models the homotopy fiber products of affine spaces, as taken in the category of local C^∞-ringed spaces.

A derived manifold is hence a smooth, homotopical version of a scheme. In Section 5, we define our basic objects, the lax C^∞-rings, and prove some lemmas about their relationship with C^∞-rings (in the usual sense) and with commutative rings. In Section 6, we define local C^∞-ringed spaces and derived manifolds (see Definition 6.15). We then must discuss cotangent complexes for derived manifolds in Section 7. In Section 8, we give the proofs of several technical results. Finally, in Section 9 we prove that our category of derived manifolds is good for doing intersection theory on manifolds, in the sense of Definition 2.1.

Convention 4.1
In this article, we rely heavily on the theory of model categories and their localizations (see [11] or [12] for a good introduction to this subject).

If X is a category, then there are two common model structures on the simplicial presheaf category $\mathbf{sSets}^X$ called the injective and the projective model structures. In this article, *we always use the injective model structure*. If we speak of a model structure on $\mathbf{sSets}^X$ without specifying it, we mean the injective model structure. The injective model structure on $\mathbf{sSets}^X$ is given by object-wise cofibrations, object-wise weak equivalences, and fibrations determined by the right lifting property with respect to acyclic cofibrations.

With the injective model structure, $\mathbf{sSets}^X$ is a left proper, combinatorial, simplicial model category. As stated above, weak equivalences and cofibrations are determined object-wise; in particular, every object in $\mathbf{sSets}^X$ is cofibrant (see, e.g., [4] or [20] for further details).

5. C^∞-rings

Let $\mathbb{E}$ denote the full subcategory of **Man** spanned by the Euclidean spaces $\mathbb{R}^i$, for $i \in \mathbb{N}$; we refer to $\mathbb{E}$ as *the Euclidean category*. Lawvere, Dubuc, Moerdijk and Reyes, and others have studied $\mathbb{E}$ as an algebraic theory (see [17], [9], [22]). The $\mathbb{E}$-algebras, which are defined as the product preserving functors from $\mathbb{E}$ to sets, are called C^∞-rings. We use a homotopical version, in which we replace sets with simplicial sets, and strictly product preserving functors with functors which preserve products up to weak equivalence.

Let $\mathbf{sSets}^{\mathbb{E}}$ denote the simplicial model category of functors from $\mathbb{E}$ to the category of simplicial sets. As usual (see Convention 4.1), we use the injective model structure on $\mathbf{sSets}^{\mathbb{E}}$, in which cofibrations and weak equivalences are determined object-wise, and fibrations are determined by the right lifting property.

For $i \in \mathbb{N}$, let $H_i \in \mathbf{sSets}^{\mathbb{E}}$ denote the functor $H_i(\mathbb{R}^n) = \mathrm{Hom}_{\mathbb{E}}(\mathbb{R}^i, \mathbb{R}^n)$. For each $i, j \in \mathbb{N}$, let

$$p_{i,j} \colon H_i \amalg H_j \to H_{i+j} \tag{5.0.1}$$

denote the natural map induced by coordinate projections $\mathbb{R}^{i+j} \to \mathbb{R}^i$ and $\mathbb{R}^{i+j} \to \mathbb{R}^j$. We also refer to H_i as $H_{\mathbb{R}^i}$ (see Example 5.6).

Note that if $F \colon \mathbb{E} \to \mathbf{sSets}$ is any functor, then the Yoneda lemma gives a natural isomorphism of simplicial sets

$$\mathrm{Map}(H_i, F) \cong F(\mathbb{R}^i).$$

Definition 5.1

With notation as above, define the category of *lax simplicial C^∞-rings*, denoted $s\mathbf{C}^\infty$, to be the localization of $\mathbf{sSets}^{\mathbb{E}}$ at the set $P = \{p_{i,j} | i, j \in \mathbb{N}\}$ (see [11, Definition 3.1.1]).

We often refer to the objects of $s\mathbf{C}^\infty$ simply as C^∞-rings, dropping the words *lax simplicial*. We can identify a discrete object in $s\mathbf{C}^\infty$ with a (strict) C^∞-ring in the classical sense (see [22]), and thus we refer to these objects as *discrete C^∞-rings.*

The model category $s\mathbf{C}^\infty$ is a left proper, cofibrantly-generated simplicial model category (see [11, Proposition 3.4.4]), in which all objects are cofibrant. Let $F \in \mathbf{sSets}^{\mathbb{E}}$ be a fibrant object, considered as an object of $s\mathbf{C}^\infty$. Then F is fibrant in $s\mathbf{C}^\infty$ if and only if it is local with respect to all the $p_{i,j}$; that is, for each $i, j \in \mathbb{N}$ the natural map

$$\mathrm{Map}(H_{i+j}, F) \to \mathrm{Map}(H_i \amalg H_j, F)$$

is a weak equivalence. Since each H_i is the functor represented by $\mathbb{R}^i$, this is equivalent to the condition that the natural map

$$F(\mathbb{R}^{i+j}) \to F(\mathbb{R}^i) \times F(\mathbb{R}^j)$$

be a weak equivalence. In other words, $F: \mathbb{E} \to \mathbf{sSets}$ is fibrant in $s\mathbf{C}^\infty$ if and only if it is weakly product preserving (and fibrant as an object of $\mathbf{sSets}^{\mathbb{E}}$).

Remark 5.2
There is a model category of strict simplicial C^∞-rings consisting of (strictly) product preserving functors from $\mathbb{E}$ to **sSets**. It is Quillen equivalent to our $s\mathbf{C}^\infty$ (see [19, Corollary 13.3]). The reason we use the lax version is that, as a localization of a left proper model category, it is clearly left proper. (Note that, by [24], we could have used a simplicial version of the theory to obtain a proper model of strict algebras, but this model is more difficult to use for our purposes.)

This left properness of $s\mathbf{C}^\infty$ is necessary for the further localization we use to define the category of homotopy sheaves of C^∞-rings. The use of homotopy sheaves is one of the key ways in which our theory differs from the synthetic differential geometry literature.

PROPOSITION 5.3
Suppose that $\phi: F \to G$ is a morphism of fibrant C^∞-rings. Then ϕ is a weak equivalence if and only if $\phi(\mathbb{R}): F(\mathbb{R}) \to G(\mathbb{R})$ is a weak equivalence of simplicial sets.

Proof
Follows from basic facts about Bousfield localization. □

Note that if $F \in s\mathbf{C}^\infty$ is fibrant and if for all $i \in \mathbb{N}$ the simplicial set $F(\mathbb{R}^i)$ is a discrete set, then F is a C^∞-ring in the classical sense; that is, it is a strictly product preserving functor from $\mathbb{E}$ to **Sets**.

LEMMA 5.4
The functor $\pi_0: \mathbf{sSets}^{\mathbb{E}} \to \mathbf{Sets}^{\mathbb{E}}$ sends fibrant objects in $s\mathbf{C}^\infty$ to C^∞-rings (in the classical sense). It sends object-wise fibrant homotopy pushout diagrams in $s\mathbf{C}^\infty$ to pushouts of C^∞-rings.

Proof
If $F \in s\mathbf{C}^\infty$ is fibrant, then $\mathrm{Map}(p_{i,j}, F)$ is a weak equivalence of simplicial sets for each $p_{i,j} \in P$. Hence $\pi_0\mathrm{Map}(p_{i,j}, F)$ is a bijection of sets. Since each $p_{i,j}$ is a map between discrete objects in $s\mathbf{C}^\infty$, we have $\pi_0\mathrm{Map}(p_{i,j}, F) = \mathrm{Hom}(p_{i,j}, \pi_0 F)$, so $\pi_0 F$ is indeed a C^∞-ring.

To prove the second assertion, suppose that

$$\Psi = (A \xleftarrow{f} B \xrightarrow{g} C)$$

is a diagram of fibrant objects in $s\mathbf{C}^\infty$. Factor f as a cofibration $B \to A'$ followed by an acyclic fibration $A' \to A$. The homotopy colimit of Ψ is given by the usual colimit of the diagram Ψ' of functors $A' \leftarrow B \to C$. Applying π_0 commutes with taking colimits. Since A' and A are both fibrant and weakly equivalent in $s\mathbf{C}^\infty$, they are weakly equivalent in $\mathbf{sSets}^{\mathbb{E}}$, so $\pi_0 A' \to \pi_0 A$ is an isomorphism in $\mathbf{Sets}^{\mathbb{E}}$. Hence we have

$$\pi_0 \operatorname{hocolim}(\Psi) \cong \pi_0 \operatorname{colim}(\Psi') \cong \operatorname{colim}(\pi_0 \Psi') \cong \operatorname{colim}(\pi_0 \Psi),$$

completing the proof. □

The following lemma is perhaps unnecessary, but we include it to give the reader more of an idea of how classical (i.e., discrete) C^∞-rings work.

LEMMA 5.5
Let F be a discrete fibrant C^∞-ring. Then the set $F(\mathbb{R})$ naturally has the structure of a ring.

Proof
Let $R = F(\mathbb{R}) \in \mathbf{Sets}$, and note that $F(\mathbb{R}^i) \cong R^i$, and that in particular $F(\mathbb{R}^0) = \{*\}$ is a singleton set. Let $0, 1\colon \mathbb{R}^0 \to \mathbb{R}$ denote the additive and multiplicative units, let $+, \times\colon \mathbb{R}^2 \to \mathbb{R}$ denote the addition and multiplication functions, and let $\iota\colon \mathbb{R} \to \mathbb{R}$ denote the additive inverse. All of these are smooth functions and are hence morphisms in $\mathbb{E}$. Applying F, we obtain elements $F(0), F(1)\colon \{*\} \to R$, two binary functions $F(+), F(\times)\colon R \times R \to R$, and a function $F(\iota)\colon R \to R$. Since all of the ring axioms can be written as the commutativity condition on a diagram, and since F is a functor and preserves commutative diagrams, one sees that the operations $F(0)$, $F(1)$, $F(+)$, $F(\times)$, and $F(\iota)$ satisfy all the axioms for R to be a ring. □

If $F \in s\mathbf{C}^\infty$ is fibrant but not necessarily discrete, then the ring axioms hold up to homotopy. Note, however, that even in the discrete case, $R = F(\mathbb{R})$ has much more structure than just that of a ring. Any smooth function $\mathbb{R}^n \to \mathbb{R}^m$ gives rise to a function $R^n \to R^m$, satisfying all appropriate commutative diagrams (e.g., the functions 3^{a+b} and $3^a 3^b$ taking $\mathbb{R}^2 \to \mathbb{R}$ are equal, so they are sent to equal vertices of the mapping space $\operatorname{Map}(R^2, R)$).

Example 5.6
Let M be a manifold. Let $H_M\colon \mathbb{E} \to \mathbf{sSets}$ be defined by $H_M(\mathbb{R}^i) := \operatorname{Hom}_{\mathbf{Man}}(M, \mathbb{R}^i)$. One checks easily that H_M is a discrete fibrant C^∞-ring. We usually denote H_M by $C^\infty(M)$, though this notation tends to obscure the role of H_M as a functor, instead highlighting the value of H_M on $\mathbb{R}$, the set of C^∞-functions $M \to \mathbb{R}$.

Since $s\mathbf{C}^\infty$ is a model category, it is closed under homotopy colimits. The homotopy colimit of the diagram $C^\infty(\mathbb{R}^0) \leftarrow C^\infty(\mathbb{R}) \rightarrow C^\infty(\mathbb{R}^0)$, induced by the diagram of manifolds $\mathbb{R}^0 \xrightarrow{0} \mathbb{R} \xleftarrow{0} \mathbb{R}^0$, is an example of a nondiscrete C^∞-ring.

Definition 5.7
Let $\mathbb{E}^{\text{alg}}$ denote the subcategory of $\mathbb{E}$ whose objects are the Euclidean spaces $\mathbb{R}^i$, but in which we take as morphisms only those maps $\mathbb{R}^i \rightarrow \mathbb{R}^j$ which are given by j polynomials in the i coordinate functions on $\mathbb{R}^i$. For each $i, j \in \mathbb{N}$, let $H_i(\mathbb{R}^j) = \mathrm{Hom}_{\mathbb{E}\text{-alg}}(\mathbb{R}^i, \mathbb{R}^j)$, and let $p_{i,j} \colon H_i \amalg H_j \rightarrow H_{i+j}$ be as above (see (5.0.1)).

Let $s\mathbb{R}$ denote the localization of the injective model category $\mathbf{sSets}^{\mathbb{E}^{\text{alg}}}$ at the set $\{p_{i,j} | i, j \in \mathbb{N}\}$. We call $s\mathbb{R}$ the category of *(lax) simplicial* $\mathbb{R}$*-algebras*.

The functor $\mathbb{E}^{\text{alg}} \rightarrow \mathbb{E}$ induces a functor $U \colon s\mathbf{C}^\infty \rightarrow s\mathbb{R}$, which we refer to as *the underlying* $\mathbb{R}$*-algebra functor*. It is a right Quillen functor, and as such preserves fibrant objects. Similarly, the functor $s\mathbf{C}^\infty \rightarrow \mathbf{sSets}$ given by $F \mapsto F(\mathbb{R})$ is a right Quillen functor, which we refer to as *the underlying simplicial set functor*. Both of these right Quillen functors preserve *cofibrations* as well, because cofibrations in all three model categories are monomorphisms.

COROLLARY 5.8
The functor $U \colon s\mathbf{C}^\infty \rightarrow s\mathbb{R}$ preserves and reflects weak equivalences between fibrant objects.

Proof
Since U is a right Quillen functor, it preserves fibrant objects. The result now follows from Proposition 5.3 and the corresponding fact about $s\mathbb{R}$. □

We do not need the following proposition, but we include it for the reader's convenience and edification. For example, it may help orient the reader to Section 7 on cotangent complexes.

PROPOSITION 5.9
The model category $s\mathbb{R}$ of simplicial $\mathbb{R}$-algebras is Quillen equivalent to the model category of connective commutative differential graded $\mathbb{R}$-algebras.

Proof
In [26, Theorem 1.1(3)], Schwede and Shipley prove that the model category of connective commutative differential grade $\mathbb{R}$-algebras is Quillen equivalent to the model category of strict simplicial commutative $\mathbb{R}$-algebras. This is in turn Quillen equivalent to the category of lax simplicial commutative $\mathbb{R}$-algebras by [19, Corollary 13.3]. □

Definition 5.10
A C^∞-ring F is called *local* if its underlying discrete $\mathbb{R}$-algebra $\pi_0(U(F))$ is local in the usual sense, and a morphism $\phi\colon F \to G$ of local C^∞-rings is called *local* if its underlying morphism of discrete $\mathbb{R}$-algebras $\pi_0(U(\phi))$ is local in the usual sense.

We give a more intuitive version of the locality condition in Proposition 8.10.

Remark 5.11
We regret the overuse of the word local; when we "localize" a model category to add weak equivalences, when we demand a "locality condition" on the stalks of a ringed space, and later when we talk about derived manifolds as having "local models" (as the local models for manifolds are Euclidean spaces), we are using the word local in three different ways. But each is in line with typical usage, so it seems that overloading the word could not be avoided.

6. Local C^∞-ringed spaces and derived manifolds

In this section, we define the category of (lax simplicial) C^∞-ringed spaces, a subcategory called the category of local C^∞-ringed spaces, and a full subcategory of that called the category of derived manifolds. These definitions resemble those of ringed spaces, local ringed spaces, and schemes from algebraic geometry. Ordinary C^∞-ringed spaces and C^∞-schemes have been studied for quite a while (see [17], [9], [22]).

Let X be a topological space, and let $\mathrm{Op}(X)$ denote the category of open inclusions in X. A homotopy sheaf of simplicial sets on X is a functor $F\colon \mathrm{Op}(X)^{\mathrm{op}} \to \mathbf{sSets}$ which is fibrant as an object of $\mathbf{sSets}^{\mathrm{Op}(X)^{\mathrm{op}}}$ and which satisfies a *homotopy descent condition.* Roughly, the descent condition says that given open sets U and V, a section of F over each one, and a *choice of homotopy* between the restricted sections on $U \cap V$, there is a homotopically unique section over $U \cup V$ which restricts to the given sections. Jardine showed in [14] that there is a model category structure on $\mathbf{sSets}^{\mathrm{Op}(X)^{\mathrm{op}}}$ in which the fibrant objects are homotopy sheaves on X; we denote this model category $\mathbf{Shv}(X, \mathbf{sSets})$.

In [10], the authors show that $\mathbf{Shv}(X, \mathbf{sSets})$ is a localization of the injective model structure on $\mathbf{sSets}^{\mathrm{Op}(X)^{\mathrm{op}}}$ at a certain set of morphisms (called the hypercovers) to obtain $\mathbf{Shv}(X, \mathbf{sSets})$. From this, one deduces that $\mathbf{Shv}(X, \mathbf{sSets})$ is a left proper, cofibrantly generated simplicial model category in which all objects are cofibrant. A weak equivalence between fibrant objects of $\mathbf{Shv}(X, \mathbf{sSets})$ is a morphism which restricts to a weak equivalence on every open subset of X.

We wish to find a suitable category of homotopy sheaves of C^∞-rings. This is obtained as a localization of the injective model structure on $\mathbf{sSets}^{\mathrm{Op}(X)^{\mathrm{op}} \times \mathbb{E}}$.

PROPOSITION 6.1

Let A and B be categories, let P be a set of morphisms in $\mathbf{sSets}^A$, and let Q be a set of morphisms in $\mathbf{sSets}^B$. There exists a localization of $\mathbf{sSets}^{A\times B}$, called the factor-wise localization *and denoted $\mathcal{M}$, in which an object $F \in \mathcal{M}$ is fibrant if and only if*

(1) *F is fibrant as an object of $\mathbf{sSets}^{A\times B}$;*

(2) *for each $a \in A$, the induced object $F(a,-) \in \mathbf{sSets}^B$ is Q-local; and*

(3) *for each $b \in B$, the induced object $F(-,b) \in \mathbf{sSets}^A$ is P-local.*

Proof

The projection $A \times B \to A$ induces a Quillen pair

$$\mathbf{sSets}^A \underset{R_A}{\overset{L_A}{\rightleftarrows}} \mathbf{sSets}^{A\times B}$$

(see Convention 4.1). Similarly, there is a Quillen pair (L_B, R_B). The union of the image of P under L_A and the image of Q under L_B is a set of morphisms in $\mathbf{sSets}^{A\times B}$, which we denote $R = L_A(P) \amalg L_B(Q)$. Let $\mathcal{M}$ denote the localization of $\mathbf{sSets}^{A\times B}$ at R. The result follows from [11, Proposition 3.1.12]. □

Definition 6.2

Let X be a topological space. Let Q be the set of hypercovers in $\mathbf{sSets}^{\mathrm{Op}(X)^{\mathrm{op}}}$ and let $P = \{p_{i,j} | i, j \in \mathbb{N}\}$ denote the set of maps from Definition 5.1. We denote by $\mathbf{Shv}(X, s\mathbf{C}^\infty)$ the factor-wise localization of $\mathbf{sSets}^{\mathrm{Op}(X)^{\mathrm{op}}\times\mathbb{E}}$ with respect to Q and P, and refer to it as *the model category of sheaves of C^∞-rings on X.*

We refer to $F \in \mathbf{Shv}(X, s\mathbf{C}^\infty)$ as *a sheaf of C^∞-rings on X* if F is fibrant; otherwise, we simply refer to it as an object of $\mathbf{Shv}(X, s\mathbf{C}^\infty)$.

Similarly, one defines the model category of sheaves of simplicial $\mathbb{R}$-algebras, denoted $\mathbf{Shv}(X, s\mathbb{R})$, as the factor-wise localization of $\mathbf{sSets}^{\mathrm{Op}(X)^{\mathrm{op}}\times\mathbb{E}^{\mathrm{alg}}}$ with respect to the same sets, Q and P.

Given a morphism of topological spaces $f\colon X \to Y$, one has pushforward and pullback functors $f_*\colon \mathbf{Shv}(X, s\mathbf{C}^\infty) \to \mathbf{Shv}(Y, s\mathbf{C}^\infty)$ and $f^{-1}\colon \mathbf{Shv}(Y, s\mathbf{C}^\infty) \to \mathbf{Shv}(X, s\mathbf{C}^\infty)$ as usual. The functor f^{-1} is left adjoint to f_*, and this adjunction is a Quillen adjunction. Recall that when we speak of sheaves on X, we always mean fibrant objects in $\mathbf{Shv}(X)$. To that end, we write $f^*\colon \mathbf{Shv}(Y, s\mathbf{C}^\infty) \to \mathbf{Shv}(X, s\mathbf{C}^\infty)$ to denote the composition of f^{-1} with the fibrant replacement functor.

Definition 6.3

Let $X \in \mathbf{CG}$ be a topological space. An object $F \in \mathbf{Shv}(X, s\mathbf{C}^\infty)$ is called *a sheaf of local C^∞-rings on X* if

(1) F is fibrant, and
(2) for every point $p\colon \{*\} \to X$, the stalk p^*F is a local C^∞-ring.
In this case, the pair (X, F) is called a *local C^∞-ringed space*.

Let F and G be sheaves of local C^∞-rings on X. A morphism $a\colon F \to G$ is called *a local morphism* if, for every point $p\colon \{*\} \to X$, the induced morphism on stalks $p^*(a)\colon p^*(F) \to p^*(G)$ is a local morphism (see Definition 5.10). We denote by $\mathrm{Map}_{\mathrm{loc}}(F, G)$ the simplicial subset of $\mathrm{Map}(F, G)$ spanned by the vertices $a \in \mathrm{Map}(F, G)_0$ which represent local morphisms $a\colon F \to G$.

Let $(X, \mathcal{O}_X)$ and $(Y, \mathcal{O}_Y)$ denote local C^∞-ringed spaces. A *morphism of local C^∞-ringed spaces*

$$(f, f^\sharp)\colon (X, \mathcal{O}_X) \to (Y, \mathcal{O}_Y)$$

consists of a map of topological spaces $f\colon X \to Y$ and a morphism $f^\sharp\colon f^*\mathcal{O}_Y \to \mathcal{O}_X$, such that $f^\sharp$ is a local morphism of sheaves of C^∞-rings on X.

More generally, we define a simplicial category **LRS** whose objects are the local C^∞-ringed spaces and whose mapping spaces have as vertices the morphisms of local C^∞-ringed spaces. Precisely, we define for local C^∞-ringed spaces $(X, \mathcal{O}_X)$ and $(Y, \mathcal{O}_Y)$ the mapping space

$$\mathrm{Map}_{\mathbf{LRS}}\big((X, \mathcal{O}_X), (Y, \mathcal{O}_Y)\big) := \coprod_{f \in \mathrm{Hom}_{\mathbf{CG}}(X,Y)} \mathrm{Map}_{\mathrm{loc}}(f^*\mathcal{O}_Y, \mathcal{O}_X).$$

Example 6.4
Given a local C^∞-ringed space $(X, \mathcal{O}_X)$, any subspace is also a local C^∞-ringed space. For example, a manifold with boundary is a local C^∞-ringed space. We do not define derived manifolds with boundary in this article, but we could do so inside the category of local C^∞-ringed spaces. In fact, that was done in the author's dissertation (see [27]).

If $i\colon U \subset X$ is the inclusion of an open subset, then we let $\mathcal{O}_U = i^*\mathcal{O}_X$ be the restricted sheaf, and we refer to the local C^∞-ringed space $(U, \mathcal{O}_U)$ as the *open subobject of $\mathcal{X}$ over $U \subset X$*.

Definition 6.5
A map $(f, f^\sharp)\colon (X, \mathcal{O}_X) \to (Y, \mathcal{O}_Y)$ is an *equivalence of local C^∞-ringed spaces* if $f\colon X \to Y$ is a homeomorphism of topological spaces and if $f^\sharp\colon f^*\mathcal{O}_Y \to \mathcal{O}_X$ is a weak equivalence in $\mathbf{Shv}(X, s\mathbf{C}^\infty)$.

Remark 6.6
The relation which we called equivalence of local C^∞-ringed spaces in Definition 6.5 is clearly reflexive and transitive. Since the sheaves $\mathcal{O}_X$ and $\mathcal{O}_Y$ are assumed cofibrant-fibrant, it is symmetric as well (see, e.g., [11, Theorem 7.5.10]).

Remark 6.7

Note that the notion of equivalence in Definition 6.5 is quite strong. If $(f, f^\sharp)$ is an equivalence, the underlying map f of spaces is a homeomorphism, and the morphism $\pi_0(f^\sharp)$ is an isomorphism of the underlying sheaves of C^∞-rings. Therefore, equivalence in this sense should not be thought of as a generalization of homotopy equivalence, but instead a generalization of diffeomorphism.

In the next lemma, we need to use the *mapping cylinder* construction for a morphism $f\colon A \to B$ in a model category. To define it, factor the morphism $f \amalg \mathrm{id}_B\colon A \amalg B \to B$ as a cofibration followed by an acyclic fibration,

$$A \amalg B \rightarrowtail \mathrm{Cyl}(f) \overset{\simeq}{\twoheadrightarrow} B;$$

the intermediate object $\mathrm{Cyl}(f)$ is called the mapping cylinder. Note that if f is a weak equivalence, then so are the induced cofibrations $A \to \mathbf{Cyl}(f)$ and $B \to \mathbf{Cyl}(f)$; consequently, A and B are strong deformation retracts of $\mathbf{Cyl}(f)$.

LEMMA 6.8

Suppose that $\mathcal{X} = (X, \mathcal{O}_X)$ and $\mathcal{Y} = (Y, \mathcal{O}_Y)$ are local C^∞-ringed spaces and that $\mathcal{U} = (U, \mathcal{O}_U)$ and $\mathcal{U}' = (U', \mathcal{O}_{U'})$ are open subobjects of $\mathcal{X}$ and $\mathcal{Y}$, respectively, and suppose that $\mathcal{U}$ and $\mathcal{U}'$ are equivalent. Then in particular there is a homeomorphism $U \cong U'$.

If the union $X \amalg_U Y$ of underlying topological spaces is Hausdorff, then there is a local C^∞-ringed space denoted $\mathcal{X} \cup \mathcal{Y}$ with underlying space $X \amalg_U Y$ and structure sheaf $\mathcal{O}$, such that the diagram

$$\begin{array}{ccc} \mathcal{U} & \longrightarrow & \mathcal{Y} \\ \downarrow & \searrow^{k} & \downarrow{\scriptstyle j} \\ \mathcal{X} & \underset{i}{\longrightarrow} & \mathcal{X} \cup \mathcal{Y} \end{array} \tag{6.8.1}$$

commutes (up to homotopy), and such that the natural maps $i^*\mathcal{O} \to \mathcal{O}_X$, $j^*\mathcal{O} \to \mathcal{O}_Y$, and $k^*\mathcal{O} \to \mathcal{O}_U$ are weak equivalences.

Proof

We define the structure sheaf $\mathcal{O}$ as follows. First, let $V = U$, let $\mathcal{O}_V$ denote the mapping cylinder for the equivalence $\mathcal{O}_U \to \mathcal{O}_{U'}$, and let $\mathcal{V} = (V, \mathcal{O}_V)$. Then the natural maps $\mathcal{U}, \mathcal{U}' \to \mathcal{V}$ are equivalences, and we have natural maps $\mathcal{V} \to \mathcal{X}$ and $\mathcal{V} \to \mathcal{Y}$ which are equivalent to the original subobjects $\mathcal{U} \to \mathcal{X}$ and $\mathcal{U}' \to \mathcal{Y}$.

Redefine k to be the natural map $k\colon V \to X \amalg_V Y$. We take $\mathcal{O}$ to be the homotopy limit in $\mathbf{Shv}(X \amalg_V Y, s\mathbf{C}^\infty)$, given by the diagram

$$\begin{array}{ccc} \mathcal{O} & \longrightarrow & j_*\mathcal{O}_Y \\ \downarrow & \diamond & \downarrow \\ i_*\mathcal{O}_X & \longrightarrow & k_*\mathcal{O}_V \end{array}$$

On the open set $X \subset X \cup Y$, the structure sheaf $\mathcal{O}$ restricts to $\mathcal{O}_X(X) \times_{\mathcal{O}_V(V)} \mathcal{O}_Y(V)$. Since $\mathcal{V}$ is an open subobject of $\mathcal{Y}$, we in particular have an equivalence $\mathcal{O}_Y(V) \simeq \mathcal{O}_V(V)$. This implies that $\mathcal{O}(X) \to \mathcal{O}_X(X)$ is a weak equivalence. The same holds for any open subset of X, so we have a weak equivalence $i^*\mathcal{O} \cong \mathcal{O}_X$. Symmetric reasoning implies that $j^*\mathcal{O} \to \mathcal{O}_Y$ is also a weak equivalence.

The property of being a local sheaf is local on $X \amalg_V Y$, and the open sets in X and in Y together form a basis for the topology on $X \amalg_V Y$. Thus, $\mathcal{O}$ is a local sheaf on $X \amalg_V Y$. □

PROPOSITION 6.9

Let $\mathcal{X}$, $\mathcal{Y}$, $\mathcal{U}$, and $\mathcal{U}'$ be as in Lemma 6.8. Then Diagram 6.8.1 is a homotopy colimit diagram.

Proof

Let $\mathcal{Z} \in \mathbf{LRS}$ denote a local C^∞-ringed space, and let $\mathcal{V} \simeq \mathcal{U} \simeq \mathcal{U}'$, $\mathcal{X} \cup \mathcal{Y} = (X \amalg_V Y, \mathcal{O})$ and $k\colon V \to X \amalg_V Y$ be as in the proof of Lemma 6.8. We must show that the natural map

$$\mathrm{Map}(\mathcal{X} \cup \mathcal{Y}, \mathcal{Z}) \to \mathrm{Map}(\mathcal{X}, \mathcal{Z}) \times_{\mathrm{Map}(\mathcal{V},\mathcal{Z})} \mathrm{Map}(\mathcal{Y}, \mathcal{Z}) \qquad (6.9.1)$$

is a weak equivalence.

Recall that for local C^∞-ringed spaces $\mathcal{A}$, $\mathcal{B}$, the mapping space is defined as

$$\mathrm{Map}_{\mathbf{LRS}}(\mathcal{A}, \mathcal{B}) = \coprod_{f\colon A \to B} \mathrm{Map}_{\mathrm{loc}}(f^*\mathcal{O}_B, \mathcal{O}_A);$$

that is, it is a disjoint union of spaces, indexed by maps of underlying topological spaces $A \to B$. Since $X \amalg_V Y$ is the colimit of $X \leftarrow V \to Y$ in $\mathbf{CG}$, the indexing set of $\mathrm{Map}_{\mathbf{LRS}}(\mathcal{X} \cup \mathcal{Y}, \mathcal{Z})$ is the fiber product of the indexing sets for the mapping spaces on the right-hand side of equation 6.9.1. Thus, we may apply Lemma 8.12 to reduce to the case of a single index $f\colon X \amalg_V Y \to Z$. That is, we must show that the natural map

$$\mathrm{Map}_{\mathrm{loc}}(f^*\mathcal{O}_Z, \mathcal{O}) \to \mathrm{Map}_{\mathrm{loc}}(i^*f^*\mathcal{O}_Z, \mathcal{O}_X) \times_{\mathrm{Map}_{\mathrm{loc}}(k^*f^*\mathcal{O}_Z, \mathcal{O}_V)} \mathrm{Map}_{\mathrm{loc}}(j^*f^*\mathcal{O}_Z, \mathcal{O}_Y), \tag{6.9.2}$$

indexed by f, is a weak equivalence. By definition of $\mathcal{O}$, we have the first weak equivalence in the following display

$$\begin{aligned} \mathrm{Map}_{\mathrm{loc}}(f^*\mathcal{O}_Z, \mathcal{O}) &\simeq \mathrm{Map}_{\mathrm{loc}}(f^*\mathcal{O}_Z, i_*\mathcal{O}_X \times_{k_*\mathcal{O}_V} j_*\mathcal{O}_Y) \\ &\simeq \mathrm{Map}_{\mathrm{loc}}(f^*\mathcal{O}_Z, i_*\mathcal{O}_X) \times_{\mathrm{Map}_{\mathrm{loc}}(f^*\mathcal{O}_Z, k_*\mathcal{O}_V)} \mathrm{Map}_{\mathrm{loc}}(f^*\mathcal{O}_Z, j_*\mathcal{O}_Y), \end{aligned} \tag{6.8.3}$$

and the second weak equivalence follows from the fact that the property of a map being local is itself a local property.

Note that i, j, and k are open inclusions. We have reduced to the following: given an open inclusion of topological spaces $\ell\colon A \subset B$ and given sheaves $\mathcal{F}$ on A and $\mathcal{G}$ on B, we have a solid arrow diagram

$$\begin{array}{ccc} \mathrm{Map}_{\mathrm{loc}}(\mathcal{F}, \ell_*\mathcal{G}) & \dashrightarrow & \mathrm{Map}_{\mathrm{loc}}(\ell^*\mathcal{F}, \mathcal{G}) \\ \downarrow & & \downarrow \\ \mathrm{Map}(\mathcal{F}, \ell_*\mathcal{G}) & \xrightarrow[\cong]{} & \mathrm{Map}(\ell^*\mathcal{F}, \mathcal{G}) \end{array}$$

coming from the adjointness of ℓ^* and ℓ_*. We must show that the dotted map exists and is an isomorphism.

Recall that a simplex is in $\mathrm{Map}_{\mathrm{loc}}$ if all of its vertices are local morphisms. So it suffices to show that local morphisms $\mathcal{F} \to \ell_*\mathcal{G}$ are in bijective correspondence with local morphisms $\ell^*\mathcal{F} \to \mathcal{G}$. This is easily checked by taking stalks: one simply uses the fact that for any point $a \in A$, a basis of open neighborhoods of a are sent under ℓ to a basis of open neighborhoods of $\ell(a)$, and vice versa.

Now we can see that the right-hand sides of equations 6.9.2 and 6.8.3 are equivalent, which shows that equation 6.9.1 is indeed a weak equivalence, as desired. □

Definition 6.10
Let $\mathcal{X}$, $\mathcal{Y}$, $\mathcal{U}$, and $\mathcal{U}'$ be as in Lemma 6.8. We refer to $\mathcal{X} \cup \mathcal{Y}$ as the *union* of $\mathcal{X}$ and $\mathcal{Y}$ along the equivalent local C^∞-ringed spaces $\mathcal{U} \simeq \mathcal{U}'$.

PROPOSITION 6.11
There is a fully faithful functor $\mathbf{i}\colon \mathbf{Man} \to \mathbf{LRS}$.

Proof
Given a manifold M, let $\mathbf{i}(M)$ denote the local C^∞-ringed space (M, C^∞_M) whose underlying space is that of M, and such that for any open set $U \subset M$, the discrete C^∞-ring $C^\infty_M(U)$ is the functor $\mathbb{E} \to \mathbf{Sets}$ given by

$$C^\infty_M(U)(\mathbb{R}^n) = \mathrm{Hom}_{\mathbf{Man}}(U, \mathbb{R}^n).$$

It is easy to check that C^∞_M is a fibrant object in $\mathbf{Shv}(M, s\mathbf{C}^\infty)$. It satisfies the locality condition in the sense of Definition 6.3 because, for every point p in M, the smooth real-valued functions which are defined on a neighborhood of p, but are not invertible in any neighborhood of p, are exactly those functions that vanish at p.

We need to show that $\mathbf{i}$ takes morphisms of manifolds to morphisms of local C^∞-ringed spaces. If $f: M \to N$ is a smooth map, if $p \in M$ is a point, and if one has a smooth map $g: N \to \mathbb{R}^m$ such that g has a root in every sufficiently small open neighborhood of $f(p)$, then $g \circ f$ has a root on every sufficiently small neighborhood of p.

It only remains to show that $\mathbf{i}$ is fully faithful. It is clear by definition that $\mathbf{i}$ is injective on morphisms, so we must show that any local morphism $\mathbf{i}(M) \to \mathbf{i}(N)$ in $\mathbf{LRS}$ comes from a morphism of manifolds. Suppose that $(f, f^\sharp): (M, C^\infty_M) \to (N, C^\infty_N)$ is a local morphism; we must show that $f: M \to N$ is smooth. This does not even use the locality condition: for every chart $V \subset N$ with $c: V \cong \mathbb{R}^n$, a smooth map $f^\sharp(c): f^{-1}(V) \to \mathbb{R}^n$ is determined, and these are compatible with open inclusions. □

Notation 6.12
Suppose that $(X, \mathcal{O}_X)$ is a C^∞-ringed space. Recall that $\mathcal{O}_X$ is a fibrant sheaf of (simplicial) C^∞-rings on X, so that for any open subset $U \subset X$, the object $\mathcal{O}_X(U)$ is a (weakly) product preserving functor from $\mathbb{E}$ to $\mathbf{sSets}$. The value which "matters most" is

$$|\mathcal{O}_X(U)| := \mathcal{O}_X(U)(\mathbb{R}),$$

because every other value $\mathcal{O}_X(U)(\mathbb{R}^n)$ is weakly equivalent to an n-fold product of it.

We similarly denote $|F| := F(\mathbb{R})$ for a C^∞-ring F. Lemmas 5.3 and 5.5 further demonstrate the importance of $|F|$.

The theorem below states that for local C^∞-ringed spaces $\mathcal{X} = (X, \mathcal{O}_X)$, the sheaf $\mathcal{O}_X$ holds the answer to the question, what are the real-valued functions on $\mathcal{X}$?

THEOREM 6.13
Let $(X, \mathcal{O}_X)$ be a local C^∞-ringed space, and let $(\mathbb{R}, C^\infty_{\mathbb{R}})$ denote (the image under $\mathbf{i}$ *of) the manifold* $\mathbb{R} \in \mathbf{Man}$. *There is a homotopy equivalence*

$$\mathrm{Map}_{\mathbf{LRS}}\big((X, \mathcal{O}_X), (\mathbb{R}, C^\infty_{\mathbb{R}})\big) \xrightarrow{\simeq} |\mathcal{O}_X(X)|.$$

We prove Theorem 6.13 in Section 8 as Theorem 8.11.

As in algebraic geometry, there is a *prime spectrum* functor taking C^∞-rings to local C^∞-ringed spaces. We do not use this construction in any essential way, so we present it as a remark without proof.

Remark 6.14
The global sections functor $\Gamma\colon \mathbf{LRS} \to s\mathbf{C}^\infty$, given by $\Gamma(X, \mathcal{O}_X) := \mathcal{O}_X(X)$, has a right adjoint, denoted Spec (see [9]). Given a C^∞-ring R, let us briefly explain $\mathrm{Spec}\, R = (\mathrm{Spec}\, R, \mathcal{O})$. The points of the underlying space of Spec R are the maximal ideals in $\pi_0 R$, and a closed set in the topology on Spec R is a set of points on which some element of $\pi_0 R$ vanishes. The sheaf $\mathcal{O}$ assigns to an open set $U \subset \mathrm{Spec}\, R$ the C^∞-ring $R[\chi_U^{-1}]$, where χ_U is any element of R that vanishes on the complement of U.

The unit of the (Γ, Spec) adjunction is a natural transformation, $\eta_{\mathcal{X}}\colon (X, \mathcal{O}_X) \to \mathrm{Spec}\, \mathcal{O}_X(X)$. If $\mathcal{X}$ is a manifold then $\eta_{\mathcal{X}}$ is an equivalence of local C^∞-ringed spaces (see [22, Theorem 1.2.8]).

The category of local C^∞-ringed spaces contains a full subcategory, called the category of derived smooth manifolds, which we now define. Basically, it is the full subcategory of **LRS** spanned by the local C^∞-ringed spaces that can be covered by affine derived manifolds, where an affine derived manifold is the vanishing set of a smooth function on affine space (see also Axiom (5) of Definition 2.1).

Definition 6.15
An *affine derived manifold* is a pair $\mathcal{X} = (X, \mathcal{O}_X) \in \mathbf{LRS}$, where $\mathcal{O}_X \in s\mathbf{C}^\infty$ is fibrant, and which can be obtained as the homotopy limit in a diagram of the form

$$\begin{array}{ccc} (X, \mathcal{O}_X) & \longrightarrow & \mathbb{R}^0 \\ {\scriptstyle g}\downarrow & \diamond & \downarrow{\scriptstyle 0} \\ \mathbb{R}^n & \xrightarrow[f]{} & \mathbb{R}^k \end{array}$$

We sometimes refer to an affine derived manifold as *a local model for derived manifolds*. We refer to the map $g\colon \mathcal{X} \to \mathbb{R}^n$ as the *canonical inclusion of the zero set*.

A *derived smooth manifold* (or *derived manifold*) is a local C^∞-ringed space $(X, \mathcal{O}_X) \in \mathbf{LRS}$, where $\mathcal{O}_X \in s\mathbf{C}^\infty$ is fibrant, and for which there exists an open covering $\bigcup_i U_i = X$ such that each $(U_i, \mathcal{O}_X|_{U_i})$ is an affine derived manifold.

We denote by **dMan** the full subcategory of **LRS** spanned by the derived smooth manifolds, and refer to it as the (simplicial) category of derived manifolds. A morphism of derived manifolds is called an *equivalence of derived manifolds* if it is an equivalence of local C^∞-ringed spaces.

Manifolds canonically have the structure of derived manifolds; more precisely, we have the following lemma.

LEMMA 6.16
The functor $\mathbf{i}\colon \mathbf{Man} \to \mathbf{LRS}$ *factors through* **dMan**.

Proof
Euclidean space $(\mathbb{R}^n, C^\infty_{\mathbb{R}^n})$ is a principle derived manifold. □

An imbedding of derived manifolds is a map $g\colon \mathcal{X} \to \mathcal{Y}$ that is locally a zero set. Precisely, the definition is given by Definition 2.3, with the simplicial category **dM** replaced by **dMan**. Note that if g is an imbedding, then the morphism $g^\sharp\colon g^*\mathcal{O}_Y \to \mathcal{O}_X$ is surjective on π_0, because it is a pushout of such a morphism.

7. Cotangent complexes

The idea behind cotangent complexes is as follows. Given a manifold M, the cotangent bundle $T^*(M)$ is a vector bundle on M, which is dual to the tangent bundle. A smooth map $f\colon M \to N$ induces a map on vector bundles

$$T^*(f)\colon T^*(N) \to T^*(M).$$

Its cokernel, coker $(T^*(f))$, is not necessarily a vector bundle, and is thus generally not discussed in the basic theory of smooth manifolds. However, it is a sheaf of modules on M and does have geometric meaning: its sections measure tangent vectors in the fibers of the map f. Interestingly, if $f\colon M \to N$ is an embedding, then $T^*(f)$ is a surjection and the cokernel is zero. In this case, its *kernel*, $\ker(T^*(f))$, has meaning instead—it is dual to the normal bundle to the imbedding f.

In general, the cotangent complex of $f\colon M \to N$ is a sheaf on M which encodes all of the above information (and more) about f. It is in some sense the "universal linearization" of f.

The construction of the cotangent complex L_f associated to any morphism $f\colon A \to B$ of commutative rings was worked on by several people, including André [1], Quillen [23], Lichtenbaum and Schlessinger [18], and Illusie [13]. Later, Schwede

[25] introduced cotangent complexes in more generality by introducing spectra in model categories, and by showing that L_f emerges in a canonical way when one studies the stabilization of the model category of A-algebras over B.

The cotangent complex for a morphism of C^∞-ringed spaces can be obtained using the process given in Schwede's article [25]. Let us call this the C^∞-cotangent complex. It has all of the usual formal properties of cotangent complexes (analogous to those given in Theorem 7.1). Unfortunately, to adequately present this notion requires a good bit of setup, namely the construction of spectra in the model category of sheaves of lax simplicial C^∞-rings, following [25]. Moreover, the C^∞-cotangent complex is unfamiliar and quite technical, and it is difficult to compute with.

Underlying a morphism of C^∞-ringed spaces is a morphism of ringed spaces. It too has an associated cotangent complex, which we call the ring-theoretic cotangent complex. Clearly, the C^∞-cotangent complex is more canonical than the ring-theoretic cotangent complex in the setting of C^∞-ringed spaces. However, for the reasons given in the above paragraph, we opt to use the ring-theoretic version instead. This version does not have quite as many useful formal properties as does the C^∞-version (Property (4) is weakened), but it has all the properties we need.

In [13], the cotangent complex for a morphism $A \to B$ of simplicial commutative rings is a simplicial B-module. The model category of simplicial modules over B is equivalent to the model category of nonnegatively graded chain complexes over B, by the Dold-Kan correspondence. We use the latter approach for simplicity.

Let us begin with the properties that we use about ring-theoretic cotangent complexes.

THEOREM 7.1

Let X be a topological space. Given a morphism $f\colon R \to S$ of sheaves of simplicial commutative rings on X, there exists a complex of sheaves of S-modules, denoted L_f or $L_{S/R}$, called the cotangent complex associated to f*, with the following properties.*

(1) *Let $\Omega^1_f = H_0 L_f$ be the zeroth homology group. Then, as a sheaf on X, one has that Ω^1_f is the usual S-module of Kähler differentials of S over R.*

(2) *The cotangent complex is functorially related to f in the sense that a morphism of arrows $i\colon f \to f'$; that is, a commutative diagram of sheaves in $s\mathbb{R}$,*

$$\begin{array}{ccc} R & \longrightarrow & R' \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle f'} \\ S & \longrightarrow & S' \end{array}$$

induces a morphism of S-modules

$$L_i\colon L_f \to L_{f'}.$$

(3) *If $i\colon f \to f'$ is a weak equivalence (i.e., if both the top and bottom maps in the above square are weak equivalences of simplicial commutative rings), then $L_i\colon L_f \to L_{f'}$ is a weak equivalence of S-modules, and its adjoint $L_f \otimes_S S' \to L_{f'}$ is a weak equivalence of S'-modules.*

(4) *If the diagram from Property (2) is a homotopy pushout of sheaves of simplicial commutative rings, then the induced morphsim $L_i\colon L_f \to L_{f'}$ is a weak equivalence of modules.*

(5) *To a pair of composable arrows $R \xrightarrow{f} S \xrightarrow{g} T$, one can functorially assign an exact triangle in the homotopy category of T-modules,*

$$L_{S/R} \otimes_S T \to L_{T/R} \to L_{T/S} \to L_{S/R} \otimes_S T[-1].$$

Proof

The above five properties are proved in [13, Chapter 2] as Proposition 1.2.4.2, Statement 1.2.3, Proposition 1.2.6.2, Proposition 2.2.1, and Proposition 2.1.2, respectively. □

All of the above properties of the cotangent complex for morphisms of sheaves have contravariant corollaries for morphisms of ringed spaces, and almost all of them have contravariant corollaries for maps of C^∞-ringed spaces. For example, a pair of composable arrows of C^∞-ringed spaces $\mathcal{X} \xleftarrow{f} \mathcal{Y} \xleftarrow{g} \mathcal{Z}$ induces an exact triangle in the homotopy category of $\mathcal{O}_Z$-modules,

$$g^*L_{\mathcal{Y}/\mathcal{X}} \to L_{\mathcal{Z}/\mathcal{X}} \to L_{\mathcal{Z}/\mathcal{Y}} \to g^*L_{\mathcal{Y}/\mathcal{X}}[-1].$$

It is in this sense that Properties (1), (2), (3), and (5) have contravariant corollaries.

The exception is Property (4); one asks, does a homotopy pullback in the category of C^∞-ringed spaces induce a weak equivalence of cotangent complexes? In general, the answer is *no* because the ringed space underlying a homotopy pullback of C^∞-ringed spaces is not the homotopy pullback of the underlying ringed spaces. In other words, the *underlying simplicial* $\mathbb{R}$*-algebra* functor $U\colon s\mathbf{C}^\infty \to s\mathbb{R}$ does not preserve homotopy colimits. However, there are certain types of homotopy colimits that U does preserve. In particular, if $\mathcal{X} \to \mathcal{Z} \leftarrow \mathcal{Y}$ is a diagram of C^∞-ringed spaces in which one of the two maps is an *imbedding* of derived manifolds, then taking underlying rings *does commute* with taking the homotopy colimit (this follows from Corollary 8.6), and the cotangent complexes satisfy Property (4) of Theorem 7.1. This is the only case in which cotangent complexes are necessary for our article. Thus, we restrict our attention to the ring-theoretic version rather than the C^∞-version of cotangent complexes.

Here is our analogue of Theorem 7.1.

COROLLARY 7.2
Given a local C^∞-ringed space $\mathcal{X} = (X, \mathcal{O}_X)$, let $U(\mathcal{O}_X)$ denote the underlying sheaf of simplicial commutative rings on X. Given a morphism of local C^∞-ringed spaces $f : \mathcal{X} \to \mathcal{Y}$, there exists a complex of sheaves of $U(\mathcal{O}_X)$-modules on X, denoted L_f or $L_{\mathcal{X}/\mathcal{Y}}$, called the ring-theoretic cotangent complex associated to f *(or just the* cotangent complex for f*), with the following properties.*

(1) *Let $\Omega^1_f = H_0 L_f$ be the zeroth homology group. Then Ω^1_f is the usual $\mathcal{O}_X$-module of Kähler differentials of $\mathcal{X}$ over $\mathcal{Y}$.*

(2) *The cotangent complex is contravariantly related to f in the sense that a morphism of arrows $i = (i_0, i_1) : f \to f'$; that is, a commutative diagram in* **LRS***,*

$$\begin{array}{ccc} \mathcal{Y} & \xrightarrow{i_1} & \mathcal{Y}' \\ {\scriptstyle f}\uparrow & & \uparrow{\scriptstyle f'} \\ \mathcal{X} & \xrightarrow[i_0]{} & \mathcal{X}' \end{array}$$

induces a morphism of $U(\mathcal{O}_X)$-modules

$$L_i : i_0^* L_{f'} \to L_f.$$

(3) *If $i : f \to f'$ is an equivalence (i.e., it induces equivalences on sheaves), then $L_i : i_0^* L_{f'} \to L_f$ is a weak equivalence of $U(\mathcal{O}_X)$-modules.*

(4) *If the diagram from Property (2) is a homotopy pullback in* **LRS** *and if either f' or i_1 is an immersion, then the induced morphism $L_i : L_{f'} \to L_f$ is a weak equivalence of $U(\mathcal{O}_X)$-modules.*

(5) *To a pair of composable arrows $\mathcal{X} \xrightarrow{f} \mathcal{Y} \xrightarrow{g} \mathcal{Z}$, one can functorially assign an exact triangle in the homotopy category of $U(\mathcal{O}_X)$-modules,*

$$f^* L_{\mathcal{Y}/\mathcal{Z}} \to L_{\mathcal{X}/\mathcal{Z}} \to L_{\mathcal{X}/\mathcal{Y}} \to f^* L_{\mathcal{Y}/\mathcal{Z}}[-1].$$

Proof
The map $f : \mathcal{X} \to \mathcal{Y}$ induces $U(f^\sharp) : U(f^* \mathcal{O}_Y) \to U(\mathcal{O}_X)$, and we set $L_f := L_{U(f^\sharp)}$, which is a sheaf of $U(\mathcal{O}_X)$-modules.

Let $h\colon X \to Y$ be a morphism of topological spaces, and let $\mathcal{F} \to \mathcal{G}$ be a map of sheaves of simplicial commutative rings on Y. By [13, Equation II.1.2.3.4], since inductive limits commute with the cotangent complex functor for simplicial commutative rings, one has an isomorphism

$$h^*(L_{\mathcal{G}/\mathcal{F}}) \to L_{h^*\mathcal{G}/h^*\mathcal{F}}.$$

Properties (1), (2), and (5) now follow from Theorem 7.1.

Sheaves on local C^∞-ringed spaces are assumed cofibrant-fibrant, and weak equivalences between fibrant objects are preserved by U (see Corollary 5.8). Property (3) follows from Theorem 7.1.

Finally, if f' or i_1 is an immersion, then taking homotopy pullback commutes with taking underlying ringed spaces, by Corollary 8.6, and so Property (4) also follows from Theorem 7.1. □

If $\mathcal{X}$ is a local C^∞-ringed space, then we write $L_{\mathcal{X}}$ to denote the cotangent complex associated to the unique map $t\colon \mathcal{X} \to \mathbb{R}^0$. It is called the *absolute cotangent complex* associated to $\mathcal{X}$.

COROLLARY 7.3

Let $t\colon \mathbb{R}^n \to \mathbb{R}^0$ be the unique map. Then the cotangent complex L_t is zero-truncated, and its zeroth homology group

$$\Omega^1_t \cong C^\infty_{\mathbb{R}^n}\langle dx_1, \ldots, dx_n\rangle$$

is a free $C^\infty_{\mathbb{R}^n}$ module of rank n.

Let $p\colon \mathbb{R}^0 \to \mathbb{R}^n$ be any point. Then the cotangent complex L_p on $\mathbb{R}^0$ has homology concentrated in degree one, and $H_1(L_p)$ is an n-dimensional real vector space.

Proof

The first statement follows from Property (1) of Theorem 7.1. The second statement follows from the exact triangle, given by Property (5), for the composable arrows $\mathbb{R}^0 \xrightarrow{p} \mathbb{R}^n \xrightarrow{t} \mathbb{R}^0$. □

Let $\mathcal{X} = (X, \mathcal{O}_X)$ be a derived manifold, and let $L_{\mathcal{X}}$ be its cotangent complex. For any point $x \in X$, let $L_{\mathcal{X},x}$ denote the stalk of $L_{\mathcal{X}}$ at x; it is a chain complex over the field $\mathbb{R}$ so its homology groups are vector spaces. Let $e(x)$ denote the alternating sum of the dimensions of these vector spaces. As defined, $e\colon X \to \mathbb{Z}$ is a just function between sets.

COROLLARY 7.4

Let $\mathcal{X} = (X, \mathcal{O}_X)$ be a derived manifold, let $L_{\mathcal{X}}$ be its cotangent complex, and let $e \colon X \to \mathbb{Z}$ be the pointwise Euler characteristic of $L_{\mathcal{X}}$ defined above. Then e is continuous (i.e., locally constant), and for all $i \geq 2$, we have $H_i(L_{\mathcal{X}}) = 0$.

Proof

We can assume that $\mathcal{X}$ is an affine derived manifold, that is, that there is a homotopy limit square of the form

$$\begin{array}{ccc} \mathcal{X} & \xrightarrow{t} & \mathbb{R}^0 \\ {\scriptstyle i}\downarrow & \diamond & \downarrow{\scriptstyle 0} \\ \mathbb{R}^n & \xrightarrow[f]{} & \mathbb{R}^k \end{array}$$

By Property (4), the map $L_i \colon L_t \to L_f$ is a weak equivalence of sheaves. Recall that $L_{\mathcal{X}}$ is shorthand for L_t, so it suffices to show that the Euler characteristic of L_f is constant on $\mathcal{X}$.

The composable pair of morphisms $\mathbb{R}^n \xrightarrow{f} \mathbb{R}^k \to \mathbb{R}^0$ induces an exact triangle

$$f^*L_{\mathbb{R}^k} \to L_{\mathbb{R}^n} \to L_f \to f^*L_{\mathbb{R}^k}[-1].$$

By Corollary 7.3, this reduces to an exact sequence of real vector spaces

$$0 \to H_1(L_f) \to \mathbb{R}^k \to \mathbb{R}^n \to H_0(L_f) \to 0. \qquad (7.4.1)$$

Note also that for all $i \geq 2$, we have $H_i(L_f) = 0$, proving the second assertion. The first assertion follows from the exactness of (7.4.1), because

$$\operatorname{rank}\big(H_0(L_f)\big) - \operatorname{rank}\big(H_1(L_f)\big) = n - k$$

at all points in $\mathcal{X}$. □

Definition 7.5

Let $\mathcal{X} = (X, \mathcal{O}_X)$ be a derived manifold, and let $e \colon X \to \mathbb{Z}$ be the function defined in Corollary 7.4. For any point $x \in X$, the value $e(x) \in \mathbb{Z}$ is called *the virtual dimension*, or just *the dimension*, of $\mathcal{X}$ at x, and denoted $\dim_x \mathcal{X}$.

COROLLARY 7.6

Suppose that $\mathcal{X}$ is a derived manifold, and suppose that M is a smooth manifold. If $i \colon \mathcal{X} \to M$ is an imbedding, then the cotangent complex L_i has homology concentrated in degree one.

The first homology group $H_1(L_i)$ *is a vector bundle on* $\mathcal{X}$*, called* the conormal bundle of i*, and is denoted* $\mathcal{N}_i$ *or* $\mathcal{N}_{\mathcal{X}/M}$*. The rank of* $\mathcal{N}_i$ *at a point* $x \in \mathcal{X}$ *is given by the formula*

$$\operatorname{rank}_x \mathcal{N}_i = \dim_{i(x)} M - \dim_x \mathcal{X}.$$

In case $\mathcal{X} \cong L$ *is a smooth manifold, the bundle* $\mathcal{N}_{L/M}$ *is the dual to the usual normal bundle for the imbedding. In particular, if* $i : L \to E$ *is the zero section of a vector bundle* $E \to L$*, then* $\mathcal{N}_{L/E}$ *is canonically isomorphic to the dual* $E^\vee$ *of* E*.*

Proof

All but the last claim are established locally on $\mathcal{X}$. Imbeddings i of derived manifolds are locally of the form

$$\begin{array}{ccc} \mathcal{X} & \longrightarrow & \mathbb{R}^0 \\ {\scriptstyle i}\downarrow & \diamond & \downarrow{\scriptstyle z} \\ \mathbb{R}^n & \xrightarrow[f]{} & \mathbb{R}^k \end{array}$$

By Property (4), we have a quasi isomorphism $L_i \simeq L_z$. The claim that L_i is locally free and has homology concentrated in degree one now follows from Corollary 7.3. Note that the conormal bundle $\mathcal{N}_i = H_1(L_i)$ has rank k.

The exact triangle for the composable morphisms $\mathcal{X} \xrightarrow{i} \mathbb{R}^n \xrightarrow{f} \mathbb{R}^k$ implies that the Euler characteristic of $L_{\mathcal{X}}$ is $n-k$, and the second assertion follows.

For the final assertion, we use the exact sequence

$$0 \to \mathcal{N}_{L/M} \to i^*\Omega^1_M \to \Omega^1_L \to 0.$$

□

7.1. Other calculations

In this section, we prove some results which are useful later.

LEMMA 7.7

Suppose that given a diagram

$$\begin{array}{ccc} \mathcal{X} & \xrightarrow{g} & \mathcal{X}' \\ {\scriptstyle f}\downarrow & \swarrow{\scriptstyle f'} & \\ \mathcal{Y} & & \end{array}$$

of local smooth-ringed spaces such that g, f, and f' are closed immersions, such that the underlying map of topological spaces $g : X \to X'$ is a homeomorphism, and such that the induced map $g^* L_{f'} \to L_f$ is a quasi isomorphism. Then g is an equivalence.

Proof
It suffices to prove this on stalks; thus we may assume that X and X' are points. Let U_X and $U_{X'}$ denote the local simplicial commutative rings underlying $\mathcal{O}_X$ and $\mathcal{O}_{X'}$, and let $U_g : U_{X'} \to U_X$ denote the map induced by g. By Corollary 5.8, it suffices to show that U_g is a weak equivalence.

Since g is a closed immersion, U_g is surjective; let I be the kernel of U_g. It is proved in [13, Corollary III.1.2.8.1] that the conormal bundle $H_1(L_{U_g})$ is isomorphic to I/I^2. By the distinguished triangle associated to the composition $f' \circ g = f$, we find that $L_{U_g} = 0$, so $I/I^2 = 0$. Thus, by Nakayama's lemma, $I = 0$, so U_g is indeed a weak equivalence. □

PROPOSITION 7.8
Suppose that $p\colon E \to M$ is a vector bundle. Suppose that $s : M \to E$ is a section of p such that the diagram

$$\begin{array}{ccc} (X, \mathcal{O}_X) & \xrightarrow{f} & M \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle s} \\ M & \xrightarrow[z]{} & E \end{array} \tag{7.8.1}$$

commutes and such that the underlying space X is a pullback. The diagram induces a morphism of vector bundles

$$\lambda_s : f^*(E)^\vee \to \mathcal{N}_f,$$

which is an isomorphism if and only if Diagram 7.8.1 is a pullback in **LRS**.

Proof
On any open subset U of X, we have a commutative square

$$\begin{array}{ccc} \mathcal{O}_E(U) & \longrightarrow & \mathcal{O}_M(U) \\ \downarrow & & \downarrow \\ \mathcal{O}_M(U) & \longrightarrow & \mathcal{O}_X(U) \end{array}$$

of sheaves on X. By Property (2) of cotangent complexes, this square induces a morphism $f^*L_z \to L_f$. By Corollary 7.6, we may identify $H_1(f^*L_z)$ with $f^*(E)^\vee$, and $H_1(L_f)$ with $\mathcal{N}_f$, and we let $\lambda_s\colon f^*(E)^\vee \to \mathcal{N}_f$ denote the induced map.

If Diagram 7.8.1 is a pullback, then λ_s is an isomorphism by Property (4), so we have only to prove the converse.

Suppose that λ_s is an isomorphism, let $\mathcal{X}'$ be the fiber product in the diagram

$$\begin{array}{ccc} \mathcal{X}' & \xrightarrow{f'} & M \\ {\scriptstyle f'}\downarrow & \diamond & \downarrow{\scriptstyle s} \\ M & \xrightarrow[z]{} & E \end{array}$$

and let $g : \mathcal{X} \to \mathcal{X}'$ be the induced map. Note that on underlying spaces g is a homeomorphism. Since the composition $X \xrightarrow{g} X' \xrightarrow{f'} M$, namely f, is a closed immersion, so is g. By assumption and by Property (4), g induces an isomorphism on cotangent complexes $g^*L_{f'} \xrightarrow{\cong} f^*L_z \xrightarrow{\cong} L_f$. The result follows from Lemma 7.7. □

Here is a kind of linearity result.

PROPOSITION 7.9

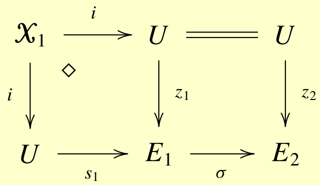
Suppose that M is a manifold, and suppose that $\mathcal{X}_1$ and $\mathcal{X}_2$ are derived manifolds which are defined as pullbacks

$$\begin{array}{ccc} \mathcal{X}_1 & \xrightarrow{j_1} & M \\ {\scriptstyle j_1}\downarrow & \diamond & \downarrow{\scriptstyle z_1} \\ M & \xrightarrow[s_1]{} & E_1 \end{array} \qquad\qquad \begin{array}{ccc} \mathcal{X}_2 & \xrightarrow{j_2} & M \\ {\scriptstyle j_2}\downarrow & \diamond & \downarrow{\scriptstyle z_2} \\ M & \xrightarrow[s_2]{} & E_2 \end{array}$$

where $E_i \to M$ is a vector bundle, s_i is a section, and z_i is the zero section, for $i = 1, 2$. Then $\mathcal{X}_1$ and $\mathcal{X}_2$ are equivalent as derived manifolds over M if and only if there exists an open neighborhood $U \subset M$ containing both $\mathcal{X}_1$ and $\mathcal{X}_2$, and an isomorphism $\sigma : E_1 \to E_2$ over U, such that $s_2|_U = \sigma \circ s_1|_U$.

Proof

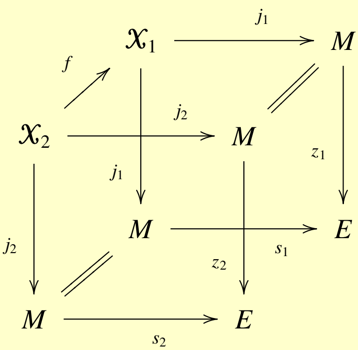
Suppose first that $U \subset M$ is an open neighborhood of both $\mathcal{X}_1$ and $\mathcal{X}_2$, and denote $E_i|_U$ and $s_i|_U$ by E_i and s_i, respectively, for $i = 1, 2$. Suppose that $\sigma\colon E_1 \to E_2$ is an isomorphism. Suppose that $s_2 = \sigma \circ s_1$. Consider the diagram



where the left-hand square is Cartesian. In fact, the right-hand square is Cartesian as well, because σ is an isomorphism of vector bundles, and in particular fixes the zero section. Thus we see that $\mathcal{X}_1$ and $\mathcal{X}_2$ are equivalent as derived manifolds over U.

For the converse, suppose that we have an equivalence $f\colon \mathcal{X}_2 \to \mathcal{X}_1$ such that $j_2 = j_1 \circ f$. In particular, on underlying topological spaces, we have $X_1 = X_2 =: X$, and on cotangent complexes, we have a quasi isomorphism $f^*L_{j_1} \xrightarrow{\cong} L_{j_2}$. In particular, this implies that $H_1(L_{z_1})$ is isomorphic to $H_1(L_{z_2})$, so E_1 and E_2 are isomorphic vector bundles on U. We can write E to denote both bundles and adjust the sections as necessary.

Consider the diagram



By the commutativity of the left-hand square of the diagram and by Property (4) of cotangent complexes, one has a chain of quasi isomorphisms

$$j_2^*L_{s_1} = f^*j_1^*L_{s_1} \xrightarrow{\cong} f^*L_{j_1} \xrightarrow{\cong} L_{j_2} \xrightarrow{\cong} j_2^*L_{s_2}.$$

We write part of the long-exact sequences coming from the composable arrows $U \xrightarrow{s_1} E \to \mathbb{R}^0$ and $U \xrightarrow{s_2} E \to \mathbb{R}^0$, and we furnish some morphisms to obtain the solid arrow diagram:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & H_1(L_{s_1}) & \longrightarrow & \Omega^1_E & \xrightarrow{s_1^*} & \Omega^1_U & \longrightarrow & 0 \\ & & \cong\big\downarrow & & \tau\big\downarrow & & \big\| & & \\ 0 & \longrightarrow & H_1(L_{s_2}) & \longrightarrow & \Omega^1_E & \xrightarrow[s_2^*]{} & \Omega^1_U & \longrightarrow & 0 \end{array}$$

By the five lemma, there is an isomorphism $\tau\colon \Omega^1_E \to \Omega^1_E$ making the diagrams commute.

Now every section $s\colon U \to E$ of a vector bundle $E \to U$ induces a pullback map $s^*\colon \Omega^1_E \to \Omega^1_U$, and two sections induce equivalent pullback maps if and only if they differ by an automorphism of E fixing U. Since s_1 and s_2 induce equivalent pullback maps, there is an automorphism $\sigma\colon E \to E$ with $s_2 = \sigma \circ s_1$. □

8. Proofs of technical results

Recall from Section 5 that $H_{\mathbb{R}^i} \in s\mathbf{C}^\infty$ is the discrete C^∞-ring corepresenting $\mathbb{R}^i \in \mathbb{E}$. A smooth map $f\colon \mathbb{R}^i \to \mathbb{R}^j$ (contravariantly) induces a morphism of C^∞-rings $H_{\mathbb{R}^j} \to H_{\mathbb{R}^i}$, which we often denote by f for convenience. Recall also that in $s\mathbf{C}^\infty$, there is a canonical weak equivalence $H_{\mathbb{R}^{i+j}} \xrightarrow{\simeq} H_{\mathbb{R}^i} \amalg H_{\mathbb{R}^j}$.

Let $U\colon s\mathbf{C}^\infty \to s\mathbb{R}$ denote the *underlying simplicial commutative ring* functor from Corollary 5.8.

LEMMA 8.1

Let $m, n, p \in \mathbb{N}$ with $p \leq m$. Let $x\colon \mathbb{R}^{n+m} \to \mathbb{R}^m$ denote the projection onto the first m coordinates, let $g\colon \mathbb{R}^p \to \mathbb{R}^m$ denote the inclusion of a p-plane in $\mathbb{R}^m$, and let Ψ be the diagram

$$\begin{array}{ccc} H_{\mathbb{R}^m} & \xrightarrow{x} & H_{\mathbb{R}^{n+m}} \\ g\big\downarrow & & \\ H_{\mathbb{R}^p} & & \end{array}$$

of C^∞-rings. The homotopy colimit of Ψ is weakly equivalent to $H_{\mathbb{R}^{n+p}}$.

Moreover, the application of $U\colon s\mathbf{C}^\infty \to s\mathbb{R}$ commutes with taking homotopy colimit of Ψ in the sense that the natural map

$$\operatorname{hocolim}(U\Psi) \to U \operatorname{hocolim}(\Psi)$$

is a weak equivalence of simplicial commutative $\mathbb{R}$-algebras.

Proof

We may assume that g sends the origin to the origin. For now, we assume that $p = 0$ and that $m = 1$.

Consider the all-Cartesian diagram of manifolds

$$\begin{array}{ccccc} \mathbb{R}^n & \longrightarrow & \mathbb{R}^{n+1} & \longrightarrow & \mathbb{R}^n \\ \downarrow & \ulcorner & \downarrow{\scriptstyle x} & \ulcorner & \downarrow \\ \mathbb{R}^0 & \xrightarrow[g]{} & \mathbb{R} & \longrightarrow & \mathbb{R}^0 \end{array}$$

Apply $H\colon \mathbf{Man}^{\mathrm{op}} \to s\mathbf{C}^\infty$ to obtain the diagram

$$\begin{array}{ccccc} H_{\mathbb{R}^n} & \longleftarrow & H_{\mathbb{R}^{n+1}} & \longleftarrow & H_{\mathbb{R}^n} \\ {\scriptstyle x}\uparrow & & \uparrow & & \uparrow \\ H_{\mathbb{R}^0} & \xleftarrow{g} & H_{\mathbb{R}} & \longleftarrow & H_{\mathbb{R}^0} \end{array}$$

Since H sends products in $\mathbb{E}$ to homotopy pushouts in $s\mathbf{C}^\infty$, the right-hand square and the big rectangle are homotopy pushouts. Hence the left square is as well, proving the first assertion.

For notational reasons, let U^i denote $U(H_{\mathbb{R}^i})$ so that U^i is the (discrete simplicial) commutative ring whose elements are smooth maps $\mathbb{R}^i \to \mathbb{R}$. Define a simplicial commutative $\mathbb{R}$-algebra D as the homotopy colimit in the diagram

$$\begin{array}{ccc} U^1 & \xrightarrow{x} & U^{n+1} \\ {\scriptstyle g}\downarrow & \diamond & \downarrow \\ U^0 & \longrightarrow & D \end{array}$$

We must show that the natural map $D \to U^n$ is a weak equivalence of simplicial commutative rings. Recall that the homotopy groups of a homotopy pushout of simplicial commutative rings are the Tor groups of the corresponding tensor product of chain complexes.

Since x is a nonzero divisor in the ring U^{n+1}, the homotopy groups of D are

$$\pi_0(D) = U^{n+1} \otimes_{U^1} U^0; \text{ and } \pi_i(D) = 0, i > 0.$$

Thus, $\pi_0(D)$ can be identified with the equivalence classes of smooth functions $f\colon \mathbb{R}^{n+1} \to \mathbb{R}$, where $f \sim g$ if $f - g$ is a multiple of x.

On the other hand, we can identify elements of the ring U^n of smooth functions on $\mathbb{R}^n$ with the equivalence classes of functions $f\colon \mathbb{R}^{n+1} \to \mathbb{R}$, where $f \sim g$ if

$$f(0, x_2, x_3, \ldots, x_{n+1}) = g(0, x_2, x_3, \ldots, x_{n+1}).$$

Thus to prove that the map $D \to U^n$ is a weak equivalence, we must only show that if a function $f(x_1, \ldots, x_{n+1})\colon \mathbb{R}^{n+1} \to \mathbb{R}$ vanishes whenever $x_1 = 0$, then x_1 divides f. This is called Hadamard's lemma, and it follows from the definition of smooth functions. Indeed, given a smooth function $f(x_1, \ldots, x_{n+1})\colon \mathbb{R}^{n+1} \to \mathbb{R}$ which vanishes whenever $x_1 = 0$, define a function $g\colon \mathbb{R}^{n+1} \to \mathbb{R}$ by the formula

$$g(x_1, \ldots, x_{n+1}) = \lim_{a \to x_1} \frac{f(a, x_2, x_3, \ldots, x_{n+1})}{a}.$$

It is clear that g is smooth and that $xg = f$.

We have now completed the case when $m = 1$; we continue to assume that $p = 0$, and we prove the result for general $m + 1$ by induction. The inductive step follows from the all-Cartesian diagram below, in which each vertical arrow is an inclusion of a plane and each horizontal arrow is a coordinate projection:

$$\begin{array}{ccccc}
\mathbb{R}^n & \longrightarrow & \mathbb{R}^0 & & \\
\downarrow & \ulcorner & \downarrow & & \\
\mathbb{R}^{m+n} & \longrightarrow & \mathbb{R}^m & \longrightarrow & \mathbb{R}^0 \\
\downarrow & \ulcorner & \downarrow & \ulcorner & \downarrow \\
\mathbb{R}^{n+m+1} & \longrightarrow & \mathbb{R}^{m+1} & \longrightarrow & \mathbb{R}
\end{array}$$

Apply H to this diagram. The arguments above imply that the two assertions hold for the right square and bottom rectangle; thus they hold for the bottom left square. The inductive hypothesis implies that the two assertions hold for the top square, so they hold for the left rectangle, as desired.

For the case of general p, let $k\colon \mathbb{R}^m \to \mathbb{R}^{m-p}$ denote the projection orthogonal to g. Applying H to the diagram

$$\begin{array}{ccccc}
\mathbb{R}^{n+p} & \longrightarrow & \mathbb{R}^p & \longrightarrow & \mathbb{R}^0 \\
\downarrow & \ulcorner & \downarrow{\scriptstyle g} & \ulcorner & \downarrow \\
\mathbb{R}^{n+m} & \xrightarrow[x]{} & \mathbb{R}^m & \xrightarrow[k]{} & \mathbb{R}^{m-p}
\end{array}$$

the result holds for the right square and the big rectangle (for both of which $p = 0$), so it holds for the left square as well. □

Remark 8.2

Note that (the nonformal part of) Lemma 8.1 relies heavily on the fact that we are dealing with smooth maps. It is this lemma which fails in the setting of topological manifolds, piecewise linear manifolds, and so on.

Let $\mathcal{M}$ be a model category, and let Δ denote the simplicial indexing category. Recall that a simplicial resolution of an object $X \in \mathcal{M}$ consists of a simplicial diagram $X'_\bullet\colon \Delta^{\mathrm{op}} \to \mathcal{M}$ and an augmentation map $X'_\bullet \to X$, such that the induced map $\operatorname{hocolim}(X') \to X$ is a weak equivalence in $\mathcal{M}$.

Conversely, the geometric realization of a simplicial object $Y_\bullet\colon \Delta^{\mathrm{op}} \to \mathcal{M}$ is the object in $\mathcal{M}$ obtained by taking the homotopy colimit of the diagram $Y_\bullet$.

PROPOSITION 8.3

*The functors $U\colon s\mathbf{C}^\infty \to s\mathbb{R}$ and $-(\mathbb{R})\colon s\mathbf{C}^\infty \to$ **sSets** each preserve geometric realizations.*

Proof

By [19, Proposition A.1], the geometric realization of any simplicial object in any of the model categories $s\mathbf{C}^\infty$, $s\mathbb{R}$, and **sSets** is equivalent to the diagonal of the corresponding bisimplicial object. Since both U and $-(\mathbb{R})$ are functors which preserve the diagonal, they each commute with geometric realization. □

Let $-(\mathbb{R})\colon s\mathbf{C}^\infty \to$ **sSets** denote the functor $F \mapsto F(\mathbb{R})$. It is easy to see that $-(\mathbb{R})$ is a right Quillen functor. Its left adjoint is $(- \otimes H_\mathbb{R})\colon K \mapsto K \otimes H_\mathbb{R}$ (although note that $K \otimes H_\mathbb{R}$ is not generally fibrant in $s\mathbf{C}^\infty$). We call a C^∞-ring *free* if it is in the essential image of this functor $- \otimes H_\mathbb{R}$, and similarly, we call a morphism of free C^∞-rings *free* if it is in the image of this functor.

Thus, a morphism $d'\colon H_{\mathbb{R}^k} \to H_{\mathbb{R}^\ell}$ is free if and only if it is induced by a function $d\colon \{1, \dots, k\} \to \{1, \dots, \ell\}$. To make d' explicit, we just need to provide a dual map $\mathbb{R}^\ell \to \mathbb{R}^k$; for each $1 \le i \le k$, we supply the map $\mathbb{R}^\ell \to \mathbb{R}$ given by projection onto the $d(i)$ coordinate.

If K is a simplicial set such that each K_i is a finite set with cardinality $|K_i| = n_i$, then $X := K \otimes H_\mathbb{R}$ is the simplicial C^∞-ring with $X_i = H_{\mathbb{R}^{n_i}}$, and for a map $[\ell] \to [k]$ in Δ, the structure map $X_k \to X_\ell$ is the free map defined by $K_k \to K_\ell$.

LEMMA 8.4

Let X be a simplicial C^∞-ring. There exists a functorial simplicial resolution $X'_\bullet \to X$ in which X'_n is a free simplicial C^∞-ring for each n.

Moreover, if $f: X \to Y$ is a morphism of C^∞-rings and if $f'_\bullet: X'_\bullet \to Y'_\bullet$ is the induced map on simplicial resolutions, then for each $n \in \mathbb{N}$, the map $f'_n: X'_n \to Y'_n$ is a morphism of free C^∞-rings.

Proof
Let R denote the underlying simplicial set functor $-(\mathbb{R}) : s\mathbf{C}^\infty \to \mathbf{sSets}$, and let F denote its left adjoint. The cotriple FR gives rise to an augmented simplicial C^∞-ring

$$\Phi = \cdots \substack{\longrightarrow\\\longrightarrow\\\longrightarrow} FRFR(X) \substack{\longrightarrow\\\longrightarrow} FR(X) \longrightarrow X.$$

By [31, Proposition 8.6.10], the induced augmented simplicial set

$$R\Phi = \cdots \substack{\longrightarrow\\\longrightarrow\\\longrightarrow} RFRFR(X) \substack{\longrightarrow\\\longrightarrow} RFR(X) \longrightarrow R(X)$$

is a weak equivalence. By Propositions 5.8 and 8.3, Φ is a simplicial resolution.

The second assertion is clear by construction. □

In the following theorem, we use a basic fact about simplicial sets: if $g: F \to H$ is a fibration of simplicial sets and if $\pi_0 g$ is a surjection of sets, then for each $n \in \mathbb{N}$ the function $g_n: F_n \to H_n$ is surjective. This is proved using the left lifting property for the cone $\Delta^0 \to \Delta^{n+1}$.

THEOREM 8.5
Let Ψ be a diagram

$$\begin{array}{ccc} F & \xrightarrow{f} & G \\ {\scriptstyle g}\downarrow & & \\ H & & \end{array}$$

of cofibrant-fibrant C^∞-rings. Assume that $\pi_0 g: \pi_0 F \to \pi_0 H$ is a surjection. Then application of $U: s\mathbf{C}^\infty \to s\mathbb{R}$ commutes with taking the homotopy colimit of Ψ in the sense that the natural map

$$\operatorname{hocolim}(U\Psi) \to U \operatorname{hocolim}(\Psi)$$

is a weak equivalence of simplicial commutative rings.

Proof

We prove the result by using simplicial resolutions to reduce to the case proved in Lemma 8.1. We begin with a series of replacements and simplifications of the diagram Ψ, each of which preserves both $\mathrm{hocolim}(U\Psi)$ and $U\,\mathrm{hocolim}(\Psi)$.

First, replace g with a fibration and replace f with a cofibration. Note that since g is surjective on π_0 and is a fibration, it is surjective in each degree; note also that f is injective in every degree.

Next, replace the diagram by the simplicial resolution given by Lemma 8.4. This is a diagram $H'_\bullet \xleftarrow{g'_\bullet} F'_\bullet \xrightarrow{f'_\bullet} G'_\bullet$, which has the same homotopy colimit. We can compute this homotopy colimit by first computing the homotopy colimits $\mathrm{hocolim}(H'_n \xleftarrow{g'_n} F'_n \xrightarrow{f'_n} G'_n)$ in each degree, and then by taking the geometric realization. Since U preserves geometric realization (Lemma 8.3), we may assume that Ψ is a diagram $H \xleftarrow{g} F \xrightarrow{f} G$, in which F, G, and H are free simplicial C^∞-rings, and in which g is surjective and f is injective. By performing another simplicial resolution, we may assume further that F, G, and H are discrete.

A free C^∞-ring is the filtered colimit of its finitely generated (free) sub-C^∞-rings; hence we may assume that F, G, and H are finitely generated. In other words, each is of the form $S \otimes H_{\mathbb{R}}$, where S is a finite set. We are reduced to the case in which Ψ is the diagram

$$\begin{array}{ccc} H_{\mathbb{R}^m} & \xrightarrow{f} & H_{\mathbb{R}^n} \\ {\scriptstyle g}\downarrow & & \\ H_{\mathbb{R}^p} & & \end{array}$$

where again g is surjective and f is injective. Since f and g are free maps, they are induced by maps of sets $f_1\colon \{1,\ldots,m\} \hookrightarrow \{1,\ldots,n\}$ and $g_1\colon \{1,\ldots,m\} \to \{1,\ldots,p\}$. The map $\mathbb{R}^n \to \mathbb{R}^m$ underlying f is given by

$$(a_1,\ldots,a_n) \mapsto (a_{f(1)},\ldots,a_{f(m)}),$$

which is a projection onto a coordinate plane through the origin, and we may arrange that it is a projection onto the first m coordinates. The result now follows from Lemma 8.1. $\square$

COROLLARY 8.6
Suppose that the diagram

$$\begin{array}{ccc} (A, \mathcal{O}_A) & \xrightarrow{G} & (Y, \mathcal{O}_Y) \\ {\scriptstyle F}\downarrow & \diamond & \downarrow{\scriptstyle f} \\ (X, \mathcal{O}_X) & \xrightarrow[g]{} & (Z, \mathcal{O}_Z) \end{array}$$

is a homotopy pullback of local C^∞-ringed spaces. If g is an imbedding, then the underlying diagram of ringed spaces is also a homotopy pullback.

Proof
In both the context of local C^∞-ringed spaces and local ringed spaces, the space A is the pullback of the diagram $X \to Z \leftarrow Y$. The sheaf on A is the homotopy colimit of the diagram

$$F^*\mathcal{O}_X \xleftarrow{F^*(g^\sharp)} F^*g^*\mathcal{O}_Z \xrightarrow{G^*(f^\sharp)} G^*\mathcal{O}_Y,$$

either as a sheaf of C^∞-rings or as a sheaf of simplicial commutative rings. Note that taking inverse image sheaves commutes with taking underlying simplicial commutative rings.

Since g is an imbedding, we have seen that $g^\sharp\colon g^*\mathcal{O}_Z \to \mathcal{O}_X$ is surjective on π_0, and thus so is its pullback $F^*(g^\sharp)$. The result now follows from Theorem 8.5. □

We now give another way of viewing the *locality condition* on ringed spaces. Considering sections of the structure sheaf as functions to affine space, a ringed space is local if these functions pull covers back to covers. This point of view can be found in [2] and [7], for example.

Recall the notation $|F| = F(\mathbb{R})$ for a C^∞-ring F (see Notation 6.12). Recall also that $C^\infty(\mathbb{R})$ denotes the free (discrete) C^∞-ring on one generator.

Definition 8.7
Let X be a topological space, and let $\mathcal{F}$ be a sheaf of C^∞-rings on X. Given an open set $U \subset \mathbb{R}$, let $\chi_U\colon \mathbb{R} \to \mathbb{R}$ denote a characteristic function of U (i.e., χ_U vanishes precisely on $\mathbb{R} - U$). Let $f \in |\mathcal{F}(X)|_0$ denote a global section. We say that an open subset $V \subset X$ is *contained in the preimage under f of U* if there exists a dotted arrow

making the diagram of C^∞-rings

$$\begin{array}{ccc} C^\infty(\mathbb{R}) & \xrightarrow{f} & \mathcal{F}(X) \\ \downarrow & & \downarrow \rho_{X,V} \\ C^\infty(\mathbb{R})[\chi_U^{-1}] & \dashrightarrow & \mathcal{F}(V) \end{array} \tag{8.7.1}$$

commute up to homotopy. We say that V *is the preimage under* f *of* U, and by abuse of notation write $V = f^{-1}(U)$, if it is maximal with respect to being contained in the preimage. Note that these notions are independent of the choice of characteristic function χ_U for a given $U \subset \mathbb{R}$. Note also that since localization is an epimorphism of C^∞-rings (as it is for ordinary commutative rings; see [22, Corollary 1.2.2, Proposition 1.2.6]), the dotted arrow is unique if it exists.

If $f, g \in |\mathcal{F}(X)|_0$ are homotopic vertices, then for any open subset $U \subset \mathbb{R}$, one has $f^{-1}(U) = g^{-1}(U) \subset X$. Therefore, this preimage functor is well defined on the set of connected components $\pi_0|\mathcal{F}(X)|$. Furthermore, if $f \in |\mathcal{F}(X)|_n$ is any simplex, all of its vertices are found in the same connected component, roughly denoted $\pi_0(f) \in \pi_0|\mathcal{F}(X)|$, so we can write $f^{-1}(U)$ to denote $\pi_0(f)^{-1}(U)$.

Example 8.8

The above definition can be understood from the viewpoint of algebraic geometry. Given a scheme $(X, \mathcal{O}_X)$ and a global section $f \in \mathcal{O}_X(X) = \pi_0\mathcal{O}_X(X)$, one can consider f as a scheme morphism from X to the affine line $\mathbb{A}^1$. Given a principle open subset $U = \text{Spec}\,(k[x][g^{-1}]) \subset \mathbb{A}^1$, we are interested in its preimage in X. This preimage is the largest $V \subset X$ on which the map $k[x] \xrightarrow{f} \mathcal{O}_X(X)$ can be lifted to a map $k[x][g^{-1}] \to \mathcal{O}_X(V)$. This is the content of Diagram 8.7.1.

Up next, we give an alternate formulation of the condition that a sheaf of C^∞-rings be a local in terms of these preimages. In the algebro-geometric setting, it comes down to the fact that a sheaf of rings $\mathcal{F}$ is a sheaf of *local rings* on X if and only if, for every global section $f \in \mathcal{F}(X)$, the preimages under f of a cover of Spec $(k[x])$ form a cover of X.

LEMMA 8.9

Suppose that $U \subset \mathbb{R}$ is an open subset of $\mathbb{R}$, and let χ_U denote a characteristic function for U. Then $C^\infty(U) = C^\infty(\mathbb{R})[\chi_U^{-1}]$.

There is a natural bijection between the set of points $p \in \mathbb{R}$ and the set of C^∞-functions $A_p\colon C^\infty(\mathbb{R}) \to C^\infty(\mathbb{R}^0)$. Under this correspondence, p is in U if and only if A_p factors through $C^\infty(\mathbb{R})[\chi_U^{-1}]$.

Proof
This follows from [22, Propositions 1.1.5, 1.1.6]. □

Recall that $F \in s\mathbf{C}^\infty$ is said to be a local C^∞-ring if the commutative ring underlying $\pi_0 F$ is a local ring (see Definition 5.10).

PROPOSITION 8.10
Let X be a space, and let $\mathcal{F}$ be a sheaf of finitely presented C^∞-rings on X. Then $\mathcal{F}$ is local if and only if, for any cover of $\mathbb{R}$ by open subsets $\mathbb{R} = \bigcup_i U_i$ and for any open $V \subset X$ and local section $f \in \pi_0|\mathcal{F}(V)|$, the preimages $f^{-1}(U_i)$ form a cover of V.

Proof
Choose a representative for $f \in \pi_0|F|$, call it $f \in |F|_0$ for simplicity, and recall that it can be identified with a map $f\colon C^\infty(\mathbb{R}) \to F$, which is unique up to homotopy.

Both the property of being a sheaf of local C^∞-rings and the above "preimage of a covering is a covering" property is local on X. Thus we may assume that X is a point. We are reduced to proving that a C^∞-ring F is a local C^∞-ring if and only if, for any cover of $\mathbb{R}$ by open sets $\mathbb{R} = \bigcup_i U_i$ and for any element $f \in |F|_0$, there exists an index i and a dotted arrow making the diagram

$$\begin{array}{ccc} C^\infty(\mathbb{R}) & \xrightarrow{f} & F \\ \downarrow & \nearrow & \\ C^\infty(\mathbb{R})[\chi_{U_i}^{-1}] & & \end{array}$$

commute up to homotopy.

Suppose first that for any cover of $\mathbb{R}$ by open subsets $\mathbb{R} = \bigcup_i U_i$ and that for any element $f \in F_0$, there exists a lift as above. Let $U_1 = (-\infty, 1/2)$, and let $U_2 = (0, \infty)$. By assumption, either f factors through $C^\infty(U_1)$ or through $C^\infty(U_2)$, and $1 - f$ factors through the other by Lemma 8.9. It is easy to show that any element of $\pi_0 F$ which factors through $C^\infty(U_2)$ is invertible (using the fact that $0 \notin U_2$). Hence, either f or $1 - f$ is invertible in $\pi_0 F$, so $\pi_0 F$ is a local C^∞-ring.

Now suppose that F is local (i.e., that it has a unique maximal ideal), and suppose that $\mathbb{R} = \bigcup_i U_i$ is an open cover. Choose $f \in |F|_0$, considered as a map of C^∞-rings $f\colon C^\infty(\mathbb{R}) \to F$. By [22, Proposition 1.3.8], $\pi_0 F$ has a unique point $F \to \pi_0 F \to C^\infty(\mathbb{R}^0)$. By Lemma 8.9, there exists i such that the composition $C^\infty(\mathbb{R}) \xrightarrow{f} F \to$

$C^\infty(\mathbb{R}^0)$ factors through $C^\infty(\mathbb{R})[\chi_{U_i}^{-1}]$, giving the solid arrow square

$$\begin{array}{ccc} C^\infty(\mathbb{R}) & \xrightarrow{f} & F \\ \downarrow & \nearrow & \downarrow \\ C^\infty(\mathbb{R})[\chi_{U_i}^{-1}] & \longrightarrow & C^\infty(\mathbb{R}^0) \end{array}$$

Since $\chi_{U_i} \in C^\infty(\mathbb{R})$ is not sent to $0 \in C^\infty(\mathbb{R}^0)$, its image $f(\chi_{U_i})$ is not contained in the maximal ideal of $\pi_0 F$, so a dotted arrow lift exists making the diagram commute up to homotopy. This proves the proposition. □

In the following theorem, we use the notation $|A|$ to denote the simplicial set $A(\mathbb{R}) = \mathrm{Map}_{s\mathbf{C}^\infty}(C^\infty(\mathbb{R}), A)$ underlying a simplicial C^∞-ring $A \in s\mathbf{C}^\infty$.

THEOREM 8.11
Let $\mathcal{X} = (X, \mathcal{O}_X)$ be a local C^∞-ringed space, and let $\mathbf{i}\mathbb{R} = (\mathbb{R}, C^\infty_{\mathbb{R}})$ denote the (image under $\mathbf{i}$ of the) real line. There is a natural homotopy equivalence of simplicial sets

$$\mathrm{Map}_{\mathbf{LRS}}\big((X, \mathcal{O}_X), (\mathbb{R}, C^\infty_{\mathbb{R}})\big) \xrightarrow{\simeq} |\mathcal{O}_X(X)|.$$

Proof
We construct morphisms

$$K\colon \mathrm{Map}_{\mathbf{LRS}}(\mathcal{X}, \mathbf{i}\mathbb{R}) \to |\mathcal{O}_X(X)| \text{ and}$$
$$L\colon |\mathcal{O}_X(X)| \to \mathrm{Map}_{\mathbf{LRS}}(\mathcal{X}, \mathbf{i}\mathbb{R}),$$

and show that they are homotopy inverses. For the reader's convenience, we recall the definition

$$\mathrm{Map}_{\mathbf{LRS}}(\mathcal{X}, \mathbb{R}) = \coprod_{f\colon X \to \mathbb{R}} \mathrm{Map}_{\mathrm{loc}}(f^* C^\infty_{\mathbb{R}}, \mathcal{O}_X).$$

The map K is fairly easy and can be defined without use of the locality condition. Suppose that $\phi\colon X \to \mathbb{R}$ is a map of topological spaces. The restriction of K to the corresponding summand of $\mathrm{Map}_{\mathbf{LRS}}(\mathcal{X}, \mathbf{i}\mathbb{R})$ is given by taking global sections

$$\mathrm{Map}_{\mathrm{loc}}(\phi^* C^\infty_{\mathbb{R}}, \mathcal{O}_X) \to \mathrm{Map}\big(C^\infty(\mathbb{R}), \mathcal{O}_X(X)\big) \cong |\mathcal{O}_X(X)|.$$

To define L is a bit harder and depends heavily on the assumption that $\mathcal{O}_X$ is a local sheaf on X. First, given an n-simplex $g \in |\mathcal{O}_X(X)|_n$, we need to define a map of topological spaces $L(g)\colon X \to \mathbb{R}$. Let $g_0 \in \pi_0|\mathcal{O}_X(X)|$ denote the connected component containing g. By Proposition 8.10, g_0 gives rise to a function from open covers of $\mathbb{R}$ to open covers of X, and this function commutes with refinement of open covers. Since $\mathbb{R}$ is Hausdorff, there is a unique map of topological spaces $X \to \mathbb{R}$, which we take as $L(g)$, consistent with such a function. Denote $L(g)$ by G, for ease of notation.

Now we need to define a map of sheaves of C^∞-rings

$$G^\flat\colon C^\infty_{\mathbb{R}} \otimes \Delta^n \to G_*(\mathcal{O}_X).$$

On global sections, we have such a function already, since $g \in |\mathcal{O}_X(X)|_n$ can be considered as a map $g\colon C^\infty_{\mathbb{R}}(\mathbb{R}) \to |\mathcal{O}_X(X)|_n$. Let $V \subset \mathbb{R}$ denote an open subset, and let $g^{-1}(V) \subset X$ denote its preimage under g. The map $\rho\colon C^\infty_{\mathbb{R}}(\mathbb{R}) \to C^\infty_{\mathbb{R}}(V)$ is a localization; hence, it is an epimorphism of C^∞-rings.

To define $G^\flat$, we need to show that there exists a unique dotted arrow making the diagram

$$\begin{array}{ccc} C^\infty_{\mathbb{R}}(\mathbb{R}) \otimes \Delta^n & \xrightarrow{g} & |\mathcal{O}_X(X)| \\ {\scriptstyle \rho\otimes\Delta^n}\downarrow & & \downarrow \\ C^\infty_{\mathbb{R}}(V) \otimes \Delta^n & \dashrightarrow & |\mathcal{O}_X(g^{-1}(V))| \end{array} \tag{8.11.1}$$

commute. Such an arrow exists by definition of g^{-1}. It is unique because ρ is an epimorphism of C^∞-rings, so $\rho \otimes \Delta^n$ is as well. We have now defined $G^\flat$, and we take $G^\sharp\colon G^*C^\infty_{\mathbb{R}} \to \mathcal{O}_X$ to be the left adjoint of $G^\flat$.

We must show that for every $g \in |\mathcal{O}_X(X)|_0$, the map $G^\sharp\colon G^*C^\infty_{\mathbb{R}} \to \mathcal{O}_X$ provided by L is local. (We can choose g to be a vertex because, by definition, a simplex in $\mathrm{Map}(G^*C^\infty_{\mathbb{R}}, \mathcal{O}_X)$ is local if all of its vertices are local.) To prove this, we may take X to be a point, $F = \mathcal{O}_X(X)$ to be a local C^∞-ring, and $x = G(X) \in \mathbb{R}$ to be the image point of X. We have a morphism of C^∞-rings $G^\sharp\colon (C^\infty_{\mathbb{R}})_x \to F$, in which both the domain and codomain are local. It is a local ring homomorphism because all prime ideals in the local ring $(C^\infty_{\mathbb{R}})_x$ are maximal.

The maps K and L have now been defined, and they are homotopy inverses by construction. □

The following is a technical lemma that allows us to take homotopy limits component-wise in the model category of simplicial sets.

LEMMA 8.12

Let **sSets** *denote the category of simplicial sets. Let I, J, and K denote sets and let*

$$A = \coprod_{i \in I} A_i, \qquad B = \coprod_{j \in J} B_j, \quad \text{and} \quad C = \coprod_{k \in K} C_k$$

denote coproducts in **sSets** *indexed by I, J, and K.*

Suppose that $f\colon I \to J$ and $g\colon K \to J$ are functions, and suppose that $F\colon A \to B$ and $G\colon C \to B$ are maps in **sSets** *that respect f and g in the sense that $F(A_i) \subset B_{f(i)}$ and $G(C_k) \subset B_{g(k)}$ for all $i \in I$ and $k \in K$. Let*

$$I \times_J K = \left\{(i, j, k) \in I \times J \times K \mid f(i) = j = g(k)\right\}$$

denote the fiber product of sets. For typographical reasons, we use $\times^h$ to denote homotopy limit in **sSets**, *and we use $\times$ to denote the one-categorical limit.*

Then the natural map

$$\Big(\coprod_{(i,j,k) \in I \times_J K} A_i \times^h_{B_j} C_k \Big) \longrightarrow A \times^h_B C$$

is a weak equivalence in **sSets**.

Proof

If we replace F by a fibration, then each component $F_i := F|_{A_i}\colon A_i \to B_{f(i)}$ is a fibration. We reduce to showing that the map

$$\Big(\coprod_{(i,j,k) \in I \times_J K} A_i \times_{B_j} C_k \Big) \longrightarrow A \times_B C \tag{8.12.1}$$

is an isomorphism of simplicial sets. Restricting to the n-simplicies of both sides, we may assume that A, B, and C are (discrete simplicial) sets. It is an easy exercise to show that the map in (8.12.1) is injective and surjective; that is, an isomorphism in **Sets**. □

PROPOSITION 8.13

Suppose that $a : M_0 \to M$ and $b : M_1 \to M$ are morphisms of manifolds, and suppose that a fiber product N exists in the category of manifolds. If $\mathcal{X} = (X, \mathcal{O}_X)$ is *the fiber product*

$$\begin{array}{ccc} \mathcal{X} & \longrightarrow & M_0 \\ \big\downarrow & \diamond & \big\downarrow a \\ M_1 & \xrightarrow[b]{} & M \end{array}$$

taken in the category of derived manifolds, then the natural map $g \colon N \to \mathcal{X}$ is an equivalence if and only if a and b are transverse.

Proof

Since limits taken in **Man** and in **dMan** commute with taking underlying topological spaces, the map $N \to X$ is a homeomorphism. We have a commutative diagram

$$\begin{array}{ccc} N & \xrightarrow{g} & \mathcal{X} \\ {\scriptstyle f}\big\downarrow & \swarrow {\scriptstyle f'} & \\ M_0 \times M_1 & & \end{array}$$

in which f and f' are closed immersions (pullbacks of the diagonal $M \to M \times M$).

If a and b are not transverse, one shows easily that the first homology group $H_1 L_{\mathcal{X}} \neq 0$ of the cotangent complex for $\mathcal{X}$ is nonzero, whereas $H_1 L_N = 0$ because N is a manifold; hence $\mathcal{X}$ is not equivalent to N.

If a and b are transverse, then one can show that g induces a quasi isomorphism $g^* L_{f'} \to L_f$. By Lemma 7.7, the map g is an equivalence of derived manifolds. □

PROPOSITION 8.14

The simplicial category **LRS** *of local C^∞-ringed spaces is closed under taking finite homotopy limits.*

Proof

The local C^∞-ringed space $(\mathbb{R}^0, C^\infty(\mathbb{R}^0))$ is a homotopy terminal object in **LRS**. Hence it suffices to show that a homotopy limit exists for any diagram

$$(X, \mathcal{O}_X) \xleftarrow{F} (Y, \mathcal{O}_Y) \xrightarrow{G} (Z, \mathcal{O}_Z)$$

in **LRS**. We first describe the appropriate candidate for this homotopy limit.

The underlying space of the candidate is $X \times_Y Z$, and we label the maps as in the diagram

$$\begin{array}{ccc} X \times_Y Z & \xrightarrow{g} & Z \\ {\scriptstyle f}\downarrow & \overset{h}{\searrow} & \downarrow{\scriptstyle G} \\ X & \xrightarrow[F]{} & Y \end{array}$$

The structure sheaf on the candidate is the homotopy colimit of pullback sheaves

$$\mathcal{O}_{X \times_Y Z} := (g^* \mathcal{O}_Z) \otimes_{(h^* \mathcal{O}_Y)} (f^* \mathcal{O}_X). \tag{8.14.1}$$

To show that $\mathcal{O}_{X \times_Y Z}$ is a sheaf of local C^∞-rings, we take the stalk at a point, apply π_0, and show that it is a local C^∞-ring. The homotopy colimit written in equation (8.14.1) becomes the C^∞-tensor product of pointed local C^∞-rings. By [22, Corollary 1.3.12], the result is indeed a local C^∞-ring.

One shows that $(X \times_Y Z, \mathcal{O}_{X \times_Y Z})$ is the homotopy limit of the diagram in the usual way. We do not prove it here, but refer the reader to [27, Proposition 2.3.21] or, for a much more general result, to [21, Proposition 2.3.21]. □

THEOREM 8.15

Let M be a manifold, let $\mathcal{X}$ and $\mathcal{Y}$ be derived manifolds, and let $f: \mathcal{X} \to M$ and $g: \mathcal{Y} \to M$ be morphisms of derived manifolds. Then a fiber product $\mathcal{X} \times_M \mathcal{Y}$ exists in the category of derived manifolds.

Proof

We showed in Proposition 8.14 that $\mathcal{X} \times_M \mathcal{Y}$ exists as a local C^∞-ringed space. To show that it is a derived manifold, we must only show that it is locally an affine derived manifold. This is a local property, so it suffices to look locally on M, $\mathcal{X}$, and $\mathcal{Y}$. We prove the result by first showing that affine derived manifolds are closed under taking products, then by showing that they are closed under solving equations, and finally by showing that these two facts combine to prove the result.

Given affine derived manifolds $\mathbb{R}^n_{a=0}$ and $\mathbb{R}^m_{b=0}$, it follows formally that $\mathbb{R}^{n+m}_{(f,g)=0}$ is their product, and it is an affine derived manifold.

Now let $\mathcal{X} = \mathbb{R}^n_{a=0}$, where $a: \mathbb{R}^n \to \mathbb{R}^m$, and suppose that $b: \mathcal{X} \to \mathbb{R}^k$ is a morphism. By Theorem 8.11, we can consider b as an element of $\mathcal{O}_X(X)(\mathbb{R}^k)$. By Lemma 5.4, it is homotopic to a composite $\mathcal{X} \to \mathbb{R}^n \xrightarrow{b'} \mathbb{R}^k$, where $\mathcal{X} \to \mathbb{R}^n$ is the canonical imbedding. Now we can realize $\mathcal{X}_{b=0}$ as the homotopy limit in the all-Cartesian diagram

$$\begin{array}{ccccc}
\mathcal{X}_{b=0} & \longrightarrow & \mathbb{R}^0 & & \\
\downarrow & \diamond & \downarrow & & \\
\mathcal{X} & \xrightarrow{b} & \mathbb{R}^k & \longrightarrow & \mathbb{R}^0 \\
\downarrow & \diamond & \downarrow & \diamond & \downarrow \\
\mathbb{R}^n & \xrightarrow[(b',a)]{} & \mathbb{R}^k \times \mathbb{R}^m & \longrightarrow & \mathbb{R}^m
\end{array}$$

Therefore, $\mathcal{X}_{b=0} = \mathbb{R}^n_{(b',a)=0}$ is affine.

Finally, suppose that $\mathcal{X}$ and $\mathcal{Y}$ are affine, and suppose that $M = \mathbb{R}^p$. Let $-\colon \mathbb{R}^p \times \mathbb{R}^p \to \mathbb{R}^p$ denote the coordinate-wise subtraction map. Then there is an all-Cartesian diagram

$$\begin{array}{ccccc}
\mathcal{X} \times_{\mathbb{R}^p} \mathcal{Y} & \longrightarrow & \mathbb{R}^p & \longrightarrow & \mathbb{R}^0 \\
\downarrow & \diamond & {\scriptstyle \mathrm{diag}}\downarrow & \diamond & \downarrow{\scriptstyle 0} \\
\mathcal{X} \times \mathcal{Y} & \longrightarrow & \mathbb{R}^p \times \mathbb{R}^p & \xrightarrow[-]{} & \mathbb{R}^p
\end{array}$$

where $\mathrm{diag}\colon \mathbb{R}^p \to \mathbb{R}^p \times \mathbb{R}^p$ is the diagonal map. We have seen that $\mathcal{X} \times \mathcal{Y}$ is affine, so since $\mathcal{X} \times_{\mathbb{R}^p} \mathcal{Y}$ is the solution to an equation on an affine derived manifold, it too is affine. This completes the proof. □

Remark 8.16

Note that Theorem 8.15 *does not* say that the category **dMan** is closed under arbitrary fiber products. Indeed, if M is not assumed to be a smooth manifold, then the fiber product of derived manifolds over M need not be a derived manifold in our sense. The cotangent complex of any derived manifold has homology concentrated in degrees zero and one (see Corollary 7.4), whereas a fiber product of derived manifolds (over a nonsmooth base) would not have that property.

Of course, using the spectrum functor Spec, defined in Remark 6.14, one could define a more general category $\mathcal{C}$ of "derived manifolds" in the usual scheme-theoretic way. Then our **dMan** would form a full subcategory of $\mathcal{C}$, which one might call the subcategory of *quasi-smooth* objects (see [27]). The reason we did not introduce this category $\mathcal{C}$ is that it does not have the general cup product formula in cobordism; that is, Theorem 1.8 does not apply to $\mathcal{C}$.

9. Derived manifolds are good for doing intersection theory

In this section, we prove that the simplicial category **dMan** of derived smooth manifolds, as defined in Definition 6.15, is good for doing intersection theory on manifolds, in the sense of Definition 2.1.

PROPOSITION 9.1

The simplicial category **dMan** *satisfies Axiom (1) of Definition 2.1. That is, in the sense of that definition,* **dMan** *is geometric.*

Proof

Given derived manifolds $\mathcal{X} = (X, \mathcal{O}_X)$ and $\mathcal{Y} = (Y, \mathcal{O}_Y)$, the mapping space $\mathrm{Map}_{\mathbf{dMan}}(\mathcal{X}, \mathcal{Y})$ is defined as a disjoint union over maps $f: X \to Y$ of simplicial sets $\mathrm{Map}_{\mathrm{loc}}(f^*\mathcal{O}_Y, \mathcal{O}_X)$, each of which is a certain subset of the components of $\mathrm{Map}(f^*\mathcal{O}_Y, \mathcal{O}_X)$. All objects of $\mathbf{Shv}(X, s\mathbf{C}^\infty)$ are cofibrant, and $\mathcal{O}_X$ is fibrant, so this mapping space is fibrant, and it follows that **dMan** has fibrant mapping spaces.

Lemma 6.16 states that there is a fully faithful functor $\mathbf{i}: \mathbf{Man} \to \mathbf{dMan}$. The fact that imbedding preserves transverse intersections is proved in Proposition 8.13. The underlying space $\mathbf{U}(\mathcal{X})$ of a derived manifold $\mathcal{X} = (X, \mathcal{O}_X)$ is just X, and $\mathbf{U}$ preserves limits (see Theorem 8.15). □

PROPOSITION 9.2

The simplicial category **dMan** *satisfies Axioms (2), (3), and (5) of Definition 2.1. That is, in the sense of that definition,* **dMan** *has enough open subobjects, has unions, and has local models.*

Proof

Axioms (2) and (5) follow directly from the definition of derived manifolds (see Definition 6.15). Axiom (3) follows from Proposition 6.9. □

PROPOSITION 9.3

The simplicial category **dMan** *satisfies Axiom (4) of Definition 2.1. That is, the fiber product of two derived manifolds over a smooth manifold is a derived manifold.*

Proof

This is Theorem 8.15. □

For the next proposition, one may recall that Axiom (6) of Definition 2.1 says roughly that maps from embedded derived submanifolds $\mathcal{X} \hookrightarrow \mathcal{Y}$ to the affine line $\mathbb{R}$ can locally be extended to maps from an open neighborhood of the imbedding.

PROPOSITION 9.4

The simplicial category **dMan** *satisfies Axiom (6) of Definition 2.1. That is, given an imbedding* $g\colon \mathcal{X} \to \mathcal{Y}$*, any real-valued function on* $\mathcal{X}$ *extends to a real-valued function on* $\mathcal{Y}$*, up to homotopy.*

Proof

By the definition of imbedding, there is a cover of $\mathcal{Y}$ by open subobjects $\mathcal{Y}_i$ such that, if we set $\mathcal{X}_i = g^{-1}(\mathcal{Y}_i)$, then for each i there is a homotopy pullback square

$$\begin{array}{ccc} \mathcal{X}_i & \longrightarrow & \mathbb{R}^0 \\ {\scriptstyle g_i}\downarrow & \diamond & \downarrow{\scriptstyle 0} \\ \mathcal{Y}_i & \underset{f_i}{\longrightarrow} & \mathbb{R}^{n_i} \end{array}$$

that is, g_i is a model imbedding. We must show that the morphism of sheaves $g^\sharp\colon g^*\mathcal{O}_Y \to \mathcal{O}_X$ is surjective after applying π_0. It suffices to work locally; hence we may drop the i-subscripts and assume that $g\colon \mathcal{X} \to \mathcal{Y}$ is a model imbedding. That is, there is some map $f\colon \mathcal{Y} \to \mathbb{R}^n$ such that $\mathcal{X} \cong \mathcal{Y}_{f=0}$. Let $X = \mathbf{U}(\mathcal{X})$ and $Y = \mathbf{U}(\mathcal{Y})$ denote the underlying spaces.

By construction (see Theorem 8.14), the structure sheaf $\mathcal{O}_X$ of $\mathcal{X}$ is the homotopy colimit in the diagram

$$\begin{array}{ccc} \mathcal{O}_{\mathbb{R}^n} & \longrightarrow & \mathcal{O}_{\mathbb{R}^0} \\ {\scriptstyle f^\sharp}\downarrow & \diamond & \downarrow \\ \mathcal{O}_Y & \underset{g^\sharp}{\longrightarrow} & \mathcal{O}_X \end{array} \tag{9.4.1}$$

of sheaves on X, where as usual we have suppressed the fact that three of these sheaves are preimage sheaves under various maps out of X.

By Definition 6.3, all of the objects in Diagram (9.4.1) are fibrant, so by Lemma 5.4, the map $g^\sharp(X)\colon g^*\mathcal{O}_Y(X) \to \mathcal{O}_X(X)$ is sent under π_0 to a pushout diagram of C^∞-rings. Thus the map $\pi_0(g^\sharp(X))$ is the pushout of a surjection, and hence is itself a surjection. □

PROPOSITION 9.5

The simplicial category **dMan** *satisfies Axiom (7) of Definition 2.1. That is, in the sense of that definition, there is a normal bundle and defining section for any imbedding* $g\colon \mathcal{X} \to M$ *of a derived manifold into a smooth manifold.*

Proof
We may assume that M and $\mathcal{X}$ are equidimensional; let $k = \dim M - \dim \mathcal{X}$ denote the codimension of $\mathcal{X}$ in M. Let L_g denote the cotangent complex for g, which is a sheaf of simplicial modules over the sheaf of simplicial commutative rings $U(\mathcal{O}_X)$ underlying the structure sheaf of $\mathcal{X}$. Its first homology sheaf $H_1(L_g)$ is a vector bundle of rank k on the underlying space X, by Corollary 7.6. Write $\mathcal{N}_g = H_1(L_g)^\vee)$ to denote its dual. By the paracompactness of M, there exists an open neighborhood $U \subset M$ of $g(X)$, a vector bundle $E \to U$, and an isomorphism $g^*E \cong \mathcal{N}_g$.

We can cover U by Euclidean charts $U_\alpha \cong \mathbb{R}^n$ over which the induced bundle $E_\alpha \cong \mathbb{R}^k$ is trivial and $\mathcal{X}_\alpha$ is cut out by k real-valued functions on U_α (see Definition 2.3). This gives a section $s_\alpha \colon U_\alpha \to E_\alpha$ for which the diagram

$$\begin{array}{ccc} \mathcal{X}_\alpha & \xrightarrow{g_\alpha} & U_\alpha \\ {\scriptstyle g_\alpha}\downarrow & \diamond & \downarrow{\scriptstyle z} \\ U_\alpha & \xrightarrow[s_\alpha]{} & E_\alpha \end{array}$$

is a pullback.

Acting on s_α by a linear change of coordinates does not change the fact that the above square is a pullback, by Proposition 7.9. Thus we can patch these sections together to get a global section of $E \to U$ which cuts out $\mathcal{X}$. We have proved the existence of a normal bundle and defining section. □

THEOREM 9.6
The simplicial category **dMan** *(see Definition 6.15) is good for doing intersection theory on manifolds, in the sense of Definition 2.1. Therefore,* **dMan** *has the general cup product in cobordism in the sense of Definition 1.7.*

Proof
The first assertion follows from Propositions 9.1, 9.2, 9.3, 9.4, and 9.5. The second assertion follows from Corollary 3.13. □

10. Relationship to similar articles

In this section, we discuss other research which is in some way related to this article. Most relevant is Section 10.1, in which we discuss the relationship to Lurie's work on derived algebraic geometry, and in particular to structured spaces. The other sections discuss manifolds with singularities, Chen spaces and diffeological spaces, synthetic differential geometry, and a catch-all section to concisely state how one might orient our article within the canon.

10.1. Lurie's structured spaces

There is a version of the above work on derived manifolds, presented in the author's Ph.D. dissertation (see [27]), which very closely follows Jacob Lurie's theory of structured spaces, as presented in [21]. Similar in spirit is Toen and Vezzosi's articles [29], [30] on homotopical algebraic geometry. In this section, we attempt to orient the reader to Lurie's theory of structured spaces.

In order to define structured spaces, Lurie begins with the definition of a *geometry*. A geometry is an ∞-category, equipped with a given choice of *admissible* morphisms that generate a Grothendieck topology, and satisfying certain conditions. For example, there is a geometry whose underlying category is the category of affine schemes Spec R, with admissible morphisms given by principal open sets Spec $R[a^{-1}] \to$ Spec R, and with the usual Grothendieck topology of open coverings.

Given a geometry G and a topological space X, a G-structure on X is roughly a functor $\mathcal{O}_X : G \to \mathbf{Shv}(X)$ which preserves finite limits and sends covering sieves on G to effective epimorphisms in $\mathbf{Shv}(X)$.

One should visualize the objects of G as spaces and visualize the admissible morphisms in G as open inclusions. In this visualization, a G-structure on X provides, for each "space" $g \in G$, a sheaf $\mathcal{O}_X(g)$, whose sections are seen as *maps* from X to g. Since a map from X to a limit of g's is a limit of maps, one sees immediately why we require $\mathcal{O}_X$ to be left exact. Given a covering sieve in G, we want to be able to say that to give a map from X to the union of the cover is accomplished by giving local maps to the pieces of the cover, such that these maps agree on overlaps. This is the covering sieve condition in the definition of G-structure.

Our approach follows Lurie's in spirit, but not in practice. The issue is in his definition (see [21, Definition 1.2.1]) of admissibility structure, which we recall here.

Definition 10.1

Let $\mathcal{G}$ be an ∞-category. An *admissibility structure* on $\mathcal{G}$ consists of the following data:

(1) A subcategory $\mathcal{G}^{\mathrm{ad}} \subset \mathcal{G}$, containing every object of $\mathcal{G}$. Morphisms of $\mathcal{G}$ which belong to $\mathcal{G}^{\mathrm{ad}}$ are called *admissible* morphisms in $\mathcal{G}$.

(2) A Grothendieck topology on $\mathcal{G}^{\mathrm{ad}}$.

These data are required to satisfy the following conditions:

(i) Let $f: U \to X$ be an admissible morphism in $\mathcal{G}$, and let g: $X' \to X$ be any morphism. Then there exists a pullback diagram

$$\begin{array}{ccc} U' & \longrightarrow & U \\ {\scriptstyle f'}\downarrow & & \downarrow{\scriptstyle f} \\ X' & \xrightarrow[g]{} & X \end{array}$$

where f' is admissible.

(ii) Suppose that given a commutative triangle

$$\begin{array}{ccccc} & & Y & & \\ & {\scriptstyle f}\nearrow & & \searrow{\scriptstyle g} & \\ X & & \xrightarrow[h]{} & & Z \end{array}$$

in $\mathcal{G}$, where g and h are admissible. Then f is admissible.

(iii) Every retract of an admissible morphism of $\mathcal{G}$ is admissible.

In our case, the role of $\mathcal{G}$ is played by $\mathbb{E}$, the category of Euclidean spaces, and the role of $\mathcal{G}^{\text{ad}}$ is played by open inclusions $\mathbb{R}^n \hookrightarrow \mathbb{R}^n$ (see Sections 5 and 6). But, as such, $\mathcal{G}^{\text{ad}}$ is not an admissibility structure on $\mathcal{G}$ because it does not satisfy condition (i): the pullback of a Euclidean open subset is not necessarily Euclidean.

However such a pullback is *locally* Euclidean. This should be enough to define something like a *preadmissibility structure*, which does the same job as an admissibility structure. In private correspondence, Lurie told me that such a notion would be useful—perhaps this issue can be resolved in a later version of [21].

In the author's dissertation, however, we did not use C^∞-rings as our basic algebraic objects. Instead, we used something called *smooth rings*. A smooth ring is a functor **Man** $\to$ **sSets** which preserves pullbacks along submersions (see [27, Definition 2.1.3]). In this case, the role of $\mathcal{G}$ is played by **Man**, and the role of $\mathcal{G}^{\text{ad}}$ is played by submersions. This is an admissibility structure in Lurie's sense, and it should not be hard to prove that the category of structured spaces one obtains in this case is equivalent to our category of local C^∞-ringed spaces.

10.2. Manifolds with singularities

A common misconception about derived manifolds is that every singular homology class $a \in H_*(M, \mathbb{Z})$ of a manifold M should be representable by an oriented derived manifold. The misconception seems to arise from the idea that manifolds with singularities, objects obtained by coning off a submanifold of M, should be examples of derived manifolds. This is not the case.

By the phrase *coning off a submanifold* $A \subset M$, one means taking the colimit of a diagram $\{*\} \leftarrow A \rightarrow M$. Derived manifolds are not closed under taking arbitrary colimits (e.g., quotients). In particular, one cannot naturally obtain a derived manifold structure on a manifold with singularities. Instead, one obtains derived manifolds from taking the zero set of a section of a smooth vector bundle (see Example 2.7). The collection of derived manifolds is quite large, but it does not include manifolds with singularities in a natural way. In order to obtain arbitrary colimits, perhaps one should consider stacks on derived manifolds, but we have not worked out this idea.

10.3. Chen spaces, diffeological spaces

Another common generalization of the category of manifolds was invented by Kuo Tsai Chen in [8]. Let Conv denote the category whose objects are convex subsets of $\mathbb{R}^\infty$, and whose morphisms are smooth maps. With the Grothendieck topology in which coverings are given by open covers in the usual sense, we can define the topos **Shv**(Conv). The category of Chen spaces is roughly this topos, the difference being that points are given more importance in Chen spaces than in **Shv**(Conv), in a sense known as *concreteness* (see [3] for a precise account). Diffeological spaces are similar—they are defined roughly as sheaves on the category of open subsets of Euclidean spaces.

The difference between these approaches and our own is that Chen spaces are based on "maps in" to the object in question (the Chen space) whereas our objects carry information about "maps out" of the object in question (the derived manifold). In other words, the simplest question one can ask about a Chen space X is, what are the maps from $\mathbb{R}^n$ to X? Since X is a sheaf on a site in which $\mathbb{R}^n$ is an object, the answer is simply $X(\mathbb{R}^n)$. In the case of derived manifolds, the simplest question one can ask is, what are the maps from X to $\mathbb{R}^n$? By the structure theorem, Theorem 8.11, information about maps from X to Euclidean spaces is carried by the structure sheaf $\mathcal{O}_X$—the answer to the question is $\mathcal{O}_X(X)^n$.

If one is interested in generalizing manifolds to better study maps in to them, one should probably use Chen spaces or diffeological spaces. In our case, we were interested in cohomological properties (intersection theory and cup product); since elements of cohomology on X are determined by maps out of X, we constructed our generalized manifolds to be well behaved with respect to maps out. It may be possible to generalize further and talk about "derived Chen spaces," but we have not yet pursued this idea.

10.4. Synthetic differential geometry

In [22], Moerdijk and Reyes discuss yet another generalization of manifolds, called smooth functors. These are functors from the category of (discrete) C^∞-rings to sets that satisfy a descent condition (see [22, Lemma 3.1.1]). A smooth functor can be considered as a patching of local neighborhoods, each of which is a formal C^∞-variety.

Neither our setup nor theirs is more general than the other. While both are based on C^∞-rings, our approach uses homotopical ideas, whereas theirs does not; their approach gives a topos, whereas ours does not. It certainly may be possible to combine these ideas into "derived smooth functors," but we have not pursued this idea either. The non-homotopical approach does not seem adequate for a general cup product formula in the sense of Definition 1.7.

10.5. Other similar articles

There has been far too much written about the intersection theory of manifolds for us to list here. In our article, we achieve an intersection pairing at the level of spaces: the intersection of two submanifolds is still a geometric object (i.e., an object in a geometric category in the sense of Definition 1.3), and this geometric object has an appropriate fundamental class in cohomology (see Definition 1.7). No "general position" requirements are necessary. We hope that this is enough to distinguish our results from previous ones.

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