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Corso di Laurea Magistrale in Matematica

ON THE GEOMETRY OVER THE FIELD WITH ONE ELEMENT

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Academic Year 2009 - 2010

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Introduction

Although the “the field with one element” $\mathbb{F}_1$ was originally mentioned in 1956 by Tits [**33**], it in fact emerged as an signficant object to investigate in the ’90s. Despite its youth, a lot of interesting constructions have been built out of studying $\mathbb{F}_1$-geometry, especially in the last decade. The interested reader may find excellent commentaries on the motivations of this theory in various papers such as [**10**], [**11**], and [**13**]. We also refer to the beautiful article of J. López Peña and O. Lorscheid [**27**], in which the whole picture of the $\mathbb{F}_1$-universe is presented. The $\mathbb{F}_1$-geometry project has been considered too ambitious by many, since none of the big aims that motivated its original introduction have been reached yet. That said, we have to specify that the theory itself has not been settled fully since a lot of different approaches have been made and thus, it is still undergoing a continuous evolution. Moreover, it seems that some results in other parts of mathematics, such as combinatorics, can really be proven using the $\mathbb{F}_1$-machinery. We also feel that some of the approaches to $\mathbb{F}_1$-geometry, such as the ones we present in here, are undoubtedly elegant as well as natural, being in turn relevant on their own.

In this thesis, we focus mainly on Deitmar’s and Toën-Vaquié’s theory. In particular, we show their equivalence, generalizing a classical result (1.3.11) to $\mathbb{F}_1$-geometry (2.3.18). The former theorem presents a complete characterization of schemes (which are often called here “schemes over $\mathbb{Z}$” in order to distinguish from the $\mathbb{F}_1$-geometry setting) inside the category of functors of points, where they are naturally embedded via the Yoneda functor. In particular, it states that the category of schemes over $\mathbb{Z}$ is equivalent to the category of presheaves on affine schemes, which are Zariski sheaves and have an affine covering, in some sense. The proof of this result was presented by Demazure and Gabriel in [**12**].

In the $\mathbb{F}_1$-universe, there is a similar duality. Kato and Deitmar’s approach to schemes over $\mathbb{F}_1$ is inspired by the usual definition of schemes over $\mathbb{Z}$. They defined schemes over $\mathbb{F}_1$ as topological spaces together with a particular structure sheaf of monoids, more or less by substituting the word “ring” with the word “monoid” at every occurrence. Despite seeming impossible at first glance, this process runs smoothly, and the category of schemes over $\mathbb{F}_1$ can be defined analogously. On the other hand, one can work on the functorial side, and start from the aforementioned characterization of schemes inside functor categories. In order to do this, however, it is vital to have a definition, which is complete category-wise of the properties that characterize schemes, so that the definition can be easily generalized substituting everywhere the word “ring” with the word “monoid”. This can also be done, and all the specific terms referred to rings can really be purged from the definitions, using in a crucial way a result by Grothendieck ([**20**] IV.17.9.1). This procedure gives rise to another definition of schemes over $\mathbb{F}_1$, firstly given by Toën and Vaquié in [**34**].

We remark that these two approaches were born using two different perspectives which are proven equivalent in the classical case. However, the question of their renewed equivalence in the $\mathbb{F}_1$-setting was still open. As a matter of fact, it has been taken for granted by many (see the map in [**27**]) and conventionally considered to be true, but there was no trace of a complete proof anywhere. In this thesis, we present a proof of this fact in Theorem 2.3.18, which states that Deitmar's $\mathbb{F}_1$-schemes are equivalent to Toën-Vaquié's. We find that the technical core of this fact (which is Theorem 2.3.12), despite having a rather elementary proof, is not trivial and it is strongly related to some facts on commutative monoids that generalize similar statements on commutative rings. However, the tools that are used are necessarily different. This is because, for instance, the category of M-modules for a given monoid M is not an abelian category. It goes without saying that in developing such theory, we were hugely inspired by the classical duality of schemes, and we also built a new proof of this equivalence that only partly overlaps with the one of [**12**].

In the first chapter, we overview some basic facts about schemes over $\mathbb{Z}$. In particular, we refer to them using two different notions: a "geometrical scheme" is a scheme as defined in [**22**] - hence a topological space with a structure sheaf - while a "functorial scheme" is a scheme as defined in [**12**] - hence a Zariski sheaf over the opposite category of rings, with an affine covering. We focus mainly on the various aspects of definitions and on those properties which are crucial in the following part. We also provide a different new proof of the equivalence between geometrical and functorial schemes, which can be summarized by saying that it is a mixture of three facts: the gluing lemma, a functorial characterization of open immersions of affine schemes, and Yoneda's lemma.

We enter the parallel world of schemes over $\mathbb{F}_1$ in the second chapter. We outline the basic points of the theory from Deitmar and Toën-Vaquié's point of view. The significance of this part is that Deitmar's schemes are like geometrical schemes over $\mathbb{F}_1$, while Toën-Vaquié's schemes are like functorial schemes over $\mathbb{F}_1$. We present all the results we use, trying to be as homologous as possible to the first part. Some other results that are cited in classical papers on $\mathbb{F}_1$ are unwound and better explained. We also add new facts on commutative algebra of monoids. Specifically, we give explicit descriptions of M-algebras of finite presentation, of flat local epimorphisms, and of flat epimorphisms of finite presentation. We then obtain a new functorial characterization of open immersions of affine geometrical schemes over $\mathbb{F}_1$, which naturally leads up to the proof of the equivalence of the two different notions of schemes. After showing the equivalence, we investigate the "base change" functor from schemes over $\mathbb{F}_1$ to schemes over $\mathbb{Z}$. We also give a functorial characterization of the n-dimensional projective space using the concepts of $\mathcal{O}_X$-modules and line bundles, which can be generalized to the $\mathbb{F}_1$-setting.

In the first appendix to this thesis, we add some of the motivations that pushed research on $\mathbb{F}_1$-geometry, and in particular, we focus on the reasons of the "monoid" approach, which is the one we fully analyzed. We also give an alternative description of the work of Kato [**24**] by introducing an analogue of the module of differentials in the $\mathbb{F}_1$-setting, giving a hint to the fact that $\operatorname{Spec}_{\mathbb{F}_1} \mathbb{Z}$ is an analogue of a denumerable-dimensional affine space $\mathbb{A}^{\infty}_{\mathbb{F}_1}$.

In the second appendix, we outline the main features of stack theory with the specific aim to give some useful criteria for proving that certain pseudo-functors are stacks. In particular, we apply these results to prove descent for morphisms of schemes à la Toën-Vaquiè, endowed with the Zariski topology, and to prove descent for modules over affine schemes à la Toën-Vaquiè, endowed with the fpqc topology. Not only are these two results crucial in the proofs of the second chapter, but they also generalize deep facts about descent for maps and for quasi-coherent sheaves of affine schemes over $\mathbb{Z}$.

Acknowledgments

I am deeply grateful to Professor Luca Barbieri Viale, who originally introduced me to the "fun of $\mathbb{F}_1$", and constantly encourages me with advices, and enriches me with teachings. I also thank Professors Bas Edixhoven and Ieke Moerdijk, whom I was honored to meet in the Netherlands, and who gave me specific support for the needs of this thesis. A specific workshop of young researchers was also organized in Granada, 2009 in order to describe what has been done so far in the field of $\mathbb{F}_1$-geometry, and jot down a plan for the future. I express my gratitude to the organizers and the participants for having invited me to such event, and enlightened me with interesting lectures and conversations. I would like to thank specifically Peter Arndt for having pointed out to me the result 1.3.4, and having elucidated many other ideas. I thank Professor Bertrand Toën who answered to my questions with courtesy and clarity, and who indicated to me the work of Florian Marty, and Florian Marty himself for proofreading and for sharing his interesting material with me, as well as his ideas on the subject. I am grateful to Anton Geraschenko, who not only shared his excellent material on stacks but also gave helpful feedback. I had the chance to talk about many aspects of this thesis in a seminar talk organized in the University of Milan. I warmly thank the organizers for this opportunity, and the participants for their interesting remarks. Specifically, I would like to thank Professor Fabrizio Andreatta, who has also been a enormous source of help and insight, as well as Riccardo Brasca, Valeria Marcucci and Ila Varma: the mathematical conversations we had and their friendly support have been precious in developing this thesis.

Notation

In all this work, a choice of a universe $\mathcal{U}$ is implicit, and all categories introduced here must be thought as $\mathcal{U}$-small categories (see also [**23**], 1.1, 1.2). We indicate categories with bold fonts. The category of sets is denoted by **Set**. For a given category $\mathbf{C}$ and an object X inside it, we write $\mathbf{Psh}(\mathbf{C})$ for the category $\mathbf{Set}^{\mathbf{C}^{\mathrm{op}}}$ of presheaves over $\mathbf{C}$, $\mathbf{C}_{/X}$ for the category of objects over X, and ${}^{X/}\mathbf{C}$ for the category of objects under X.

The word "ring" will indicate a commutative ring with unity unless otherwise specified. Also, maps of rings respect the unity elements, hence subrings have the same unity of the bigger ring. The category of rings will be denoted by **Ring**, and the opposite category $\mathbf{Ring}^{\mathrm{op}}$ will be indicated with **Aff**.

Similarly, the word "monoid" will indicate a commutative monoid unless otherwise specified. We will denote the category of monoids with **Mon**.

A closed symmetric monoidal category in the sense of [**25**] will be indicated with $(\mathbf{C}, \otimes)$ omitting all the extra structure, the unit object will be indicated with $\mathbf{1}$ and the internal Hom functor with $\underline{\mathrm{Hom}}$. The category of monoids in $(\mathbf{C}, \otimes)$ will be denoted by $\mathbf{Mon}_{\mathbf{C}}$. For a given monoid A in $(\mathbf{C}, \otimes)$, the category of modules over A will be indicated with A-**Mod**, the category ${}^{A/}\mathbf{Mon}_{\mathbf{C}}$ will be denoted with A-**Alg** and its objects will be called A-algebras.

CHAPTER 1

Schemes over $\mathbb{Z}$

In order to define schemes over $\mathbb{F}_1$, it is necessary to recall the classical definitions of a scheme. We will stress out all the fundamental properties of schemes that are needed to give an idea of the generalizations we will present right afterwards, in order to point out how natural these generalizations are, if seen properly.

The aim of the whole chapter is to present the "split personality" of classical schemes, by comparing these two possible definitions.

GEOMETRICAL SCHEMES	FUNCTORIAL SCHEMES
An affine scheme is a locally ringed space isomorphic to $\operatorname{Spec} A$ for some ring A.	Zariski opens in $\mathbf{Ring}^{\mathrm{op}}$ are induced by flat epis, of finite presentation.
A *scheme* is a locally ringed space which is locally affine.	A *scheme* is a Zariski sheaf over $\mathbf{Ring}^{\mathrm{op}}$ which is locally affine.

We will clarify all the previous notions in the following part. At the end of this chapter, we will show how to connect these two notions via an equivalence of categories. It is easy to see that Yoneda's lemma determines a functor from the left side to the other. The non-trivial fact (but still totally classical) is that it induces an equivalence. After Theorem 1.3.11, in which we present a proof of this result, we will be able to use the word "scheme" referring to the objects of both categories.

1.1. "Geometrical" schemes

Let's start with the most common definition of a scheme in algebraic geometry. Note that, unlike Grothendieck's first verision of EGA or Mumford's "Red Book" [**31**], we won't require a scheme to be separated.

1.1.1. Definition. A *ringed space* is a pair $(X, \mathcal{O}_X)$ consisting of a topological space X and a sheaf of rings $\mathcal{O}_X$ on it. A morphism of locally ringed spaces is a pair $(f, f^\sharp)$ where $f\colon X \to Y$ is a map of topological spaces and $f^\sharp\colon \mathcal{O}_Y \to f_*\mathcal{O}_X$ is a map of sheaves on Y.

A *locally ringed space* is a ringed space $(X, \mathcal{O}_X)$ such that the stalks of the sheaf $\mathcal{O}_X$ are local. A *morphism of locally ringed spaces* is a morphism of ringed spaces $(f, f^\sharp)$ such that $f^\sharp$ induces an inclusion of the local fields, in the sense that the composite map of local rings $\mathcal{O}_{Y,f(x)} \to (f_*\mathcal{O}_X)_{f(x)} \to \mathcal{O}_{X,x}$ is local, for each point x in X. The category of locally ringed spaces is denoted with **LRS**.

1.1.2. Proposition ([**12**], I.1.6). *The category of ringed spaces and the category of locally ringed spaces are cocomplete, i.e. have all small colimits. The faithful inclusions preserve colimits.*

Proof. For the first part of the claim, it suffices to show that the two categories have coequalizers and arbitrary coproducts. The definition of $\coprod(X_i, \mathcal{O}_{X_i})$ is obvious. Now consider two ringed spaces $(X, \mathcal{O}_X)$ and $(Y, \mathcal{O}_Y)$ and two maps $(f, f^\sharp), (g, g^\sharp)\colon (X, \mathcal{O}_X) \rightrightarrows (Y, \mathcal{O}_Y)$. We define Z to be the topological space which is the coequalizer of f, g. Let also p be the natural projection $Y \to Z$. We then define a sheaf of rings $\mathcal{O}_Z$ on Z by setting $\mathcal{O}_Z(W)$ as the equalizer of the two maps $f^\sharp, g^\sharp\colon \mathcal{O}_Y(V) \rightrightarrows \mathcal{O}_X(U)$ where V is the inverse image if W via p, and U is the inverse image of V via either f or g. Because the definition of $\mathcal{O}_Z$ and the sheaf property are defined through limits which commute with each other, $\mathcal{O}_Z$ is a sheaf on Z. It is easy to prove that $(Z, \mathcal{O}_Z)$ enjoys the universal property of the coequalizers in the category of ringed spaces. Now we prove that if f and g are maps of locally ringed spaces, then $\mathcal{O}_Z$ has local stalks and that the map $p^\sharp\colon \mathcal{O}_Z(W) \to \mathcal{O}_Y(V)$ induces a local morphism at the level of stalks. Fix a section $s \in \mathcal{O}_Z(W)$ and a point $y \in V$, and suppose that $p^\sharp(s)$ is invertible in $\mathcal{O}_{Y,y}$. This implies that y lies in the open set $V_{p^\sharp(s)}$ defined as the set of points in which $p^\sharp(s)$ is locally invertible. By definition of the sheaf $\mathcal{O}_Z$, we know that $f^\sharp p^\sharp(s) = g^\sharp p^\sharp(s)$. Call this section $t \in \mathcal{O}_X(U)$. Because f, g are local, we also know that $f^{-1}(V_{p^\sharp(s)}) = g^{-1}(V_{p^\sharp(s)}) = U_t$. Hence $V_{p^\sharp(s)}$ is saturated with respect to the equivalence relation $f(x) = g(x)$. This means that $V_{p^\sharp(s)}$ is the inverse image of some open subset W' of Z such that $p(y) \in W' \subset W$. We then conclude that s restricted to W' is invertible, hence s is locally invertible at $p(y)$. This proves that $(p^\sharp_y)^{-1}(\mathcal{O}^\times_{Y,y}) = \mathcal{O}^\times_{Z,p(y)}$. Now fix a point $w \in W$, and let s, s' be two sections in $\mathcal{O}_Z(W)$ which are not invertible in z. By the previous part, we can conclude that $p^\sharp(s), p^\sharp(s')$ are not invertible in any point y in $p^{-1}(z)$. Since the stalks of Y are local, we also conclude that $p^\sharp(s+s')$ is not invertible in any point of $p^{-1}(z)$. By the equality $(p^\sharp_y)^{-1}(\mathcal{O}^\times_{Y,y}) = \mathcal{O}^\times_{Z,p(y)}$, we conclude that $s + s'$ is not invertible at z. We have then proved that the stalks of Z are local and that p defines a local morphism at stalks, hence the claim. By this very construction, one can also see that the colimits built in the category of locally ringed spaces are exactly the same as those in the larger category of ringed spaces. □

1.1.3. Proposition. *Let A be a ring. There is a canonical structure of locally ringed space on the topological space* $\operatorname{Spec} A$ *such that* Spec *defines a left adjoint of the functor of global sections Γ, seen as a functor from* $\mathbf{LRS}^{op}$ *to* $\mathbf{Ring}$*. In particular, for any locally ringed space X*

$$\operatorname{Hom}_{\mathbf{Ring}}(A, \Gamma(X, \mathcal{O}_X)) = \operatorname{Hom}_{\mathbf{LRS}}(X, \operatorname{Spec} A).$$

The sheaf $\mathcal{O}_{\operatorname{Spec} A}$ is such that $\mathcal{O}_{\operatorname{Spec} M}(D(f)) = A_f$ and $\mathcal{O}_{\operatorname{Spec} A,\mathfrak{p}} = A_\mathfrak{p}$ for any element f and any prime ideal $\mathfrak{p}$ of A.

Proof. The proof is classical. See, for example, [**12**], I.2.1. □

1.1.4. Definition. Locally ringed spaces which are isomorphic to a locally ringed space of the form $(\operatorname{Spec} A, \mathcal{O}_{\operatorname{Spec} A})$ for some ring A are called *affine geometrical schemes.*

1.1.5. Corollary. *The functor* Spec *from rings to affine geometrical schemes is part of a contravariant equivalence of categories.*

1.1.6. Definition. A map $(X, \mathcal{O}_X) \to (Y, \mathcal{O}_Y)$ of $\mathbf{LRS}$ is an *open immersion* if it is the composite of an isomorphism and an open inclusion $(U, \mathcal{O}_Y|_U) \hookrightarrow (Y, \mathcal{O}_Y)$.

A family of open immersions is a *Zariski covering* if it is globally surjective on the topological spaces underneath. Zariski coverings define a Grothendieck pretopology on affine geometrical schemes.

1.1.7. DEFINITION. A *geometrical scheme* is a locally ringed space $(X, \mathcal{O}_X)$ with an affine Zariski covering. The full subcategory of geometrical schemes inside **LRS** is denoted with **Sch**.

1.1.8. PROPOSITION ([**1**], IV.6.1). *Zariski coverings define a Grothendieck pretopology on* **Sch**. *The site they form will is called the* Zariski site.

PROOF. Given an open covering $\{U_i\}$ of a scheme X and a scheme over X $f\colon Y \to X$, then $Y \times_X U_i$ equals $f^{-1}(X_i)$. This yields to the proof of all the axioms of a Grothendieck pretopology (see [**29**] III.2, for example). □

Henceforth, we will refer to schemes as just defined as *geometrical schemes*, in order to distinguish them from other definitions that will be given later on. The adjective *geometrical* is due to the fact that topological spaces and a sheaves of functions over them are the typical objects an algebraic geometer has to deal with. In fact, this definition was born as a generalization of an algebraic variety (among other things).

The category **Sch** is not cocomplete. Nonetheless, some colimits do exist. In order to prove the existence of a fixed colimit, we may just check that the colimit built in the category **LRS** is indeed a scheme.

1.1.9. PROPOSITION (Gluing lemma). *Let $\{X_i\}_{i\in I}$ be a family of geometrical schemes and let $\{U_{ij} \hookrightarrow X_i\}_{j\in I, j\neq i}$ be a family of open subschemes of X_i, for every i. If there exist isomorphisms of geometrical schemes $\varphi_{ij}\colon U_{ij} \to U_{ji}$ such that*

(i) $\varphi_{ij}^{-1} = \varphi_{ji}$;
(ii) $\varphi_{ij}(U_{ik} \cap U_{ih}) = U_{jk} \cap U_{jh}$;
(iii) $\varphi_{jk} \circ \varphi_{ij} = \varphi_{ik}$;

then there exist a geometrical scheme X and isomorphisms ψ_i of X_i onto an open subscheme of X for every i such that

(i) $\{\psi_i(X_i)\}_{i\in I}$ *is an open cover of* X
(ii) $\psi_i(U_{ij}) = \psi_i(X_i) \cap \psi_j(X_j)$;
(iii) $\psi_i|_{U_i} = \psi_j \circ \varphi_{ij}|_{U_i}$,

and such X is uniquely determined up to one isomorphism.

PROOF. The geometrical scheme X wanted is the coequalizer of the maps induced by the maps φ_{ij} and φ_{ji} defined on each U_{ij}:

$$\coprod U_{ij} \rightrightarrows \coprod X_k \to X. \tag{1}$$

We then have to prove that the locally ringed space built in this way is locally affine. This is granted because we are gluing over open subschemes. □

1.1.10. COROLLARY (Gluing lemma for morphisms). *Let $\{U_i\}$ be an open cover of a geometrical scheme X, and let $\{\varphi_i\colon U_i \to S\}_i$ be a family of morphisms of geometrical schemes, such that $\varphi_i(U_i \cap U_j) = \varphi_j(U_i \cap U_j)$. There exists a unique morphism $\varphi\colon X \to S$ such that $\varphi|_{U_i} = \varphi_i$ for every i. Equivalently, the Zariski topology on geometrical schemes is subcanonical.*

PROOF. X is obtained by gluing the U_i's together. Apply then the left exact functor $\mathrm{Hom}(\cdot, S)$ to a coequalizing diagram, which is the analogue of the one shown in (1). $\square$

The proof of the Gluing Lemma we presented here is totally equivalent to the most direct one which is usually presented on books. The use of some abstract nonsense has just simplified the problem from gluing a general family of schemes to just gluing two of them.

1.1.11. PROPOSITION. *The category of geometrical schemes has pullbacks (also called* fibered products*), and affine geometrical schemes are closed under pullbacks.*

PROOF. The category of rings has pushouts, given by tensor products. Hence the category of affine geometrical schemes has fibered products. Now consider a diagram $Y \to X \leftarrow Z$. Take an affine open covering $\{X_i\}$ of X and affine open coverings $\{Y_{ij}\}$, $\{Z_{ik}\}$ of the inverse images of each X_i. Construct the generic fibered product over X by gluing together the affine schemes obtained as fibered products of Y_{ij} and Z_{ik} over X_i. The second claim is obvious by construction. $\square$

We now want to unwind Definition 1.1.7 even more, in order to rephrase every statement that appears in terms of the category of affine geometrical schemes. Indeed, there exists a complete affine-wise characterization of open immersions due to Grothendieck. In order to state it, we recall the definitions of some basic properties of morphisms between geometrical schemes.

1.1.12. DEFINITION. Let $f\colon X \to Y$ be a map of geometrical schemes.

1) A morphism $X \to Y$ of schemes is *locally of finite presentation* if for any $x \in X$ there exist an open affine neighborhood $f(x) \in V = \mathrm{Spec}\, A$ and an open affine neighborhood $x \in U = \mathrm{Spec}\, B \subset f^{-1}(V)$ such that the induced map $A \to B$ is of finite presentation.
2) A morphism $X \to Y$ of schemes is *flat* if for any $x \in X$ the induced map of rings $\mathcal{O}_{Y,f(x)} \to \mathcal{O}_{X,x}$ is flat.

There are other characterizations of morphisms that are locally of finite presentation, which are equally important.

1.1.13. PROPOSITION. *Let $f\colon X \to Y$ a morphism of schemes. The following are equivalent*

(i) f is locally of finite presentation.
(ii) There exists a covering of affine open subschemes $\{\mathrm{Spec}\, A_i \to Y\}$ of Y and affine open coverings $\{\mathrm{Spec}\, B_{ij} \to f^{-1}(\mathrm{Spec}\, A_i)\}$ of each inverse image of $\mathrm{Spec}\, A_i$ such that the induced morphisms of rings $A_i \to B_{ij}$ are of finite presentation.
(iii) Fon any open affine subscheme $W = \mathrm{Spec}\, C$ of Y, there exists an affine open covering $\{\mathrm{Spec}\, B_i\}$ of $f^{-1}(W)$ such that each induced map $C \to D_i$ is of finite presentation.

PROOF. The implications $(ii) \Rightarrow (i)$ and $(iii) \Rightarrow (ii)$ are clear. Now suppose (i), and consider an open affine subscheme $W = \mathrm{Spec}\, C$ of Y. Let x be in $f^{-1}(W)$, and let U, V be as in the definition. Consider now another affine neighborhood $\bar{V} = \mathrm{Spec}\, C_g$ of $f(x)$ such that $\bar{V} \subset W \cap V$, and the open affine neighborhood of x defined as $\bar{U} = f^{-1}(\bar{V}) = \mathrm{Spec}(B \otimes_A C_g)$. Consider now the map of rings

$C \to C_g \to B \otimes_A C_g$. Being the composite of two maps of finite presentation, it is of finite presentation. If we now consider the collection of opens $\bar{U}$ as x runs in the set $f^{-1}(W)$, we obtain an open covering $\{\operatorname{Spec} B_i\}$ of it, and by construction all maps $C \to B_i$ are of finite presentation. This proves (iii), hence the claim. □

1.1.14. THEOREM. *A map of geometrical schemes $X \to Y$ is an open immersion if and only if it is a monomorphism which is flat and locally of finite presentation.*

PROOF. The fact that open inclusions are monomorphisms is straightforward. They are also flat and locally of finite presentation, since the maps on rings are locally identities. All these properties are stable under composition, and are obviously satisfied by isomorphisms. We conclude that any open immersion (which is a composition of an isomorphism, followed by an open inclusion) is a flat monomorphism, locally of finite presentation.

For the other implication, we show initially that a flat monomorphism locally of finite presentation is an homeomorphism onto its image. Because monomorphisms are injective, we only have to prove that such a map is open. Being open is a local property on X and Y, so we can think of them as affine schemes, say $X = \operatorname{Spec} B$, $Y = \operatorname{Spec} A$. Fix a point $x = \mathfrak{q}$ of X and its image $f(x) = \mathfrak{p}$ in Y. Consider now the local map induced at stalks $\mathcal{O}_{Y,y} = A_\mathfrak{p} \to \mathcal{O}_{X,x} = B_\mathfrak{q}$. Being flat, it induces a surjective map on spectra ([**4**], Exercise 3.16) i.e. every ideal in $A_\mathfrak{p}$ is a contracted ideal.

As a topological space, $\operatorname{Spec} B_\mathfrak{q}$ is the intersection of all open neighborhoods of x, and similarly $\operatorname{Spec} A_\mathfrak{p}$ is the intersection of all open neighborhoods of $f(x)$. Indeed we have by Proposition 1.1.3

$$\operatorname{Spec} A_\mathfrak{p} = \operatorname{Spec}(\varinjlim_{g \notin \mathfrak{p}} A_g) = \varprojlim_{g \notin \mathfrak{p}} \operatorname{Spec} A_g = \bigcap_{g \notin \mathfrak{p}} \operatorname{Spec} A_g.$$

We have then proved that the intersection of all open sets containing $f(x)$ is contained in all the images of open sets containing x. We claim that this implies that $f(x)$ is in the interior of $f(U)$ for any open neighborhood U of x, hence that the map is open at x. Fix an affine open neighborhood U of x. Because f is locally of finite presentation, by Chevalley Theorem ([**18**], I.8.4) and by [**18**], I.1.9.5 (viii)-(ix), we can conclude that there exists a map $\operatorname{Spec} C \to Y$ such that its image is $Y \setminus f(U)$. Because $\operatorname{Spec} A_\mathfrak{p}$ lies inside $f(U)$, we conclude that $\operatorname{Spec} A_\mathfrak{p} \times_Y \operatorname{Spec} C = \operatorname{Spec} A_\mathfrak{p} \otimes_A C$ is empty. This means $0 = \operatorname{Spec} A_\mathfrak{p} \otimes_A C = (\varinjlim_{g \notin \mathfrak{p}} A_g) \otimes_A C$. Because tensor products commute with colimits (left adjointness of the tensor product), we conclude $\varinjlim_{g \notin \mathfrak{p}} (A_g \otimes_A C) = 0$. But if a direct limit of A algebras is zero then $[0] = [1]$, hence one of the elements in the direct system is zero. We conclude that there exists a $g \notin \mathfrak{p}$ such that $(A_g \otimes_A C) = 0$, which means that $\operatorname{Spec} A_g = D(g)$ is included in $f(U)$.

Consider the open subscheme V of Y which is the image of f, and consider f as a map from X to U. Now that we showed that f is an homeomorphism, it suffices to show that $f^\sharp$ induces an isomorphism of sheaves. This property is local, hence we can suppose $Y = V = \operatorname{Spec} A$ affine and that f is of finite presentation. Since the property of being a flat monomorphism locally of finite presentation is stable under base change, we conclude that f is a universal homeomorphism, hence it is universally closed. We can also see that f is quasi-finite. Indeed, consider a K-point $\operatorname{Spec} K \to Y$. each map $X \times_Y \operatorname{Spec} K \to \operatorname{Spec} K$ is a monomorphism, hence either $X \times_Y \operatorname{Spec} K$ is empty, or it is reduced to a single point. In the second case, it is also affine, say equal to $\operatorname{Spec} C$. We then conclude that $K \to C$ is an epimorphism, hence the cokernel pair $C \rightrightarrows C \otimes_K C$ is constituted by isomorphisms, whose inverse is the map $C \otimes_K C \to C$. Now suppose that there exists an element $c \in C$ outside the image of K. We have that both $c \otimes 1$ and $1 \otimes c$ have the same image via $C \otimes_K C \to C$, which is an isomorphism. Then $c \otimes 1 = 1 \otimes c$ which is absurd because $1, c$ are K-linearly independent. In particular, each fiber $f^{-1}(y)$ is either empty or isomorphic to $\operatorname{Spec} k(y)$. We also conclude that for all fields K, the map $X(K) \to Y(K)$ is injective, hence the map $X(k) \times_{Y(k)} X(k)$ is a bijection. This is sufficient to prove that the diagonal morphism is surjective, hence a closed immersion ([**22**], Corollary II.4.2). The map f is then also separated. Obviously, being f of finite presentation, it is also of finite type. We then conclude that f is separated, of finite type and universally closed, hence proper. Also, it is quasi-finite, hence finite ([**19**], IV.8.11.1). In particular, it is also affine ([**17**], II.6.1.1 and [**22**], Exercise II.5.17), hence we can set $X = f^{-1}(Y) = \operatorname{Spec} B$.

We now finally show that the induced map $A \to B$ is an isomorphism. Because this map is an isomorphism if and only if it is an isomorphism as a map of A-modules, we can prove that it induces an isomorphism between $A_{\mathfrak{p}}$ and $B \otimes_A A_{\mathfrak{p}}$ for all prime ideals in $\operatorname{Spec} A$. Being a base extension of a map which has such properties, each map $A_{\mathfrak{p}} \to B \otimes_A A_{\mathfrak{p}}$ is finite and of finite presentation. We can then suppose that A is local. Let $B = A[b_1, \dots, b_n]$ where each b_i, being integral over A, satisfies a relation $F_i(b_i) = 0$ where $F_i \in A[x_i]$, a monic polynomial of positive degree. The ring $B' := A[x_1, \dots, x_n]/(F_1, \dots, F_n)$ is of finite presentation, and it is such that there exists a surjective map $B' \to B$. Notice that, as a A-module, B' is free. We claim that the kernel of the map $B' \to B$ is A-finitely generated, hence that B is a A-module of finite presentation. In order to prove this, consider the map $A[x_1, \dots, x_n] \to B$. Because B is of finite presentation, the kernel of this map is $A[x_1, \dots, x_n]$-finitely generated ([**18**], IV.1.4.4). Hence, its image via $A[x_1, \dots, x_n] \to B'$, which is exactly the kernel of $B' \to B$, is B'-finitely generated. The claim then follows from the fact that B' is a finite over A. We then conclude that B is a flat module of finite presentation over a local ring, hence it is free ([**9**], Corollary 2 to Proposition II.3.2.5), say of dimension m. Because of what already stated on the fibers, we have that $B \otimes_A A/\mathfrak{m}$ is isomorphic to $A/\mathfrak{m}$, where $\mathfrak{m}$ is the maximal ideal of A. We then conclude that $m = 1$, so that $A \to B$ is an isomorphism. This concludes the proof. $\square$

1.1.15. Corollary. *Let $f\colon \operatorname{Spec} B \to \operatorname{Spec} A$ be a map of affine geometrical schemes. It is an open immersion if and only if the induced map of rings $A \to B$ is a flat epimorphism of finite presentation.*

We claim that the previous corollary can also be used as a completely categorical definition of the topology in the category of affine geometrical schemes, in the sense that it only depends on the structure of the category **Ring**, and not on the category **LRS**. This comes from the following classical description of maps of finite presentations.

1.1.16. Proposition. *Let $f\colon A \to B$ a map of rings. The map f is of finite presentation if and only if for every direct system $\{C_i\}_{i \in I}$ of A-algebras, the canonical map*

$$\varinjlim \operatorname{Hom}_{A\text{-}\mathbf{Alg}}(B, C_i) \to \operatorname{Hom}_{A\text{-}\mathbf{Alg}}(B, \varinjlim C_i)$$

is bijective.

Proof. We will proceed in two steps. First of all, we shall see that a A-algebra B of finite presentation has the commutativity property required. Let $\{C_i, f_{ij}\}$ be a direct system of A-algebras. We can think of its direct limit as the A-algebra in which the elements are equivalence classes $[c_i]$ of elements $c_i \in C_i$, with respect to the relation that identifies $c_i \sim c_j$ if and only if there exists a $k \geq i, j$ such that $f_{ik}(c_i) = f_{jk}(c_j)$. The operations are defined acting on representatives of each class which lie in the same algebra C_i. Any A-algebra B of finite presentation is isomorphic to $A[x_1, \ldots, x_n]/(p_1, \ldots, p_m)$. In particular, giving a map $B \to \varinjlim C_i$ is equivalent to give an n-tuple of elements $[c_i], \ldots, [c_n]$ such that $p_j([c_1], \ldots, [c_n]) = 0$ for all i, j. We can set an index i such that all the representatives c_i are in C_i (now we are using the finite generation property). Because of the definition of the A-algebra structure defined on the limit, we have then $0 = p_j([c_1], \ldots, [c_n]) = [p_j(c_1, \ldots, c_n)]$. Hence each $p_j(c_1, \ldots, c_n)$ is zero at some level. Now let k be an index such that $f_{ik}(p_j(c_1, \ldots, c_n)) = 0$ for all j (now we are using the finite presentation property). Because the transition maps are A-morphisms, we conclude that $p_j(f_{ik}(c_1), \ldots, f_{ik}(c_n)) = 0$. By definition of B then, we can define a unique map $B \to C_k$ which is represented by the n-tuple $(f_{ik}(c_1), \ldots, f_{ik}(c_n))$, hence an element of $\varinjlim_{i \in I} \mathrm{Hom}_A(B, C_i)$. This splitting is unique. Indeed, let $[f], [g]$ two elements in the direct limit splitting the same map. We can assume that they are represented by two maps f_k, g_k in $\mathrm{Hom}(B, C_k)$, i.e. by two n-tuples $(x_{k,1}, \ldots, x_{k,n})$, $(y_{k,1}, \ldots, y_{k,n})$ of elements in C_k such that they are zeros of the m polynomials. Because they both split the map to the direct limit, we get that each $[x_{k,i}]$ is equal to $[y_{k,i}]$, hence there exists an index r in which the two n-tuples coincide. This means that $f_r = g_r$, and so $[f] = [g]$.

Now we prove that the commutativity condition is also sufficient. We will show initially that a A-algebra B that satisfies the commutativity condition has to be finitely generated, then we will show that the kernel of the presentation is finitely generated as well. It is straightforward that B can be expressed as the direct limit of its finitely generated sub-A-algebras. By hypothesis then, there exists a splitting of the identity map $id_B \in \mathrm{Hom}_A(B, B)$ in $B \to A[b_1, \ldots, b_n] \hookrightarrow B$, where $A[b_1, \ldots, b_n]$ is a finitely generated sub-A-algebra of B. We conclude that the inclusion map $A[b_1, \ldots, b_n] \hookrightarrow B$ is surjective, hence an identity. B is finitely generated.

Let I be the kernel of a presentation $A[x] := A[x_1, \ldots, x_n] \to B$. It is the direct limit of its finitely generated sub-A-algebras $\{I_i\}$. Because the direct limit is right exact (it is the left adjoint of the Δ functor, see **[37]**, Exercise 2.6.4), we can conclude that B is the direct limit of the direct system $\{A[x]/I_i, p_{ij} \colon A[x]/I_i \to A[x]/I_j\}$. In particular, the identity map of B splits $B \to A[x]/I_i \to B$ for some index i. Let's give a name to all the maps involved. We will call g_i the induced map $g_i \colon B \to A[x]/I_i$, and we will refer to the projections with the following notations $p_i \colon A[x]/I_i \to B$, $\pi \colon A[x] \to B$ and $\pi_i \colon A[x] \to A[x]/I_i$. We know that $\pi = p_i \pi_i$ for any i, and that $p_k p_{ik} = p_i$, $p_{ik}\pi_i = \pi_k$ for all $k \geq i$. In particular, calling g_k the map $p_{ik} g_i$ for any index $k \geq i$, we have another splitting of the identity $id_B = p_k g_k \colon B \to A[x]/I_k \to B$.

The map we obtain composing $g_i \pi$ needs not to be the same projection map π_i. However, we claim that there exists a suitable index $k \geq i$ such that the map $g_k \pi$ is indeed the same projection map π_k. Because $p_i g_i = id_B$, we know that

$p_i(g_i p_i \pi_i - \pi_i)(x_j) = 0$ for all $j = 1, \dots, n$. Because there are just finitely many j's, we can hence suppose that all the elements $(g_i p_i \pi_i - \pi_i)(x_j)$ lie in I_k/I_i for some index $k \geq i$. This is equivalent to say that the two maps $p_{ik} g_i p_i \pi_i = g_k \pi$ and $p_{ik}\pi_i = \pi_k$ are indeed the same map, as claimed. Now we can write a commutative square of $A[x]$-algebras

$$\begin{tikzcd} A[x] \arrow[r, "\pi"] \arrow[d, equal] & B \arrow[d, "g_k"] \\ A[x] \arrow[r, "\pi_k"] & A[x]/I_k \end{tikzcd}$$

which fits into the following commutative diagram.

$$\begin{tikzcd} A[x] \arrow[r, "\pi_k"] \arrow[d, equal] & A[x]/I_k \arrow[d, "p_k"] \\ A[x] \arrow[r, "\pi"] \arrow[d, equal] & B \arrow[d, "g_k"] \\ A[x] \arrow[r, "\pi_k"] & A[x]/I_k \end{tikzcd}$$

We deduce that the composite map $g_k p_k$ is then another splitting of the map π_k through π_k. By universal property, we then deduce it has to be the identity map. We conclude that the maps g_k and p_k are one the inverse of the other, hence they define an isomorphism $B = A[x]/I_k$. We conclude that B is of finite presentation. □

1.1.17. PROPOSITION. *A collection of maps $\{\operatorname{Spec} A_i \to \operatorname{Spec} A\}_{i \in I}$ is a Zariski covering if and only each map is an open immersion and if there is a finite subset $J \subset I$ such that the collection $\{\operatorname{Spec} A_i \to \operatorname{Spec} A\}_{j \in J}$ reflects isomorphisms of modules, in the sense that a map of A-modules $M \to N$ is an isomorphism if and only if each of the induced maps $M \otimes_A A_j \to N \otimes_A A_j$ is an isomorphism, for all $j \in J$.*

PROOF. We start by proving the necessary condition. We can assume that all the maps are inclusions, since isomorphism don't interfere with the cited properties. Also, because $\operatorname{Spec} A$ is quasi-compact, we can think from the very beginning that the collection is finite. We know that each map $M \otimes_A A_i \to N \otimes_A A_i$ is an isomorphism if and only if it is an isomorphism when localized to all points of $\operatorname{Spec} A_i$ ([**4**], 3.9). Because the subschemes $\operatorname{Spec} A_i$'s form a covering, the collection of all their points is the whole set of points of $\operatorname{Spec} A$. We conclude that all the maps $M \otimes_A A_i \to N \otimes_A A_i$ are isomorphisms if and only if the localizations $M \otimes_A A_{\mathfrak{p}} \to N \otimes_A A_{\mathfrak{p}}$ are isomorphisms, which is equivalent to say that the map $M \to N$ is an isomorphism, as claimed.

As for the sufficient condition, suppose there is a point $\mathfrak{p}$ which is not included in any of the $\operatorname{Spec} A_i$'s. Because each $\operatorname{Spec} A_i$ is open, also $\{\bar{\mathfrak{p}}\} = \{\mathfrak{p}' : \mathfrak{p} \subset \mathfrak{p}'\} =: V(\mathfrak{p})$ is not included in any of those subsets. Now for any $\mathfrak{q}$ which is not in $V(\mathfrak{p})$, we have $A/\mathfrak{p} \otimes_A A_{\mathfrak{q}} = A_{\mathfrak{q}}/\mathfrak{p}A_{\mathfrak{q}} = 0$, where the last equality follows because $\mathfrak{p}$ is not contained in $\mathfrak{q}$. In particular, following the same idea as before, we conclude that all $A/\mathfrak{p} \otimes_A A_i$ are isomorphic to $0 \otimes_A A_i = 0$. But $A/\mathfrak{p} \to 0$ is not an isomorphism of

A modules. In case the spectrum is empty, then $A = 0$ in which case the statement is obvious. $\square$

We conclude this section with a nice set-wise characterization of the spectrum of a ring. It tells us that the Yoneda embedding can be used to reconstruct the spectrum of a ring, at least as a set.

1.1.18. PROPOSITION. *Let A be a ring. Then the set* $\operatorname{Spec} A$ *equals the set*

$$\operatorname{colim} \operatorname{Hom}(\operatorname{Spec} K, \operatorname{Spec} A)$$

as K varies in the category of fields and inclusions.

PROOF. We recall that by Corollary 1.1.5, there is a canonical isomorphism $\operatorname{Hom}(\operatorname{Spec} K, \operatorname{Spec} A) = \operatorname{Hom}(A, K)$. Consider the map $\operatorname{Hom}(A, K) \to \operatorname{Spec} A$ which associates to each morphism its kernel. Because inclusions of fields have a trivial kernel, it naturally induces a map $\varphi\colon \operatorname{colim} \operatorname{Hom}(\operatorname{Spec} K, \operatorname{Spec} A) \to \operatorname{Spec} A$. Given a prime ideal $\mathfrak{p}$ of A, the morphism $A \to \operatorname{Frac}(A/\mathfrak{p})$ has $\mathfrak{p}$ as kernel. This proves that φ is surjective. Now suppose that two maps $f\colon A \to K$ and $f'\colon A \to K'$ have the same kernel $\mathfrak{p}$. Hence, they both split over $A \to \operatorname{Frac}(A/\mathfrak{p})$. In particular, they coincide in the colimit. This proves that φ is injective. $\square$

1.2. "Functorial" schemes

We now change out perspective completely. It is well known (see [**28**], for example) that any category $\mathcal{C}$ can be embedded in the category of presheaves of sets over $\mathcal{C}$ via the Yoneda embedding $X \mapsto \operatorname{Hom}(\cdot, X)$. In particular, geometrical schemes can be seen as particular presheaves of sets over the category of geometrical schemes. By restricting these presheaves to the full subcategory of affine geometrical schemes, we can say that a geometrical scheme is a particular presheaf of sets over affine geometrical schemes. This change of perspective is totally natural. "Considering a scheme X as a presheaves over affine schemes" is an abstract-nonsensical way to define simply the research of A-points in a specified scheme X, by letting A vary in the category of rings. The next result, completely classical, can be formulated by saying that a scheme is fully identified by knowing all its A-points for every ring A. By what we stated about Yoneda embedding, and because each scheme is nothing but a gluing of affine ones, this result is not totally unexpected.

1.2.1. PROPOSITION. *The functor from* $\mathbf{Sch}$ *to* $\mathbf{Psh}(\mathbf{Aff})$ *defined by the Yoneda embedding*

$$X \mapsto h_X = \operatorname{Hom}(\cdot, X)$$

is fully faithful. In particular, we can embed the category of geometrical schemes in $\mathbf{Psh}(\mathbf{Aff})$*, which is equivalent to the category of functors from rings to sets.*

In order to prove the previous proposition, we introduce some basic terminology in category theory which will turn out to be extremely useful also in the upcoming part.

1.2.2. DEFINITION ([**28**], X.6). A subcategory $\mathbf{D}$ of $\mathbf{C}$ is *dense* if for any object X of $\mathbf{C}$ the colimit of the forgetful functor from $\mathbf{D}/X$ to $\mathbf{C}$ exists, and is equal to X.

1.2.3. PROPOSITION. *Let* $\mathbf{D}$ *be a subcategory of* $\mathbf{C}$*. The following are equivalent:*

(i) The category $\mathbf{D}$ *is dense in* $\mathbf{C}$.

(ii) Any object of $\mathbf{C}$ *is a colimit of a cone in which each arrow is in* $\mathbf{D}$.
(iii) The functor $\mathbf{C} \to \mathbf{Psh}(\mathbf{D})$ *defined as* $X \mapsto \mathrm{Hom}_{\mathbf{C}}(\cdot, X)$ *is fully faithful.*

PROOF. The implication $(i) \Rightarrow (ii)$ is clear. Let now $G\colon \mathrm{Hom}_{\mathbf{C}}(\cdot, X) \to \mathrm{Hom}_{\mathbf{C}}(\cdot, Y)$ be a map in $\mathbf{Psh}(\mathbf{D})$, and suppose that X is the colimit of a functor $F\colon J \to \mathbf{D} \hookrightarrow \mathbf{C}$. We can create an arrow from F to ΔY using the canonical arrow $F \to \Delta X$ and the transformation G. This lifts to a unique arrow from X to Y, by universal property of the colimit. This proves (iii). The implication $(iii) \Rightarrow (i)$ follows by the aforementioned fact that a map of functors $(\mathbf{D}/X \to \mathbf{C}) \to \Delta Y$ induces a map in $\mathbf{Psh}(\mathbf{D})$ from $\mathrm{Hom}_{\mathbf{C}}(\cdot, X) \to \mathrm{Hom}_{\mathbf{C}}(\cdot, Y)$, hence, by hypothesis, it induces a unique map from $X \to Y$. In particular, X is the colimit of the forgetful functor $\mathbf{D}/X \to \mathbf{C}$, as claimed (see [**28**], X.6 Proposition 2) □

PROOF OF PROPOSITION 1.2.1. Using Proposition 1.2.3, it suffices to say that each scheme is colimit of affine ones, namely it is the result of gluing the affines which form an affine covering of it. □

1.2.4. PROPOSITION (The co-Yoneda's lemma). *Let* $\mathbf{C}$ *be a category. Every presheaf* $\mathcal{F}$ *in* $\mathbf{Psh}(\mathbf{C})$ *is the colimit of the functor* $\mathbf{C}_{/\mathcal{F}} \to \mathbf{Psh}(\mathbf{C})$ *defined as*

$$F_{\mathcal{F}}\colon (\mathrm{Hom}(\cdot, X) \to \mathcal{F}) \mapsto \mathrm{Hom}(\cdot, X).$$

In particular, the category of representable functors is dense in $\mathbf{Psh}(\mathbf{C})$.

PROOF. We have to prove that $\mathrm{Hom}(F_{\mathcal{F}}, \Delta\mathcal{G}) = \mathrm{Hom}(\mathcal{F}, \mathcal{G})$. An arrow from $F_{\mathcal{F}}$ to $\Delta\mathcal{G}$ is a collection of arrows $\mathrm{Hom}(\cdot, X) \to \mathcal{G}$ for every arrow $\mathrm{Hom}(\cdot, X) \to \mathcal{F}$. By Yoneda's lemma, this defines in particular maps from $\mathcal{F}(X) \to \mathcal{G}(X)$ for every X, and vice versa. The compatibility conditions translate exactly in the property of these maps of defining an arrow from $\mathcal{F}$ to $\mathcal{G}$, which is functorial. □

The following general statements introduce some of the basic properties of topoi which are illustrated in [**29**]. Recall that in a category $\mathbf{C}$ with fibered products, the kernel pair of a map f is the pullback of f with itself while the cokernel pair of f is the pushout of f with itself.

1.2.5. LEMMA. *In a topos, the pullback of an epimorphism is an epimorphism.*

PROOF. Let $f\colon \mathcal{F} \to \mathcal{G}$ be a map in a topos $\mathbf{E}$. A map is an epimorphism if and only if its cokernel pair is constituted by identities. We can then prove that the pullback functor $f^*\colon \mathbf{E}_{/\mathcal{G}} \to \mathbf{E}_{/\mathcal{F}}$ preserves colimits. This is true because it has a right adjoint, namely the dependent product (see [**29**], IV.7.2). □

1.2.6. LEMMA. *In a topos, every epimorphism is the coequalizer of its kernel pair.*

PROOF. Let $e\colon \mathcal{F} \to \mathcal{G}$ be an epimorphism in a topos $\mathbf{E}$. Moving to the topos $\mathbf{E}_{/\mathcal{G}}$, we can suppose that $\mathcal{G}$ is the terminal object $\mathbf{1}$ of $\mathbf{E}$. We then have to prove that $\mathcal{F} \times \mathcal{F} \rightrightarrows \mathcal{F} \to \mathbf{1}$ is a coequalizing diagram. Let $q\colon \mathcal{F} \to \mathcal{C}$ be the coequalizer of the projection maps π_1, π_2 from $\mathcal{F} \times \mathcal{F}$ to $\mathcal{F}$. Since $\mathcal{F} \to \mathbf{1}$ is an epimorphism, also $\mathcal{C} \to \mathbf{1}$ is an epimorphism. We now prove it is also a monomorphism, hence an isomorphism by [**29**] IV.1.2. Consider two maps f, g from an object $\mathcal{X}$ to $\mathcal{C}$. They induce a map $(f, g)\colon \mathcal{X} \to \mathcal{C} \times \mathcal{C}$ such that $\pi'_1(f, g) = f$ and $\pi'_2(f, g) = g$ where π'_i are the two natural projections. We claim that $\pi'_1 = \pi'_2$, hence $f = g$. Let $q \times q$ be the map $\mathcal{F} \times \mathcal{F} \to \mathcal{C} \times \mathcal{C}$. We have $\pi'_1(q \times q) = q\pi_1 = q\pi_2 = \pi'_2(q \times q)$. The map

$q \times q$ is the composite of $q \times id_{\mathcal{F}}$ and $id_{\mathcal{C}} \times q$, each of which is the pullback of an epimorphism via a projection map. By the previous lemma, we deduce that $q \times q$ is an epimorphism, hence $\pi_1' = \pi_2'$ as wanted. □

1.2.7. LEMMA. *The category of sheaves over a site* $(\mathbf{C}, \tau)$ *is distributive, i.e.* $(\coprod X_i) \times_Z Y = \coprod (X_i \times_Z Y)$.

PROOF. Since **Set** is distributive, also $\mathbf{Psh}(\mathbf{C})$ is distributive, being limits and colimits computed componentwise. In the full subcategory of sheaves colimits are constructed applying the sheafification functor sh to colimits built in $\mathbf{Psh}(\mathbf{C})$, while limits stay the same. We also recall that sh preserves finite limits, by its very construction ([**29**], III.5.1). Denoting by $\coprod_{\mathbf{Psh}}$ and $\coprod_{\mathbf{Sh}}$ the coproduct in the category of presheaves and sheaves respectively, we conclude

$$\left(\coprod_{\mathbf{Sh}} X_i\right) \times_Z Y = \mathrm{sh}\left(\coprod_{\mathbf{Psh}} X_i\right) \times_Z Y = \mathrm{sh}\left(\left(\coprod_{\mathbf{Psh}} X_i\right) \times_Z Y\right) =$$
$$= \mathrm{sh}\left(\coprod_{\mathbf{Psh}} (X_i \times_Z Y)\right) = \coprod_{\mathbf{Sh}} (X_i \times_Z Y)$$

hence the claim. □

We now investigate more the category $\mathbf{Psh}(\mathbf{Aff})$. Our ultimate goal is to find a complete characterization of geometrical schemes inside it. We will see that the two characteristic properties will be the Gluing Lemma, and to be locally affine. In the geometrical case, this last property is equivalent to having an affine open covering. We shall now define a suitable notion of open covering in the category of presheaves on affine geometrical schemes.

1.2.8. DEFINITION. An *affine functorial scheme* is an object of $\mathbf{Psh}(\mathbf{Aff})$ represented by an affine geometrical scheme. By Corollary 1.1.10, it is a Zariski sheaf.

1.2.9. DEFINITION. Let $f \colon \mathcal{F} \to \mathcal{G}$ a morphism of Zariski sheaves.

(1) Suppose that $\mathcal{G} = h_{\mathrm{Spec}\, A}$ is an affine functorial scheme. Then f is an *open immersion* if there is a family of open immersions of affine geometrical schemes $\{\mathrm{Spec}\, A_i \to \mathrm{Spec}\, A\}_{i \in I}$ such that $\mathcal{F}$ is the image of the induced map of sheaves $\coprod_{i \in I} h_{\mathrm{Spec}\, A_i} \to h_{\mathrm{Spec}\, A}$.
(2) The map f is an *open immersion* if for any map $h_{\mathrm{Spec}\, A} \to \mathcal{G}$ from an affine functorial scheme to $\mathcal{G}$, the induced morphism $\mathcal{F} \times_{\mathcal{G}} h_{\mathrm{Spec}\, A} \to h_{\mathrm{Spec}\, A}$ is an open immersion.
(3) A family of morphisms $\{\mathcal{F}_i \to \mathcal{F}\}$ is a *Zariski open covering* of $\mathcal{F}$ if each map is an open immersion and the induced morphism $\coprod \mathcal{F}_i \to \mathcal{F}$ is an epimorphism of Zariski sheaves.
(4) An *affine open covering* of $\mathcal{F}$ is an open covering $\{\mathcal{F}_i \to \mathcal{F}\}$ such that each $\mathcal{F}_i$ is an affine functorial scheme.

We remark that if $\mathcal{F} \to h_{\mathrm{Spec}\, A}$ is an open immersion, then $\mathcal{F}$ is a subsheaf of $\mathcal{G}$ (see, for example, [**5**], Sheaves 16.3).

The next two results state that the definitions given are equivalent to the one in [**12**]. We preferred to stick to the one we presented in order to have more symmetry with the second part of this paper.

1.2.10. PROPOSITION. *A family $\{\mathcal{F}_i \to \mathcal{F}\}$ of open immersions of Zariski sheaves over* **Aff** *is a Zariski open covering of $\mathcal{F}$ if and only if for every field K, the induced map $\coprod \mathcal{F}_i(\operatorname{Spec} K) \to \mathcal{F}(\operatorname{Spec} K)$ is a surjection.*

PROOF. By [**29**] III.7.6, the map of sheaves $\coprod \mathcal{F}_i \to \mathcal{F}$ is an epimorphism if and only if for each object $\operatorname{Spec} A$ and each element $\rho \in \mathcal{F}(\operatorname{Spec} A)$, there exists a covering $\{U_j \to \operatorname{Spec} A\}$ such that each $\rho|_{U_j}$ lies in the image of the map $\mathcal{F}_i(U_j) \to \mathcal{F}(U_j)$. Because coverings of $\operatorname{Spec} K$ are trivial, we conclude the necessary condition.

We now turn to the sufficient condition. First of all, we claim that a map of Zariski sheaves $f\colon \mathcal{F} \to \mathcal{G}$ is an epimorphism if and only if it is an epimorphism via any affine base change. One implication is clear from 1.2.5. For the other implication, suppose to have two maps $g_1, g_2\colon \mathcal{G} \to \mathcal{X}$ such that $g_1 f = g_2 f$. Because of the co-Yoneda lemma (Proposition 1.2.4), $\mathcal{G}$ can be written as a colimit of affine schemes, say $\mathcal{G} = \operatorname{colim} X_i$. Suppose that each pullback $\mathcal{F} \times_{\mathcal{G}} h_{X_i} \to h_{X_i}$ is an epimorphism. In particular, we obtain that the two composite maps $h_{X_i} \to \mathcal{G} \to \mathcal{X}$ are the same. Because they uniquely determine the maps g_1 and g_2, we conclude that they have to be equal, as claimed.

In particular, in order to prove that $\coprod \mathcal{F}_i \to \mathcal{F}$ is an epimorphism we can suppose that $\mathcal{F}$ is an affine functorial scheme $h_{\operatorname{Spec} A}$. By substituting each $\mathcal{F}_i$ with its affine parts, we can also suppose that each $\mathcal{F}_i$ is affine, say $h_{\operatorname{Spec} A_i}$. We can also suppose that each A_i is a localization over an element f_i in A. We now claim that the collection of open immersions $\{\operatorname{Spec} A_i \to \operatorname{Spec} A\}$ is a covering in **Aff**. Let $\mathfrak{m}$ be a maximal ideal of A. From the hypothesis, the map $A \to A/\mathfrak{m}$ splits over $A_i = A_{f_i}$ for some i. Hence, the element f_i is not in $\mathfrak{m}$ and therefore $\mathfrak{m}$ lies in $\operatorname{Spec} A_i$. This implies that any prime ideal $\mathfrak{p}$ lies in A_i for some i, hence the claim.

In order to conclude the argument, we are left to prove that if $\{\operatorname{Spec} A_i \to \operatorname{Spec} A\}$ is a covering of $\operatorname{Spec} A$, then $\coprod h_{\operatorname{Spec} A_i} \to h_{\operatorname{Spec} A}$ is an epimorphism of sheaves. Let us call φ_i each map $h_{\operatorname{Spec} A_i} \to h_{\operatorname{Spec} A}$. Let $\mathcal{F}$ be another sheaf, and let f, g be maps from $h_{\operatorname{Spec} A} \to \mathcal{F}$ such that $f\varphi_i = g\varphi_i$ for every i. Hence, by Yoneda's lemma, the maps f, g translate into two elements ρ, σ in $\mathcal{F}(\operatorname{Spec} A)$ such that $\mathcal{F}(\varphi_i)(\rho) = \mathcal{F}(\varphi_i)(\sigma)$ for every i. Since $\mathcal{F}$ is a sheaf and because the φ_i's define a covering, this implies that $\rho = \sigma$, hence $f = g$. This concludes the proof. $\square$

1.2.11. PROPOSITION. *Let $f\colon \mathcal{F} \to \mathcal{G}$ be a morphism of Zariski sheaves, and let $\mathcal{G} = h_{\operatorname{Spec} A}$ be affine. Then f is an open immersion if and only if $\mathcal{F}$ is isomorphic over $\mathcal{G}$ to h_U where U is an open geometrical subscheme of* $\operatorname{Spec} A$.

PROOF. This amounts to prove that for a given family $\{\operatorname{Spec} A_i\}$ of affine subschemes of $\operatorname{Spec} A$, the image of the map $\coprod h_{\operatorname{Spec} A_i} \to h_{\operatorname{Spec} A}$ is h_U, where U is the union of the $\operatorname{Spec} A_i$'s. Since each $\operatorname{Spec} A_i$ is included in U, it is clear that such map splits over h_U. Using [**29**], III.7.7, we also conclude that the map $\coprod h_{\operatorname{Spec} A_i} \to h_U$ is an epimorphism, hence the claim. $\square$

We remark that the first and the second part of the definition agree when taking $\mathcal{G}$ to be affine. Still, this definition overlaps with the definition of open immersions in the geometrical sense, in case we consider affine geometrical schemes. We hence have to check that they do agree in this case.

1.2.12. Proposition. *Let f be a map of affine geometrical (resp. functorial) schemes. It is an open immersion if and only if the induced map of affine functorial (resp. geometrical) schemes is an open immersion.*

Proof. Suppose that $f\colon \operatorname{Spec} B \to \operatorname{Spec} A$ is an open immersion of affine geometrical schemes. To see that the induced map $h_{\operatorname{Spec} B} \to h_{\operatorname{Spec} A}$ is an open immersion, it is sufficient to choose the single-element family $\{\operatorname{Spec} B \to \operatorname{Spec} A\}$.

Consider now an open immersion of affine functorial schemes $f\colon h_{\operatorname{Spec} B} \to h_{\operatorname{Spec} A}$. From the definitions, we conclude that there exists a family $\{A \to B_i\}$ of maps of rings that define open immersions, that split over $A \to B$ and such that $\coprod h_{B_i} \to h_B$ is an epimorhphism. Since each $\operatorname{Spec} B_i \to \operatorname{Spec} A$ splits over $\operatorname{Spec} B \to \operatorname{Spec} A$, we deduce that $\operatorname{Spec} B_i \times_{\operatorname{Spec} A} \operatorname{Spec} B = \operatorname{Spec} B_i$, hence that $\operatorname{Spec} B_i \to \operatorname{Spec} B$ is an open immersion by base change. Using [**29**], III.7.7, we then conclude that the maps $\{B \to B_i\}$ induce a Zariski covering of $\operatorname{Spec} B$. We can also suppose that such family is finite because of the quasi-compactness property of affine schemes. Since each $B \to B_i$ is flat and because of Proposition 1.1.17, we conclude that the map $A \to B$ is flat. Also, since $\operatorname{Spec} B \to \operatorname{Spec} A$ is a monomorphism, we also deduce that $A \to B$ is an epimorphism. We now prove it is of finite presentation. Let then $\{A \to C_\alpha\}$ be a directed system of A-algebras, and let $C = \varinjlim C_\alpha$. Because $A \to B$ is an epimorphism, for any A-algebra X, two A-linear maps $B \rightrightarrows X$ must be equal. Hence, we deduce that both sets $\varinjlim \operatorname{Hom}_A(B, C_\alpha)$ and $\operatorname{Hom}_A(B, C)$ are either empty, or constituted of one element. It suffices to prove that if there exists a A-linear map $B \to C$, then there exists a map $B \to C_{\alpha_0}$ for some index α_0.

Consider the diagrams

$$\prod B_i \rightrightarrows \prod B_i \otimes_B B_j$$
$$\prod C \otimes_A B_i \rightrightarrows \prod C \otimes_A B_i \otimes_B B_j$$

which are connected by vertical arrows induced by the map $B \to C$. Since each B_i and $B_i \otimes_B B_j$ are A-algebras of finite presentation, and since the covering family is finite, we deduce that these vertical arrows split at some level α_0, i.e. on some diagram

$$\prod C_{\alpha_0} \otimes_A B_i \rightrightarrows \prod C_{\alpha_0} \otimes_A B_i \otimes_B B_j.$$

Let now E, F and F_0 be the equalizers of the diagrams presented above, in the category A-**Alg**. By Theorem B.3.10, the quasi-functor that associates to an affine scheme $\operatorname{Spec} B$ over $\operatorname{Spec} A$ the category of B-modules is a stack with respect to the Zariski topology induced in $\mathbf{Aff}_{/A}$. Also, the forgetful functor A-**Alg** $\to$ A-**Mod** commutes with limits, hence, the A-linear maps $B \to E$, $C \otimes_A B \to F$ and $C_{\alpha_0} \otimes_A B \to F_0$ induced by universal property become isomorphisms in the category A-**Mod**. Because a map in A-**Alg** is an isomorphism if and only if it is an isomorphism in the category A-**Mod**, we conclude that the equalizers in the category A-**Alg** of the diagrams presented are B, $C \otimes_A B$ and $C_{\alpha_0} \otimes_A B$ respectively. Hence, the arrows of diagrams induce a A-linear map $B \to C_{\alpha_0}$ which splits $B \to C$, as wanted.

□

1.2.13. DEFINITION. A *functorial scheme* is a Zariski sheaf of sets over affine geometrical schemes that has an affine open covering. The full subcategory of functorial schemes inside the category of Zariski sheaves is denoted with **FSch**.

1.2.14. PROPOSITION ([**34**], Proposition 2.18). *The category* **FSch** *inside the category of Zariski sheaves is stable under open immersions, disjoint unions and fibered products.*

PROOF. Suppose that $\mathcal{F} \to \mathcal{G}$ is an open immersion, and let $\mathcal{G}$ be a scheme with an affine covering $\{\mathcal{G}_i \to \mathcal{G}\}$. We denote with $\mathcal{F}_i$ the sheaf $\mathcal{F} \times_{\mathcal{G}} \mathcal{G}_i$. Since $\mathcal{F} \to \mathcal{G}$ is open, we conclude that $\mathcal{F}_i$ is the image of a map of sheaves $\coprod \mathcal{G}_{ij} \to \mathcal{G}_i$. Because of Lemma 1.2.5, the map $\coprod \mathcal{G}_{ij} \to \mathcal{F}$ is an epimorphism. Because open immersions are stable under base change (this comes directly from the definition) and because $\mathcal{G}_i \to \mathcal{G}$ is open, also $\mathcal{F}_i \to \mathcal{F}$ is open. Also, because the map $\mathcal{G}_{ij} \to \mathcal{G}_i$ is open and has image inside the subsheaf $\mathcal{F}_i$, we conclude that also $\mathcal{G}_{ij} \to \mathcal{F}_i$ is open, hence also the composite map $\mathcal{G}_{ij} \to \mathcal{F}$ is open. We conclude that $\mathcal{F}$ is a scheme that has an affine covering constituted by the collection $\{\mathcal{G}_{ij}\}$.

Now we consider disjoint unions. Suppose that $\{\mathcal{F}_i\}_{i \in I}$ is a family of object in **FSch**, with chosen affine coverings $\{\mathcal{F}_{ij}\}$, and call $\mathcal{F} := \coprod \mathcal{F}_i$ The map induced from $\coprod \mathcal{F}_{ij}$ to $\mathcal{F}$ is obviously epimorphic. We have to prove that each map $\mathcal{F}_{ij} \to \mathcal{F}_i \to \mathcal{F}$ is a Zariski open immersion. We are then left to prove that $\mathcal{F}_i \to \mathcal{F}$ is a Zariski open immersion. Let $\mathcal{G} = h_{\operatorname{Spec} A}$ be an affine functorial scheme over $\mathcal{F}$. This gives a section $f \in \mathcal{F}(\operatorname{Spec} A)$. Because of the sheafification process ([**2**], Lemma II.3.1), the sheaf $\mathcal{F}$ is locally isomorphic to the coproduct of presheaves $\coprod_{\mathbf{Psh}} \mathcal{F}_i$. This means that for a fixed section f in $\mathcal{F}(\operatorname{Spec} A)$, there exists a covering (which we can think to be finite because of the quasi-compactness of affines) $\{\operatorname{Spec} A_j \operatorname{Spec} A\}$ of $\operatorname{Spec} A$ such that the sections $f|_{\operatorname{Spec} A_j} \in \mathcal{F}(\operatorname{Spec} A_j)$ lies in $\coprod \mathcal{F}_i(\operatorname{Spec} A_j)$. Let $a(j)$ be the index of I such that $f|_{\operatorname{Spec} A_j}$ lies inside $\mathcal{F}_{a(j)}(\operatorname{Spec} A_j)$. We can draw the following commutative diagrams.

$$\begin{tikzcd} h_{\operatorname{Spec} A_j} \arrow[r] \arrow[d] & h_{\operatorname{Spec} A} \arrow[d] \\ \mathcal{F}_{a(j)} \arrow[r] & \mathcal{F} \end{tikzcd}$$

We then conclude that $\mathcal{F}_i \times_{\mathcal{F}} h_{\operatorname{Spec} A} \to h_{\operatorname{Spec} A}$ consists of the image of the sheaf map

$$\coprod_{j \in J_i} h_{\operatorname{Spec} A_j} \to h_{\operatorname{Spec} A}$$

where J_i is the subset of J constituted of those $j \in J$ such that $a(j) = i$.

As for fibered products, consider the following diagram in **FSch**.

$$\begin{tikzcd} & \mathcal{F} \arrow[d] \\ \mathcal{G} \arrow[r] & \mathcal{H} \end{tikzcd}$$

We have to conclude that the sheaf $\mathcal{F} \times_{\mathcal{H}} \mathcal{G}$ is a scheme. Let also $\{\mathcal{F}_i \to \mathcal{F}\}$, $\{\mathcal{G}_j \to \mathcal{G}\}$, $\{\mathcal{H}_k \to \mathcal{H}\}$ be affine coverings of $\mathcal{F}$, $\mathcal{G}$ and $\mathcal{H}$ respectively. Because $(\mathcal{F} \times_{\mathcal{H}} \mathcal{G}) \times_{\mathcal{F}} \mathcal{F}_i = \mathcal{F}_i \times_{\mathcal{H}} \mathcal{G}$, we can invoke Lemma B.3.5 and suppose that $\mathcal{F}$ is affine. Analogously, we can also suppose that $\mathcal{G}$ is affine. Now consider the map $\mathcal{F} \times_{\mathcal{H}} \mathcal{H}_k \to \mathcal{F}$. It is an open immersion according to Definition 1.2.9, hence it defines a scheme according to the first part of this proposition. Suppose now that $\{\mathcal{F}_i\}$ is an affine covering of $\mathcal{F}$, consituted by the affine coverings of each $\mathcal{F} \times_{\mathcal{H}} \mathcal{H}_k$. In the same way, we can define an affine covering $\{\mathcal{G}_j\}$ of $\mathcal{G}$. By their very construction, the maps $\mathcal{F}_i \to \mathcal{H}$ and $\mathcal{G}_j \to \mathcal{H}$ split via some map $\mathcal{H}_k \to \mathcal{H}$, which is an monomorphism. In particular, $\mathcal{F}_i \times_{\mathcal{G}} \mathcal{G}_j = \mathcal{F}_i \times_{\mathcal{H}_k} \mathcal{G}_j$, which is an affine scheme because the category of affines has pullbacks. Since each $\mathcal{F}_i \times_{\mathcal{H}} \mathcal{G}_j$ is isomorphic to $(\mathcal{F} \times_{\mathcal{H}} \mathcal{G}_j) \times_{\mathcal{F}} \mathcal{F}_i$ and $\mathcal{F} \times_{\mathcal{H}} \mathcal{G}_j$ is isomorphic to $(\mathcal{F} \times_{\mathcal{H}} \mathcal{G}) \times_{\mathcal{G}} \mathcal{G}_j$, we can use Lemma B.3.5 twice and conclude that $\mathcal{F} \times_{\mathcal{H}} \mathcal{G}$ is a scheme. □

The next proposition clarifies that Zariski coverings define a topology on functorial schemes.

1.2.15. PROPOSITION. *Zariski coverings define a Grothendieck pretopology on functorial schemes. The site they form is again called the* Zariski site.

PROOF. We prove stability under base change. Suppose that $\{\mathcal{F}_i \to \mathcal{F}\}$ is a covering family for $\mathcal{F}$ and that $\mathcal{G} \to \mathcal{F}$ is a functorial scheme over $\mathcal{F}$. By Lemma 1.2.6, the induced map $\coprod \mathcal{F}_i \times_{\mathcal{F}} \mathcal{G} \to \mathcal{G}$ is the map $(\coprod \mathcal{F}_i) \times_{\mathcal{F}} \mathcal{G}$ which is a pullback of an epimorphism, hence an epimorphism by Lemma 1.2.5. Since the definition of an open immersion is given through affine base change, the other axioms of a Grothendieck pretopology are easily verified. □

We now want to analyze further the properties of the Zariski topology we defined. Once again, our primary source of inspiration is the category of geometrical schemes.

1.2.16. PROPOSITION. *Let $\{h_{\mathrm{Spec}\, A_i}\}$ be an affine open covering of a functorial scheme $\mathcal{F}$, and let $\{h_{\mathrm{Spec}\, A_{ijk}}\}$ be an affine open covering of the functorial scheme $h_{\mathrm{Spec}\, A_i} \times_{\mathcal{F}} h_{\mathrm{Spec}\, A_j}$ (Proposition 1.2.14). Then $\mathcal{F}$ is the coequalizer in the diagram below*

$$\coprod h_{\mathrm{Spec}\, A_{ijk}} \rightrightarrows \coprod h_{\mathrm{Spec}\, A_i} \to \mathcal{F}$$

PROOF. We now work in the category of sheaves over **Aff**, where **FSch** is a full subcategory. By the Lemma 1.2.6, we know that $\mathcal{F}$ is the coequalizer in the diagram

$$\left(\coprod h_{\mathrm{Spec}\, A_i}\right) \times_{\mathcal{F}} \left(\coprod h_{\mathrm{Spec}\, A_i}\right) \rightrightarrows \coprod h_{\mathrm{Spec}\, A_i} \to \mathcal{F}$$

However, by the Lemma 1.2.7 we know that

$$\left(\coprod h_{\mathrm{Spec}\, A_i}\right) \times_{\mathcal{F}} \left(\coprod h_{\mathrm{Spec}\, A_i}\right) = \coprod \left(h_{\mathrm{Spec}\, A_i} \times_{\mathcal{F}} h_{\mathrm{Spec}\, A_j}\right).$$

We remark that if f is an epimorphism, then the coequalizer of g, h is the same as the coequalizer of gf, hf since $agf = bhf \Leftrightarrow ag = bh$. We then conclude that $\mathcal{F}$ is also the coequalizer in the diagram

$$\coprod h_{\mathrm{Spec}\, A_{ij}} \to \coprod (h_{\mathrm{Spec}\, A_i} \times_{\mathcal{F}} h_{\mathrm{Spec}\, A_j}) \rightrightarrows \coprod h_{\mathrm{Spec}\, A_i} \to \mathcal{F}$$

as wanted. □

With the notations introduced in 1.2.3, the previous proposition has immediate corollaries.

1.2.17. COROLLARY. *The category of affine schemes is dense in* **FSch**.

PROOF. This amounts to notice that, by the previous proposition, each functorial scheme is a colimit of a diagram constituted of only affines. $\square$

1.2.18. COROLLARY. *The Zariski topology restricted to* **FSch** *is subcanonical.*

PROOF. Let $\mathcal{G}$ be a functorial scheme. Apply the left-exact functor $\mathrm{Hom}(\cdot, \mathcal{G})$ to the diagram shown in the proof of Proposition 1.2.16. $\square$

1.3. Geometrical - functorial equivalence

Up to now, we presented the two notions of geometrical schemes and functorial schemes. We have also introduced the Yoneda functor from **Sch** to **Psh**(**Aff**). We will now prove that it respects the two topologies we introduced, and use this fact to see that its image lies in **FSch**. We will also construct another functor from **FSch** to **LRS**, and prove that it determines an equivalence (1.3.11). This main result will finally let us use the term "scheme" referring to both geometrical and functorial ones.

We now define the functor from functorial schemes to geometrical schemes that realizes the equivalence together with Yoneda functor. Its definition may seem over complicated at first glance. The idea behind it is fairly simple though. Consider the category of geometrical schemes over X. Here X is obviously the terminal object. Consider now the subcategory of affine schemes over X. Here there is no terminal object in general (if X is not affine), but X can be reconstructed as the colimit of the natural functor from this category to the whole category of geometrical schemes.

1.3.1. DEFINITION. Let $\mathcal{F}$ be a presheaf over affine geometrical schemes. Consider the category $(* \downarrow \mathcal{F})$ whose objects are couples $(\mathrm{Spec}\, A, \rho)$ where ρ is an element of $\mathcal{F}(\mathrm{Spec}\, A)$, and whose arrows between $(\mathrm{Spec}\, A, \rho)$ and $(\mathrm{Spec}\, B, \sigma)$ are arrows $f\colon \mathrm{Spec}\, A \to \mathrm{Spec}\, B$ such that $\mathcal{F}(f)(\sigma) = \rho$. Let $d_{\mathcal{F}}$ be the forgetful functor from this category to locally ringed spaces $(\mathrm{Spec}\, A, \rho) \mapsto \mathrm{Spec}\, A$. The *geometrical realization* $|\mathcal{F}|$ of $\mathcal{F}$ is the colimit of $d_{\mathcal{F}}$. Now let f be a map $\mathcal{F} \to \mathcal{G}$. We define $|f|$ to be the map $|\mathcal{F}| \to |\mathcal{G}|$ induced by the composition with f.

This explicit definition is classical, and appears in [**12**]. As already noted in that book, it is indeed a special case of a more general setting in which a neat property of the Yoneda embedding is used.

1.3.2. PROPOSITION. *Let F be a functor for a category* **C** *to a cocomplete category* **D**. *Then F splits as the Yoneda embedding* $\mathbf{C} \to \mathbf{Psh}(\mathbf{C})$ *followed by a colimit-preserving functor* $\tilde{F}\colon \mathbf{Psh}(\mathbf{C}) \to \mathbf{D}$. *Any other colimit-preserving splitting $\tilde{F}'$ is canonically isomorphic to $\tilde{F}$.*

PROOF. Suppose **D** is a cocomplete category, and let $F\colon \mathbf{C} \to \mathbf{D}$ be a functor. In order to for a commutative diagram with the Yoneda embedding, we must put $\mathrm{Hom}(\cdot, X) \mapsto F(X)$. Since the full subcategory of representable presheaves is dense in **Psh**(**C**), this functor extends naturally to the whole category of presheaves by posing $\mathrm{colim}_X \mathrm{Hom}(\cdot, X) \mapsto \mathrm{colim}_X F(X)$. Any other choice of functor is related to another choice of colimits, hence it is canonically isomorphic to the preceding one. By [**23**] Theorem 2.7.1, set-theoretical questions do not arise, since there is no need to enlarge the chosen universe (see also [**23**], Theorem 2.3.3). $\square$

We cannot call the previous property "universal" since the induced functors are not uniquely defined, but just up to a unique isomorphism of functors. Still, we may refer to it as a universal property in the sense of 2-categories, or pseudo-universal property.

1.3.3. PROPOSITION. *The geometric realization is the functor defined by the pseudo-universal property of Proposition 1.3.2 and the inclusion* $\mathbf{Aff} \to \mathbf{LRS}$.

PROOF. We claim that any presheaf of affines $\mathcal{F}$ is the colimit of the following functor.

$$(* \downarrow \mathcal{F}) \to \mathbf{Psh}(\mathbf{Aff})$$
$$(\operatorname{Spec} A, \rho \in \mathcal{F}(\operatorname{Spec} A)) \mapsto \operatorname{Hom}(\cdot, \operatorname{Spec} A)$$

Indeed, by Yoneda's lemma, this functor is exactly the one defined in Proposition 1.2.4. Applying the functor defined by the inclusion $\mathbf{Aff} \to \mathbf{LRS}$, we then obtain exactly the geometric realization functor $|\cdot|$, and the proposition is proven. □

The pseudo-universal property of the Yoneda embedding has another crucial consequence. The following result also appears in [**3**], Proposition 1.45.

1.3.4. PROPOSITION. *Let F be a functor from a category $\mathbf{C}$ to a cocomplete category $\mathbf{D}$. The functor $\tilde{F}\colon \mathbf{Psh}(\mathbf{C}) \to \mathbf{D}$ defined by the pseudo-universal property (1.3.2) has a right adjoint $D \mapsto \operatorname{Hom}_{\mathbf{D}}(F(\cdot), D)$.*

PROOF. Suppose $C = \operatorname{colim} \operatorname{Hom}(\cdot, X)$ is a generic element in $\mathbf{Psh}(\mathbf{C})$. By the definition of the functor $\tilde{F}$ and the strong version of Yoneda's lemma ([**28**], III.2) we obtain the following equalities

$$\operatorname{Hom}_{\mathbf{Psh}(\mathbf{C})}(C, \operatorname{Hom}_{\mathbf{D}}(F(\cdot), D)) =$$
$$\lim \operatorname{Hom}_{\mathbf{Psh}(\mathbf{C})}(\operatorname{Hom}_{\mathbf{C}}(\cdot, X), \operatorname{Hom}_{\mathbf{D}}(F(\cdot), D)) =$$
$$\lim \operatorname{Hom}_{\mathbf{D}}(F(X), D) = \operatorname{Hom}_{\mathbf{D}}(\operatorname{colim} F(X), D) = \operatorname{Hom}_{\mathbf{D}}(\tilde{F}(X), D).$$

which end the proof. □

1.3.5. COROLLARY. *The Yoneda functor h and the geometrical realization functor $|\cdot|$ form an adjoint couple from $\mathbf{LRS}$ to $\mathbf{Psh}(\mathbf{Aff})$. In particular, for any locally ringed space X and any presheaf $\mathcal{F}$*

$$\operatorname{Hom}(\mathcal{F}, h_X) = \operatorname{Hom}(|\mathcal{F}|, X).$$

PROOF. It follows from the previous proposition and Proposition 1.3.3. □

We now want to specify better what we meant in our introductory motivation of the definition of the functor $|\cdot|$, where we claimed that the definition made sense since any geometrical scheme can be reconstructed as a special colimit of affine ones. In explaining this in the proof, we also prove easily a step towards the main theorem of the section.

1.3.6. PROPOSITION. *For any geometrical scheme X, there is a natural isomorphism $|h_X| \cong X$. In particular, for any affine geometrical scheme X and any affine functorial scheme $\mathcal{F}$, there are natural isomorphisms $X \cong |h_X|$ and $\mathcal{F} \cong h_{|\mathcal{F}|}$.*

PROOF. We consider the image of a representable functor h_X via $|\cdot|$

$$h_X \mapsto \operatorname{colim}(\mathbf{Aff}_{/X} \to \mathbf{LRS}) = |h_X|.$$

By Proposition 1.2.3, we know that $\operatorname{colim}(\mathbf{Aff}_{/X} \to \mathbf{Sch})$ exists and it is equal to X, hence we have a map $|h_X| \to X$. We also know that X is the colimit in $\mathbf{LRS}$ of the gluing diagram induced by an affine open covering, which is embedded in the colimiting diagram $\mathbf{Aff}_{/X} \to \mathbf{LRS}$. Hence we have also a map $X \to |h_X|$, which determines an isomorphism. $\square$

1.3.7. PROPOSITION. *Let $f\colon X \to Y$ be a map of geometrical schemes. If f is an open immersion, then $h_f\colon h_X \to h_Y$ is an open immersion.*

PROOF. Since f is an open immersion, also the map obtained by base change $X \times_Y \operatorname{Spec} A \to \operatorname{Spec} A$ via any affine scheme $\operatorname{Spec} A$ over Y is an open immersion. Because the Yoneda functor commutes with limits (Corollary 1.3.5) and because of of Proposition 1.2.11, we conclude that all maps $h_X \times_{h_Y} h_{\operatorname{Spec} A} \to h_{\operatorname{Spec} A}$ are open immersions, hence h_f is an open immersion, as claimed. $\square$

1.3.8. PROPOSITION. *Let X be a geometrical scheme. Then the associated presheaf $h_X = \operatorname{Hom}(\cdot, X)$ is a functorial scheme.*

PROOF. The first condition of the definition of a functorial scheme follows by applying Corollary 1.1.10. The second part follows by applying Yoneda's lemma to the surjection $\coprod \operatorname{Spec} A_i \to X$. Indeed, each of the maps $h_{\operatorname{Spec} A_i} \to h_X$ is an open immersion, thanks to Proposition 1.3.7. We are left to prove that $\{h_{\operatorname{Spec} A_i} \to h_X\}$ is a covering. Let $x \in X$ and let $k(x) \hookrightarrow K$ be an inclusion of fields associated to a map f in $\operatorname{Hom}(\operatorname{Spec} K, X)$. Let $\operatorname{Spec} A_i$ be one affine open set of the covering that contains x. Then the map f pulls back to $\operatorname{Spec} A_i$. Hence $\coprod h_{\operatorname{Spec} A_i}(\operatorname{Spec} K) \to h_X(\operatorname{Spec} K)$ is surjective. By Proposition 1.2.10, we conclude the claim. $\square$

1.3.9. THEOREM. *If $\mathcal{F}$ is a functorial scheme with an affine open covering $\{f_i\colon h_{\operatorname{Spec} A_i} \to \mathcal{F}\}$, then $|\mathcal{F}|$ is a geometrical scheme with an affine open covering $\{|f_i|\colon \operatorname{Spec} A_i \to |\mathcal{F}|\}$.*

PROOF. By Proposition 1.2.16, we can write $\mathcal{F}$ as a colimit of the diagram

$$\coprod h_{\operatorname{Spec} A_i} \times_{\mathcal{F}} h_{\operatorname{Spec} A_j} \rightrightarrows \coprod h_{\operatorname{Spec} A_i} \to \mathcal{F}$$

where all the arrows are open immersions. Indeed, by their very definition, open immersions are stable under affine base change, hence $h_{\operatorname{Spec} A_i} \times_{\mathcal{F}} h_{\operatorname{Spec} A_j} \to h_{\operatorname{Spec} A_i}$ is an open immersion. Hence, because of Proposition 1.2.11, we can write a map $h_{\operatorname{Spec} A_i} \times_{\mathcal{F}} h_{\operatorname{Spec} A_j} \to h_{\operatorname{Spec} A_i}$ as $h_{U_{ij}} \to h_{\operatorname{Spec} A_i}$. Also, since the geometrical realization functor commutes with colimits, we obtain the following coequalizing diagram

$$\coprod |h_{U_{ij}}| \rightrightarrows \coprod |h_{\operatorname{Spec} A_i}| \to |\mathcal{F}|$$

which is in turn equivalent to the following diagram, by Proposition 1.3.6

$$\coprod U_{ij} \rightrightarrows \coprod \operatorname{Spec} A_i \to |\mathcal{F}|.$$

We then conclude that $|\mathcal{F}|$ is a geometrical scheme that has $\{\operatorname{Spec} A_i\}$ as affine open covering. $\square$

1.3.10. THEOREM. *If $\mathcal{F}$ is a functorial scheme, then the map $\mathcal{F} \to h_{|\mathcal{F}|}$ is invertible.*

PROOF. Let $\{h_{\operatorname{Spec} A_i}\}$ be an affine covering of $\mathcal{F}$. Because of Proposition 1.3.7 and Theorem 1.3.9, it is also a covering of $h_{|\mathcal{F}|}$. Because the Zariski topology is subcanonical (Corollary 1.2.18), a map $h_{|\mathcal{F}|} \to \mathcal{F}$ can be defined as a collection of maps $h_{\operatorname{Spec} A_i} \to \mathcal{F}$ which is compatible with the topology. It is then sufficient to choose the covering maps $h_{\operatorname{Spec} A_i} \to \mathcal{F}$. $\square$

1.3.11. THEOREM. *The Yoneda functor is part of an equivalence of categories between geometrical schemes and functorial schemes.*

PROOF. This follows by Proposition 1.3.8, Theorem 1.3.9, Proposition 1.3.6 and Theorem 1.3.10. □

It is easy to see that the equivalence of categories respects the topology of the two sites.

1.3.12. PROPOSITION. *A morphism in* **Sch** *is an open immersion of geometrical schemes if and only the induced morphism in* **FSch** *is an open immersion of geometrical schemes. For a fixed geometrical scheme X, a collection of maps in over* $\mathbf{Sch}_{/X}$ *is an open Zariski covering of X with respect to Definition 1.1.6 if and only if the collection of induced maps in* $\mathbf{FSch}_{/h_X}$ *is an open Zariski covering of* h_X *with respect to Definition 1.2.9.*

PROOF. The first claim follows from the fact that open coverings in both cases can be defined as maps that are open immersions by any affine base change (use [**16**], I.4.2.4 and Definition 1.2.9), and in the affine case the two notions do agree. For coverings, it suffices to write down the associate coequalizing diagrams and use the gluing lemma. □

We conclude presenting a nice characterization of the underlying set of the geometrical realization of a presheaf. It can be considered an extension of Proposition 1.1.18; which is in fact the only non-trivial fact involved in its proof.

1.3.13. PROPOSITION. *Let $\mathcal{F}$ a presheaf over affine geometrical schemes. Let ^{e} be the functor that associates to each ringed space its underlying set and let $\mathcal{K}$ the subcategory of affine schemes constituted of spectra of fields. Then* $|\mathcal{F}|^e \cong \operatorname{colim} \mathcal{F}|_{\mathcal{K}}$.

PROOF. The functor ^{e} can be seen as a composite of two forgetful functors. The first one is the one that forgets the sheaf of rings over a ringed space, and the second one is the one that forgets the topology over a set. Both these functors have right adjoints, given respectively by the functor $X \mapsto (X, \mathbb{Z})$ that adds to constant sheaf $\mathbb{Z}$, and the functor that provides a set with the discrete topology. Because of Proposition 1.1.2, a colimit in the category of locally ringed spaces can be equally seen as a colimit in the category of ringed spaces. We can then conclude that the functor ^{e} commutes with colimits taken in the category of locally ringed spaces. We can then write

$$|\mathcal{F}|^e = (\operatorname{colim} d_{\mathcal{F}})^e = \operatorname*{colim}_{(\operatorname{Spec} A, \rho)} (\operatorname{Spec} A)^e \xrightarrow{\sim} \operatorname*{colim}_{(\operatorname{Spec} A, \rho)} \operatorname*{colim}_{K} \operatorname{Hom}(\operatorname{Spec} K, \operatorname{Spec} A)$$

$$\xrightarrow{\sim} \operatorname*{colim}_{K} \operatorname*{colim}_{(\operatorname{Spec} A, \rho)} \operatorname{Hom}(\operatorname{Spec} K, \operatorname{Spec} A) \xrightarrow{\sim} \operatorname*{colim}_{K} \mathcal{F}(\operatorname{Spec} K) = \operatorname{colim} \mathcal{F}|_{\mathcal{K}}$$

where the natural isomorphisms are deduced by commutativity of ^{e} with colimits, by Proposition 1.1.18, by the commutativity of colimits, and the definition of $|\cdot|$. □

The functorial perspective might seem at first glance overcomplicated, especially when compared to the equivalent geometrical approach. We now want to present an example in which the functorial point of view really adds some taste to the theory. We will define the scheme $\mathbb{P}^1_{\mathbb{Z}}$ as a functorial scheme. As a matter of fact, the functorial perspective is used by Grothendieck himself in [**21**], I.9.7.4 in order to define Grassmanians and projective spaces.

1.3.14. EXAMPLE. Let $\mathbb{P}^1$ be the functor from **Ring** to **Set** that associates to each ring A the set of the submodules L of $A \times A$ such that $(A \times A)/L$ is locally free of rank one, and to a map of rings $A \to B$, the map of sets that associates to a module $L \subset A \times A$ the image of $L \otimes_A B \to B \times B$. We claim it is a functorial sheaf.

Since line bundles can be glued over Zariski topology, it is immediate to see that the functor just defined is indeed a Zariski sheaf. We are then left to prove that $\mathbb{P}^1$ has an open affine covering in the sense of Definition 1.2.9.

We denote with $\mathbb{G}_a$ the forgetful functor from rings to sets, i.e. the affine functorial scheme represented by $\operatorname{Spec} \mathbb{Z}[t]$. We can define two arrows f_1, f_2 from $\mathbb{G}_a$ to $\mathbb{P}^1_{\mathbb{Z}}$ in the following way: for each ring A, we consider the maps $f_{i,A} \colon A \to \mathbb{P}^1$ such that $f_{1,A}(a) = \langle (1,a) \rangle$ and $f_{2,A}(a) = \langle (a,1) \rangle$ respectively. We claim that these coincide with the subfunctors of $\mathbb{P}^1$ that associate a ring A to those A-modules L of $\mathbb{P}^1(A)$ such that the composite map

$$p_i \colon L \hookrightarrow A \times A \xrightarrow{\pi_i} A$$

is an isomorphism, where π_1 and π_2 are the natural projections. Indeed, any such isomorphism has an inverse $A \to L$ which induces a map $A \to A \times A$ that splits π_i, hence it is determined by an element of $A \times A$ that has 1 at the i-th place. We also claim that these subfunctors coincide with those that associate to a ring A the set of A-modules L of $\mathbb{P}^1(A)$ such that p_i is an epimorphism. Indeed, if $p_i \colon L \to A$ is surjective, then $L \cong A \oplus \ker p_i$, because there is a splitting of p_i. Because both A and L have the same rank at each prime, by Nakayama lemma we conclude that $\ker p_i = 0$.

We can now prove that the two maps f_i are open immersions. In order to do this, we have to prove that base change with any affine functorial scheme $h_{\operatorname{Spec} A}$ is an open immersion, i.e. it is isomorphic to a map $h_U \to h_{\operatorname{Spec} A}$ where U is an open subscheme of $\operatorname{Spec} A$. Suppose then to have a map $h_{\operatorname{Spec} A} \to \mathbb{P}^1$. This is equivalent to choose an element of $\mathbb{P}^1(A)$, hence a submodule L of $A \times A$. The pullback of this map with π_i defines a subfunctor $\mathcal{F}_i$ of $h_{\operatorname{Spec} A}$ that associates to each affine scheme $\operatorname{Spec} B$ the set of maps of rings $\varphi \colon A \to B$ such that $p_i \colon L \otimes_A B \to B$ is surjective. This is equivalent to say that each prime ideal $\mathfrak{p}$ of B does not contain the set $p_i(L \otimes A)$, hence that the ideal $\varphi^{-1}(\mathfrak{p})$ does not contain the set $p_i(L)$. Since $p_i(L)$ is an ideal of A, we conclude that the subfunctor $\mathcal{F}_i$ is indeed represented by the open subscheme of $\operatorname{Spec} A$ which is the complement of the closed subset $\operatorname{Spec}(A/p_i(L))$.

Now we want to show that $\mathbb{P}^1$ is not affine. First of all, let's compute the pullback of the maps f_1, f_2. Call this functor $\mathcal{F}$. We then have the following cartesian diagram.

$$\begin{tikzcd} \mathcal{F} \arrow[r, "g_1"] \arrow[d, "g_2"'] & \mathbb{G}_a \arrow[d, "f_1"] \\ \mathbb{G}_a \arrow[r, "f_2"] & \mathbb{P}^1 \end{tikzcd}$$

Fix a ring A. Since the section functor $\Gamma(\cdot, A)$ is left exact ([**29**], III.6), we conclude by the previous diagram that the submodule $\langle (g_1(x), 1) \rangle$ is equal to the submodule $\langle (1, g_2(x)) \rangle$ in $A \times A$, for all $x \in \mathcal{F}(A)$. This is equivalent to say that $g_1(x)g_2(x) = 1$. We conclude that $\mathcal{F} = \mathbb{G}_m$, the functor that associates to each ring its multiplicative group, i.e the affine functorial subscheme represented by the ring $\mathbb{Z}[t, t^{-1}]$. Also, the maps g_1 and g_2 are the maps that act on sections by sending an invertible element x to x itself, and to x^{-1} respectively. The previous diagram can be hence rewritten in the following way.

$$\begin{tikzcd} \mathbb{G}_m \arrow[r, "t\mapsto t"] \arrow[d, "t\mapsto t^{-1}"'] & \mathbb{G}_a \arrow[d, "f_1"] \\ \mathbb{G}_a \arrow[r, "f_2"] & \mathbb{P}^1 \end{tikzcd}$$

We also notice that the pullback of each f_i with itself is constituted by identities. This is because the maps f_i define subfunctors of $\mathbb{P}^1$, hence monomorphisms. We can now use the abstract nonsense introduced in the previous section. In particular, because of Lemmas 1.2.7 and 1.2.6, we conclude that $\mathbb{P}^1$ is the coequalizer in the following diagram

$$(\mathbb{G}_{a,1}\times_{\mathbb{P}^1}\mathbb{G}_{a,1})\sqcup(\mathbb{G}_{a,1}\times_{\mathbb{P}^1}\mathbb{G}_{a,2})\sqcup(\mathbb{G}_{a,2}\times_{\mathbb{P}^1}\mathbb{G}_{a,1})\sqcup(\mathbb{G}_{a,2}\times_{\mathbb{P}^1}\mathbb{G}_{a,2}) \rightrightarrows \mathbb{G}_{a,1}\sqcup\mathbb{G}_{a,2} \to \mathbb{P}^1$$

where we introduce the indexes $1, 2$ in order to refer to the maps f_1, f_2. By what we just said about the pullbacks of f_i with itself, we conclude that the maps starting from $\mathbb{G}_{a,1} \times_{\mathbb{P}^1} \mathbb{G}_{a,1} \cong \mathbb{G}_{a,2} \times_{\mathbb{P}^1} \mathbb{G}_{a,2} \cong \mathbb{G}_a$ are identities. Also, the two couples of maps starting from each copy of

$$\mathbb{G}_m \cong \mathbb{G}_{a,1} \times_{\mathbb{P}^1} \mathbb{G}_{a,2} \cong \mathbb{G}_{a,2} \times_{\mathbb{P}^1} \mathbb{G}_{a,1}$$

are the same. Hence, the previous coequalizing diagram can be rewritten in the following way

$$\mathbb{G}_m \rightrightarrows \mathbb{G}_{a,1} \sqcup \mathbb{G}_{a,2} \to \mathbb{P}^1$$

which is equivalent to say that the square

$$\begin{tikzcd} \mathbb{G}_m \arrow[r, "t\mapsto t"] \arrow[d, "t\mapsto t^{-1}"'] & \mathbb{G}_a \arrow[d, "f_1"] \\ \mathbb{G}_a \arrow[r, "f_2"] & \mathbb{P}^1 \end{tikzcd}$$

not only is cartesian, but it is also cocartesian. We now apply to this square the functor $\mathrm{Hom}(\cdot, \mathbb{G}_a)$. We will call this functor $\mathcal{O}$. Indeed, beacause of Proposition 1.1.3 and Corollary 1.3.5, there holds the following series of natural isomorphisms, for any functorial scheme $\mathcal{F}$.

$$\mathcal{O}(\mathcal{F}) = \mathrm{Hom}(\mathcal{F}, h_{\mathrm{Spec}\,\mathbb{Z}[t]}) \cong \mathrm{Hom}(|\mathcal{F}|, \mathrm{Spec}\,\mathbb{Z}[t]) \cong \mathrm{Hom}(\mathbb{Z}[t], \mathcal{O}(|\mathcal{F}|)) \cong \mathcal{O}(|\mathcal{F}|)$$

Because $\mathcal{O}$ takes colimits to limits, we conclude that the following square of rings is cartesian.

$$\begin{tikzcd} \mathcal{O}(\mathbb{P}^1) \arrow[r] \arrow[d] & \mathbb{Z}[t] \arrow[d, "t\mapsto t"] \\ \mathbb{Z}[t] \arrow[r, "t\mapsto t^{-1}"] & \mathbb{Z}[t, t^{-1}] \end{tikzcd}$$

Hence, $\mathcal{O}(\mathbb{P}^1)$ is $\mathbb{Z}$. Suppose now $\mathbb{P}^1$ is affine. Hence, $\mathbb{P}^1 = h_{\mathrm{Spec}\,\mathbb{Z}}$. This is absurd because $\sharp\mathbb{P}^1(k) > 1$ for any field k. We then proved that $\mathbb{P}^1$ is a functorial scheme which is not affine.

Notice that the functorial scheme just defined does coincide with the functor $h_{\mathbb{P}^1_{\mathbb{Z}}}$, defined using the standard (geometrical) construction of $\mathbb{P}^1_{\mathbb{Z}}$. This is because defining a map to $\mathbb{P}^1_{\mathbb{Z}}$ is equivalent to choosing two sections of a line bundle which are not simultaneously zero at any point, i.e. that generate the line bundle at every point ([**22**], Theorem II.7.1). Since surjectivity of modules is a local property, and because line bundles of an affine scheme $\operatorname{Spec} A$ are just A-modules which are locally free of rank one, this implies that defining a map from $\operatorname{Spec} A$ to $\mathbb{P}^1_{\mathbb{Z}}$ is equivalent to choosing a surjective map $A \times A \to P$ where P is locally free of rank one, up to isomorphisms. Hence, considering kernels, it is equivalent to choose submodules of $A \times A$ whose associated quotients are locally free of rank one, as claimed.

We want now to summarize all the work that has been done up to now. This pattern will be essential in the future, and can be considered as the real motivation of all the statements proved in the previous sections. It is a complete translation of the classical definition of a scheme into purely categorical notions, which can be easily generalized changing the "affines category" beneath.

1.3.15. Scholium. A scheme is characterized by the three following properties.

(1) It is a presheaf on the category of affine schemes, which is equivalent to $\mathbf{Ring}^{\mathrm{op}}$.
(2) It is a sheaf with respect to the Zariski topology. We can define a Zariski affine open of $\operatorname{Spec} A$ as an affine scheme $\operatorname{Spec} B$ with a map $\operatorname{Spec} B \to \operatorname{Spec} A$ such that the induced map of rings has the following three properties: (i) is epi, (ii) is such that base extension of modules is exact, (ii) $\operatorname{Hom}(B, \cdot)$ commutes with direct limits in the category of A-algebras, which is the category ${}^{A/}\mathbf{Ring}$.
(3) It has a open affine Zariski covering. We can define a covering to be a set of maps that is globally an epimorphism of sheaves. We can define an open subscheme of an affine scheme as a subsheaf which is the image of the map $\coprod h_{\operatorname{Spec} A_i} \to h_{\operatorname{Spec} A}$ induced by some Zariski opens $\operatorname{Spec} A_i$ of $\operatorname{Spec} A$; we can then define an open subscheme of X as a map that is an open subscheme via any affine base change.

CHAPTER 2

Schemes over $\mathbb{F}_1$

2.1. Schemes over $\mathbb{F}_1$ à la Dietmar

We want to present the notion of affine scheme over $\mathbb{F}_1$, keeping in mind the classical definition of "geometrical" schemes over $\mathbb{Z}$. The following definitions were presented by Kato in [**24**] and Deitmar in [**11**]. In the latter paper, the author shows that the operation of the sum in rings can be overlooked for many purposes, and some of the basic notions and facts can be straightforwardly generalized to a broader context.

2.1.1. Definition. Let M be a monoid.

(1) A subset I of M is an *ideal* if the set
$$IM := \{xm\colon x \in I, m \in M\}$$
equals I.

(2) An ideal I is *prime* if $M \setminus I$ is a submonoid of M.

(3) The *radical* of an ideal I is the set of all elements $x \in M$ for which there exists a positive integer k such that $x^k \in I$. It is an ideal of M, and it is denoted with $\sqrt{I}$.

(4) The *prime spectrum of M over* $\mathbb{F}_1$ is the topological set of all prime ideals $\mathfrak{p}$ of M, with the topology generated by the closed sets $V(I) := \{\mathfrak{p}\colon I \subset \mathfrak{p}\}$, where I is a subset of M. It is indicated with $\operatorname{Spec}_{\mathbb{F}_1}(M)$ (or simply with $\operatorname{Spec} M$ if the context is clear). The topology on $\operatorname{Spec}_{\mathbb{F}_1}(M)$ is called the *Zariski topology.*

2.1.2. Proposition. *Let $\varphi\colon M \to N$ be a map of monoids.*

(1) If I is an ideal of N, then $\varphi^{-1}(I)$ is an ideal of M.

(2) If $\mathfrak{p}$ is a prime ideal of N, then $\varphi^{-1}(\mathfrak{p})$ is a prime ideal of M.

Proof. Let I be an ideal of N and $x \in M$ such that $\varphi(x) \in I$, and let $m \in M$. Then $\varphi(xm) = \varphi(x)\varphi(m) \in I$. This proves the first claim. Let now $\mathfrak{p}$ be a prime ideal of N and $x, y \in M$ such that $\varphi(x) \notin \mathfrak{p}$, $\varphi(y) \notin \mathfrak{p}$. Then $\varphi(xy) = \varphi(x)\varphi(y) \notin \mathfrak{p}$. This proves the second claim. $\square$

We remark that in case we consider a ring A, an additive subgroup I is an ideal of $(A, +, \cdot)$ if and only if it is an ideal of $(A, \cdot)$; and the two notions of prime ideals do coincide. However, the set $\operatorname{Spec}_{\mathbb{F}_1}(A)$ is strictly bigger that $\operatorname{Spec}(A)$. Indeed, the empty set is always an element of the first set, and never an element of the second. In general, the empty set always constitute the minimum element with respect to inclusion in the set $\operatorname{Spec}_{\mathbb{F}_1}(M)$, for any monoid M. On the contrary, the whole set M is never a prime ideal, because the empty set has no unity, hence it is never a submonoid of M. Still, we can state that any monoid is local, in the sense specified by the following proposition.

2.1.3. PROPOSITION. *Every monoid M has a unique maximal ideal, namely the subset $M \setminus M^\times$. It is a prime ideal.*

PROOF. Every ideal I is contained in $M \setminus M^\times$. Indeed, if $a \in I$ and $a \in M^\times$, then $aa^{-1} = 1 \in I$ hence $I = M$, which is absurd. We are left to prove that $M \setminus M^\times$ is a prime ideal. Let a and b be in M. Suppose that $ab \in M^\times$. Then also a is invertible and $a^{-1} = b(ab)^{-1}$. This proves that $M \setminus M^\times$ is an ideal. It is also prime since $M^\times$ contains 1 and it is closed under multiplication. $\square$

2.1.4. DEFINITION. A map of monoids $f\colon M \to N$ is called *local* if $f(M \setminus M^\times) \subset N \setminus N^\times$. Equivalently, if $f^{-1}(N^\times) = M^\times$.

2.1.5. PROPOSITION. *Let I be a proper ideal of a monoid M. The intersection of all prime ideals that contain I equals the radical of I.*

PROOF. Let us call the first set A. Notice that it is not empty since $I \subseteq (M \setminus M^\times)$. Suppose that $a^k \in I$, and let $\mathfrak{p}$ be a prime ideal that contains I. We conclude that $a^k \in \mathfrak{p}$, hence $a \in \mathfrak{p}$. This proves $\sqrt{I} \subset A$. Now let $a \notin \sqrt{I}$. Then I is in the set

$$\Sigma := \{J\colon J \text{ ideal of } M, I \subset J, a^k \notin J \text{ for all positive integers } k\}$$

and we can then apply Zorn's lemma on Σ and get a maximal element $\mathfrak{p}$. Suppose that x, y are not in $\mathfrak{p}$. Then $\mathfrak{p} \cup xM$ and $\mathfrak{p} \cup yM$ are ideals which are strictly bigger than $\mathfrak{p}$ and contain I. Hence, since $\mathfrak{p}$ is maximal in Σ, they both must contain a positive power of a, say a^h and a^k respectively. Since neither of these two elements is in $\mathfrak{p}$, we conclude that they are in xM and yM, respectively. Then a^{h+k} is in xyM. Because a^{h+k} does not lie in $\mathfrak{p}$, we deduce that xy is not in $\mathfrak{p}$. It is obvious that $1 \notin \mathfrak{p}$, hence $\mathfrak{p}$ is prime. This concludes the proof. $\square$

Not surprisingly, the topology introduced on spectra of monoids enjoys various properties which demonstrate its similarity to the topology on prime spectra of rings.

2.1.6. PROPOSITION. *Let M be a monoid. The closed sets of $\operatorname{Spec}_{\mathbb{F}_1} M$ are exactly the subsets $V(I)$ defined in 2.1.1, and there exists a basis constituted by the empty set and the open subsets $D(a) := \{\mathfrak{p}\colon a \notin \mathfrak{p}\}$ where a is an element of M.*

PROOF. It is straightforward that $\bigcap_i V(I_i) = V\left(\bigcup I_i\right)$. In particular

$$X \setminus V(I) = X \setminus V\left(\bigcup_{a \in I} \{a\}\right) = \bigcup_{a \in I} (X \setminus V(\{a\})) = \bigcup_{a \in I} D(a).$$

Now let $\mathfrak{p}$ be a prime ideal. Then $\mathfrak{p}$ contains a subset I if and only if it contains the ideal IM generated by I. We can then consider only the subsets $V(I)$ where I is an ideal. It is easy to see that $V(I) \cup V(J) \subset V(I \cap J)$. Now suppose there is a prime ideal $\mathfrak{p}$ in $V(I \cap J) \setminus (V(I) \cup V(J))$. There must exist an element x in $I \setminus \mathfrak{p}$ and an element y in $J \setminus \mathfrak{p}$. However, xy is in $I \cap J$, hence in $\mathfrak{p}$. But $\mathfrak{p}$ is prime, and this is a contradiction. Hence, $V(I) \cup V(J) = V(I \cap J)$. Also, $\emptyset = V(1)$ and $\operatorname{Spec}_{\mathbb{F}_1} M = V(\emptyset)$. These facts prove that the subsets $V(I)$ themselves enjoy the properties of closed sets, as wanted. By the first equality we proved, we also conclude that the sets $D(a)$ form a basis of open sets. $\square$

Not only has a monoid a maximal prime ideal, but also a minimal prime ideal, namely the empty set. This fact has an immediate corollary.

2.1.7. Corollary. *Let M be a monoid. The space* $\operatorname{Spec}_{\mathbb{F}_1} M$ *is irreducible, i.e. all non-empty open subsets are dense.*

Proof. It is easy to see that $D(a) \cap D(b) = D(ab)$. An open subset $D(x)$ is never empty since it cointains the point $\emptyset$. We then proved that all couples of non-empty basis open sets intersects, hence the claim. $\square$

2.1.8. Proposition. *Let M be a monoid. There are no nontrivial coverings of* $\operatorname{Spec}_{\mathbb{F}_1}(M)$ *with respect to the Zariski topology, i.e. every open covering includes the open subset* $\operatorname{Spec}_{\mathbb{F}_1}(M)$ *itself.*

Proof. Let $\{U_i\}$ be a covering of $\operatorname{Spec}_{\mathbb{F}_1}(M)$. Because of Proposition 2.1.6, we can assume that all U_i are base open sets $U_i = D(a_i)$ with $a_i \in M$. Let I be the submonoid generated by the elements a_i. By the previous proposition, we conclude that

$$\operatorname{Spec}_{\mathbb{F}_1}(M) = D(1) \subset \bigcup D(a_i) = X \setminus V(I).$$

We then deduce the implication $I \subset \mathfrak{p} \Rightarrow 1 \in \mathfrak{p}$ for every prime ideal $\mathfrak{p}$. Using Proposition 2.1.5, we conclude that $1 \in I$. Since $I = \bigcup a_i M$, there exists an index j such that $1 \in a_j M$. Hence a_j is invertible, and $D(a_j) = \operatorname{Spec}_{\mathbb{F}_1}(M)$.

Alternatively, consider the point of $\operatorname{Spec}_{\mathbb{F}_1}(M)$ associated to the maximal ideal $\mathfrak{m}$. If $\mathfrak{m} \in D(a)$, then $a \notin \mathfrak{m}$, i.e. a is invertible. Hence, the only (basis) open subset which contains $\mathfrak{m}$ is $D(1) = \operatorname{Spec}_{\mathbb{F}_1}(M)$. $\square$

One of the main special features of prime spectra of rings is the structural sheaf, defined via localizations. Also in this setting, localizations can be defined using similar techniques.

2.1.9. Definition. Let M be a monoid, and let S be a subset of M. We call *localization of M at S* and indicate it with $S^{-1}M$ the monoid such that there exists a canonical map $M \to S^{-1}M$ which has the following universal property. For every map of monoids $f\colon M \to N$ such that $f(S) \subset N^{\times}$, there exists a unique map $S^{-1}M \to N$ such that the following diagram commutes.

$$\begin{tikzcd} M \arrow[r] \arrow[d] & S^{-1}M \arrow[dl, dotted] \\ N & \end{tikzcd}$$

If $S = \{a\}$, we indicate $S^{-1}M$ with M_a. If $S = M \setminus \mathfrak{p}$ where $\mathfrak{p}$ is a prime ideal, we indicate $S^{-1}M$ with $M_{\mathfrak{p}}$.

We remark that if two elements of M are sent to units in N, so it is their product. Also, the unity of M is always mapped to the unity of N. We can then restrict ourselves to considering localizations with respect to submonoids of M. Indeed, by what just said, $S^{-1}M$ equals $T^{-1}M$ where T is the submonoid of M generated by S.

2.1.10. Proposition. *Let M be a monoid, and S be a submonoid of M. The localization $S^{-1}M$ is well defined, and has the following explicit description. As a set, $S^{-1}M$ is the set of formal fractions*

$$\left\{ \frac{a}{x} : a \in M, x \in S \right\} \Big/ \sim$$

where $\frac{a}{x} \sim \frac{b}{y}$ *if there exists an element* $t \in S$ *such that* $ayt = bxt$. *The monoid operation in* $S^{-1}M$ *is defined as* $\frac{a}{x} \cdot \frac{b}{y} = \frac{ab}{xy}$ *and the map of monoids* $M \to S^{-1}M$ *is the map* $a \mapsto \frac{a}{1}$.

PROOF. The proof is analogous to the one for rings ([**4**] 3.1, for example). $\square$

2.1.11. DEFINITION. A *monoidal space* is a pair $(X, \mathcal{O}_X)$ consisting of a topological space X and a sheaf of monoids $\mathcal{O}_X$ on it. A *morphism of monoidal spaces* is a pair $(f, f^\sharp)$ where $f\colon X \to Y$ is a map of topological spaces and $f^\sharp\colon \mathcal{O}_Y \to f_*\mathcal{O}_X$ is a map of sheaves on Y such that for every $x \in X$, the induced morphism of stalks $f^\sharp_x\colon \mathcal{O}_{Y,f(x)} \to \mathcal{O}_{X,x}$ is local. The category of monoidal spaces is denoted by **MS**.

We remark that the definition of stalks is well-posed because the category of monoids has directed colimits, built in the usual way (e.g. [**28**], XI.2). Notice that the parallelism with the classical case of locally ringed spaces seems to be broken here since we do not explicitly require that the stalks are local. However, this is automatically granted since every monoid is local (2.1.3).

2.1.12. PROPOSITION. *The category of monoidal spaces* **MS** *is cocomplete.*

PROOF. We have to prove that it has arbitrary coproducts and coequalizers. We can define $\coprod (X_i, \mathcal{O}_{X_i})$ in the obvious way. Now consider two monoidal spaces $(X, \mathcal{O}_X)$ and $(Y, \mathcal{O}_Y)$ and two maps $(f, f^\sharp), (g, g^\sharp)\colon (X, \mathcal{O}_X) \rightrightarrows (Y, \mathcal{O}_Y)$. We define Z to be the topological space which is the coequalizer of f, g. Let also p be the natural projection $Y \to Z$. We then define a sheaf of monoids $\mathcal{O}_Z$ on Z by setting $\mathcal{O}_Z(W)$ as the equalizer of the two maps $f^\sharp, g^\sharp\colon \mathcal{O}_Y(V) \rightrightarrows \mathcal{O}_X(U)$ where V is the inverse image if W via p, and U is the inverse image of V via either f or g. Because the definition of $\mathcal{O}_Z$ and the sheaf property are both defined through limits which commute with each other, $\mathcal{O}_Z$ is a sheaf on Z. We are left to prove that the map $p^\sharp\colon \mathcal{O}_Z(W) \to \mathcal{O}_Y(V)$ induces a local morphism at the level of stalks. Indeed, once we show this fact, it is immediate that $(Z, \mathcal{O}_Z)$ enjoys the universal property of the coequalizers. Fix a section $s \in \mathcal{O}_Z(W)$ and a point $y \in V$, and suppose that $p^\sharp(s)$ is invertible in $\mathcal{O}_{Y,y}$. This implies that y lies in the open set $V_{p^\sharp(s)}$ defined as the set of points in which $p^\sharp(s)$ is locally invertible. By definition of the sheaf $\mathcal{O}_Z$, we know that $f^\sharp p^\sharp(s) = g^\sharp p^\sharp(s)$. Call this section $t \in \mathcal{O}_X(U)$. Because f, g are local, we also know that $f^{-1}(V_{p^\sharp(s)}) = g^{-1}(V_{p^\sharp(s)}) = U_t$. Hence $V_{p^\sharp(s)}$ is saturated with respect to the equivalence relation $f(x) = g(x)$. This means that $V_{p^\sharp(s)}$ is the inverse image of some open subset W' of Z such that $p(y) \in W' \subset W$. Also, because of the sheaf property and the uniqueness of the inverse, we know that both $p^\sharp(s)$ and t have global inverses in $V_{p^\sharp(s)}$ and in U_t respectively. We can then write down the equalizing diagram

$$\begin{tikzcd} \mathcal{O}_W(W') \arrow[r, "p^\sharp"] & \mathcal{O}_Y(V_{p^\sharp(s)}) \arrow[r, shift left, "f^\sharp"] \arrow[r, shift right, "g^\sharp"'] & \mathcal{O}_X(U_t) \end{tikzcd}$$

and consider the element $p^\sharp(s)^{-1} \in \mathcal{O}_Y(V_{p^\sharp(s)})$. It is mapped via $f^\sharp$ and $g^\sharp$ to t^{-1}, hence it lifts to an element s^{-1} in $\mathcal{O}_Z(W')$. We then conclude that s restricted to W' is invertible, hence s is locally invertible at $p(y)$. This proves that $(p^\sharp_y)^{-1}(\mathcal{O}^\times_{Y,y}) = \mathcal{O}^\times_{Z,p(y)}$, hence the claim. $\square$

2.1.13. PROPOSITION. *Let* M *be a monoid. There is a canonical structure of monoidal space on* $\operatorname{Spec}_{\mathbb{F}_1} M$ *such that* $\operatorname{Spec}_{\mathbb{F}_1}$ *defines a left adjoint of the functor* *of global sections Γ, seen as a functor from $\mathbf{MS}^{op}$ to monoids. In particular, for any monoidal space $(X, \mathcal{O}_X)$*

$$\mathrm{Hom}_{\mathbf{Mon}}(M, \Gamma(X, \mathcal{O}_X)) = \mathrm{Hom}_{\mathbf{MS}}(X, \mathrm{Spec}_{\mathbb{F}_1} M).$$

The sheaf $\mathcal{O}_{\mathrm{Spec}_{\mathbb{F}_1} M}$ is such that $\mathcal{O}_{\mathrm{Spec}_{\mathbb{F}_1} M}(D(a)) = M_a$ for any element a in M and $\mathcal{O}_{\mathrm{Spec}_{\mathbb{F}_1} M, \mathfrak{p}} = M_\mathfrak{p}$ for any prime ideal $\mathfrak{p}$ of M.

PROOF. Firstly, we define on $\mathrm{Spec}_{\mathbb{F}_1} M$ a structure of monoidal space. Since the sets $D(a)$ form a basis for the topology, and since the category of monoids has all small limits, it suffices to define a presheaf of monoids over these sets with the induced topology, and then to extend it assigning to each colimit of basis open sets the limit of the associated monoids. Because of Corollary 2.1.8, any presheaf over the basis which assigns 1 to the empty set is automatically a sheaf. We then put $\mathcal{O}_{\mathrm{Spec}_{\mathbb{F}_1} M}(D(a)) := M_a$ and $\mathcal{O}_{\mathrm{Spec}_{\mathbb{F}_1} M}(\emptyset) := 1$. Now suppose that $D(a) \subset D(b)$. This means that a is in the intersection of all prime ideals that contain b, hence in the intersection of all prime ideals that contain the ideal bM. Because of Proposition 2.1.5, we conclude that there exists a positive integer k such that $a^k = bx$ for some x in M. Hence, in the monoid M_a, the element $\frac{b}{1}$ has $\frac{x}{a^k}$ as its inverse. By universal property, there exists a natural map $M_b \to M_a$, which defines a good restriction map for a presheaf of monoids $\mathcal{O}_{\mathrm{Spec}_{\mathbb{F}_1} M}$. Let then $\mathcal{O}_{\mathrm{Spec}_{\mathbb{F}_1} M}$ be the sheaf of monoids on $\mathrm{Spec}_{\mathbb{F}_1} M$ generated by it. Fix now a prime ideal $\mathfrak{p}$. It is straightforward to see that the direct limit of the system $\{M_a\}_{a \notin \mathfrak{p}}$ is $M_\mathfrak{p}$. Indeed, all the maps $\{M_a \to M_\mathfrak{p}\}_{a \notin \mathfrak{p}}$ are induced by universal property hence compatible with the restriction maps. Also, a collection of maps $\{M_a \to N\}_{a \notin \mathfrak{p}}$ compatible with the system is equivalent to a map $M \to N$ such that all the elements a become invertible. Hence this map factors through $M_\mathfrak{p}$. This proves the second claim.

We now turn to prove the adjunction property. Any map of monoidal spaces $X \to \mathrm{Spec}_{\mathbb{F}_1} M$ defines in particular a map on global sections $M \to \Gamma(X)$. Hence, it suffices to build a map $\mathrm{Hom}_{\mathbf{Mon}}(M, \Gamma(X)) \to \mathrm{Hom}_{\mathbf{MS}}(X, \mathrm{Spec}_{\mathbb{F}_1} M)$ which defines an inverse. Let φ be in $\mathrm{Hom}_{\mathbf{Mon}}(M, \Gamma(X))$ and let x be a point of X. Define $f(x)$ to be the inverse image of the maximal ideal of $\mathcal{O}_{X,x}$ via the map $M \to \mathcal{O}(X) \to \mathcal{O}_{X,x}$. It is a prime ideal of M, hence a point in $\mathrm{Spec}_{\mathbb{F}_1} M$. The points of $f^{-1}(D(a))$ are exactly the points x in which $\varphi(a)$ is locally invertible. If $\varphi(a)$ is invertible in $\mathcal{O}_{X,x} = \varinjlim_{x \in U} \mathcal{O}_X(U)$, then it is also invertible in some $\mathcal{O}_X(U_x)$ with $x \in U_x$. Hence, this set is open. Also, by uniqueness of the inverse and the sheaf property, the elements $\varphi(a)^{-1}$ defined in U_x for different x in $f^{-1}(D(a))$ glue together forming a global inverse $\varphi(a)^{-1}$ on $f^{-1}(D(a))$. Hence, the map $M \to \mathcal{O}_X(X) \to \mathcal{O}_X(f^{-1}(D(a)))$ is such that the image of a is invertible, so it splits through $M_a = \mathcal{O}_{\mathrm{Spec}_{\mathbb{F}_1} M}(D(a))$. These maps define a map of sheaves $\mathcal{O}_{\mathrm{Spec}_{\mathbb{F}_1} M} \to f_* \mathcal{O}_X$, which together with f form a map of monoidal spaces $X \to \mathrm{Spec}_{\mathbb{F}_1} M$. Indeed, the induced map $M_{f(x)} \to \mathcal{O}_{X,x}$ is local by the very definition of f. It is easy to check that the map $\mathrm{Hom}_{\mathbf{Mon}}(M, \Gamma(X)) \to \mathrm{Hom}_{\mathbf{MS}}(X, \mathrm{Spec}_{\mathbb{F}_1} M)$ defined this way is the inverse wanted. $\square$

2.1.14. COROLLARY. *Let M be a monoid. The sheaf $\mathcal{O}_{\mathrm{Spec}_{\mathbb{F}_1} M}$ associates to each open set U the set of those functions s from U to $\coprod_{\mathfrak{p} \in U} M_\mathfrak{p}$ such that for every $\mathfrak{p}$ in U the following two conditions hold.*

(i) The element $s(\mathfrak{p})$ lies in $M_\mathfrak{p}$.

(ii) *There exists a base open set* $D(a)$ *with* $\mathfrak{p} \in D(a) \subset U$ *and an element* $t \in M_a$ *such that the restriction of* s *to* $D(a)$ *equals* t.

The monoid structure is the one induced by the inclusion in $\prod_{\mathfrak{p}\in U} M_{\mathfrak{p}}$.

PROOF. Writing U as the colimit of opens of the form $D(a)$, it is easy to see that the monoid just defined is the corresponding limit of the monoids M_a. □

2.1.15. DEFINITION. Monoidal spaces which are isomorphic to a monoidal space of the form $(\mathrm{Spec}_{\mathbb{F}_1} M, \mathcal{O}_{\mathrm{Spec}_{\mathbb{F}_1} M})$ for some M are called *affine geometrical* $\mathbb{F}_1$*-schemes.*

2.1.16. COROLLARY. *The functor* $\mathrm{Spec}_{\mathbb{F}_1}$ *from monoids to affine geometrical* $\mathbb{F}_1$*-schemes is part of a contravariant equivalence of categories.*

2.1.17. DEFINITION. A map $(X, \mathcal{O}_X) \to (Y, \mathcal{O}_Y)$ of **MS** is an *open immersion* if it is the composite of an isomorphism and an open inclusion $(U, \mathcal{O}_Y|_U) \hookrightarrow (Y, \mathcal{O}_Y)$. A family of open immersions is a *Zariski covering* if it is globally surjective on the topological spaces underneath. Zariski coverings define a Grothendieck pretopology on affine geometrical $\mathbb{F}_1$-schemes, and the site they form is called the *Zariski site.*

2.1.18. DEFINITION. A *geometrical* $\mathbb{F}_1$*-scheme* (or a *scheme over* $\mathbb{F}_1$ *à la Deitmar*) is a monoidal space $(X, \mathcal{O}_X)$ with an affine Zariski covering.

2.1.19. DEFINITION. Zariski coverings define a Grothendieck pretopology on geometrical $\mathbb{F}_1$-schemes. The site they form is again called the *Zariski site.*

2.1.20. PROPOSITION. *Let* $\{X_i\}_{i\in I}$ *be a family of geometrical* $\mathbb{F}_1$*-schemes and let* $\{U_{ij} \hookrightarrow X_i\}_{j\in I, j\neq i}$ *be a family of open subschemes of* X_i*, for every* i*. If there exist isomorphisms of geometrical* $\mathbb{F}_1$*-schemes* $\varphi_{ij}\colon U_{ij} \to U_{ji}$ *such that*

(i) $\varphi_{ij}^{-1} = \varphi_{ji}$;
(ii) $\varphi_{ij}(U_{ik} \cap U_{ih}) = U_{jk} \cap U_{jh}$;
(iii) $\varphi_{jk} \circ \varphi_{ij} = \varphi_{ik}$;

then there exist a geometrical $\mathbb{F}_1$*-scheme* X *and isomorphisms* ψ_i *of* X_i *onto an open subscheme of* X *for every* i *such that*

(i) $\{\psi_i(X_i)\}_{i\in I}$ *is an open cover of* X
(ii) $\psi_i(U_{ij}) = \psi_i(X_i) \cap \psi_j(X_j)$;
(iii) $\psi_i|_{U_i} = \psi_j \circ \varphi_{ij}|_{U_i}$,

and such X *is uniquely determined up to one isomorphism.*

PROOF. The geometrical $\mathbb{F}_1$-scheme X wanted is the coequalizer of the maps induced by the maps φ_{ij} and φ_{ji} defined on each U_{ij}:

$$\coprod U_{ij} \rightrightarrows \coprod X_k \to X. \tag{2}$$

We then have to prove that the monoidal space built in this way is locally affine. This is granted because we are gluing over open subschemes. □

2.1.21. COROLLARY. *Let* $\{U_i\}$ *be an open cover of a geometrical* $\mathbb{F}_1$*-scheme* X*, and let* $\{\varphi_i\colon U_i \to S\}_i$ *be a family of morphisms of geometrical* $\mathbb{F}_1$*-schemes, such that* $\varphi_i(U_i \cap U_j) = \varphi_j(U_i \cap U_j)$*. There exists a unique morphism* $\varphi\colon X \to S$ *such that* $\varphi|_{U_i} = \varphi_i$ *for every* i*. Equivalently, the Zariski topology on geometrical* $\mathbb{F}_1$*-schemes is subcanonical.*

PROOF. X is obtained by gluing the U_i's together. Apply then the left exact functor $\mathrm{Hom}(\cdot, S)$ to a coequalizing diagram, which is the analogue of the one in (2). □

2.1.22. PROPOSITION ([**11**], Proposition 3.1). *The category of geometrical $\mathbb{F}_1$-schemes has pullbacks (also called* fibered products*), and affine geometrical $\mathbb{F}_1$-schemes are closed under pullbacks.*

PROOF. First we show that the category of monoids has pushouts. Let $f\colon M \to N$ and $g\colon M \to P$ be maps of monoids. Consider the two maps $M \rightrightarrows N \times P$ defined as $m \mapsto (f(m), 1)$ and $m \mapsto (1, g(m))$, and the minimal submonoidal equivalence relation E which includes $(f(m)n, p) \sim (m, g(m)p)$. There is a natural monoidal structure on $N \sqcup_M P := (P \oplus N)/E$, and a map of monoids $P \oplus N \to P \sqcup_M N$. It is easy to see that the induced diagram below is a pushout.

$$\begin{tikzcd} M \arrow[r] \arrow[d] & N \arrow[d] \\ P \arrow[r] & N \sqcup_M P \end{tikzcd}$$

We then conclude that the category of affine geometrical $\mathbb{F}_1$-schemes has pullbacks. Now consider a diagram $Y \to X \leftarrow Z$. Take an affine open covering $\{X_i\}$ of X and affine open coverings $\{Y_{ij}\}$, $\{Z_{ik}\}$ of the inverse images of each X_i. Construct the generic fibered product over X by gluing together the affine schemes obtained as fibered products of Y_{ij} and Z_{ik} over X_i. The second claim is obvious by construction. □

2.1.23. DEFINITION. Let M be a monoid. We call it a *monoid with zero* if there exists an element 0 such that $\{0\}$ is an ideal. Arrows between monoids with zero are arrows of monoids that send 0 to 0. We call the category they form with $\mathbf{Mon}_0$. The forgetful functor $\mathbf{Mon}_0 \to \mathbf{Mon}$ has a left adjoint that sends M to $M_0 := M \sqcup \{0\}$, with the obvious operation.

In the classical case of schemes, we proved that the spectrum of a ring can be defined though a colimit using K-points, as K varies among the fields (1.1.18). In the case of monoids, the naive attempt would be to consider the G-points as G runs through the groups. This does not work, as the following remark specifies.

2.1.24. PROPOSITION. *Let G be an abelian group and X a monoidal space. Defining a G-point on X is the same as considering a point x of X such that $\mathcal{O}_{X,x}$ is a group, together with a group homomorphism $\mathcal{O}_{X,x} \to G$.*

PROOF. Suppose that f is a map from $\mathrm{Spec}_{\mathbb{F}_1} G$ to X. Since a group has only one prime ideal $\emptyset$, this maps defines automatically a point $x = f(\emptyset)$ in X. Adding to this, it defines a local map of monoids $\mathcal{O}_{X,x} \to G$. The fact that this map is local implies that all elements of $\mathcal{O}_{X,x}$ are invertible, as wanted. Conversely, given a point x define the map between topological spaces that sends $\emptyset$ to x. Note that by defining a map $\mathcal{O}_{X,x} = \varinjlim_{x \in U} \mathcal{O}_X(U) \to G$, we also define maps $\mathcal{O}_X(U) \to G$ for every U such that $x \in U$. Together with the trivial maps $\mathcal{O}_X(U) \to 1$ for those U that do not contain x, they define a map of sheaves $\mathcal{O}_X \to f_* \mathrm{Spec}_{\mathbb{F}_1} G$, as wanted. □

In particular, we conclude that G-points on monoidal spaces are rare to find, so that there is no possibility to recover the topological space beneath just by using them.

2.2. Schemes over $\mathbb{F}_1$ à la Toën-Vaquié

We now want to present one of the first generalizations of the concept of scheme that has been introduced by Toën and Vaquié in their paper [**34**]. The main advantage of this approach is its generality. The way new schemes are introduced is purely categorical and the case of $\mathbb{F}_1$ is just a particular case of a more general picture, in which the protagonists are well-behaved monoidal categories. Adding to this, the construction made by these authors leads immediately to the notion of homotopy over $\mathbb{F}_1$, one of the major results of their paper, and hence to the definition of the "brave new schemes" (from the French expression "schemas courageux").

In this section we will use the Remark 1.3.15 as inspiring model to introduce all the definitions. First of all we will define the category of affine schemes, then put a topology on it and define schemes as particular sheaves over that category.

2.2.1. DEFINITION. A *monoidal category* is a category $\mathbf{C}$ equipped with the structure $(\otimes, \mathbf{1}, a, l, r)$ where $\otimes\colon \mathbf{C} \times \mathbf{C} \to \mathbf{C}$ is a functor, $\mathbf{1}$ is an object of $\mathbf{C}$, and a, l, r are natural isomorphisms of functors $a\colon (\cdot \otimes \cdot) \otimes \cdot \to \cdot \otimes (\cdot \otimes \cdot)$, $l\colon \mathbf{1} \otimes \cdot \to id_{\mathbf{C}}$, $r\colon \cdot \otimes \mathbf{1} \to id_{\mathbf{C}}$ such that three coherence diagrams commute. These conditions can be written shortly as

$$r_{\mathbf{1}} = l_{\mathbf{1}}\colon \mathbf{1} \otimes \mathbf{1} \to \mathbf{1};$$
$$(id \otimes a) \circ a \circ (a \otimes id) = a \circ a\colon ((\cdot \otimes \cdot) \otimes \cdot) \otimes \cdot \to \cdot \otimes (\cdot \otimes (\cdot \otimes \cdot));$$
$$(id \otimes l) \circ a = r \otimes id\colon (\cdot \otimes \mathbf{1}) \otimes \cdot \to \cdot \otimes \cdot.$$

A *symmetric monoidal category* is a monoidal category together with a natural transformation T of functors from $\mathbf{C} \times \mathbf{C}$ to $\mathbf{C}$ defined as $T_{A,B}\colon A \otimes B \to B \otimes A$ which is coherent with the monoidal structure, in the sense that $T^2 = id$, $Tl = r$, $T_{\mathbf{1},\mathbf{1}} = id_{\mathbf{1}\otimes\mathbf{1}}$, $a \circ (T \otimes id) = a \circ T \circ a$. A *closed symmetric monoidal category* is a closed monoidal category $\mathbf{C}$ together with a functor $\underline{\mathrm{Hom}}\colon \mathbf{C}^{\mathrm{op}} \times \mathbf{C} \to \mathbf{C}$ and a natural isomorphism of functors φ defined as $\varphi_{A,B,C}\colon \mathrm{Hom}_{\mathbf{C}}(A \otimes B, C) \to \mathrm{Hom}_{\mathbf{C}}(A, \underline{\mathrm{Hom}}(B, C))$.

From now on, we will consider $(\mathbf{C}, \otimes)$ to be a closed symmetric monoidal category with all limits and colimits, omitting all the extra structure. Being closed, we obtain in particular that the tensor product commutes with colimits, because it has a right adjoint.

Given such a monoidal category, it is possible to define monoids in it, and modules over a monoid.

2.2.2. DEFINITION. Let $\mathbf{C}$ be a symmetric monoidal category.

A *(unitary, associative and commutative) monoid* is an object R of $\mathbf{C}$, together with a "multiplication" map $\mu\colon R \otimes R \to R$ and a "unit" map $\eta\colon \mathbf{1} \to R$ such that the two following diagrams commute.

$$\begin{tikzcd}
R \otimes (R \otimes R) \arrow[r, "a"] \arrow[d, "1\otimes\mu"'] & (R \otimes R) \otimes R \arrow[r, "\mu\otimes 1"] & R \otimes R \arrow[d, "\mu"] \\
R \otimes R \arrow[rr, "\mu"] & & R
\end{tikzcd}$$

$$\begin{tikzcd} \mathbf{1}\otimes R \arrow[r, "\eta\otimes 1"] \arrow[dr, "l"'] & R\otimes R \arrow[d, "\mu"] & R\otimes R \arrow[l, "T_{R,R}"'] \arrow[dl, "\mu"] \\ & R & \end{tikzcd}$$

A *R-module over a monoid* R is an object N together with an "action" map $R\otimes N \to N$ such that the two possible ways to map $R\otimes(R\otimes N)$ to N coincide, and the two possible ways to map $\mathbf{1}\otimes N$ to N coincide.

Monoids and modules over them constitute respectively two categories: a *morphism between monoids* is a map in $\mathbf{C}$ which is compatible with multiplication and unit maps, in the obvious way. We indicate the category of monoids with $\mathbf{Mon_C}$. A *morphism between R-modules* is a map in $\mathbf{C}$ which is compatible with the action map, in the obvious way. The category of A-modules is indicated with A-$\mathbf{Mod}$. Given a monoid A, the category ${}^{A/}\,\mathbf{Mon_C}$ is denoted with A-$\mathbf{Alg}$, and its objects are called *A-algebras*.

2.2.3. Definition. Let A be an object of $\mathbf{Mon_C}$, and let M, N be objects of A-$\mathbf{Mod}$ with actions φ, ψ respectively. We define the *tensor product of M and N over A*, and we indicate it with $M\otimes_A N$, the coequalizer in the diagram

$$A\otimes M\otimes N \underset{\psi\otimes M}{\overset{\varphi\otimes N}{\rightrightarrows}} M\otimes N$$

with the natural A-module structure $A\otimes(M\otimes_A N)\to(M\otimes_A N)$ induced by the arrow $A\otimes M\otimes N\to M\otimes_A N$ and the fact that $A\otimes(M\otimes_A N)\to(M\otimes_A N)$ is the coequalizer of the diagram

$$A\otimes A\otimes M\otimes N \underset{A\otimes\psi\otimes M}{\overset{A\otimes\varphi\otimes N}{\rightrightarrows}} M\otimes N.$$

since tensors are right exact.

2.2.4. Proposition. *Consider a map $f\colon A\to B$ in $\mathbf{Mon_C}$.*

(1) There is a natural forgetful functor B-$\mathbf{Mod}\to A$-$\mathbf{Mod}$ that sends an object N to N itself, considered as a A-module with the action defined as the composite

$$A\otimes N\to B\otimes N\to N.$$

In particular, the map f naturally defines a structure of A-module on B, with the action defined as above.

(2) The forgetful functor has a left adjoint, indicated with $\otimes_A B$, which sends a A-module M to $M\otimes_A B$, with a suitable B-action.

(3) The forgetful functor has a right adjoint, which sends a A-module M to $\underline{\mathrm{Hom}}(B,M)$, with a suitable B-action.

(4) The pushout in the category of monoids of a diagram $B\leftarrow A\to C$ is isomorphic as C-module to $B\otimes_A C$.

Proof. The first claim is clear. Let's turn to the second. We give to the object $M\otimes_A B$ the structure of B-module induced in the following way: since $\otimes B$ commutes with colimits, $(M\otimes_A B)\otimes B$ is the colimit of the diagram

$$A\otimes M\otimes B\otimes B \underset{\psi\otimes M\otimes B}{\overset{\varphi\otimes B\otimes B}{\rightrightarrows}} M\otimes B\otimes B$$

where φ and ψ are the action maps of A on M and B respectively. Now consider the following diagram

$$\begin{tikzcd} A\otimes M\otimes B\otimes B \arrow[r, shift left, "\varphi\otimes B\otimes B"] \arrow[r, shift right, "\psi\otimes M\otimes B"'] \arrow[d, "A\otimes M\otimes\mu"'] & M\otimes B\otimes B \arrow[d, "M\otimes\mu"] \\ A\otimes M\otimes B \arrow[r, shift left, "\varphi\otimes B"] \arrow[r, shift right, "\psi\otimes M"'] & M\otimes B \end{tikzcd}$$

which induces a map between colimits $(M\otimes_A B)\otimes B\to M\otimes_A B$, as wanted. Let M be a A-module and N be a B-module. We call with φ the A-action map on M and with ψ the B-action map on N. Now we show how to associate to a map α in $\mathrm{Hom}_{A\text{-}\mathbf{Mod}}(M,N)$ a map in $\mathrm{Hom}_{B\text{-}\mathbf{Mod}}(M\otimes_A B,N)$. Consider the commutative diagram below (we indicate each map omitting tensors with the identity)

$$\begin{tikzcd} M\otimes A\otimes B \arrow[rr, "f"] \arrow[dr, "\alpha"] \arrow[dd, "\varphi"] & & M\otimes B\otimes B \arrow[r, "\mu"] \arrow[d, "\alpha"] & M\otimes B \arrow[d, "\alpha"] \\ & N\otimes A\otimes B \arrow[r, "f"] & N\otimes B\otimes B \arrow[r, "\mu"] \arrow[d, "\psi"] & N\otimes B \arrow[d, "\psi"] \\ M\otimes B \arrow[rr, "\alpha"] & & N\otimes B \arrow[r, "\psi"] & N \end{tikzcd}$$

where the commutativity of the lower left sub-diagram comes from the commutative diagram which expresses the A-linearity of α, tensored with B; the lower right sub-diagram commutes because N is a B-module; and the upper ones because of the functoriality of the tensor. The external square induces a map from $M\otimes_A B\to N$. We leave to the reader to check it is B linear. Now we show how to associate to a map β in $\mathrm{Hom}_{B\text{-}\mathbf{Mod}}(M\otimes_A B,N)$ a map in $\mathrm{Hom}_{A\text{-}\mathbf{Mod}}(M,N)$. It suffices to consider the compositions

$$M\xrightarrow{\sim} M\otimes\mathbf{1}\to M\otimes B\to M\otimes_A B\to N.$$

We leave to the reader to check that this map is A-linear, and together with the previous definition it defines a canonical isomorphism $\mathrm{Hom}_{A\text{-}\mathbf{Mod}}(M,N)\cong \mathrm{Hom}_{B\text{-}\mathbf{Mod}}(M\otimes_A B,N)$.

We give to the object $\underline{\mathrm{Hom}}(B,M)$ the structure of B-module induced in the following way: consider the composition

$$\underline{\mathrm{Hom}}(B,M)\otimes B\otimes B\to\underline{\mathrm{Hom}}(B,M)\otimes B\to M$$

where the first arrow is $id\otimes\mu$ and the second is the one associated by adjunction to the identity in $\mathrm{Hom}(\underline{\mathrm{Hom}}(B,M),\underline{\mathrm{Hom}}(B,M))$. This map defines, again by adjunction, a map $\underline{\mathrm{Hom}}(B,M)\otimes B\to\underline{\mathrm{Hom}}(B,M)$, as wanted. In this case, the maps which define a canonical isomorphism between $\mathrm{Hom}_{A\text{-}\mathbf{Mod}}(N,M)$ and $\mathrm{Hom}_{B\text{-}\mathbf{Mod}}(N,\underline{\mathrm{Hom}}(B,M))$ are simply given by the composition to the right with the maps $N\xrightarrow{\sim} N\otimes\mathbf{1}\to N\otimes B$ and $N\otimes B\to N$. We leave to the reader to verify all the details.

As for the last point, the claim follows easily after having equipped $B\otimes_A C$ with a suitable monoid structure. We now contemplate the following diagram, in which $(B\otimes_A C)\otimes(B\otimes_A C)$ is the "bottom right" colimit, because of its definition and commutation of tensors with colimits (we use different symbols for the two copies of each monoid, in order to emphasize what the various maps are).

$$\begin{tikzcd} A \otimes B \otimes C \otimes \bar{A} \otimes \bar{B} \otimes \bar{C} \arrow[r, shift left] \arrow[r, shift right] \arrow[d, shift left] \arrow[d, shift right] & B \otimes C \otimes \bar{A} \otimes \bar{B} \otimes \bar{C} \arrow[d, shift left] \arrow[d, shift right] \\ A \otimes B \otimes C \otimes \bar{B} \otimes \bar{C} \arrow[r] & B \otimes C \otimes \bar{B} \otimes \bar{C} \end{tikzcd}$$

Hence, we can define a map $(B \otimes_A C) \otimes (B \otimes_A C) \to B \otimes_A C$ starting from the map $B \otimes C \otimes B \otimes C \to B \otimes_A C$, which is induced by the two multiplications. $\square$

2.2.5. Definition. The category A-**Mod** has a natural structure of tensor category induced by $\otimes_A$ as defined in Proposition 2.2.4. We call $M \otimes_A N$ the *tensor product* of M and N in A-**Mod**.

2.2.6. Corollary. *Let A be a monoid and M be a A-module. Then $M \otimes_A A$ is canonically isomorphic to M.*

Proof. Set $A = B$ in the previous theorem. Both $\otimes_A A$ and the identity itself are left adjoint of the identity, hence they are isomorphic. $\square$

2.2.7. Corollary. *Let $M \to N$ be a map of $\mathbf{Mon_C}$. The forgetful functor N-**Alg** $\to M$-**Alg** has a left adjoint, which maps $M \to P$ to $N \to N \otimes_M P$ with the monoid structure induced by the isomorphism as modules of $N \otimes_M P$ with the coproduct of monoids $N \sqcup_M P$.*

Proof. Clear by universal property of the coproduct. $\square$

We introduce another important property of forgetful functors in monoidal categories.

2.2.8. Definition. A functor $F\colon \mathbf{C} \to \mathbf{D}$ *reflects isomorphisms* if for any arrow f of $\mathbf{C}$, the fact that $F(f)$ is an isomorphism implies that f itself is an isomorphism.

2.2.9. Proposition. *Let $(\mathbf{C}, \otimes)$ be a closed symmetric monoidal category with all small limits and colimits. The forgetful functor $\mathbf{Mon_C} \to \mathbf{C}$ commutes with limits and reflects isomorphisms.*

Proof. Since $\mathbf{C}$ has in particular denumberable coproducts, one can define the functor $\mathbf{C} \to \mathbf{Mon_C}$ that sends an object A to $F(A) := \coprod_{n \in \mathbb{Z}_{\geq 0}} A^n$ where $A^0 = \mathbf{1}$ and $A^n = A^{n-1} \otimes A$. Since the monoidal category is closed, tensors distribute over coproducts. Hence, $F(A)$ has a natural structure of monoid, and F defines a left adjoint of the forgetful functor (see also [**28**] Theorem VII.3.2), which then commutes with small limits.

The fact that it reflects isomorphisms comes from the fact that the adjoint couple just presented is monadic, and Beck's monadicity theorem ([**6**], Theorem 3.14). Explicitly, call $\Sigma\colon \mathbf{C} \to \mathbf{C}$ the composite of F and the forgetful functor, and consider the functor $\mathbf{Mon_C} \to \mathbf{C}^\Sigma$ defined over $\mathbf{C}$, where $\mathbf{C}^\Sigma$ is the category of objects A of $\mathbf{C}$ with a map $\alpha\colon \Sigma(A) \to A$ such that $A \to \Sigma(A) \to A$ is the identity of A, and the two possible ways to define a map $(\Sigma \circ \Sigma)(A) \to A$ are the same. Monadicity means that this functor is an equivalence. In our case, this means that an arrow $F(A) \to A$ is uniquely determined by the map $A \otimes A \to A$. Because equivalences of categories reflect isomorphisms, we are left to prove that the arrow $\mathbf{C}^\Sigma \to \mathbf{C}$ reflects isomorphisms, and this is done in [**6**], Proposition 3.2. $\square$

2.2.10. COROLLARY. *Let $(\mathbf{C}, \otimes)$ be a closed symmetric monoidal category with all small limits and colimits, and let A be a monoid of it. The forgetful functor A-$\mathbf{Alg} \to A$-$\mathbf{Mod}$ commutes with limits and reflects isomorphisms.*

PROOF. This comes directly from the previous proposition, because the category of monoids in the tensor category A-$\mathbf{Mod}$ is equivalent to the category A-$\mathbf{Alg}$. $\square$

2.2.11. PROPOSITION. *Let A, B be two monoids and let M be a A-module, P a B-module and N both a A-module and a B-module such that the diagram*

$$\begin{tikzcd} A \otimes B \otimes N \arrow[r] \arrow[d] & A \otimes N \arrow[d] \\ B \otimes N \arrow[r] & N \end{tikzcd}$$

is commutative.

(1) *The module $M \otimes_A N$ has a natural structure of B-module and $N \otimes_B P$ has a natural structure of A-module.*
(2) *There holds an isomorphism $(M \otimes_A N) \otimes_B P \cong M \otimes_A (N \otimes_B P)$ of A-modules and B-modules.*

PROOF. We start with the first point. It suffices to prove the claim for $M \otimes_A N$. Since tensoring is right exact, $(M \otimes_A N) \otimes B$ can be considered as the coequalizer of the diagram

$$A \otimes B \otimes M \otimes N \rightrightarrows B \otimes M \otimes N.$$

From the commutative diagram depicted in the hypothesis, we deduce that the map $B \otimes N \to N$ induces a map of diagrams

$$\begin{tikzcd} A \otimes B \otimes M \otimes N \arrow[r, shift left] \arrow[r, shift right] \arrow[d] & B \otimes M \otimes N \arrow[d] \\ A \otimes M \otimes N \arrow[r, shift left] \arrow[r, shift right] & M \otimes N \end{tikzcd}$$

which in turn induces a map of coequalizers $(M \otimes_A N) \otimes B \to M \otimes_A N$, as wanted. Some diagram chasing, based on the corresponding diagrams on N, ensures that this map defines a B-module structure on $M \otimes_A N$.

We now turn to the second claim. Associativity and the exactness of the sequence $B \otimes N \otimes P \rightrightarrows N \otimes_B P \to N \otimes_B P$ imply the commutativity of the following diagram, in which all horizontal lines are exact, and in which vertical columns coequalize.

$$\begin{tikzcd} (A \otimes M \otimes N) \otimes B \otimes P \arrow[r, shift left] \arrow[r, shift right] \arrow[d, shift left] \arrow[d, shift right] & (M \otimes N) \otimes B \otimes P \arrow[r] \arrow[d, shift left] \arrow[d, shift right] & (M \otimes_A N) \otimes B \otimes P \\ (A \otimes M \otimes N) \otimes P \arrow[r, shift left] \arrow[r, shift right] \arrow[d] & (M \otimes N) \otimes P \arrow[r] \arrow[d] & (M \otimes_A N) \otimes P \\ A \otimes M \otimes (N \otimes_B P) \arrow[r, shift left] \arrow[r, shift right] & M \otimes (N \otimes_B P) \arrow[r] & M \otimes_A (N \otimes_B P) \end{tikzcd}$$

The previous diagram induces the following one

$$(M \otimes_A N) \otimes B \otimes P \rightrightarrows (M \otimes_A N) \otimes P \to M \otimes_A (N \otimes_B P)$$

hence a B-linear arrow $(M \otimes_A N) \otimes_B P \to M \otimes_A (N \otimes_B P)$. We leave to the reader to prove that it is also A-linear, and that the analogous map $M \otimes_A (N \otimes_B P) \to (M \otimes_A N) \otimes_B P$ defines an inverse of it. $\square$

2.2.12. DEFINITION. Let $(\mathbf{C}, \otimes)$ be a closed symmetric monoidal category with all small limits and colimits. The opposite category of the category of monoids in $\mathbf{C}$ is denoted by $\mathbf{Aff}_{\mathbf{C}}$, and its objects are called *affine schemes relative to* $\mathbf{C}$. We call $\operatorname{Spec} A$ the object in $\mathbf{Aff}_{\mathbf{C}}$ which corresponds to the monoid A of $\mathbf{C}$.

It is now high time to introduce the topology we put on the category of affine schemes. Being a generalization of the Zariski topology defined in Chapter 1, we refer to it as the Zariski topology. For its definition, we will simply mimic Corollary 1.1.15 and Propositions 1.1.16, 1.1.17.

2.2.13. DEFINITION. Let $f\colon A \to B$ be a map in $\mathbf{Mon}_{\mathbf{C}}$.

(1) The map f is *flat* if the functor $\otimes_A B$ from A-modules to B-modules is exact (equivalently, left exact) in the sense that it commutes with finite limits and colimits.
(2) The map f is *of finite presentation* if if for every direct system $\{C_i\}_{i \in I}$ of A-algebras, the canonical map
$$\varinjlim \operatorname{Hom}_{A\text{-}\mathbf{Alg}}(B, C_i) \to \operatorname{Hom}_{A\text{-}\mathbf{Alg}}(B, \varinjlim C_i)$$
is bijective.
(3) A map $\operatorname{Spec} B \to \operatorname{Spec} A$ is an *open immersion* if the correspondent map $A \to B$ is a flat epimorphism of finite presentation.
(4) A collection of maps $\{\operatorname{Spec} A_i \to \operatorname{Spec} A\}_{i \in I}$ is a *Zariski covering* if each map is an open immersion and if there is a finite subset $J \subset I$ such that any map of A-modules $M \to N$ is an isomorphism if and only if each of the induced maps $M \otimes_A A_j \to N \otimes_A A_j$ is an isomorphism.

Note that, in particular, we are now posing as part of the definition the fact that affine schemes are quasi-compact (a finite sub-covering is indexed by J), while that was granted by the explicit definition of the Zariski topology in the case of rings.

2.2.14. PROPOSITION. *Zariski coverings define a Grothendieck pretopology on* $\mathbf{Aff}_{\mathbf{C}}$. *The site they form is called the* Zariski site.

PROOF. The only non-trivial part is preservation under base change. We note that being flat, of finite presentation and monomorphism are all stable properties under base change. This comes from the fact that for any A in $\mathbf{Mon}_{\mathbf{C}}$, the functor $\otimes_A A$ is equivalent to the identity and the fact that tensors are "associative" in the obvious sense, due to the uniqueness of the adjoint. We now prove that the covering property is inherited under base change. Consider now a Zariski covering $\{\operatorname{Spec} A_i \to \operatorname{Spec} A\}$ of $\operatorname{Spec} A$, and let J is the subset of indexes as in the definition. Consider also a map $\operatorname{Spec} C \to \operatorname{Spec} A$ in $\mathbf{Aff}_{\mathbf{C}}$ and two C-modules M, N. If the maps $M \otimes_C (C \otimes_A A_i) \to N \otimes_C (C \otimes_A A_i)$ are isomorphisms of $(C \otimes_A A_i)$-modules, then also the maps $M \otimes_A A_i \to N \otimes_A A_i$ are isomorphisms of A_i-modules, hence $M \to N$ is an isomorphism of A-modules. We deduce that $M \otimes_A C \to N \otimes_A C$ is an isomorphism of C-modules, therefore $M \to N$ is an isomorphism of C-modules by Corollary 2.2.6. $\square$

Now that we created a topology on affine schemes, we can define sheaves over affine schemes. In the case of rings, the functor represented by any affine scheme was also a sheaf. In this more general setting, this fact is still unexpectedly true, and it needs indeed a rather complicated proof.

2.2.15. THEOREM. *Let X be an object of $\mathbf{Aff_C}$. Then the functor $h_X = \mathrm{Hom}(\cdot, X)$ is a Zariski sheaf, i.e. the Zariski topology is subcanonical.*

PROOF. Let $\{\operatorname{Spec} A_i \to \operatorname{Spec} A\}$ be a covering of $\operatorname{Spec} A$ in $\mathbf{Aff_C}$. Consider the following diagram in A-$\mathbf{Alg}$

$$\prod A_i \rightrightarrows \prod A_i \otimes_A A_j$$

and suppose that E is the equalizer. We have a map $A \to E$ induced by universal property. When passing to the category A-$\mathbf{Mod}$ via the forgetful functor, this map becomes an isomorphism because the forgetful functor preserves limits (Corollary 2.2.10), and because of Theorem B.3.10. Since the forgetful functor also reflects isomorphisms (Corollary 2.2.10), we conclude that the map $A \to E$ is an isomorphism also in the category A-$\mathbf{Alg}$. We deduce the following coequalizing diagram in the category $\mathbf{Aff_C}$.

$$\coprod \operatorname{Spec}(A_i \otimes_A A_j) \rightrightarrows \coprod \operatorname{Spec} A_i \to \operatorname{Spec} A$$

In order to prove the claim, it is then sufficient to apply the functor $\mathrm{Hom}(\cdot, \operatorname{Spec} B)$ to this diagram. $\square$

We then use the word "affine scheme" to refer both to objects of $\mathbf{Aff_C}$ and also the functors represented by them.

In order to define a scheme as we did in Chapter 1, we still have to define open coverings of sheaves, so to have a good definition of "being locally affine" also for a sheaf. Not surprisingly, we are again using the classical case as a model (see Definition 1.2.9).

2.2.16. DEFINITION. Let $f\colon \mathcal{F} \to \mathcal{G}$ be a map of Zariski sheaves over $\mathbf{Aff_C}$.

(1) Suppose that $\mathcal{G} = h_X$ is affine. Then f is *an open immersion* if there exists a family of open immersions $\{X_i \to X\}_{i \in I}$ such that $\mathcal{F}$ is isomorphic over h_X to the image of the induced map of sheaves $\coprod_{i\in I} h_{X_i} \to h_X$.
(2) The map f is *an open immersion* if for every affine scheme h_X over $\mathcal{G}$, the induced morphism $\mathcal{F} \times_{\mathcal{G}} h_X \to h_X$ is an open immersion.
(3) A collection of maps $\{X_i \to \mathcal{F}\}_{i\in I}$ is a *Zariski covering* if each map is an open immersion and the induced map $\coprod_{i\in I} X_i \to \mathcal{F}$ is an epimorphism.

Also in this case, we have to check that all the definitions given agree on affine schemes. This is again something completely not trivial.

2.2.17. PROPOSITION. *Let $f\colon X \to Y$ be a map in $\mathbf{Aff_C}$. It is an open immersion in the sense of Definition 2.2.13 if and only if it is an open immersion in the sense of Definition 2.2.16.*

PROOF. The proof is analogous to the one of Proposition 1.2.12, using Corollary 2.2.10 whenever the exactness property of the forgetful functor from algebras to modules is used. $\square$

We are now ready to give the definition of a scheme in this new setting.

2.2.18. DEFINITION. A *scheme relative to* $\mathbf{C}$ (or a *scheme à la Toën-Vaquié relative to* $\mathbf{C}$) is a Zariski sheaf over affine schemes in the sense of Definition 2.2.12, which has a covering made by open immersions of affine schemes.

2.2.19. PROPOSITION ([**34**], Proposition 2.18). *The category of schemes relative to* $\mathbf{C}$ *inside the category of Zariski sheaves is stable under open immersions, disjoint unions and fibered products.*

PROOF. The proof is the same as the one of Proposition 1.2.14. □

2.2.20. PROPOSITION. *Zariski coverings define a Grothendieck pretopology on schemes relative to* $\mathbf{C}$*. The site they form is again called the* Zariski site*.*

PROOF. Since the definition of an open immersion in given through affine base change, the axioms of a Grothendieck pretopology are easily verified. □

We now want to build up a geometrical object out of the very abstract definition of a scheme à la Toën-Vaquié. Let us consider what we did in the case of traditional schemes. In that setting, we defined the geometrical realization as a colimit of a forgetful functor, built in the category of locally ringed spaces. We see then that in this case we cannot simply replicate the procedures we presented, because we don't have at our disposal a (cocomplete) category of "spaces" as an environment where to look for a geometrical structure. We then have to use a totally different perspective, which is highly nonsensical, but that at the end would give a topological space. This perspective fully uses the ideas of Grothendieck about topoi, and specifically coherent topoi. The principal reference for this part is the book [**29**].

2.2.21. THEOREM. *Let* $\mathcal{F}$ *be a scheme relative to* $\mathbf{C}$*. The small site of* $\mathcal{F}$ *is isomorphic to the site of open subsets of a topological space* $|\mathcal{F}|$*. The isomorphism is functorial.*

PROOF. We are considering the category $\mathbf{Zar}(\mathcal{F})$ of Zariski open immersions into a scheme $\mathcal{F}$, equipped with the topology induced by the inclusion in the site of sheaves over affine schemes, with the canonical topology. We want to prove it comes from a topological space. The proof is divided in several steps.

(1) The category of sheaves over $\mathbf{Zar}(\mathcal{F})$ is canonically equivalent to the category of sheaves over $\mathbf{Aff}\,\mathbf{Zar}(\mathcal{F})$, the category of open immersions from affines to $\mathcal{F}$. Indeed, the inclusion $\mathbf{Aff}\,\mathbf{Zar}(\mathcal{F}) \hookrightarrow \mathbf{Zar}(\mathcal{F})$ is continuous and induces an equivalence of sheaves ([**2**], III.4).
(2) The category of sheaves over $\mathbf{Aff}\,\mathbf{Zar}(\mathcal{F})$ is a coherent topos ([**29**], Definition IX.11) because affines are still Zariski quasi-compact.
(3) We conclude that the topos of sheaves over $\mathbf{Aff}\,\mathbf{Zar}(\mathcal{F})$ has enough points by Deligne's Theorem ([**29**], IX.11.3). Because of the first point, also the topos of sheaves over $\mathbf{Zar}(\mathcal{F})$ has enough points.
(4) The topos of sheaves over $\mathbf{Zar}(\mathcal{F})$ is localic.
(5) If localic topos has enough points, then the site itself has enough points, as a locale.
(6) We conclude that $\mathbf{Zar}(\mathcal{F})$ is a locale with enough points, then by means of [**29**], Corollary IX.3.4, it is equivalent to a locale of open subsets of a topological space $|\mathcal{F}|$. All the steps are functorial.

□

2.2.22. DEFINITION. Let $\mathcal{F}$ be a scheme relative to **C**. Consider the contravariant functor from the category of open immersions of affine schemes into $\mathcal{F}$ to the category $\mathbf{Mon_C}$ that associates $h_{\operatorname{Spec} A} \to \mathcal{F}$ to A. This is a sheaf ([**34**], 2.11), hence a sheaf over $\mathbf{Zar}(\mathcal{F})$. By the previous theorem, it defines a sheaf (called *structure sheaf*) over $|\mathcal{F}|$, which we denote with $\mathcal{O}_\mathcal{F}$. The couple $(|\mathcal{F}|, \mathcal{O}_\mathcal{F})$ is called the *geometric realization of* $\mathcal{F}$.

A special case of schemes à la Toën-Vaquié is the case in which we consider the closed monoidal category $(\mathbf{Ab}, \otimes_\mathbb{Z}, \mathbb{Z})$. In this case, because of Corollary 1.1.15 and Proposition 1.1.17, the definitions presented in this last part overlap with those of the previous section: the category of affines is exactly the category $\mathbf{Ring}^{\mathrm{op}}$, and the category of schemes is exactly the classical category of schemes. In particular, given a classical scheme there are two different ways of constructing a topological space. One is the trivial one: consider the topological space beneath. The other one is presented in the previous proposition. It is not difficult to show that they both coincide, so that the definition of the geometrical realization is indeed well-posed.

2.2.23. PROPOSITION. *Let X be a scheme in the sense of Definition 1.1.7. Then $|h_X|$ as defined in Theorem 2.2.21 is canonically homeomorphic to the topological space beneath X and their structure sheaf is the same. In particular, the two functors of geometrical realization from functorial schemes to ringed spaces are isomorphic.*

PROOF. This follows from the fact that the equivalence of [**29**], IX.3.4 is such that a topos over a topological space is sent to the topological space itself. $\square$

Up to now, we presented the whole picture of generalized schemes à la Toën-Vaquié. It is now time to focus on schemes over $\mathbb{F}_1$ which is just a special case of this more general theory.

2.2.24. DEFINITION. A $\mathbb{F}_1$*-scheme* or a *scheme over* $\mathbb{F}_1$ is a scheme relative to the monoidal category $(\mathbf{Set}, \times, \{*\})$. The category of $\mathbb{F}_1$-schemes is denoted with $\mathbf{Sch}_{\mathbb{F}_1}$.

In particular, since monoids in $(\mathbf{Set}, \times, \{*\})$ are just ordinary commutative monoids with unity, the category **Aff** is the category $\mathbf{Mon}^{\mathrm{op}}$. Also, for a fixed monoid M, the category of M-modules is the category of M-sets, i.e. sets with an action of M. It is not an abelian category, since the initial object $\emptyset$ it is not the final object $\{*\}$.

We also note that for a couple of M-modules S and T, $S \otimes_M T$ is the set $S \times T$ modulo the equivalence relation generated by the relation $(m \cdot s, t) \sim (s, m \cdot t)$. In case S and T are M-algebras, by Proposition 2.2.4, the module $S \otimes_M T$ inherits a M-algebra structure, and it is isomorphic to $S \sqcup_M T$ in the category M-**Alg**, according to Proposition 2.2.4.

2.3. Deitmar - Toën-Vaquié equivalence

We now want to prove the equivalence of categories between the two different notions of schemes over $\mathbb{F}_1$ that we introduced so far. We will follow the idea of the proof of Theorem 1.3.11. However, we do not yet dispose of all the commutative algebra results that were used, sometimes very subtly, in the proof of that theorem. Hence, we dedicate a large part of this section to introduce some propositions that would lay some other foundations for commutative algebra of monoids, trying to set up an environment similar to the classical one of commutative rings.

2.3.1. PROPOSITION. *Let M be a monoid. The forgetful functor from M-algebras to monoids has a left adjoint which sends a monoid N to $M \times N$ with the natural M-action. In particular, the forgetful functor from M-algebras to sets has a left adjoint that sends a set S to the monoid*

$$M[S] := \{m \cdot s_1^{d_1} s_2^{d_2} \dots s_k^{d_k} : k \in \mathbb{Z}_{\geq 0}, m \in M, s_i \in S, d_i \in \mathbb{Z}_{\geq 0}\}$$

with the obvious operation and M-action. We shall indicate $M[\{x_1, \dots, x_n\}]$ with $M[x_1, \dots, x_n]$.

PROOF. We denote with $\mathbb{F}_1$ the trivial monoid $\{1\}$. The category of monoids is the category of $\mathbb{F}_1$-algebras, and for any couple of monoids M and N, we have $M \otimes_{\mathbb{F}_1} N = M \times N$. The result then follows from Corollary 2.2.7. □

2.3.2. EXAMPLE. Consider the monoid $(\mathbb{Z}_{\geq 1}, \cdot)$. It is isomorphic to $\mathbb{F}_1[x_1, x_2, \dots]$ through the map $x_1 \mapsto p_1$, where the p_i's are the positive primes.

2.3.3. DEFINITION. Let N be M-algebra, and let φ be the map $M \to N$. An equivalence relation $\sim$ on N is called *monoidal* if the following implication holds

$$a \sim a', b \sim b' \Rightarrow ab \sim a'b'.$$

It is called *M-linear* if the following implication holds

$$a \sim a' \Rightarrow m \cdot a \sim m \cdot a'.$$

Equivalently, a monoidal M-linear equivalence relation on N is a subset of $N \times N$ which defines an equivalence relation, and that is a sub-M-algebra with respect to the natural operation and the diagonal M-action. Given a monoidal M-linear equivalence relation on N, it is possible to define a structure of M-algebra on $N/{\sim}$ via the map $m \mapsto [\varphi(m)]$.

2.3.4. DEFINITION. Let M be a monoid. A M-algebra N is called *finitely generated* if there exists an integer n and a surjective arrow of M-algebras from $M[x_1, \dots, x_n]$ to N. Equivalently, if it is isomorphic as M-algebra to the monoid $M[x_1, \dots, x_n]/{\sim}$ for a suitable monoidal M-linear equivalence relation $\sim$.

2.3.5. PROPOSITION. *Let M be a monoid and N be a M-algebra. Then N is of finite presentation if and only if it is isomorphic as a M-algebra to the monoid $M[x_1, \dots, x_n]/{\sim}$ where $\sim$ is a finitely generated sub-$M[x_1, \dots, x_n]$-algebra of the monoid $M[x_1, \dots, x_n] \times M[x_1, \dots, x_n]$ i.e. it is the coequalizer in the category of $M[x_1, \dots, x_n]$-algebras of a diagram*

$$M[x_1, \dots, x_n][y_1, \dots, y_m] \rightrightarrows M[x_1, \dots, x_n]$$

for some suitable $n, m \in \mathbb{N}$.

PROOF. We will proceed in two steps. First of all, we shall see that such a M-algebra $N = M[x_1, \dots, x_n]/\sim$ is of finite presentation. We indicate with (p_j, q_j) the generators of $\sim$, and treat them as couples of monomials. Let $\{C_i, f_{ij}\}$ be a direct system of M-algebras. We can think of its direct limit as the M-algebra in which the elements are equivalence classes $[c_i]$ of elements $c_i \in C_i$, with respect to the relation that identifies $c_i \sim c_j$ if and only if there exists a $k \geq i, j$ such that $f_{ik}(c_i) = f_{jk}(c_j)$. The operations are defined acting on representatives of each class which lie in the same algebra C_i. Giving a map $N \to \varinjlim C_i$ is equivalent to give an n-tuple of elements $[c_i], \ldots, [c_n]$ such that $p_j([c_1], \ldots, [c_n]) = q_j([c_1], \ldots, [c_n])$ for all j. We can set an index i such that all the representatives c_i are in C_i (now we are using the finite generation property). Because of the definition of the M-algebra structure defined on the limit, we have then $[p_j(c_1, \ldots, c_n)] = p_j([c_1], \ldots, [c_n]) = q_j([c_1], \ldots, [c_n]) = [q_j(c_1, \ldots, c_n)]$. Hence each $p_j(c_1, \ldots, c_n)$ is equal to $q_j(c_1, \ldots, c_n)$ at some level. Now let k be an index such that $f_{ik}(p_j(c_1, \ldots, c_n))$ equals $f_{ik}(q_j(c_1, \ldots, c_n))$ for all j (now we are using the finite generation of $\sim$). Because the transition maps are M-linear, we conclude that $p_j(f_{ik}(c_1), \ldots, f_{ik}(c_n)) = q_j(f_{ik}(c_1, \ldots, c_n))$. By definition of N then, we can define a unique map $N \to C_k$ which is represented by the n-tuple $(f_{ik}(c_1), \ldots, f_{ik}(c_n))$, hence an element of $\varinjlim_{i \in I} \mathrm{Hom}_M(N, C_i)$. This splitting is unique. Indeed, let $[f], [g]$ two elements in the direct limit splitting the same map. We can assume that they are represented by two maps f_k, g_k in $\mathrm{Hom}(N, C_k)$, i.e. by two n-tuples $(x_{k,1}, \ldots, x_{k,n})$, $(y_{k,1}, \ldots, y_{k,n})$ of elements in C_k such that they satisfy m equations. Because they both split the map to the direct limit, we get that each $[x_{k,i}]$ is equal to $[y_{k,i}]$, hence there exists an index r in which the two n-tuples coincide. This means that $f_r = g_r$, and so $[f] = [g]$.

Now we prove the other implication. It is straightforward that N can be expressed as the direct limit of its finitely generated sub-M-algebras. By hypothesis then, there exists a splitting of the identity map $id_N \in \mathrm{Hom}_M(N, N)$ in $N \to M[b_1, \ldots, b_n] \hookrightarrow N$, where $M[b_1, \ldots, b_n]$ is the (finitely generated) sub-M-algebra of N generated by the elements $b_1, \ldots, b_n$. We conclude that the inclusion map $M[b_1, \ldots, b_n] \hookrightarrow N$ is surjective, hence an identity. N is finitely generated.

Let $\sim$ be the relation of a presentation $M[x] := M[x_1, \ldots, x_n] \to N$. It is the direct limit of its finitely generated sub-$M[x_1, \ldots, x_n]$-algebras $\{\sim_i\}$. Because the direct limit is right exact (it is the left adjoint of the Δ functor, see **[37]**, Exercise 2.6.4), we can conclude that N is the direct limit of the direct system $\{M[x]/\sim_i, p_{ij} \colon (M[x]/\sim_i) \to (M[x]/\sim_j)\}$. In particular, the identity map of N splits $N \to (M[x]/\sim_i) \to N$ for some index i. Let's give a name to all the maps involved. We call g_i the induced map $g_i \colon N \to (M[x]/\sim_i)$, and we refer to the projections with the following notations $p_i \colon (M[x]/\sim_i) \to N$, $\pi \colon M[x] \to N$ and $\pi_i \colon M[x] \to (M[x]/\sim_i)$. We also call with $\sim_{ij}$ the equivalence relation associated to the structure of $(M[x]/\sim_i)$-algebra of $M[x]/\sim_j$ induced by p_{ij}. We know that $\pi = p_i \pi_i$ for any i, and that $p_k p_{ik} = p_i$, $p_{ik}\pi_i = \pi_k$ for all $k \geq i$. In particular, calling g_k the map $p_{ik} g_i$ for any index $k \geq i$, we have another splitting of the identity $id_N = p_k g_k \colon N \to (M[x]/\sim_k) \to N$.

The map we obtain composing $g_i \pi$ needs not to be the same projection map π_i. However, we claim that there exists a suitable index $k \geq i$ such that the map $g_k \pi$ is indeed the same projection map π_k. Since $p_i g_i = id_N$, we have $p_i(g_i p_i \pi_i)(x_j) = p_i(\pi_i)(x_j)$ for all $j = 1, \ldots, n$. Hence, the couples $((g_i p_i \pi_i)(x_j), \pi_i(x_j))$ lie in the equivalence relation associated to the $(M[x]/\sim_i)$-algebra structure of N defined by p_i. Because there are just finitely many j's, we can hence suppose that all the couples $((g_i p_i \pi_i)(x_j), \pi_i(x_j))$ lie in $\sim_{ik}$ for some index $k \geq i$. This is equivalent to say that the two maps $p_{ik} g_i p_i \pi_i = g_k \pi$ and $p_{ik}\pi_i = \pi_k$ are indeed the same map, as claimed. Now we can write a commutative square of $M[x]$-algebras

$$\begin{tikzcd} M[x] \arrow[r, "\pi"] \arrow[d, equal] & N \arrow[d, "g_k"] \\ M[x] \arrow[r, "\pi_k"] & M[x]/\sim_k \end{tikzcd}$$

which fits into the following commutative diagram.

$$\begin{tikzcd} M[x] \arrow[r, "\pi_k"] \arrow[d, equal] & M[x]/\sim_k \arrow[d, "p_k"] \\ M[x] \arrow[r, "\pi"] \arrow[d, equal] & N \arrow[d, "g_k"] \\ M[x] \arrow[r, "\pi_k"] & M[x]/\sim_k \end{tikzcd}$$

We deduce that the composite map $g_k p_k$ is then another splitting of the map π_k through π_k. By universal property, we then deduce it has to be the identity map. We conclude that the maps g_k and p_k are one the inverse of the other, hence they define an isomorphism $N = M[x]/\sim_k$. □

Let $\{p_i, q_i\}_{i\in I}$ be elements of $M[S]$. From now on, we indicate with $(p_i = q_i)_{i\in I}$ the monoidal M-linear equivalence relation on $M[S]$ generated by the couples (p_i, q_i).

2.3.6. Example. Using the notation of Definition 2.1.23, the monoid $(\mathbb{Z}, \cdot)$ is isomorphic to the monoid

$$\left(\mathbb{F}_1[u, x_1, x_2, \ldots] \Big/ (u^2 = 1)\right)_0$$

via the map $u \mapsto -1$, $x_1 \mapsto p_1$, where the p_i's are the positive primes.

2.3.7. Corollary. *A localization of a monoid over a finite set of elements is of finite presentation.*

Proof. Considering the localization over the product of the elements, we can reduce ourselves to consider the case in which we localize over a single element a. It is straightforward that $M_a = M[x]/(ax = 1)$. We can then apply the previous proposition and conclude the claim. □

2.3.8. Proposition. *Localizations of monoids are flat.*

Proof. Let M be a monoid, and S a multiplicatively closed subset. We have to prove that the operation of tensoring over M with $S^{-1}M$ commutes with equalizers and finite products in the category of M-modules. In this category, both these limits are built over the the limits in the category of sets, with the obvious M-action induced. Let T be a M-module. Since the equivalence relation that defines the tensor is hard to handle with, we now give an alternative explicit description of the $S^{-1}M$-module $T \otimes_M S^{-1}M$. Consider the set

$$S^{-1}T := \left\{\frac{t}{s} : t \in T, s \in S\right\} \Big/ \sim$$

where $\sim$ is the equivalence relation that identifies $\frac{t}{s}$ and $\frac{t'}{s'}$ if there exists an element $s'' \in S$ such that $s''s' \cdot t = s''s \cdot t'$. The set $S^{-1}T$ has a natural structure of $S^{-1}M$-module defined by $\frac{m}{s} \cdot \frac{t}{s'} := \frac{m \cdot t}{ss'}$. The following map

$$\begin{aligned} T \times S^{-1}M &\to S^{-1}T \\ \left(t, \frac{m}{s}\right) &\mapsto \frac{m}{s} \cdot t \end{aligned}$$

coequalizes the two maps $M \times T \times S^{-1}M \rightrightarrows T \times S^{-1}M$, hence it induces a map $T \otimes_M S^{-1}M \to S^{-1}T$ that sends $t \otimes \frac{m}{s}$ to $\frac{m \cdot t}{s}$. It is obviously $S^{-1}M$-linear. We can also define an inverse of this map by sending $\frac{t}{s}$ to $t \otimes \frac{1}{s}$. It can be checked that this map is indeed well-defined and $S^{-1}M$-linear. The fact that it is an inverse of the previous map comes from the following equalities.

$$\frac{m}{s} \cdot t \otimes 1 = \frac{m}{s} \cdot t \otimes \frac{s}{s} = t \otimes \frac{m}{s}.$$

From now on, we can then consider $S^{-1}T$ as the module $T \otimes_M S^{-1}M$.

We can now prove the commutation properties. Let T and U be M-modules. We define the following two maps.

$$\begin{aligned} S^{-1}(T \times U) &\to S^{-1}T \times S^{-1}U \\ \frac{(t,u)}{s} &\mapsto \left(\frac{t}{s}, \frac{u}{s}\right) \\ S^{-1}T \times S^{-1}U &\to S^{-1}(T \times U) \\ \left(\frac{t}{s}, \frac{u}{s'}\right) &\mapsto \frac{(s' \cdot t, s \cdot u)}{ss'} \end{aligned}$$

They are obviously M-linear, and they define an isomorphism of M-modules from $S^{-1}(T \times U)$ to $S^{-1}T \times S^{-1}U$, as wanted.

We now prove the commutation with equalizers. Suppose we have two arrows of M-modules $\varphi, \psi \colon T \rightrightarrows U$ whose equalizer is E. There is a natural map from $S^{-1}E$ to the equalizer E' of the induced couple of arrows $S^{-1}T \rightrightarrows S^{-1}U$, which sends the element $\frac{x}{s}$ to $\frac{x}{s}$, seen as an element of $S^{-1}T$. We claim that this map is an isomorphism. The injectivity is clear. Suppose now that $\frac{t}{s}$ is in E'. This means that $\frac{\varphi(t)}{s} = \frac{\psi(t)}{s}$, hence that there exists an element $s' \in S$ such that

$$\varphi(s's \cdot t) = s's \cdot \varphi(t) = s's \cdot \psi(t) = \psi(s's \cdot t).$$

We then conclude that $\frac{t}{s} = \frac{s's \cdot t}{s's^2}$ and $s's \cdot t \in E$. This proves the surjectivity, hence the claim. $\square$

2.3.9. Proposition. *Let M be a monoid. Monomorphisms in M-**Mod** are injections, and epimorphisms are surjections.*

Proof. An injection of M-modules is obviously a monomorphism. For the contrary, consider a monomorphism $S \to T$ and suppose that both s, s' have the same image t. Composing on the left with the two maps $M \to S$ induced respectively by $1 \mapsto s$ and $1 \mapsto s'$, we obtain the same map $1 \mapsto t$, hence $s = s'$. A surjection of M-modules is obviously an epimorphism. For the contrary, consider an epimorphism $\varphi \colon S \to T$, and consider the set $T/\!\sim$ where $\sim$ identifies the elements of $\varphi(S)$. We can give to $T/\!\sim$ a M-module structure induced by the one on T. It is well defined. Indeed, if $t = \varphi(s)$, then $m \cdot t = \varphi(m \cdot s)$. Consider now the two maps $T \to T/\!\sim$ defined by the natural projection, and by the constant map $[\varphi(s)]$. Since S has the same image via both maps, we conclude that $\varphi(S) = T$, as wanted. □

The following two results concern flat epimorphisms of monoids. In particular, we would like to conclude that local flat epimorphisms are isomorphisms. Stenström in [**32**] refers to the work of Roos and he states that flat epimorphisms of (non necessarily commutative) monoids can be characterized as localizations over Gabriel topologies, using the tools of torsion theory developed in [**15**] by Gabriel. In particular, any epimorphism of monoids $M \to N$ induces a full embedding of categories $N\,\mathbf{Mon} \to M\text{-}\mathbf{Mod}$ via the forgetful functor. Due to the flatness property, this forgetful functor has also an exact left adjoint, hence it defines a localization (i.e. a full subcategory with a left exact left adjoint of the inclusion functor) of $M\text{-}\mathbf{Mod}$. However, the proof of the fact that such reflective subcategories are all localization with respect to some Gabriel topologies of monoids is very hard to find (it is not present in [**32**]), and it is not a direct corollary of the general results of Gabriel, who considered abelian categories. Therefore, since in our case $M\text{-}\mathbf{Mod}$ is not abelian, we prefer to follow a more explicit approach, which is in turn valid just for our specific setting.

Analogous results on the comparison of the two topologies on $\mathbf{Mon}^{\mathrm{op}}$ have been proved independently by Florian Marty, who used a more abstract and general approach, based on Gabriel filters. All the details can be found in his article [**30**].

2.3.10. Lemma. *A local epimorphism of monoids is surjective on invertible elements.*

Proof. Let $\varphi\colon M \to N$ be a local epimorphism of monoids. Consider the set $N/\!\sim_{\mathfrak{m}}$, where $\sim_{\mathfrak{m}}$ identifies the elements of the maximal ideal $\mathfrak{m} := N \setminus N^{\times}$. It has a natural monoid structure induced by the one in N, and it is isomorphic to the monoid with zero $(N^{\times})_0$. We also consider the subgroup $\varphi(M^{\times})$ in $N^{\times}$, and the quotient taken in the category of groups $T := N^{\times}/\varphi(M^{\times})$. We add the 0 to T and obtain a monoid $T \sqcup \{0\}$. We can now consider two maps $N^{\times} \sqcup \{0\} \rightrightarrows T \sqcup \{0\}$: the first one is induced by the projection, the second is induced by the constant map $N^{\times} \mapsto 1_T$. Since φ is local, the image of an element in M via the two composite maps $N \to N^{\times} \sqcup \{0\} \rightrightarrows T \sqcup \{0\}$ is the same. Hence, because φ is an epimorphism, we conclude that $\varphi(M^{\times}) = N^{\times}$. □

2.3.11. Proposition (see [**26**], Lemma IV.1.2 for rings). *Let $\varphi\colon M \to N$ a map of monoids.*

1. *If φ is flat and local, then it is injective.*
2. *If φ is a local flat epimorphism, then it is an isomorphism.*

Proof. We initially prove the first claim. Suppose that $\varphi(a) = \varphi(b) = t$. Consider the two maps of M-modules $M \to M$, $1 \mapsto a$ and $1 \mapsto b$, and let E be their equalizer. By using the isomorphisms of M-modules $m \otimes n \mapsto \varphi(m)n$ from $M \otimes N$ to N, we conclude that the two maps tensored with N are both equal to the map $N \to N$, $n \mapsto tn$. In particular, the equalizer of the two is the whole of N. By the flatness property, we then deduce that the map $E \otimes N \to N$, $x \otimes n \mapsto \varphi(x)n$ is an isomorphism. In particular, there exists an element $x \in E$ and an element $n \in N$ such that $\varphi(x)n = 1$. Because the map is local, we conclude that x is invertible. Since $ax = bx$, this implies that $a = b$.

Now we turn to the second claim. Because we already know that φ is injective, we consider M as a submonoid of N, and consider φ as the inclusion. We recall that a map is an epimorphism if and only if its cokernel pair is constituted by identities. Because $N \otimes_M N$ is the cokernel pair of φ in the category of monoids (Proposition 2.2.4), we conclude that the two maps $N \to N \otimes_M N$ defined as $n \mapsto 1 \otimes n$ and $n \mapsto n \otimes 1$ are isomorphisms. Now consider the M-module $N/\sim_M$, defined as the quotient of N with respect to the equivalence relation which identifies the elements of M. It has a well-defined M-module structure induced by the one of N, and a natural projection map $\pi\colon N \to N/\sim_M$. This projection has the following universal property: any map of M-modules $N \to T$ such that the image of M is constant, splits uniquely through π. In other words, π is the pushout of the diagram below.

$$\begin{tikzcd} M \arrow[r, "\varphi"] \arrow[d] & N \\ \{*\} & \end{tikzcd}$$

Because of the flatness property, $\otimes_M N$ commutes with small products, hence it preserves the terminal object $\{*\}$ (the empty product). Also, because it commutes with colimits and $\varphi \otimes_M N = id_N$, we conclude that $(N/\sim_M) \otimes_M N$ is the pushout of the diagram

$$\begin{tikzcd} N \arrow[r, "="] \arrow[d] & N \\ \{*\} & \end{tikzcd}$$

hence it is the trivial module $\{*\}$.

We now inspect the kernel pair K of the projection $\pi\colon N \to N/\sim_M$. It is constituted by the couples (x, y) in $N \times N$ such that $\pi(x) = \pi(y)$. Since $(N/\sim_M) \otimes_M N$ is the terminal object, the kernel pair of the tensored map is the product of two copies of $N \otimes_M N = N$. Because of the flatness property, we then conclude that the map $K \otimes_M N \to N \times N$, $(x, y) \otimes n \mapsto (xn, yn)$ is an isomorphism. Fix now an element $\bar{n}$ of N. In particular, the couple $(1, \bar{n})$ has to be reached by the previous map, hence there is a couple $(x, y) \in K$ and an element $n \in N$ such that $xn = 1$ and $yn = \bar{n}$. We then conclude that n and x are invertible, hence they are elements of M by Lemma 2.3.10. Because the couple (x, y) lies in K and x is in M, we conclude that also y is in M. Therefore, $\bar{n}$ is an element of M. This holds for any $\bar{n}$, hence $M = N$. We then showed that φ is also surjective. Because any bijective map of monoids is an isomorphism, the claim is proven. □

2.3.12. THEOREM (see Theorem 1.1.14 for rings). *Let $\varphi\colon M \to N$ a morphism of monoids. The following are equivalent.*

- *(i) The map φ is a flat epimorphism, of finite presentation.*
- *(ii) The map φ is isomorphic to a localization over an element of M.*
- *(iii) The map φ defines an open immersion of affine geometrical $\mathbb{F}_1$-schemes.*

PROOF. The fact that (ii) implies (iii) is obvious. It is also easy to show that (iii) implies (ii). Indeed, suppose that $\mathrm{Spec}_{\mathbb{F}_1} N$ is an open geometrical $\mathbb{F}_1$-subscheme of $\mathrm{Spec}_{\mathbb{F}_1} M$. Cover it with base open sets $\{\mathrm{Spec}_{\mathbb{F}_1} M_{a_i}\}$, and cover each of these with base open sets $\{\mathrm{Spec}_{\mathbb{F}_1} N_{b_{ij}}\}$. By Corollary 2.1.8, we conclude that $\operatorname{Spec}_{\mathbb{F}_1} N_{b_{ij}}$ equals $\operatorname{Spec}_{\mathbb{F}_1} N$ for some couple (i, j), and in particular $\operatorname{Spec}_{\mathbb{F}_1} N$ equals $\operatorname{Spec}_{\mathbb{F}_1} M_{a_i}$. The fact that (ii) implies (i) comes from Corollary 2.3.7, Proposition 2.3.8 and the universal property of localizations. We are then left to prove that (i) implies (ii). By universal property, the map φ splits over the monoid $\varinjlim_{a_i \in \varphi^{-1}(N^\times)} M_{a_i} = M_{\mathfrak{p}}$ where $\mathfrak{p}$ is $\varphi^{-1}(N \setminus N^\times)$. The induced map $M_{\mathfrak{p}} \to N$ is local, and still an epimorphism. We now prove it is also flat. Suppose that S is a $M_{\mathfrak{p}}$-module. We claim that $S = S \otimes_M M_{\mathfrak{p}}$. Indeed, the map $x \mapsto x \otimes 1$ defines an inverse of the natural map $x \otimes \frac{m}{f} \mapsto \frac{m}{f} \cdot x$. This is because of the following equalities.

$$\frac{m}{f} \cdot x \otimes 1 = \frac{m}{f} \cdot x \otimes \frac{f}{f} = x \otimes \frac{m}{f}.$$

We remark that, more generally, whenever $A \to B$ is an epimorphism of monoids and M is a B-module, then by using 2.2.6 and 2.2.11, we conlude

$$M \cong M \otimes_B B \cong M \otimes_B (B \otimes_A B) \cong M \otimes_A B$$

as wanted. Also, by the essential uniqueness of the adjoint functor, whenever we have a composite map of monoids $M \to N \to P$, then the functor $(\otimes_M N) \otimes_N P$ is canonically isomorphic to the functor $\otimes_M P$. We then write $S \otimes_M N \otimes_N P$ without using brackets, and consider it equal to $S \otimes_M P$, for any M-module S. Now consider a small limit $\lim S_i$ of $M_{\mathfrak{p}}$-modules. We write $\hat{S}_i$ whenever we consider them as M-modules. Using the flatness of φ and of localizations (Proposition 2.3.8), we then conclude the following sequence of equivalences

$$\begin{aligned}
&(\lim S_i) \otimes_{M_{\mathfrak{p}}} N = (\lim \hat{S}_i \otimes_M M_{\mathfrak{p}}) \otimes_{M_{\mathfrak{p}}} N = (\lim \hat{S}_i) \otimes_M M_{\mathfrak{p}} \otimes_{M_{\mathfrak{p}}} N = \\
&= (\lim \hat{S}_i) \otimes_M N = \lim(\hat{S}_i \otimes_M N) = \lim(\hat{S}_i \otimes_M M_{\mathfrak{p}} \otimes_{M_{\mathfrak{p}}} N) = \\
&= \lim(S_i \otimes_{M_{\mathfrak{p}}} N)
\end{aligned}$$

which proves that $M_{\mathfrak{p}} \to N$ is flat.

By Proposition 2.3.11, we conclude that $M_{\mathfrak{p}} \to N$ is an isomorphism. Because of the finite presentation property, the identity map $N \to M_{\mathfrak{p}}$ has to split over some M_a with $a \in \varphi^{-1}(N^\times)$. Because all the maps involved are maps of M-algebras, we also conclude that the composite $\psi \colon M_a \to N \to M_a$ is a map of M-algebras, hence it is the identity by universal property. We then conclude that $N = M_a$, as wanted. □

2.3.13. Corollary. *Let $\varphi \colon M \to N$ a map of monoids. The induced map* $\operatorname{Spec} N \to \operatorname{Spec} M$ *is an open Zariski immersion in the sense of Definition 2.2.13 if and only if the induced map* $\operatorname{Spec}_{\mathbb{F}_1} N \to \operatorname{Spec}_{\mathbb{F}_1} M$ *is an open Zariski immersion in the sense of Definition 2.1.17.*

Proof. This comes by definition of open immersions of $\mathbb{F}_1$-schemes, and the equivalence of (i) and (iii) in the previous theorem. □

2.3.14. Theorem. *The Zariski site of affine geometrical $\mathbb{F}_1$-schemes is equivalent to the Zariski site of* $\mathbf{Mon}^{\mathrm{op}}$.

Proof. The two categories underneath are equivalent because of Corollary 2.1.16. By the previous corollary, we also know that open inclusions are the same. We have to prove that coverings are the same. Let M be a monoid. In the case of affine geometrical $\mathbb{F}_1$-schemes, coverings must include the trivial immersion $\operatorname{Spec}_{\mathbb{F}_1} M \to \operatorname{Spec}_{\mathbb{F}_1} M$ (Corollary 2.1.8). We now prove that this is also true for the Zariski topology. Let $\{\operatorname{Spec} M_{a_i}\}$ be a Zariski covering. Suppose that none of those open immersions is trivial, i.e. that none of the a_i's is invertible. Consider the M-module $M/{\sim_{\mathfrak{m}}}$ where $\sim_{\mathfrak{m}}$ identifies the non-invertible elements in M. We claim that $(M/{\sim_{\mathfrak{m}}}) \otimes_M M_{a_i}$ is isomorphic to the trivial M-module $\{*\}$, for all a_i's. Indeed, since a_i is not invertible, we conclude the following sequence of equalities for any element $[x] \otimes \frac{m}{a_i^k}$ in $(M/{\sim_{\mathfrak{m}}}) \otimes_M M_{a_i}$:

$$[x] \otimes \frac{m}{a_i^k} = [mx] \otimes \frac{a_i}{a_i^{k+1}} = [a_i] \otimes \frac{1}{a_i^{k+1}} = [a_i^{k+1} a_i] \otimes \frac{1}{a_i^{k+1}} = [a_i] \otimes 1.$$

However, the morphism $(M/{\sim_{\mathfrak{m}}}) \to \{*\}$ is never an isomorphism, unless M is the trivial group in which case the statement is obvious. We then conclude that any Zariski covering must include the trivial open immersion, as claimed. □

2.3.15. Warning. From now on, we will then drop the subscript when referring to affine geometrical $\mathbb{F}_1$-schemes, and just write $\operatorname{Spec} M$. Also, we won't refer to any specific definition when considering open immersions of affine $\mathbb{F}_1$-schemes. It is also legitimate to refer to the site we built on $\mathbf{Mon}^{\mathrm{op}}$ as the *Zariski site*, without specifying which definition we are using at every occurrence.

2.3.16. Proposition. *Let $f\colon \mathcal{F} \to \mathcal{G}$ be a morphism of Zariski sheaves over $\mathbf{Mon}^{\mathrm{op}}$, and let $\mathcal{G} = h_{\operatorname{Spec} M}$ be affine. Then f is an open immersion if and only if $\mathcal{F}$ is isomorphic over $\mathcal{G}$ to $h_U := \operatorname{Hom}(\cdot, U)$ where U is an open geometrical $\mathbb{F}_1$-subscheme of* $\operatorname{Spec} M$.

Proof. Because of Proposition 2.2.17, this amounts to say that for a given family of affine open geometrical $\mathbb{F}_1$-subschemes $\operatorname{Spec} M_i$ of $\operatorname{Spec} M$, the image of the sheaf map $\coprod h_{\operatorname{Spec} M_i} \to h_{\operatorname{Spec} M}$ is h_U where U is the open geometrical $\mathbb{F}_1$-subschemes constituted by the union of the $\operatorname{Spec} M_i$'s. This is analogue to the proof of Proposition 1.2.11. □

2.3.17. Lemma. *A map $X \to Y$ of geometrical $\mathbb{F}_1$-schemes is an open immersion if and only if for any affine scheme* $\operatorname{Spec} M$ *over Y, the induced arrow* $X \times_Y \operatorname{Spec} M \to \operatorname{Spec} M$ *is an open immersion.*

Proof. This follows in the same way as in [**16**], I.4.2.4. □

2.3.18. Theorem. *The category of $\mathbb{F}_1$-schemes is equivalent to the category of geometrical $\mathbb{F}_1$-schemes.*

Proof. Since the category of locally monoidal spaces is cocomplete (Proposition 2.1.12) , the immersion $\mathbf{Aff} \to \mathbf{MS}$ induces an adjoint pair $\mathbf{Psh}(\mathbf{Aff}) \rightleftarrows \mathbf{MS}$ by means of Proposition 1.3.2 in which the left adjoint is the functor $|\cdot|\colon \mathbf{Psh}(\mathbf{Aff}) \to \mathbf{MS}$ that sends each object $\operatorname{colim} h_{\operatorname{Spec} M}$ to $\operatorname{colim} \operatorname{Spec} M$ and the right adjoint is the functor $h\colon \mathbf{MS} \to \mathbf{Psh}(\mathbf{Aff})$ that sends X to $\operatorname{Hom}(\cdot, X)$. Let now X be a geometrical $\mathbb{F}_1$-scheme, and let $\{\operatorname{Spec} M_i \to X\}$ be an affine Zariski covering of it. Because the Zariski topology is subcanonical (Corollary 2.1.21), we conclude that h_X is indeed a sheaf over $\mathbf{Aff}$. Fix now an affine $\mathbb{F}_1$-scheme $h_{\operatorname{Spec} N}$ over h_X. By Lemma 2.3.17, the morphism $\operatorname{Spec} M_i \times_X \operatorname{Spec} N \to \operatorname{Spec} N$ is an open immersion. Because of Definition 2.2.16, Proposition 2.3.16, and the fact that h is a right adjoint, we can also conclude that the map

$$h(\operatorname{Spec} M_i \times_X \operatorname{Spec} N \to \operatorname{Spec} N) = h_{\operatorname{Spec} M_i} \times_{h_X} h_{\operatorname{Spec} N} \to h_{\operatorname{Spec} N}$$

is an open immersion. This proves that each map $h_{\operatorname{Spec} M_i} \to h_{\operatorname{Spec} M}$ is an open immersion. Now we also prove that $\coprod h_{\operatorname{Spec} M_i} \to h_X$ is an epimorphism. Indeed, let $\mathcal{F}$ be another sheaf, and let f, g be maps from $h_X \to \mathcal{F}$ such that $f\varphi_i = g\varphi_i$ for every i. Note that, using [**2**] III.4, $\mathcal{F}$ can be seen not only as a sheaf over affines, but also as a sheaf over geometrical schemes over $\mathbb{F}_1$. Hence, by Yoneda's lemma, the maps f, g translate into two elements ρ, σ in $\mathcal{F}(X)$ such that $\mathcal{F}(\varphi_i)(\rho) = \mathcal{F}(\varphi_i)(\sigma)$ for every i. Since $\mathcal{F}$ is a sheaf and because the φ_i's define a covering, this implies that $\rho = \sigma$, hence $f = g$. We then conclude that h_X is a $\mathbb{F}_1$-scheme.

By the co-Yoneda's lemma (Proposition 1.2.4), we can write any presheaf of affines $\mathcal{F}$ as the colimit of the functor

$$\begin{aligned} \mathbf{Aff}_{/\mathcal{F}} &\to \mathbf{Psh}(\mathbf{C}) \\ (\operatorname{Hom}(\cdot, A) \to \mathcal{F}) &\mapsto \operatorname{Hom}(\cdot, A). \end{aligned}$$

In particular, the geometrical realization of h_X is the colimit of the functor

$$\begin{aligned} \mathbf{Aff}_{/X} &\to \mathbf{MS} \\ (\operatorname{Hom}(\cdot, A) \to \operatorname{Hom}(\cdot, X)) &\mapsto A \end{aligned}$$

which is, by Yoneda's lemma, the same colimit as the one of the functor

$$\begin{aligned} \mathbf{Aff}_{/X} &\to \mathbf{MS} \\ (A \to X) &\mapsto A. \end{aligned}$$

Since affine geometrical $\mathbb{F}_1$-schemes are dense in geometrical $\mathbb{F}_1$-schemes (Proposition 1.2.3), the colimit of the functor from $\mathbf{Aff}_{/X}$ to geometrical $\mathbb{F}_1$-schemes is exactly X, hence there is a natural map $|h_X| \to X$. We also know that X is the colimit in $\mathbf{MS}$ of the gluing diagram induced by an affine open covering, which is embedded in the colimiting diagram $\mathbf{Aff}_{/X} \to \mathbf{MS}$. Hence we have also a map $X \to |h_X|$, which determines an isomorphism.

Now suppose that $\mathcal{F}$ is a $\mathbb{F}_1$-scheme with an open affine covering $\{h_{\operatorname{Spec} M_i}\}$. Because $\mathbb{F}_1$-schemes have fibered products (Proposition 2.2.19), we can also consider affine open coverings $\{h_{\operatorname{Spec} M_{ijk}}\}$ of the $\mathbb{F}_1$-schemes $h_{\operatorname{Spec} M_i} \times_X h_{\operatorname{Spec} M_j}$. By Lemmas 1.2.6 and 1.2.7, then $\mathcal{F}$ is the coequalizer in the diagram below.

$$\coprod h_{\operatorname{Spec} M_i} \times_{\mathcal{F}} h_{\operatorname{Spec} M_j} \rightrightarrows \coprod h_{\operatorname{Spec} M_i} \to \mathcal{F}$$

Note that all these maps are open immersions. Indeed, by their very definition, open immersions are stable under affine base change, hence $h_{\operatorname{Spec} M_i} \times_{\mathcal{F}} h_{\operatorname{Spec} M_j} \to h_{\operatorname{Spec} M_i}$ is an open immersion. In particular, by Proposition 2.3.16, these maps can be written as $h_{U_{ij}} \to h_{\operatorname{Spec} M_i}$ induced by open immersions $U_{ij} \to \operatorname{Spec} M_i$. We then conclude that $|\mathcal{F}|$ is the coequalizer of a diagram

$$\coprod U_{ij} \rightrightarrows \coprod \operatorname{Spec} M_i \to |\mathcal{F}|$$

so that it is a gluing of affines on open subsets, hence a geometrical $\mathbb{F}_1$-scheme. By letting $\mathcal{G}$ be another $\mathbb{F}_1$-scheme, we can also construct the equalizing diagram

$$\operatorname{Hom}(\mathcal{F}, \mathcal{G}) \to \coprod \operatorname{Hom}(h_{\operatorname{Spec} M_i}, \mathcal{G}) \rightrightarrows \coprod \operatorname{Hom}(h_{\operatorname{Spec} M_i} \times_X h_{\operatorname{Spec} M_j}, \mathcal{G})$$

and hence conclude that the Zariski topology restricted to sheaves over affines is subcanonical. We can then define an inverse of the map $\mathcal{F} \to h_{|\mathcal{F}|}$ by gluing the maps $h_{\operatorname{Spec} M} \to h_{|\mathcal{F}|}$, hence $\mathcal{F} = h_{|\mathcal{F}|}$. This concludes the proof. $\square$

With the previous theorem we have defined in particular a way to associate to a $\mathbb{F}_1$-scheme a topological space: it suffices to consider the space beneath the associated geometrical $\mathbb{F}_1$-scheme. Also in this case, we have to prove that this procedure gives the same result as the one presented in Theorem 2.2.21.

2.3.19. PROPOSITION. *Let X be a geometrical $\mathbb{F}_1$-scheme. Then $|h_X|$ as defined in Theorem 2.2.21 is canonically homeomorphic to the topological space beneath X and their structure sheaf is the same. In particular, the two functors of geometrical realization from $\mathbb{F}_1$-schemes to* **MS** *are isomorphic.*

PROOF. This follows from the fact that the equivalence of [**29**], IX.3.4 is such that a topos over a topological space is sent to the topological space itself. $\square$

Also for schemes over $\mathbb{F}_1$, it is easy to see that the equivalence of categories respects the topology of the two sites.

2.3.20. PROPOSITION. *A morphism of geometrical $\mathbb{F}_1$-schemes is an open immersion if and only the induced morphism of $\mathbb{F}_1$-schemes is an open immersion. Let now X be a fixed geometrical $\mathbb{F}_1$-scheme. A collection of geometrical $\mathbb{F}_1$-schemes over X is an open Zariski covering of X if and only if the induced collection of $\mathbb{F}_1$-schemes over h_X is an open Zariski covering of h_X.*

PROOF. The first claim follows from the fact that open coverings in both cases can be defined as maps that are open immersion after any affine base change (use Lemma 2.3.17 and Definition 2.2.16), and in the affine case the two notions do agree. For coverings, it suffices to write down the associate coequalizing diagrams and use the gluing lemma. $\square$

2.4. Base change functors

After having defined schemes over $\mathbb{F}_1$, the natural question is how to lift them to classical schemes over $\mathbb{Z}$. We want to consider this process like a base change with $\mathbb{Z}$ over $\mathbb{F}_1$. This can be done starting from the functor that lifts a monoid M to the ring $\mathbb{Z}[M]$. However, the two approaches to $\mathbb{F}_1$ we presented in the past sections have different ways to generalize this functor from affines to arbitrary schemes. Not surprisingly, Deitmar's definition ([**11**], Section 2) is more "geometric", while Toën-Vaquié's approach ([**34**], Section 2.5) is more "functorial". Given that the two perspectives on schemes are equivalent, we have to prove that also the two ways of base-changing are naturally equivalent.

2.4.1. PROPOSITION. *The forgetful functor* **Ring** $\to$ **Mon** *has a left adjoint* **Mon** $\to$ **Ring** *that sends a monoid M to the ring $\mathbb{Z}[M]$. We indicate this functor with the notation $\otimes_{\mathbb{F}_1}\mathbb{Z}$.*

2.4.2. LEMMA. *Let $\operatorname{Spec} N \to \operatorname{Spec} M$ be an open immersion of affine schemes over $\mathbb{F}_1$. Then the induced map*

$$\operatorname{Spec}(N \otimes_{\mathbb{F}_1} \mathbb{Z}) \to \operatorname{Spec}(M \otimes_{\mathbb{F}_1} \mathbb{Z})$$

is an open immersion of affine schemes over $\mathbb{Z}$.

PROOF. By Theorem 2.3.12, it suffices to show that, for a given element $a \in M$, the following equality holds

$$\mathbb{Z}[M_a] = M_a \otimes_{\mathbb{F}_1} \mathbb{Z} \cong (M \otimes_{\mathbb{F}_1} \mathbb{Z})_a = \mathbb{Z}[M]_a$$

where the second localization is taken in the category of rings. A map $\mathbb{Z}[M_a] \to \mathbb{Z}[M]_a$ is induced by the map of monoids $M_a \to \mathbb{Z}[M]_a$, which is in turn induced by the natural map $M \to \mathbb{Z}[M]_a$. A map $\mathbb{Z}[M]_a \to \mathbb{Z}[M_a]$ is induced by the map $\mathbb{Z}[M] \to \mathbb{Z}[M_a]$, which is in turn induced by the natural map $M \to M_a$. It is easy to see that these two maps are inverse one of the other. $\square$

2.4.3. DEFINITION. Let X be a geometrical scheme over $\mathbb{F}_1$ and let $\{\operatorname{Spec} M_i\}$ be an affine covering of it. Fix now a family of affine open coverings $\{\operatorname{Spec} M_{ijk}\}$ for each $\operatorname{Spec} M_i \times_X \operatorname{Spec} M_j$. By Lemma 2.4.2, we can define a scheme over $\mathbb{Z}$ by gluing the affine schemes $\operatorname{Spec}(M_i \otimes_{\mathbb{F}_1} \mathbb{Z})$ over $\operatorname{Spec}(M_{ijk} \otimes_{\mathbb{F}_1} \mathbb{Z})$. The scheme over $\mathbb{Z}$ we obtain is called *base change of X, with respect to the covering* $\{\operatorname{Spec} M_{ijk}\}$.

2.4.4. DEFINITION. As described in [**34**], the adjoint couple from **Mon** to **Ring** induces a functor from Zariski sheaves on affine schemes over $\mathbb{Z}$ to Zariski sheaves on affine schemes over $\mathbb{F}_1$, which has a left adjoint $\otimes_{\mathbb{F}_1}\mathbb{Z}$. Also, the functor $\otimes_{\mathbb{F}_1}\mathbb{Z}$ is such that $\mathbb{F}_1$-schemes are mapped to schemes. Hence, its restriction defines a functor

$$\begin{aligned} \mathbf{Sch}_{\mathbb{F}_1} &\to \mathbf{Sch} \\ X &\mapsto X \otimes_{\mathbb{F}_1} \mathbb{Z}. \end{aligned}$$

called *base change functor*.

2.4.5. PROPOSITION. *Base change of geometrical $\mathbb{F}_1$-schemes does not depend on the covering and is canonically equivalent to base change of $\mathbb{F}_1$-schemes.*

PROOF. We remark that the base change functor is automatically defined from the adjoint couple from **Mon** to **Ring**. Let X be an arbitrary scheme over $\mathbb{F}_1$. We can then write X as the coequalizer of an affine diagram

$$\coprod \operatorname{Spec} M_{ijk} \rightrightarrows \coprod \operatorname{Spec} M_i \to X.$$

Since $\otimes_{\mathbb{F}_1}\mathbb{Z}$ is a left adjoint, we conclude that $X \otimes_{\mathbb{F}_1} \mathbb{Z}$ is the coequalizer of the diagram

$$\coprod \operatorname{Spec}(M_{ijk} \otimes_{\mathbb{F}_1} \mathbb{Z}) \rightrightarrows \coprod \operatorname{Spec}(M_i \otimes_{\mathbb{F}_1} \mathbb{Z}) \to X \otimes_{\mathbb{F}_1} \mathbb{Z}$$

which is exactly the image of X via base change with respect to the fixed covering. $\square$

We can hence summarize what we have done so far by saying that the part of the $\mathbb{F}_1$-map in [**27**] that concerns Deitmar's and Toën-Vaquié's schemes is correct, in the sense that both the equivalence between the two notions and the commutativity of the base change functors have been proven.

2.4.6. EXAMPLE. Let G be an abelian group. We can consider it as a commutative monoid. The scheme $\mathbb{D}(G) := \operatorname{Spec} G \otimes_{\mathbb{F}_1} \mathbb{Z}$ is the group scheme associated to G. Indeed, we have the following sequence of natural isomorphisms, for each scheme $(X, \mathcal{O}_X)$:

$$\operatorname{Hom}_{\mathbf{Sch}}(X, \mathbb{D}(G)) \cong \operatorname{Hom}_{\mathbf{Ring}}(\mathbb{Z}[G], \mathcal{O}_X(X)) \cong \operatorname{Hom}_{\mathbf{Mon}}(G, \mathcal{O}_X(X)).$$

In particular, there holds the isomorphism

$$\operatorname{Spec}_{\mathbb{F}_1}(\mathbb{Z}/n\mathbb{Z}) \otimes_{\mathbb{F}_1} \mathbb{Z} \cong \mu_n.$$

2.4.7. PROPOSITION. *Let X be a $\mathbb{F}_1$-scheme. We indicate with $\mathbf{Sh}(X)$ and $\mathbf{Sh}(X \otimes_{\mathbb{F}_1} \mathbb{Z})$ the categories of sheaves over X and $X \otimes_{\mathbb{F}_1} \mathbb{Z}$, respectively.*

(1) *There is a functor* υ_* *from* $\mathbf{Sh}(X \otimes_{\mathbb{F}_1} \mathbb{Z})$ *to* $\mathbf{Sh}(X)$ *that maps a sheaf* $\mathcal{F}$ *to the sheaf defined as follows*
$$\upsilon_*\mathcal{F}(U) := \mathcal{F}(U \otimes_{\mathbb{F}_1} \mathbb{Z} \hookrightarrow X \otimes_{\mathbb{F}_1} \mathbb{Z}).$$
(2) *The functor* υ_* *has a left adjoint* υ^{-1}.
(3) *There is a natural map* $\mathcal{O}_X \to \upsilon_*\mathcal{O}_{X\otimes_{\mathbb{F}_1}\mathbb{Z}}$ *in* $\mathbf{Sh}(X)$.

PROOF. We remark that the functor from the small site on X to the small site on $X \otimes_{\mathbb{F}_1} \mathbb{Z}$ that maps U to $U \otimes_{\mathbb{F}_1} \mathbb{Z}$ is well defined (in the sense that it preserves open immersions), and it is continuous (in the sense that it preserves coverings and intersections). Epimorphisms are preserved since $\otimes_{\mathbb{F}_1}\mathbb{Z}$ is right exact, hence coverings are preserved. Since X and any open subscheme U are obtained by gluing affines, it suffices to prove the claim about open immersions and intersections on affine schemes. The claim on open immersions follows from Lemma 2.4.2. The claim on intersections comes from the fact that $\mathbb{Z}[M_{ab}] \cong \mathbb{Z}[M_a] \otimes_{\mathbb{Z}[M]} \mathbb{Z}[M_b]$, since $\mathbb{Z}[\cdot]$ is left exact. Indeed, this yields to the following isomorphisms
$$\begin{aligned}(\operatorname{Spec} M_a \cap \operatorname{Spec} M_b) \otimes_{\mathbb{F}_1} \mathbb{Z} = \operatorname{Spec}(M_a \otimes_M M_b) \otimes_{\mathbb{F}_1} \mathbb{Z} \cong \\ \cong \operatorname{Spec}(\mathbb{Z}[M_a] \otimes_{\mathbb{Z}[M]} \mathbb{Z}[M_b]) = \operatorname{Spec} \mathbb{Z}[M_a] \cap \operatorname{Spec} \mathbb{Z}[M_b].\end{aligned}$$
By means on [**5**], Sites ad Sheaves 13.3, we then conclude that there is a natural adjoint couple from the two categories of sheaves, in which the right adjoint is exactly the one described in the first claim.

We are left to prove the third claim. For any open subscheme U of X with an affine open covering $\{\operatorname{Spec} M_i\}$, we have the following two coequalizing diagrams
$$\coprod \operatorname{Spec} M_{ijk} \rightrightarrows \coprod \operatorname{Spec} M_i \to U$$
$$\coprod \operatorname{Spec} \mathbb{Z}[M_{ijk}] \rightrightarrows \coprod \operatorname{Spec} \mathbb{Z}[M_i] \to U \otimes_{\mathbb{F}_1} \mathbb{Z}$$
Apply the functor $\operatorname{Hom}(\cdot, \operatorname{Spec}_{\mathbb{F}_1}\mathbb{F}_1[t]) = \Gamma(\cdot, \mathcal{O}_X)$ to the first, and $\operatorname{Hom}(\cdot, \operatorname{Spec} \mathbb{Z}) = \Gamma(\cdot, \mathcal{O}_{X\otimes_{\mathbb{F}_1}\mathbb{Z}})$ to the second. We then obtain the following map of diagrams
$$\begin{tikzcd} \Gamma(U) \arrow[r] & \prod M_i \arrow[r, shift left] \arrow[r, shift right] \arrow[d] & \prod M_{ijk} \arrow[d] \\ \Gamma(U \otimes_{\mathbb{F}_1} \mathbb{Z}) \arrow[r] & \prod \mathbb{Z}[M_i] \arrow[r, shift left] \arrow[r, shift right] & \prod \mathbb{Z}[M_{ijk}] \end{tikzcd}$$
where vertical arrows are defined via the unit of the adjunction $\mathbf{Mon} \leftrightarrows \mathbf{Ring}$. It induces a map
$$\mathcal{O}_X(U) \to \mathcal{O}_{X\otimes_{\mathbb{F}_1}\mathbb{Z}}(U \otimes \mathbb{Z}) = \upsilon_*\mathcal{O}_X(U)$$
as wanted. □

2.5. $\mathcal{O}_X$-modules and projective morphisms

In this section we introduce the fundamental definition of $\mathcal{O}_X$-modules for schemes over $\mathbb{F}_1$ and we use it in order to inspect the functor represented by $\mathbb{P}^n_{\mathbb{F}_1}$, as we did in 1.3.14 for the case $n = 1$. Not surprisingly, all the upcoming results are generalizations of classical results, such as [**22**] II.7.1 and II.6.17. In this section, a scheme over $\mathbb{F}_1$ is thought as a scheme à la Deitmar, unless otherwise specified. Nonetheless, we constantly refer to the functorial perspective, especially when using M-modules, tensor products and other tools which are typical of Toën and Vaquié's description.

2.5.1. DEFINITION. Let X be a scheme over $\mathbb{F}_1$. A *$\mathcal{O}_X$-module* $\mathcal{F}$ is a sheaf of sets on X such that for each open subscheme U of X, the set $\Gamma(U,\mathcal{F})$ has a structure of $\Gamma(U,\mathcal{O}_X)$-module, which is compatible with restriction maps, in the sense that for each inclusion of open subschemes $V \subset U$, the following square

$$\begin{array}{ccc} \mathcal{O}_X(U) \times \mathcal{F}(U) & \longrightarrow & \mathcal{F}(U) \\ \downarrow & & \downarrow \\ \mathcal{O}_X(V) \times \mathcal{F}(V) & \longrightarrow & \mathcal{F}(V) \end{array}$$

is commutative.

A *morphism of $\mathcal{O}_X$-modules* from $\mathcal{F}$ to $\mathcal{G}$ is a morphism of sheaves, such that for each open subscheme U of X, the map $\mathcal{F}(U) \to \mathcal{G}(U)$ is $\mathcal{O}_X(U)$-linear. The category of $\mathcal{O}_X$-modules is denoted with $\mathcal{O}_X$ **-Mod**.

The *tensor product* of two $\mathcal{O}_X$-modules is the sheaf associated to the the presheaf $U \mapsto \mathcal{F}(U) \otimes_{\mathcal{O}_X(U)} \mathcal{G}(U)$. It has a natural structure of $\mathcal{O}_X$-module.

2.5.2. PROPOSITION. *Let $f\colon X \to Y$ be a map of schemes, let $\mathcal{F}$ be a $\mathcal{O}_X$-module and let $\mathcal{G}$ be a $\mathcal{O}_Y$-module.*

(1) The sheaf $f_*\mathcal{F}$ has a natural structure of $\mathcal{O}_Y$-module.
(2) The functor f_* from $\mathcal{O}_X$-modules to $\mathcal{O}_Y$-modules has a left adjoint f^*.
(3) The functor f^* induces a map of global sections $\Gamma(Y,\mathcal{G}) \to \Gamma(X, f^*\mathcal{G})$.

PROOF. The functor f_* induces a local action of $f_*\mathcal{O}_X$ on $f_*\mathcal{F}$. The structure of $\mathcal{O}_Y$-module is then induced via the map $f^\sharp\colon \mathcal{O}_Y \to f_*\mathcal{O}_X$. This proves the first claim.

Let now $\mathcal{G}$ be a $\mathcal{O}_Y$-module. The functor f^{-1} induces a local action of $f^{-1}\mathcal{O}_Y$ on $f^{-1}\mathcal{G}$. Indeed, because of the definition of f^{-1} ([**22**], II.1) and because filtered colimits commute with finite products ([**28**], IX.2.1), we can define a map

$$(f^{-1}\mathcal{O}_Y)(U) \times (f^{-1}\mathcal{F})(U) \cong \varinjlim_{V \supset f(U)} (\mathcal{O}_Y(V) \times \mathcal{F}(V)) \to \varinjlim_{V \supset f(U)} \mathcal{F}(V) \cong (f^{-1}\mathcal{F})(U)$$

which is easily proven to be compatible with restrictions. Since f_* and f^{-1} are adjoint functors, the map $f^\sharp$ induces a map $f^{-1}\mathcal{O}_Y \to \mathcal{O}_X$. We define $f^*\mathcal{G}$ to be the $\mathcal{O}_X$-module $f^*\mathcal{G} \otimes_{f^{-1}\mathcal{O}_Y} \mathcal{O}_X$, defined in the obvious way. The adjoint property is a straightforward corollary of universal properties of sheafification, tensors and the adjunction f_*, f^{-1}.

We are left to prove the third claim. The map $s \mapsto s \otimes 1$ defines a map from $\mathcal{G}$ to the presheaf $U \mapsto \mathcal{F}(U) \otimes_{\mathcal{O}_X(U)} \mathcal{G}(U)$. By composing it with the sheafification map, we end up with the desired morphism. □

2.5.3. DEFINITION. A *line bundle* is a $\mathcal{O}_X$-module such that each point $x \in X$ has an open neighborhood U such that $\mathcal{F}|_U$ is isomorphic to $\mathcal{O}_X|_U$ as a $\mathcal{O}_X|_U$-module.

2.5.4. PROPOSITION. *The set of isomorphism classes of line bundles form an abelian group with respect to the tensor product.*

PROOF. The fact that the operation is well defined comes from the natural isomorphism $\mathcal{O}_X \otimes_{\mathcal{O}_X} \mathcal{O}_X \cong \mathcal{O}_X$. Associativity and commutativity come from pseudo-associativity and pseudo-commutativity of tensor products (see 2.2.4). Also, $\mathcal{O}_X$ is the identity element since $\mathcal{O}_X(U) \otimes_{\mathcal{O}_X(U)} \mathcal{F}(U) \cong \mathcal{F}(U)$ (see 2.2.6).

We are left to define an inverse for each line bundle $\mathcal{F}$. Since $\mathcal{F}$ is a sheaf, then the presheaf $U \mapsto \mathrm{Hom}_{\mathcal{F}_X(U)}(\mathcal{F}(U), \mathcal{O}_X(U))$ is a sheaf as well, which is locally isomorphic to $\mathcal{O}_X$. Let us call it $\mathcal{F}^{-1}$. The natural map $\mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{F}^{-1} \to \mathcal{O}_X$ is locally an isomorphism, hence it defines an isomorphism of line bundles, as wanted. $\square$

2.5.5. DEFINITION. For a scheme X, the *Picard group* of X is the group of isomorphic classes of line bundles with respect to the tensor product. It is denoted with $\mathrm{Pic}(X)$.

2.5.6. PROPOSITION. *The functor f^* associated to a map of schemes $f\colon X \to Y$ induces a homomorphism of groups $\mathrm{Pic}(Y) \to \mathrm{Pic}(X)$. In particular,* Pic *is a contravariant functor from $\mathbf{Sch}_{\mathbb{F}_1}$ to the category of abelian groups.*

PROOF. We have to prove that $f^*\mathcal{O}_Y \cong \mathcal{O}_X$ and that f^* respects tensor products. The first equality follows easily from the definition of f^*.

Since finite limits of sets commute with filtered colimits ([**28**] IX.2.1) and because of the definition of f^{-1} (see [**22**], II.1), we conclude that

$$f^{-1}(\mathcal{F} \times \mathcal{G}) \cong f^{-1}\mathcal{F} \times f^{-1}\mathcal{G}$$

for any couple of sheaves of sets $\mathcal{F}$, $\mathcal{G}$. Now apply the right exact functor f^{-1} to the following coequalizing sequence

$$\mathcal{O}_X \times \mathcal{L} \times \mathcal{L}' \rightrightarrows \mathcal{L} \times \mathcal{L}' \to \mathcal{L} \otimes_{\mathcal{O}_X} \mathcal{L}'$$

to conclude that $f^{-1}(\mathcal{L} \otimes_{\mathcal{O}_X} \mathcal{L}') \cong (f^{-1}\mathcal{L}) \otimes_{f^{-1}\mathcal{O}_X} (f^{-1}\mathcal{L}')$. The fact that f^* is a homomorphism then follows from the following sequence of isomorphisms.

$$\begin{aligned} f^{-1}(\mathcal{L} \otimes_{\mathcal{O}_Y} \mathcal{L}') \otimes_{f^{-1}\mathcal{O}_Y} \mathcal{O}_X &\cong (f^{-1}\mathcal{L} \otimes_{f^{-1}\mathcal{O}_Y} f^{-1}\mathcal{L}') \otimes_{f^{-1}\mathcal{O}_Y} \mathcal{O}_X \cong \\ &\cong (f^{-1}\mathcal{L} \otimes_{f^{-1}\mathcal{O}_Y} \mathcal{O}_X) \otimes_{\mathcal{O}_X} (f^{-1}\mathcal{L}' \otimes_{f^{-1}\mathcal{O}_Y} \mathcal{O}_X) \end{aligned}$$

$\square$

2.5.7. PROPOSITION. *The Picard group of an affine scheme over $\mathbb{F}_1$ is trivial.*

PROOF. Let M be an ideal. Consider the point of $\operatorname{Spec} M$ associated to the maximal ideal $\mathfrak{m}$ of M. If $\mathfrak{m}$ lies in a basis open set $D(a)$, then $a \notin \mathfrak{m}$ which implies $D(a) = \operatorname{Spec} M$. The only open neighborhood of $\mathfrak{m}$ is then the whole space. In particular, by definition, any line bundle $\mathcal{L}$ is isomorphic as $\mathcal{O}_{\operatorname{Spec} M}$-module to $\mathcal{O}_{\operatorname{Spec} M}$. $\square$

We now investigate how the base change functor described in Section 2.4 acts on $\mathcal{O}_X$-modules and Picard groups. From now on, we use the notation of Proposition 2.4.7.

2.5.8. DEFINITION. We denote with $\mathbb{Z}[\cdot]$ both the functor from **Mon** to **Ring** and the functor from **Set** to abelian groups, which are left adjoints of the two forgetful functors.

2.5.9. PROPOSITION. *Let S, T be two sets. The group $\mathbb{Z}[S \times T]$ is isomorphic to $\mathbb{Z}[S] \otimes_{\mathbb{Z}} \mathbb{Z}[T]$. In particular, for a given monoid M and a M-module S, the group $\mathbb{Z}[S]$ has a natural $\mathbb{Z}[M]$-module structure. We will again indicate with $\mathbb{Z}[\cdot]$ the induced functor from M-**Mod** to $\mathbb{Z}[M]$-**Mod**.*

PROOF. The two invertible arrows are induced as follows. The map

$$S \times T \to \mathbb{Z}[S] \otimes_{\mathbb{Z}} \mathbb{Z}[T]$$
$$(s,t) \mapsto s \otimes t$$

induces a morphism $\mathbb{Z}[S \times T] \to \mathbb{Z}[S] \otimes_{\mathbb{Z}} \mathbb{Z}[T]$. Also, the map

$$\mathbb{Z}[S] \times \mathbb{Z}[T] \to \mathbb{Z}[S \times T]$$
$$\left(\sum \lambda_i s_i, \sum \mu_j t_j\right) \mapsto \sum \lambda_i \mu_j (s_i, t_j).$$

coequalizes the two maps $\mathbb{Z} \times \mathbb{Z}[S] \times \mathbb{Z}[T] \rightrightarrows \mathbb{Z}[S] \times \mathbb{Z}[T]$, hence it induces a morphism $\mathbb{Z}[S] \otimes_{\mathbb{Z}} \mathbb{Z}[T] \to \mathbb{Z}[S \times T]$, as wanted. It is easy to prove that these maps are inverse one of the other, by checking the claim on the elements of the form (s,t) and $s \otimes t$. □

2.5.10. PROPOSITION. *Let X be a scheme over $\mathbb{F}_1$, and let $X \otimes_{\mathbb{F}_1} \mathbb{Z}$ the scheme over $\mathbb{Z}$ obtained by base change.*

(1) For any $\mathcal{O}_{X \otimes_{\mathbb{F}_1} \mathbb{Z}}$-module $\mathcal{F}$, the sheaf $\upsilon_* \mathcal{F}$ is naturally a $\mathcal{O}_X$-module.
(2) The functor υ_* from $\mathcal{O}_{X \otimes_{\mathbb{F}_1} \mathbb{Z}}$-modules to $\mathcal{O}_X$-modules has a left adjoint functor υ^*.
(3) The functor υ^* induces a homomorphism $\operatorname{Pic}(X) \to \operatorname{Pic}(X \otimes_{\mathbb{F}_1} \mathbb{Z})$.

PROOF. We start by proving the first claim. The sheaf $\upsilon_* \mathcal{F}$ has a structure of $(\upsilon_* \mathcal{O}_{X \otimes_{\mathbb{F}_1} \mathbb{Z}})$-module. By Proposition 2.4.7, it inherits a structure of $\mathcal{O}_X$-module, induced by the map $\mathcal{O}_X \to \mathcal{O}_{X \otimes_{\mathbb{F}_1} \mathbb{Z}}$.

We now turn to the second claim. We remark that there is an explicit definition of $\upsilon^{-1}\mathcal{F}$ (see [**5**], Sites and Sheaves 13.3) as the sheaf associated to the presheaf

$$U \mapsto \varinjlim_{V \otimes_{\mathbb{F}_1} \mathbb{Z} \supset U} \mathcal{F}(V).$$

Using the same proof as Proposition 2.5.6, we conclude that for a $\mathcal{O}_X$-module $\mathcal{F}$, there is an action of $\upsilon^{-1}\mathcal{O}_X$ on $\upsilon^{-1}\mathcal{F}$. In particular, the group $\mathbb{Z}[(\upsilon^{-1}\mathcal{F})(U)]$ is naturally a module over the ring $\mathbb{Z}[(\upsilon^{-1}\mathcal{O}_X)(U)]$, for any open subscheme U of $X \otimes_{\mathbb{F}_1} \mathbb{Z}$. We define $\upsilon^*\mathcal{F}$ to be the sheaf associated to the presheaf

$$U \mapsto \mathbb{Z}[(\upsilon^{-1}\mathcal{F})(U)] \otimes_{\mathbb{Z}[(\upsilon^{-1}\mathcal{O}_X)(U)]} \mathcal{O}_{X \otimes_{\mathbb{F}_1} \mathbb{Z}}(U).$$

It is easy to see that υ^* constitutes a left adjoint of the functor υ_* restricted to modules, as wanted.

For the third claim, we are left to prove that $\upsilon^*\mathcal{O}_X = \mathcal{O}_{X \otimes_{\mathbb{F}_1} \mathbb{Z}}$, and that $\upsilon^*(\mathcal{L} \otimes \mathcal{L}') = \upsilon^*\mathcal{L} \otimes \upsilon^*\mathcal{L}'$. The first equality follows easily from the definition of υ^*.

Using the same proof of Proposition 2.5.6, we can conclude the following isomorphism

$$\upsilon^{-1}(\mathcal{L} \otimes_{\mathcal{O}_X} \mathcal{L}') \cong (\upsilon^{-1}\mathcal{L}) \otimes_{\upsilon^{-1}\mathcal{O}_X} (\upsilon^{-1}\mathcal{L}')$$

where tensors are taken in the categories of modules over monoids.

Because of Proposition 2.5.9 and because $\mathbb{Z}[\cdot]$ is a left adjoint functor, we conclude that for any monoid M and any couple of M-modules S, T, the coequalizing diagram

$$M \times S \times T \rightrightarrows S \times T \to S \otimes_M T$$

induces the following coequalizing diagram

$$\mathbb{Z}[M] \otimes_{\mathbb{Z}} \mathbb{Z}[S] \otimes_{\mathbb{Z}} \mathbb{Z}[T] \rightrightarrows \mathbb{Z}[S] \otimes_{\mathbb{Z}} \mathbb{Z}[T] \to \mathbb{Z}[S \otimes_M T].$$

In particular, we conclude that $\mathbb{Z}[S \otimes_M T] \cong \mathbb{Z}[S] \otimes_{\mathbb{Z}[M]} \mathbb{Z}[T]$.

As a whole, we obtain the following chain of isomorphisms

$$\begin{aligned}
&\mathbb{Z}[\upsilon^{-1}(\mathcal{L} \otimes_{\mathcal{O}_X} \mathcal{L}')] \otimes_{\mathbb{Z}[\upsilon^{-1}\mathcal{O}_X]} \mathcal{O}_{X \otimes_{\mathbb{F}_1} \mathbb{Z}} \cong \\
&\qquad \cong \left(\mathbb{Z}[\upsilon^{-1}\mathcal{L}] \otimes_{\mathbb{Z}[\upsilon^{-1}\mathcal{O}_X]} \mathbb{Z}[\upsilon^{-1}\mathcal{L}']\right) \otimes_{\mathbb{Z}[\upsilon^{-1}\mathcal{O}_X]} \mathcal{O}_{X \otimes_{\mathbb{F}_1} \mathbb{Z}} \cong \\
&\qquad \cong \left(\mathbb{Z}[\upsilon^{-1}\mathcal{L}] \otimes_{\mathbb{Z}[\upsilon^{-1}\mathcal{O}_X]} \mathcal{O}_{X \otimes_{\mathbb{F}_1} \mathbb{Z}}\right) \otimes_{\mathcal{O}_{X \otimes_{\mathbb{F}_1} \mathbb{Z}}} \left(\mathbb{Z}[\upsilon^{-1}\mathcal{L}'] \otimes_{\mathbb{Z}[\upsilon^{-1}\mathcal{O}_X]} \mathcal{O}_{X \otimes_{\mathbb{F}_1} \mathbb{Z}}\right)
\end{aligned}$$

which implies that υ^* respects tensor products, hence the claim. ☐

Let A be a ring, and let $X = \operatorname{Spec} A$ the associated affine scheme. Any A-module P naturally induces a $\mathcal{O}_X$-module $\tilde{P}$, as described in [**22**], II.5. This fact has a natural analogue in the $\mathbb{F}_1$-setting.

2.5.11. Definition. Let $X = \operatorname{Spec} M$ be an affine scheme over $\mathbb{F}_1$ and let S be a M-module. We define $\tilde{S}$ to be the sheaf obtained by posing $\tilde{S}(\operatorname{Spec} M_a) = S \otimes_M M_a$ for any element $a \in M$. It is naturally a $\mathcal{O}_X$-module, and it is called the *$\mathcal{O}_X$-module associated to S.*

2.5.12. Proposition. *Let $X = \operatorname{Spec} M$ be an affine scheme over $\mathbb{F}_1$, and let* $\operatorname{Spec} \mathbb{Z}[M]$ *be the associated scheme over $\mathbb{Z}$.*

(1) For any $\mathbb{Z}[M]$-module P, there holds an isomorphism $\upsilon_* \tilde{P} \cong \tilde{P}$.

(2) For any M-module S, there holds an isomorphism $\upsilon^* \tilde{S} \cong \widetilde{\mathbb{Z}[S]}$.

In particular, the adjoint couple υ_*, υ^* restricts to sheaves associated to modules, and its restriction is canonically equivalent to the adjoint couple M-**Mod** $\leftrightarrows$ $\mathbb{Z}[M]$-**Mod** *constituted of the forgetful functor and the functor $\mathbb{Z}[\cdot]$.*

Proof. The first claim is obvious from the definition of υ_*. We now turn to the second. We claim that the functor M-**Mod** $\to \mathcal{O}_X$-**Mod** defined as $S \mapsto \tilde{S}$ is a left adjoint of the functor of global sections. Suppose that $\mathcal{F}$ is a $\mathcal{O}_X$-module. A map of M-modules $S \to \Gamma(X, \mathcal{F})$ induces in particular a M-linear map from S to the M_a-module $\Gamma(\operatorname{Spec} M_a, \mathcal{F})$, obtained by composing it with the restriction $\Gamma(X, \mathcal{F}) \to \Gamma(\operatorname{Spec} M_a, \mathcal{F})$. Therefore, it induces a M_a-linear map

$$\Gamma(\operatorname{Spec} M_a, \tilde{S}) = S \otimes_M M_a \to \Gamma(\operatorname{Spec} M_a, \mathcal{F})$$

for all elements $a \in M$. These maps are also compatible with the restriction maps, because of the universal property of base change. Since the affine subschemes of X determine a basis of open sets, the previous maps define a morphism of $\mathcal{O}_X$-modules $\tilde{S} \to \mathcal{F}$, as wanted. Conversely, any map of $\mathcal{O}_X$-modules $\tilde{S} \to \mathcal{F}$ induces in particular a M-linear map $S \to \Gamma(X, \mathcal{F})$. It is easy to prove that these maps determine a natural bijection $\operatorname{Hom}_{\mathcal{O}_X}(\tilde{S}, \mathcal{F}) \cong \operatorname{Hom}_M(S, \Gamma(X, \mathcal{F}))$. The same proof works also for the case of modules over rings ([**22**], Exercise II.5.3).

We then have the following (a priori non-commutative) square of adjoint couples (we indicate with F the forgetful functor).

$$\begin{tikzcd}
\mathcal{O}_X\text{-}\mathbf{Mod} \arrow[r, shift left, "\Gamma"] \arrow[d, shift left, "\upsilon^*"] & M\text{-}\mathbf{Mod} \arrow[l, shift left, "(\cdot)^\sim"] \arrow[d, shift left, "\mathbb{Z}[\cdot]"] \\
\mathcal{O}_{X \otimes_{\mathbb{F}_1} \mathbb{Z}}\text{-}\mathbf{Mod} \arrow[r, shift left, "\Gamma"] \arrow[u, shift left, "\upsilon_*"] & \mathbb{Z}[M]\text{-}\mathbf{Mod} \arrow[l, shift left, "(\cdot)^\sim"] \arrow[u, shift left, "F"]
\end{tikzcd}$$

In order to prove $\upsilon^*\tilde{S} \cong \widetilde{\mathbb{Z}[S]}$, we can then alternatively prove that for a given $\mathcal{O}_{X\otimes_{\mathbb{F}_1}\mathbb{Z}}$-module $\mathcal{F}$, there holds $F(\Gamma(X \otimes_{\mathbb{F}_1} \mathbb{Z}, \mathcal{F})) \cong \Gamma(X, \upsilon_*\mathcal{F})$. This is clear from the definition of υ_*. □

We now build the projective space, following Example 1.3.14. We do not introduce explicitly the Proj construction, even though our construction of the sheaf $\mathcal{O}(1)$ may remind of it.

2.5.13. Definition. The *n-dimensional affine space over* $\mathbb{F}_1$, denoted with $\mathbb{A}^n_{\mathbb{F}_1}$, is the affine scheme $\operatorname{Spec}\mathbb{F}_1[x_1, \dots, x_n]$.

2.5.14. Proposition. *The scheme $\mathbb{A}^n_{\mathbb{F}_1}$ is the scheme that represents the functor*

$$\begin{aligned}\mathbf{Mon} &\to \mathbf{Set}\\ M &\mapsto M^n\end{aligned}$$

Proof. Clear by 2.1.13. □

2.5.15. Definition. The *projective space of dimension n over* $\mathbb{F}_1$, denoted with $\mathbb{P}^n_{\mathbb{F}_1}$, is the scheme obtained as the coequalizer of the diagram

$$\coprod_{0\le i<j\le n} \operatorname{Spec} M_{ij} \rightrightarrows \coprod_{0\le i\le n} \operatorname{Spec} M_i$$

where M_i and M_{ij} are the following submonoids of $\mathbb{F}_1[x_0, x_0^{-1} \dots, x_n, x_n^{-1}]$:

$$\begin{aligned}M_i &= \mathbb{F}_1\left[\frac{x_0}{x_i}, \dots, \frac{x_n}{x_i}\right],\\ M_{ij} &= \mathbb{F}_1\left[\frac{x_0}{x_i}, \dots, \frac{x_n}{x_i}, \frac{x_i}{x_j}\right] = \mathbb{F}_1\left[\frac{x_0}{x_j}, \dots, \frac{x_n}{x_j}, \frac{x_j}{x_i}\right].\end{aligned}$$

The two arrows are induced by the localizations $M_i \to M_{ij}$ and $M_j \to M_{ij}$. Each open subscheme $U_i := \operatorname{Spec} M_i$ is isomorphic to $\mathbb{A}^n_{\mathbb{F}_1}$. We denote by U_{ij} the open subscheme $U_i \cap U_j$.

Notice that this definition is exact analogue of the classical case of schemes over $\mathbb{Z}$ (see [**31**], II.2 Example J). We will sometimes indicate an element of M_i as a fraction of monomials $\frac{f}{g}$, where f and g have the same degree. The 1-dimensional case is particularly simple, and it can be defined more explicitly in the following way.

2.5.16. Definition. The *projective line over* $\mathbb{F}_1$, denoted with $\mathbb{P}^1_{\mathbb{F}_1}$ is the scheme obtained as the coequalizer of the diagram

$$\operatorname{Spec}\mathbb{F}_1[t, t^{-1}] \rightrightarrows \mathbb{A}^1_{\mathbb{F}_1} \sqcup \mathbb{A}^1_{\mathbb{F}_1}$$

where the two arrows are induced by two maps $\mathbb{F}_1[x] \to \mathbb{F}_1[t, t^{-1}]$ defined by $x \mapsto t$ and $x \mapsto t^{-1}$, respectively. Being localizations, they are open immersions.

2.5.17. Proposition. *The scheme $\mathbb{A}^n_{\mathbb{F}_1} \otimes_{\mathbb{F}_1} \mathbb{Z}$ is $\mathbb{A}^n_{\mathbb{Z}}$ and the scheme $\mathbb{P}^n_{\mathbb{F}_1} \otimes_{\mathbb{F}_1} \mathbb{Z}$ is $\mathbb{P}^n_{\mathbb{Z}}$.*

Proof. The first claim is a direct corollary of the fact that $\mathbb{Z}[x_1, \dots, x_n]$ is the ring generated by the monoid $\mathbb{F}_1[x_1, \dots, x_n]$.

As described in [**31**], II.2 Example J, the scheme $\mathbb{P}^n_{\mathbb{Z}}$ is obtained by a union of affine schemes of the form $A_i = \operatorname{Spec} \mathbb{Z}[x_0/x_i, \ldots, x_n/x_i]$, with the identification of the couples of open subschemes

$$\operatorname{Spec} \mathbb{Z}\left[\frac{x_0}{x_i}, \ldots, \frac{x_n}{x_i}, \frac{x_i}{x_j}\right] = \operatorname{Spec} \mathbb{Z}\left[\frac{x_0}{x_j}, \ldots, \frac{x_n}{x_j}, \frac{x_j}{x_i}\right].$$

In other words, $\mathbb{P}^n_{\mathbb{Z}}$ is the coequalizer of the diagram

$$\coprod_{0 \le i < j \le n} \operatorname{Spec} \mathbb{Z}[M_{ij}] \rightrightarrows \coprod_{0 \le i \le n} \operatorname{Spec} \mathbb{Z}[M_i].$$

This proves the second claim. □

2.5.18. PROPOSITION. *$\mathbb{P}^n_{\mathbb{F}_1}$ is not an affine scheme.*

PROOF. Apply the functor $h_{\mathbb{A}^1_{\mathbb{F}_1}}$ to the coequalizing diagram which defines $\mathbb{P}^n_{\mathbb{F}_1}$. We remark that, because of Proposition 2.1.13, for any scheme X we have

$$\operatorname{Hom}(X, \mathbb{A}^1_{\mathbb{F}_1}) = \operatorname{Hom}(\mathbb{F}_1[x], \Gamma(X, \mathcal{O}_X)) = \Gamma(X, \mathcal{O}_X).$$

We then obtain the following equalizing diagram

$$\Gamma(\mathbb{P}^n_{\mathbb{F}_1}, \mathcal{O}_{\mathbb{P}^n_{\mathbb{F}_1}}) \to \prod M_i \rightrightarrows \prod M_{ij}$$

which implies that the monoid $\Gamma(\mathbb{P}^n_{\mathbb{F}_1}, \mathcal{O}_{\mathbb{P}^n_{\mathbb{F}_1}})$ is isomorphic to $\mathbb{F}_1$. Indeed, an element of $\prod M_i$ is a tuple

$$\left(\frac{f_i}{x_i^{\deg f_i}}\right)_{i=0,\ldots,n}$$

where f_i is an element in $\mathbb{F}_1[x_0, \ldots, x_n]$, and we can suppose that it does not contain any power of x_i. Its image via the two maps has in the (i,j)-th place either

$$\frac{f_i}{x_i^{\deg f_i}}$$

or

$$\frac{f_j}{x_j^{\deg f_j}}.$$

If the two images coincide, then $f_i = 1$ for all i, as wanted. Because $\mathbb{P}^n_{\mathbb{F}_1}$ has more than one point (see the construction of coequalizers in 2.1.12), we conclude that it is not isomorphic to $\operatorname{Spec} \mathbb{F}_1$, hence it is not affine. □

2.5.19. PROPOSITION. *Consider the scheme $\mathbb{P}^n_{\mathbb{F}_1}$. For any integer k, let $\mathcal{O}(k)|_{U_i}$ be the $\mathcal{O}_{U_i}$-module whose sections are*

$$\mathcal{O}(k)|_{U_i}(V) = \left\{x_i^k \frac{f}{g} : \frac{f}{g} \in \mathcal{O}_X(V)\right\}$$

for each open subscheme $V \subset U_i$, where the sets introduced are subsets of the monoid $\mathbb{F}_1[x_0, x_0^{-1}, \ldots, x_n, x_n^{-1}]$, and where the action of $\mathcal{O}_{U_i}$ is defined via multiplication. The sheaves $\mathcal{O}(k)|_{U_i}$ glue together via the identity map to form a line bundle $\mathcal{O}(k)$ on $\mathbb{P}^n_{\mathbb{F}_1}$. Also, the sheaf $\mathcal{O}(k)$ equals $\mathcal{O}(1)^{\otimes k}$ in $\operatorname{Pic}(\mathbb{P}^n_{\mathbb{F}_1})$.

PROOF. Consider an open subset $V \subset U_i \cap U_j$. We want to prove that $\mathcal{O}(k)|_{U_i}(V) = \mathcal{O}(k)|_{U_j}(V)$ as a subset of $\mathbb{F}_1[x_0, x_0^{-1}, \dots, x_n, x_n^{-1}]$. Since $V \subset U_i \cap U_j$, both $\frac{x_i}{x_j}$ and $\frac{x_j}{x_i}$ are in $\mathcal{O}_X(V)$. Let $x_i^k \frac{f}{g}$ be an element of $\mathcal{O}(k)|_{U_i}(V)$. Then it is equal to $x_j^k \frac{f x_i^k}{g x_j^k}$. Notice that $\frac{f x_i^k}{g x_j^k}$ is an element of $\mathcal{O}_X(U_j)$, hence $x_j^k \frac{f x_i^k}{g x_j^k}$ is an element of $\mathcal{O}(k)|_{U_j}(V)$, as wanted. The sheaf $\mathcal{O}(k)$ is a line bundle since multiplication with x_i^k defines an isomorphism of $\mathcal{O}_X$-modules $\mathcal{O}_X \to \mathcal{O}(k)$.

The second claim being local, it is obvious from the explicit description of $\mathcal{O}(k)$. Specifically, the equality

$$x_i^{a_1} \frac{f_1}{x_i^{\deg f_1}} \otimes \dots \otimes x_i^{a_k} \frac{f_k}{x_i^{\deg f_k}} = x_i^{a_1} \frac{f_1 \cdots f_k}{x_i^{\sum \deg f_i}} \otimes x_i^{a_2} \otimes \dots \otimes x_i^{a_k}$$

implies that the multiplication map defines an isomorphism $\mathcal{O}(1)^{\otimes k} \to \mathcal{O}(k)$ for positive integers k. For negative integers, the claim follows from the fact that, due to the previous equality, the map

$$x_i^k \frac{f}{g} \otimes \frac{1}{x^k} \frac{f'}{g'} \mapsto \frac{f f'}{g g'}$$

defines an isomorphism $\mathcal{O}(k) \otimes \mathcal{O}(-k) \to \mathcal{O}_X$, as wanted. □

2.5.20. DEFINITION. The sheaf $\mathcal{O}(1)$ on the scheme $\mathbb{P}^n_{\mathbb{F}_1}$ of the previous proposition is called the *twisting sheaf of* $\mathbb{P}^n_{\mathbb{F}_1}$.

2.5.21. PROPOSITION. *The Picard group of* $\mathbb{P}^n_{\mathbb{F}_1}$ *is isomorphic to* $\mathbb{Z}$ *via the map* $k \mapsto \mathcal{O}(k)$.

PROOF. Let $\mathcal{F}$ be an invertible sheaf on $\mathbb{P}^n_{\mathbb{F}_1}$. Let α_i be the isomorphisms of $\mathcal{O}_{U_i}$-modules $\mathcal{F}|_{U_i} \to \mathcal{O}_{U_i}$. Let also $\mathcal{G}$ be another invertible sheaf, with local isomorphisms β_i. One can define an isomorphism $\mathcal{G} \to \mathcal{F}$ via $\alpha_i^{-1}\beta_i$ if and only if these maps coincide on U_{ij}, i.e. if and only if

$$\alpha_i|_{U_{ij}}^{-1} \beta_i|_{U_{ij}} = \alpha_j|_{U_{ij}}^{-1} \beta_j|_{U_{ij}}$$

which is equivalent to ask that the two automorphisms of $\mathcal{O}_{U_{ij}}$ as $\mathcal{O}_{U_{ij}}$-module $\alpha_i|_{U_{ij}} \alpha_j|_{U_{ij}}^{-1}$ and $\beta_i|_{U_{ij}} \beta_j|_{U_{ij}}^{-1}$ are the same. Automorphisms of $\mathcal{O}_{U_{ij}}$ as a $\mathcal{O}_{U_{ij}}$-module are the automorphisms of M_{ij} as a M_{ij}-module, hence they are represented by invertible elements of M_{ij}. By definition (see 2.5.15), invertible elements of M_{ij} are fractions of the kind

$$\frac{x_i^a x_j^b}{x_i^c x_j^d}$$

with $a + b = c + d$. These fractions are nothing but the elements $(\frac{x_j}{x_i})^{k_{ij}}$, where k_{ij} is an integer in $\mathbb{Z}$. Because of the cocycle condition on α_i and β_i, we also know that for any triple i, j, k of indexes, there must hold the equality

$$\left(\frac{x_j}{x_i}\right)^{k_{ij}} \cdot \left(\frac{x_k}{x_j}\right)^{k_{jk}} = \left(\frac{x_k}{x_i}\right)^{k_{ik}}$$

which is equivalent to say $k_{ij} = k_{jk} = k_{ik}$. Hence the integer k does not depend on the choice of the couple (i, j). Therefore, we conclude that there is a well-defined injective map $\mathrm{Pic}(\mathbb{P}^n_{\mathbb{F}_1}) \to \mathbb{Z}$, which associates to $\mathcal{F}$ the integer k correlated to the automorphism $\alpha_i|_{U_{ij}} \alpha_j|_{U_{ij}}^{-1}$.

We prove that this map is also an homomorphism of groups. Let $\mathcal{F}$, $\mathcal{G}$, α_i, β_i be as before. We are left to prove that the automorphism of $\mathcal{O}(U_{ij})$ induced by $(\alpha_i|_{U_{ij}}\otimes \beta_i|_{U_{ij}})(\alpha_j|_{U_{ij}} \otimes \beta_j|_{U_{ij}})^{-1}$ is defined through $1 \mapsto (\alpha_i|_{U_{ij}}\alpha_j|^{-1}_{U_{ij}})(1)(\beta_i|_{U_{ij}}\beta_j|^{-1}_{U_{ij}})(1)$. This comes from the fact that the natural isomorphism $\mathcal{O}(U_{ij}) \otimes_{\mathcal{O}(U_{ij})} \mathcal{O}(U_{ij}) \cong \mathcal{O}(U_{ij})$ is defined via multiplication.

In order to conclude the claim, it suffices to check that the local isomorphisms of $\mathcal{O}(1)$ with $\mathcal{O}_X$ induce the automorphism $1 \mapsto \frac{x_i}{x_j}$ on U_{ij}. By definition (see 2.5.21), the $\mathcal{O}_X$-linear maps that produce the local isomorphisms $\mathcal{O}_X \to \mathcal{O}(1)$ are defined by multiplication with x_i. We then conclude the following equality

$$\alpha_i|_{U_{ij}}\alpha_j|^{-1}_{U_{ij}}(1) = \alpha_i|_{U_{ij}}(x_j) = \alpha_i|_{U_{ij}}\left(x_i\frac{x_j}{x_i}\right) = \frac{x_j}{x_i}$$

hence the integer associated to $\mathcal{O}(1)$ is 1, as wanted. $\square$

It is straightforward to prove that the map from $\mathrm{Pic}(\mathbb{P}^n_{\mathbb{F}_1})$ to $\mathrm{Pic}(\mathbb{P}^n_{\mathbb{Z}})$ (see Proposition 2.5.10) is determined by $\mathcal{O}(1) \mapsto \mathcal{O}(1)$, where $\mathcal{O}(1)$ is defined on $\mathbb{P}^n_{\mathbb{Z}}$ as in [**22**], II.5.12.

2.5.22. PROPOSITION. *The set of global sections $\Gamma(\mathbb{P}^n_{\mathbb{F}_1}, \mathcal{O}(k))$ is isomorphic to the subset of $\mathbb{F}_1[x_0, \dots, x_n]$ constituted by the polynomials of degree k.*

PROOF. First of all, we remark that the claim can't be ameliorated, in the sense that $\Gamma(\mathbb{P}^n_{\mathbb{F}_1}, \mathcal{O}(n))$ has no extra structure. Indeed, it is a module over the monoid $\Gamma(\mathbb{P}^n_{\mathbb{F}_1}, \mathcal{O}_X) = \mathbb{F}_1$, hence it is nothing but a set.

Global sections constitute the equalizer on the following diagram

$$\mathcal{O}(k)(\mathbb{P}^n_{\mathbb{F}_1}) \to \prod \mathcal{O}(k)(U_i) \rightrightarrows \prod \mathcal{O}(k)(U_{ij})$$

If we consider the explicit description of $\mathcal{O}(k)$ given in 2.5.19, we conclude that a global section is a $(n+1)$-tuple

$$\left(x_i^k \frac{f_i}{x_i^{\deg f_i}}\right)_{i=0,\dots,n}$$

that satisfies the following condition in $\mathbb{F}_1[x_0, x_0^{-1}, \dots, x_n, x_n^{-1}]$, for each couple i, j:

$$x_i^{k-\deg f_i} f_i = x_j^{k-\deg f_j} f_j$$

which implies in particular that $\deg f_i \leq k$. We can then define the map

$$\left(x_i^k \frac{f_i}{x_i^{\deg f_i}}\right) \mapsto x_i^{k-\deg f_i} f_i$$

which constitutes an isomorphism from $\Gamma(\mathbb{P}^1_{\mathbb{F}_1}, \mathcal{O}(k))$ to the polynomials of degree k in $\mathbb{F}_1[x_0, \dots, x_n]$ $\square$

2.5.23. COROLLARY. *The global sections of the sheaves $\mathcal{O}(k)$ satisfy the following properties.*

1. *The twisting sheaf $\mathcal{O}(1)$ has $n+1$ global sections $\{x_0, \dots, x_n\}$.*
2. *The sheaves $\mathcal{O}(k)$ have no global sections if $n < 0$.*
3. $\coprod_{k\in\mathbb{Z}} \Gamma(\mathbb{P}^n_{\mathbb{F}_1}, \mathcal{O}(k)) \cong \mathbb{F}_1[x_0, \dots, x_n]$.

2.5.24. DEFINITION. Let $\mathcal{F}$ be a $\mathcal{O}_X$-module on a scheme X. We say that a set of global sections S of $\mathcal{F}$ *locally generates* $\mathcal{F}$ if for each point x, there exists an open neighborhood U of x such that S generates $\mathcal{F}(U)$ as a $\Gamma(U, \mathcal{O}_X)$-module.

2.5.25. PROPOSITION. *The scheme $\mathbb{P}^n_{\mathbb{F}_1}$ represents the functor that associates to a scheme X the set of data $(\mathcal{L}, s_0, \ldots, s_n)$ where $\mathcal{L}$ is a line bundle over X and $\{s_0, \ldots, s_n\}$ is a set of global sections of $\mathcal{L}$ that locally generate $\mathcal{L}$, modded out by the relation $\sim$ where $(\mathcal{L}, s_0, \ldots, s_n) \sim (\mathcal{L}', s'_0, \ldots, s'_n)$ if and only if there exists an isomorphism of $\mathcal{O}_X$-modules $\alpha\colon \mathcal{L} \to \mathcal{L}'$ such that $\alpha s_i = s'_i$ for $i = 0, \ldots, n$.*

PROOF. Suppose to have a map of schemes $X \to \mathbb{P}^n_{\mathbb{F}_1}$. As in the proof of 2.5.6, the $\mathcal{O}_X$-module $f^*\mathcal{O}(1)$ is a line bundle on X. By definition, we know that in $f^{-1}(U_i)$ the section f^*x_i is locally equal to $x_i \otimes 1$, hence it is locally invertible with respect to the isomorphism with $\mathcal{O}_X$, which is defined by multiplication. Because $\{f^{-1}(U_i)\}$ form an open covering of X, we conclude that the data $(f^*\mathcal{O}(1), f^*x_i)$ satisfies the hypothesis of the claim.

Conversely, suppose to have a line bundle $\mathcal{L}$ and $n+1$ global sections $s_0, \ldots, s_n$ that generate it locally. We define X_i to be the subset of X on which s_i is locally invertible (considering the local isomorphism with $\mathcal{O}_X$). These subsets are open, and they cover X. Indeed, on each affine subscheme U of X, $\mathcal{L}$ is isomorphic to $\mathcal{O}_X|_U$ (because of 2.5.7), so the hypothesis implies that on each affine subscheme, at least one section among $\{s_0, \ldots, s_n\}$ is invertible. In order to define a map $X \to \mathbb{P}^n_{\mathbb{F}_1}$, we can then define $n+1$ maps $X_i \to U_i$, and show that they agree on $X_i \cap X_j$. Fix an index i, and consider the n sections σ_{ij} in $\Gamma(X_i, \mathcal{O}_X)$ obtained by gluing the elements $(\varphi s_j)(\varphi s_i)^{-1}$ where φ is a local isomorphism with $\mathcal{O}_X$. The sections σ_{ij} do not depend on the choice of φ. Indeed, any automorphism of $\mathcal{O}_X(U)$ as $\mathcal{O}_X(U)$-module is induced by multiplication with an invertible element u, which is erased when taking the fraction. The map $M_i \to \mathcal{O}_X(X_i)$ induced by $\frac{x_j}{x_i} \mapsto \sigma_{ij}$ defines a map $X_i \to U_i$, as wanted. These maps agree on the intersection $U_i \cap U_j$, since $(\varphi s_j)(\varphi s_i)^{-1} = ((\varphi s_i)(\varphi s_j)^{-1})^{-1}$, hence they glue together to form a map $f\colon X \to \mathbb{P}^1_{\mathbb{F}_1}$.

We also remark that the map f built in this way does not depend on the representative $(\mathcal{L}, s_i)$ chosen in the equivalence class with respect to $\sim$. Indeed, suppose that $(\mathcal{L}, s_i) \sim (\mathcal{L}', s'_i)$ with respect to an isomorphism α. We call ψ the local isomorphisms of $\mathcal{L}'$ with $\mathcal{O}_X$. Then the map $\psi\alpha$ is another local isomorphism of $\mathcal{L}$ with $\mathcal{O}_X$, and we have already seen that the choice of the local isomorphism does not influence the construction of the global sections σ_{ij}. Therefore the section obtained by gluing $(\varphi s_i)(\varphi s_j)^{-1}$ is the same as the one obtained by gluing $(\psi\alpha s_i)(\psi\alpha s_j)^{-1} = (\psi s'_i)(\psi s'_j)^{-1}$.

Suppose now to start from the data $(\mathcal{L}, s_i)$, and to build the map $f\colon X \to \mathbb{P}^n_{\mathbb{F}_1}$ as before. Consider the $\mathcal{O}_{\mathbb{P}^n_{\mathbb{F}_1}}$-linear map $\mathcal{O}(1) \to f_*\mathcal{L}$ induced by $x_i \mapsto s_i$. It is well defined, and it defines a map $\alpha\colon f^*\mathcal{O}(1) \to \mathcal{L}$. Therefore, in order to prove that $(f^*\mathcal{O}(1), f^*x_i) \sim (\mathcal{L}, s_i)$ with respect to α, we can suppose that $X = \operatorname{Spec} M$ is affine, and that it is mapped to one of the U_i's, say U_0. Suppose that the module $\mathcal{L}$ is a M-module S isomorphic to M with respect to the map $1 \mapsto s_0$. The map α is then defined as

$$x_0 \frac{x_1^{a_1} \cdots x_n^{a_n}}{x_0^{\sum a_i}} \otimes m = x_0 \otimes (\sigma_{01}^{a_1} \cdots \sigma_{0n}^{a_n}) m \mapsto (\sigma_{01}^{a_1} \cdots \sigma_{0n}^{a_n} m) s_0.$$

It is equal to the composition of the two isomorphisms $f^*\mathcal{O}(1) \cong \mathcal{O}_X$ and $\mathcal{O}_X \cong \mathcal{L}$, hence an isomorphism. Indeed, the composition of these two isomorphisms defines the following map

$$x_0 \frac{x_1^{a_1} \cdots x_n^{a_n}}{x_0^{\sum a_i}} \otimes m \mapsto (\sigma_{01}^{a_1} \cdots \sigma_{0n}^{a_n}) m \mapsto (\sigma_{01}^{a_1} \cdots \sigma_{0n}^{a_n} m) s_0$$

as claimed. From the explicit description just given, it is also easy to see that $\alpha(f^*x_i) = s_i$. Hence we conclude $(f^*\mathcal{O}(1), f^*x_i) \sim (\mathcal{L}, s_i)$.

Conversely, consider a map $f\colon X \to \mathbb{P}^n_{\mathbb{F}_1}$. In order to prove that f is the same map as the one defined by $(f^*\mathcal{O}(1), f^*x_i)$, it is again sufficient to suppose that $X = \operatorname{Spec} M$ is affine, and that it is mapped inside U_0. Suppose that f is defined by a map of monoids $\mathbb{F}_1[\frac{x_1}{x_0}, \ldots, \frac{x_n}{x_0}] \to M$, $\frac{x_i}{x_0} \mapsto m_i$. Since $f^*x_0 = x_0 \otimes 1 \cong 1$ and $f^*x_i = x_0 \frac{x_i}{x_0} \otimes 1 \cong m_i$, then $\sigma_{0i} = m_i$. This proves that the two maps are equal, and concludes the proof. $\square$

2.5.26. Corollary. *The scheme $\mathbb{P}^n_{\mathbb{F}_1}$ is the scheme that represents the functor*

$$\begin{aligned} &\mathbf{Mon} \to \mathbf{Set} \\ &M \mapsto \left\{(m_0, \ldots, m_n) \in M^{n+1} \colon m_i \in M^\times \text{ for some } i\right\} \big/ M^\times \end{aligned}$$

where $M^\times$ acts via multiplication on each component.

Proof. Because of Propositions 2.5.25 and 2.5.7, $\mathbb{P}^n_{\mathbb{F}_1}$ represents the functor that associates to each monoid the set of $(n+1)$-tuples $(s_0, \ldots, s_n) \in M^{n+1}$ that generate M as M-module, up to the equivalence relation $\sim$, where $(s_0, \ldots, s_n) \sim (s'_0, \ldots, s'_n)$ if and only if there exists an M-linear automorphism α of M such that $\alpha s_i = s'_i$ for all i. The claim then follows from the fact that a tuple $(s_0, \ldots, s_n)$ generates M as a M-module if and only if at least one of the elements is invertible, and from the fact that automorphisms of M as a M-module are multiplications by invertible elements. $\square$

Because of the previous corollary, we can then consider $\mathbb{P}^n(M)$ for a monoid M as the set of $(n+1)$-tuples $(m_0\colon \ldots \colon m_n)$, where the elements $m_i \in M$ are not all non-invertible, with the relation $(m_0\colon \ldots \colon m_n) = (um_0\colon \ldots \colon um_n)$ for any invertible element $u \in M$. This is not a direct generalization of the case of rings. Indeed, for a ring A, $\mathbb{P}^n(A)$ may contain more elements than the set of representatives $(a_i\colon \ldots \colon a_n)$ with a_i invertible for some i, due to the fact that projective A-modules which are locally of rank one need not to be free.

APPENDIX A

Motivations

Being $\mathbb{F}_1$ a relatively recent object which mathematicians have started working on, we would like to give some ideas about the reasons that are pushing research in this field, and the aims which are sought. Similar commentaries can be found in any paper discussing the topic, and we will refer specifically to [**10**], [**11**], [**13**] and [**27**], as well as to other motivations presented by the community of researchers in the workshop on $\mathbb{F}_1$-geometry in Granada, November 2009. Nonetheless, our introduction is far from being complete, since there are various motivations we decided to overlook, and details which are better explained in other works. Indeed, we feel that the main focus of this thesis is independent from some of the original motivations, and even from some of the definitions given of "schemes over $\mathbb{F}_1$". The interested reader is advised to refer to the beautiful article of J. López Peña and O. Lorscheid [**27**], in which the whole picture is presented in a complete way.

We start by saying that $\mathbb{F}_1$, the elusive "field with one element", does not exist. Indeed, a field is usually defined as a ring such that the set of non zero elements is an abelian group with respect to multiplication. In particular, a standard axiom is $1 \neq 0$, hence the natural candidate of the trivial ring does not work for our purposes. Needless to say, there is more than a subtlety in the definition than makes things go wrong. Indeed, even if we accepted the trivial ring 0 as a good model for $\mathbb{F}_1$, we would end up with a field with no non-zero vector spaces, because $a = 1a = 0a = 0$ for all elements a of a 0-module. Despite this fact, mathematicians have a clear idea of what projective or affine spaces of finite dimension over $\mathbb{F}_1$ should be: finite sets. In this way, a lot of combinatorics can be read in a geometrical perspective. Also, it is likely (and hoped) that the geometry over the field with one element could be defined even without defining the field itself: it is a common belief that considering schemes over $\mathbb{F}_1$ would mean enlarging the standard picture, allowing in the scene objects with less structure than rings, and forgetting the extra structure (namely, the addition) when present. The main point of this idea is the attempt to go "under $\operatorname{Spec}\mathbb{Z}$", i.e. consider a different category of geometrical objects such that $\operatorname{Spec}\mathbb{Z}$ would no longer be a final element, but hopefully an object over some sort of an affine spectrum of a field, which, not surprisingly, would be indicated with $\operatorname{Spec}\mathbb{F}_1$. Again, this is not an affine spectrum with respect to the usual definitions, but the ultimate hope is that the techniques used to define compactifications, zeta functions and cohomology of schemes over finite fields in the classical case would have an analogue also over $\mathbb{F}_1$, so to generalize the results obtained for the geometry over finite fields (one for all: the Riemann hypothesis) to a more general setting. We also have to remark that there are other approaches which in turn describe the $\mathbb{F}_1$-structure as an enrichment, instead of a privation. One of the most notable ones is based on Λ-ring structures and is discussed by J. Borger in [**8**].

A.1. Why $\mathbb{P}^n(\mathbb{F}_1)$ is merely a set

Initially motivated by some analogies between general linear groups and symmetric groups, mathematicians have collected some evidence that hints towards the possible definition of modules, affine or projective spaces over $\mathbb{F}_1$ as sets. In here, we summarize briefly the theory presented by Cohn [**10**], which leads to a geometrical interpretation of this fact.

Following a paper of Birkhoff [**7**], and omitting the hypothesis $q > 1$, Cohn introduces this (almost) standard definition.

A.1.1. Definition. A *projective geometry of order* q is a finite set P (whose elements are called *points*), a set L of subsets of P (whose elements are called *subspaces*), and a function $\dim\colon L \to \mathbb{Z}_{\geq -1}$, called *dimension*, satisfying the following axioms (which are not minimal):

(i) The set L forms a lattice when partially ordered by inclusion. If S and T are in L, we indicate with $S \wedge T$ and $S \vee T$ the lower bound and the upper bound of S and T, respectively.
(ii) If S and T are in L and $S \subsetneq T$, then $\dim(S) < \dim(T)$.
(iii) The empty set and the whole P are in L.
(iv) For all x in P, the singleton $\{x\}$ is in L.
(v) For S in L, $\dim(S) = -1$ if and only if $S = \emptyset$, and $\dim(S) = 0$ if and only if $S = \{x\}$ for some x in P.
(vi) For S and T in L, $\dim(S) + \dim(T) = \dim(S \wedge T) + \dim(S \vee T)$.
(vii) If S is in L and $\dim(S) = 1$, then $\sharp S = q + 1$.

The dimension of P is called the *dimension of the geometry*.

It has been proved (see [**36**]) that whenever Desargues theorem holds for a finite projective geometry of order q, where q is a prime power, then the finite projective geometry is equivalent to $\mathbb{P}^n(\mathbb{F}_q)$, but there are still cases which are not of this kind (see [**35**]). Recall that in this language, the Desargues theorem can be stated as follows: let $\{A, B, C\}$ and $\{A', B', C'\}$ be two sets of distinct points such that $A \neq A'$, $B \neq B'$, $C \neq C'$. Then the space

$$(A \vee A') \wedge (B \vee B') \wedge (C \vee C')$$

is a point if and only if the space

$$((A \vee B) \wedge (A' \vee B')) \vee ((A \vee C) \wedge (A' \vee C')) \vee ((B \vee C) \wedge (B' \vee C'))$$

is a line (i.e a 1-dimensional space). This is a translation of the short statement: two triangles are in perspective axially if and only if they are in perspective centrally.

Suppose now to set $q = 1$, hence to talk about some analogue of projective geometries over $\mathbb{F}_1$. Some easy checks give the following result.

A.1.2. Proposition. *Any projective geometry of order* 1 *is equivalent to a triple* $(P, \wp(P), \dim)$ *where* $\dim(S) = \sharp S - 1$.

Proof. We claim that $\sharp P = \dim(P) + 1$. We prove this using induction on $n := \dim P$. If $n = 0$, then P is a singleton from the axioms. For the general case, suppose that $S \in L$, and that x is a point which is not in S. Then $S \wedge \{x\}$ is the empty set, and the axioms imply that

$$\dim(S \vee \{x\}) = \dim(S) + \dim(\{x\}) - \dim(S \wedge \{x\}) = \dim(S) + 1. \tag{3}$$

We deduce by induction that the join of r distinct points of P has dimension at most $r-1$, hence $\sharp P \geq n+1$. In particular, since $q=1$, the axioms imply that the 1-dimensional subspaces are exactly the sets $\{x,y\}$ constituted by two distinct points. Also, we conclude that we can construct subspaces of arbitrary dimension, by adjoining a point at a time. Let now S be a subspace of dimension $n-1$. It inherits the structure of a projective geometry, by considering subspaces contained in$S$ and the same dimension function. By induction hypothesis, we conclude that $\sharp S = n-1$. Suppose there are two distinct points x, y in $P \setminus S$. Since $\{x,y\}$ is a 1-dimensional subspace, we conclude that

$$\dim(S \vee \{x,y\}) = \dim(S) + \dim(\{x,y\}) - \dim(S \wedge \{x,y\}) = n+1$$

which is a contradiction. We conclude that $\sharp P = n+1$, as claimed. Applying this result to the projective geometry induced on any subspace S of P, we conclude that $\sharp S = \dim S + 1$. In particular, using the equation (3), we conclude that $\sharp(S \vee \{x\}) = \sharp S + 1$ whenever x is not in S. Since $S \vee \{x\}$ includes both S and $\{x\}$, we deduce that it is their union. We conclude that arbitrary unions, hence arbitrary subsets, are subspaces, so $L = \wp(P)$. □

It is hence legitimate to call $\mathbb{P}^n(\mathbb{F}_1)$ the finite set $\{0, 1, \ldots, n\}$, thought as the Boolean algebra associated to its power set.

Up to now, it may seem that the definition of the projective space over $\mathbb{F}_1$ is merely a fancy definition. However, the geometrical feeling of it is somehow enforced by the following property.

A.1.3. Definition. Let q, n and k be natural integers, with $q > 0$.

(1) The *q-analogue of n* is the number

$$[n]_q := 1 + q + q^2 + \ldots + q^{n-1}.$$

(2) The *q-factorial of n* is the number

$$[n]_q! := [1]_q [2]_q \ldots [n]_q.$$

(3) The *q-binomial coefficient of n over k* is the number

$$\begin{bmatrix} n \\ k \end{bmatrix}_q := \frac{[n]_q!}{[k]_q! [n-k]_q!}.$$

We remark that if $q=1$ the previous definitions overlap with n itself, and the usual notions of factorial and binomial coefficient respectively.

A.1.4. Theorem ([**10**], Theorem 5). *Every projective geometry of order q and dimension n contains $\begin{bmatrix} n+1 \\ k+1 \end{bmatrix}_q$ subspaces of dimension k.*

In this sense, combinatorics can be seen as part of projective geometry over $\mathbb{F}_1$. There are also other connections between these two worlds that arise from the equality $\mathbb{P}^n(\mathbb{F}_1) = \{0, 1, \ldots, n\}$, such as considering the simple groups of the form A_n as simple groups of the form $\mathrm{PSL}_n(\mathbb{F}_1)$ ([**10**], Section 4). Another typical example of this phenomenon is the equality $G(\mathbb{F}_1) = W_G$ for every split affine reduced group scheme, where W_G is the Weil group associated to it ([**33**]). This was the very first equality which made Tits wonder about the existence of $\mathbb{F}_1$, so that some well-behaved group schemes would have $\mathbb{F}_1$-valued points, which would form groups with a double algebraic-geometrical meaning.

A.2. Why addition is forgotten

We now give two different possible answers to the question of the title. Firstly, we give some hints of the following idea: if we forget addition, then we can consider $\operatorname{Spec}\mathbb{Z}$ as an analogue of $\operatorname{Spec}\mathbb{F}_1[x_1, x_2, \ldots]$. The whole point is to interpret in a geometrical sense the straightforward fact that $(\mathbb{Z}, \cdot)$ is the free commutative monoid with zero generated by a countable set $\{-1, p_1, p_2, \ldots\}$, with the relation $(-1)^2 = 1$. This way, it can be considered an analogue of the free k-algebra $k[x_1, x_2, \ldots]$. We will refer primarily to **[24]**, adding some definitions just to underline the analogies. In particular, we will try to adapt the definition of the Kähler differentials in a more general setting.

A.2.1. Definition. Let M and N be commutative monoids with unity. Use the multiplicative notation for M and the additive notation for N. A *linear M-action* on N is a map of monoids $\varphi\colon M \to \operatorname{Hom}_{\mathbf{Mon}}(N, N)$, giving to $\operatorname{Hom}_{\mathbf{Mon}}(N, N)$ the (generally non-commutative) monoidal structure induced by composition.

We write $\varphi(m)(n)$ simply with $m \cdot n$, hence the hypothesis can be translated into the equalities

$$m \cdot (n + n') = m \cdot n + m \cdot n'$$
$$1_M \cdot n = n$$
$$(mm') \cdot n = m \cdot (m' \cdot n)$$

for all m, m' in M and n, n' in N.

The objects just defined are not automatically M-algebras in the sense of Definition 2.2.2. Indeed, we do not require any bilinearity condition so that $m \cdot 1 + n \cdot 1$ needs not to be $mn \cdot 1$.

A.2.2. Definition. An *M-linear map* between two monoids N, N' that have a linear M-action is a map of monoids $f\colon N \to N'$ such that $f(m \cdot n) = m \cdot f(n)$. Commutative monoids with a linear M-action and M-linear maps form a category, which is indicated with M-**Mon**.

A.2.3. Proposition. *Let M be a monoid.*

*(1) The forgetful functor from M-**Mod** to **Set** has a left adjoint, that sends a set S to $M \times S$, with the obvious M-action.*
*(2) The forgetful functor from M **Mon** to M-**Mod** has a left adjoint, that sends a M-module S to the commutative monoid generated by it and the obvious M-action.*
*(3) The forgetful functor from M **Mon** to **Set** has a left adjoint, that sends a set S to the additive monoid*

$$\left\{ \sum^{<\infty} n_i(m_i, s_i) \colon n_i \in \mathbb{N}, m_i \in M, s_i \in S \right\}$$

with the M-action defined in the following way

$$m \cdot \left(\sum^{<\infty} n_i(m_i, s_i) \right) = \sum^{<\infty} n_i(mm_i, s_i).$$

Proof. The first claim comes from Proposition 2.2.4, considering **Set** as the category $\mathbb{F}_1$-**Mod**, where $\mathbb{F}_1$ is the trivial monoid. The second claim comes from the fact that the adjunction $\mathbf{Mon} \leftrightarrows \mathbf{Set}$ preserves M-linearity. The third claim comes from the other two. $\square$

A.2.4. Definition. Let now $(N, +)$ be a monoid with a linear M-action. A map $d\colon M \to N$ is an *absolute derivation* if for all m, m' in M the following equality holds

$$d(mm') = m \cdot d(m') + m' \cdot d(m).$$

We indicate with $\mathrm{Der}_{\mathbb{F}_1,M}(M,N)$ the set of such maps. It has a natural structure of commutative monoid with M-linear action with operations defined pointwise. Let R be a ring. Then $(R, +)$ has a natural structure of commutative monoid with $(R, \cdot)$-structure. We call *absolute derivations of* R the absolute derivations from $(R, \cdot)$ to $(R, +)$, and we indicate the set they form with $\mathrm{Der}_{\mathbb{F}_1}(R)$. It has a natural structure of R-module with operations defined pointwise.

A.2.5. Proposition. *Let R be a ring, and let d be an absolute derivation of R. Then $d(1) = 0$.*

Proof. This comes from the equalities

$$d(1) = d(1 \cdot 1) = 1 \cdot d(1) + 1 \cdot d(1) = d(1) + d(1).$$

$\square$

A.2.6. Example. Consider the ring of integers $\mathbb{Z}$ and let p be prime. The map $\frac{\partial}{\partial p}$ defined as

$$\frac{\partial}{\partial p}(a) := \begin{cases} 0 & \text{if } p \nmid a \text{ or } a = 0 \\ kp^{k-1}m & \text{if } a = p^k m \text{ with } p \nmid m \end{cases}$$

is an absolute derivation of $\mathbb{Z}$.

A.2.7. Definition. The *absolute differentials module of* M is a monoid $\Omega_{M/\mathbb{F}_1}$ with a linear M-action and an absolute derivation $d\colon M \to \Omega_{M/\mathbb{F}_1}$ such that for every monoid with a linear M-action N and every absolute derivation $\partial\colon M \to N$, there exists a unique M-linear map $f\colon \Omega_{M/\mathbb{F}_1} \to N$ such that $\partial = fd$.

Intersections of equivalence relations which are monoidal and M-linear in the sense of Definition 2.3.3 are again equivalence relations with both properties. Since the total relation is always of this kind, we can talk about the *monoidal M-linear equivalence relation generated by a subset R of $M \times M$*, defined as the intersection of relations with such properties that contain R.

A.2.8. Proposition. *Consider the free commutative monoid with a linear M-action F generated by the set of symbols $\{dm\}$ as m varies in M, then consider the monoidal and M-linear equivalence relation generated by $\{d(mm') \sim mdm' + m'dm\}$. The monoid $F/\!\sim$ together with the derivation $m \mapsto dm$ constitutes the absolute differentials module $\Omega_{M/\mathbb{F}_1}$.*

Proof. This is a straightforward consequence of universal properties. $\square$

A.2.9. Proposition. *Suppose that M is the free monoid generated by a set S, with multiplicative notation. Then $\Omega_{M/\mathbb{F}_1}$ is the free commutative monoid with a linear M-action Ω generated by the set $\{dx\colon x \in S\}$, together with a map d which sends each x to dx.*

Proof. We notice that an absolute derivation from M is uniquely determined by the images of the elements $x \in S$, since any other element $\prod_i x_i^{n_i}$ is mapped to $\sum_i n_i \left(\prod_{j \neq i} x_j^{n_j}\right) x_i^{n_i - 1} \cdot dx_i$. Conversely, any map $f\colon S \to N$ gives rise to a derivation, posing $x \mapsto f(x)$. This determines a canonical isomorphism $\mathrm{Der}_{\mathbb{F}_1,M}(M,N) = \mathrm{Hom}_{\mathbf{Set}}(S,N) = \mathrm{Hom}_{M\text{-}\mathbf{Mon}}(\Omega,N)$ as wanted. $\square$

A.2.10. EXAMPLE. Consider the multiplicative monoid $(\mathbb{Z}_{\geq 1}, \cdot)$. Recalling Example 2.3.2, we conclude that for any commutative monoid N with a $\mathbb{Z}_{\geq 1}$-linear action N there is a canonical isomorphism in M-**Mon**

$$\mathrm{Der}_{\mathbb{F}_1,\mathbb{Z}_{\geq 1}}(\mathbb{Z}_{\geq 1}, N) = \prod_{p \text{ prime}} N.$$

A.2.11. PROPOSITION. *Let M be a finitely generated monoid and let R be a generating set for the relation $\sim$ such that $M = \mathbb{F}_1[S]/\sim$. Then $\Omega_{M/\mathbb{F}_1}$ is the free commutative monoid with a linear M-action Ω generated by the set $\{dx \colon x \in S\}$, modulo the M-linear monoidal relation generated by dR, together with a map d which sends each x to dx, where dR is the subset that contains all the relations*

$$\sum_i n_i \left(\prod_{j \neq i} x_j^{n_j}\right) x_i^{n_i - 1} \cdot dx_i = \sum_{i'} n_{i'} \left(\prod_{j' \neq i'} x_{j'}^{n_{j'}}\right) x_{i'}^{n_{i'} - 1} \cdot dx_{i'}$$

$$\sum_h n_h \left(\prod_{k \neq h} x_k^{n_k}\right) x_h^{n_h - 1} \cdot dx_h = 0$$

as $(\prod x_i^{n_i}, \prod x_{i'}^{n_{i'}})$ and $(\prod x_h^{n_h}, 1)$ vary in R.

PROOF. This is analogous to the proof of the previous proposition. $\square$

A.2.12. EXAMPLE. Consider the multiplicative monoid $(\mathbb{Z}, \cdot)$. Recalling Example 2.3.6, we conclude that the monoid with a $\mathbb{Z}$-action $\Omega_{\mathbb{Z}/\mathbb{F}_1}$ is isomorphic to the free commutative monoid with a $\mathbb{Z}$-action generated by the set $\{d0, d(-1)\} \cup \{dp_i\}$ where the p_i's run through the primes, modulo the relation generated by

$$\{(-1) \cdot d(-1) + (-1) \cdot d(-1) = 0, \quad d0 = p_i \cdot d0 + 0 \cdot dp_i\}.$$

In particular, there is an isomorphism of abelian groups

$$\mathrm{Der}_{\mathbb{F}_1}(\mathbb{Z}) = \prod_{p \text{ prime}} \mathbb{Z}\frac{\partial}{\partial p_i}.$$

Indeed, since $\mathbb{Z}$ is torsion free, the relation

$$0 = (-1) \cdot d(-1) + (-1) \cdot d(-1) = -2d(-1)$$

implies that $d(-1) = 0$. Also, the relation

$$d0 = 2 \cdot d0 + 0 \cdot d2 = 2d0$$

implies that $d0 = 0$. We also remark that the derivations $\frac{\partial}{\partial p_i}$ are uniquely determined by the equalities

$$\frac{\partial}{\partial p_i}(p_j) = \delta_{ij}.$$

There is another intriguing reason why it seems conceivable to consider monoidal structures rather than ring structures in order to inspect $\mathbb{F}_1$. It comes from the attempt to define the compactification of $\mathrm{Spec}\,\mathbb{Z}$, indicated with $\widehat{\mathrm{Spec}\,\mathbb{Z}}$. As explained in [**13**], Section 1.1, such a definition would fill a gap that exists from the number fields case and the function fields case. Suppose to consider a field of functions $K = k(t)$. Then equivalence classes of norms on K which are trivial on k are in natural bijection with closed points of $\mathbb{P}^1_K$. In particular, each of them can be defined as the norm $|\cdot|_p$ induced by the discrete valuation $v_p\colon K \to \mathbb{Z} \cup \{+\infty\}$ defined by a closed point p, which has as valuation ring the local ring $\mathcal{O}_{\mathbb{P}^1_K,p}$. The valuation associated to "the point at infinity" (the only one not included in $\mathbb{A}^1_K$) acts as $v_\infty(f(t)/g(t)) = \deg(f) - \deg(g)$. The completeness (or, better saying, the properness) of $\mathbb{P}^1_K$ gives rise also to the powerful product formula (we indicate with ξ the generic point of $\mathbb{P}^1_K$):

$$\prod_{p \in \mathbb{P}^1_K, p \neq \xi} |f|_p = 1$$

which is true for any $f \in k[t]$, after a good choice of representatives for each class of norms.

If we now turn to the case $K = \mathbb{Q}$, we notice that there exists a similar bijection between non-Archimedean norms of K and closed points of $\operatorname{Spec} \mathbb{Z}$, and again the associated discrete valuations have as local rings the rings $\mathbb{Z}_{(p)}$, whose completions are the rings $\mathbb{Z}_p \subset \mathbb{Q}_p$. It is then natural to consider $\operatorname{Spec} \mathbb{Z}$ as an analogue of the affine line, and to try to attach to it some data "at infinity" which would encode the information related to the Archimedean norm v_∞. Nevertheless, in the Archimedean case, the analogue of the valuation ring would be $\{x \in \mathbb{Q}\colon |x| \leq 1\} = [-1,1] \cap \mathbb{Q}$, and the analogue of the maximal ideal would be $\{x \in \mathbb{Q}\colon |x| < 1\} = (-1,1) \cap \mathbb{Q}$. Both these sets come together with a natural monoidal structure, but not with a ring structure that includes it. In his thesis [**13**], Durov starts from the fact that the completion of $\mathbb{Q}$ with respect to the Archimedean norm $\mathbb{Q}_\infty$ is $\mathbb{R}$ and gives a well-posed definition of the classically missing objects $\mathbb{Z}_{(\infty)}$, $\mathbb{Z}_\infty$ in terms of monads. Not surprisingly, the object $\mathbb{Z}_\infty$ has as underlying monoid the monoid $[-1,1] \subset \mathbb{R}$. The interested reader may find full details in [**13**], Paragraph 3.4.12.

We can then resume saying that the reasonableness of "forgetting addition" is due to the attempt of inspecting the nature of $\operatorname{Spec} \mathbb{Z}$, which on the one hand can be considered similar to an infinite-dimensional affine space over a mythical field of one element, and on the other hand as a non-completed smooth model for $\mathbb{Q}$ which encrypts the information on its non-Archimedean norms.

APPENDIX B

Stacks

In this chapter, we will present some basic facts of stack theory, with the specific aim to have a complete proof for Propositions 1.2.12, 1.2.14 and 2.2.17. The interested reader may find exhausting descriptions of fibered categories and stacks in other texts such as Vistoli's notes [**14**] or the Stacks Project [**5**].

B.1. Pseudo-functors and fibered categories

It is very tempting to consider the data

$$\begin{aligned}\mathbf{Aff} &\to \mathbf{Cat}\\ \operatorname{Spec} A &\mapsto A\text{-}\mathbf{Mod}\\ (\operatorname{Spec} B \to \operatorname{Spec} A) &\mapsto (\otimes_A B \colon A\text{-}\mathbf{Mod} \to B\text{-}\mathbf{Mod})\end{aligned}$$

as a presheaf of categories. This is not allowed, though, because such a map does not fully agree with the 2-category structure of **Cat**, in the sense that for a couple of arrows $\operatorname{Spec} C \to \operatorname{Spec} B \to \operatorname{Spec} A$, the functor $(\otimes_A B) \otimes_B C$ is simply isomorphic, but not equal, to $\otimes_A C$. Also, the functor $\otimes_A A$ associated to id_A is simply isomorphic, but not equal, to the functor $id_{A\text{-}\mathbf{Mod}}$. The concept of pseudo-functors, and hence of fibered categories, can be initially considered as an attempt to build a theory similar to the one of presheaves of categories, even in this laxer context.

B.1.1. Definition. A *pseudo-functor* Φ from a category $\mathbf{C}$ consists of a category $\Phi(X)$ for each object X of $\mathbf{C}$, a functor $f^* \colon \Phi(Y) \to \Phi(X)$ for each arrow $f \colon X \to Y$ in $\mathbf{C}$, and isomorphisms of functors $\varepsilon_X \colon id_X^* \to id_{\Phi(X)}$ and $\alpha_{f,g} \colon f^* g^* \to (gf)^*$ for each object X and each pair of composable arrows f, g of $\mathbf{C}$ that satisfy the following coherence conditions.

(i) For each arrow $f \colon X \to Y$, the two ways of getting an isomorphism from $id_X^* f^*$ to f^* coincide.
(ii) For each triple of composable arrows f, g, h, the two ways of getting an isomorphism from $f^* g^* h^*$ to $(hgf)^*$ coincide.

There is a natural notion of map between pseudo-functors from $\mathbf{C}$, which we do not explicitly describe.

B.1.2. Example. Suppose that $\mathbf{C}$ is a category with fibered products. The data constituted by the map

$$\begin{aligned}\mathbf{C} &\to \mathbf{Cat}\\ X &\mapsto \mathbf{C}_{/X}\\ (f \colon X \to Y) &\mapsto (\times_Y X \colon \mathbf{C}_{/Y} \to \mathbf{C}_{/X})\end{aligned}$$

together with the isomorphismss of functors $\times_X X \xrightarrow{\sim} id_{\mathbf{C}_{/X}}$, $(\times_X Y)\times_Y Z \xrightarrow{\sim} \times_X Z$, defines a pseudo-functor on the category $\mathbf{C}$. It is not, however, a presheaf of categories.

B.1.3. Example. The data consituted by the map

$$\begin{aligned} \mathbf{Aff} &\to \mathbf{Cat} \\ \operatorname{Spec} A &\mapsto A\text{-}\mathbf{Mod} \\ (\operatorname{Spec} B \to \operatorname{Spec} A) &\mapsto (\otimes_A B \colon A\text{-}\mathbf{Mod} \to B\text{-}\mathbf{Mod}) \end{aligned}$$

together with the isomorphisms of functors $\otimes_A A \xrightarrow{\sim} id_{A\text{-}\mathbf{Mod}}$, $(\otimes_A B)\otimes_B C \xrightarrow{\sim} \otimes_A C$, defines a pseudo-functor on the category $\mathbf{Aff}$. This is not, however, a presheaf of categories.

Just as in the case of vector bundles and locally free sheaves, there is a way to "glue together" the categories $\Phi(X)$ in order to form a category $\mathbf{F}$ and a functor $p_{\mathbf{F}} \colon \mathbf{F} \to \mathbf{C}$, so that each fiber $p_{\mathbf{F}}^{-1}(X)$ is canonically isomorphic to $\Phi(X)$. This gives rise to the concept of fibered categories.

B.1.4. Definition. A *fibered category* over $\mathbf{C}$ is a category $\mathbf{F}$ with a functor $p_{\mathbf{F}} \colon \mathbf{F} \to \mathbf{C}$ such that for each arrow $f \colon X \to Y$ of $\mathbf{C}$, and for each ξ in $p_{\mathbf{F}}^{-1}(Y)$, there is an arrow $\eta \to \xi$ in $\mathbf{F}$ that maps to f via $p_{\mathbf{F}}$ which is such that the square

$$\begin{tikzcd} \eta \arrow[r] \arrow[d, mapsto] & \xi \arrow[d, mapsto] \\ X \arrow[r] & Y \end{tikzcd}$$

is Cartesian, in the sense that the map

$$\operatorname{Hom}_{\mathbf{F}}(\zeta, \eta) \to \operatorname{Hom}_{\mathbf{C}}(p_{\mathbf{F}}(\zeta), X) \times_{\operatorname{Hom}_{\mathbf{C}}(p_{\mathbf{F}}(\zeta), Y)} \operatorname{Hom}_{\mathbf{F}}(\zeta, \xi)$$

is a natural isomorphism, for each object ζ of $\mathbf{F}$. The element η is often called a *pullback of ξ via f*.

A *cleavage* (or also a *choice of pullbacks*) of a fibered category is the choice of a Cartesian square as above, for each map $f \colon X \to Y$ of $\mathbf{C}$ and each object $\xi \in p_{\mathbf{F}}^{-1}(Y)$. Once a cleavage is chosen, the pullback element η that completes the square is then called $f^*(\xi)$.

A *map of fibered categories* over $\mathbf{C}$ is an arrow of $\mathbf{Cat}_{/\mathbf{C}}$ that sends Cartesian arrows to Cartesian arrows. A *map of fibered categories with a cleavage* is a map of fibered categories that sends the cleavage of the first one to the cleavage of the second one.

By the definition and the axiom of choice, each fibered category has a cleavage. Remark that, even after choosing a cleavage, it is not true in general that $f^*g^*\xi$ equals $(gf)^*\xi$, nor that $id_X^*\xi$ equals ξ. However, the universal property implies (also in this setting) that pullbacks are unique up to unique isomorphisms, hence the natural arrows $\xi \to id_X^*\xi$ and $f^*g^*\xi \to (gf)^*\xi$ are isomorphisms.

In fact, this hints to part of the following proposition, whose proof is omitted.

B.1.5. Definition. Let $p_{\mathbf{F}} \colon \mathbf{F} \to \mathbf{C}$ be a fibered category over $\mathbf{C}$. A cleavage of $\mathbf{F}$ is *split* if the isomorphisms $\xi \to id_X^*\xi$ and $f^*g^*\xi \to (gf)^*\xi$ are identities, for each map f of $\mathbf{C}$ and each object ξ of $p_{\mathbf{F}}^{-1}(Y)$.

B.1.6. PROPOSITION ([**14**], 3.1.2, 3.1.3). *Let* $\mathbf{C}$ *be a category, let* $p_{\mathbf{F}}\colon \mathbf{F} \to \mathbf{C}$ *be a fibered category over* $\mathbf{C}$ *with a cleavage, and let* Φ *be a pseudo-functor from* $\mathbf{C}$*.*

(1) For any object X *of* $\mathbf{C}$*, we call* $p_{\mathbf{F}}^{-1}(X)$ *or* $\mathbf{F}(X)$ *the subcategory of* $\mathbf{F}$ *whose maps are those that are sent to* id_X *via* $p_{\mathbf{F}}$*. The data constituted by the map*

$$\mathbf{C} \to \mathbf{Cat}$$
$$X \mapsto \mathbf{F}(X)$$
$$(f\colon X \to Y) \mapsto (f^*\colon \mathbf{F}(Y) \to \mathbf{F}(X))$$

together with the isomorphisms $f^*g^* \overset{\sim}{\to} (gf)^*$*,* $id_{\mathbf{F}(X)} \overset{\sim}{\to} id_X^*$*, defines a pseudo-functor on the category* $\mathbf{C}$*. It is a presheaf of categories if and only if the cleavage is split.*

(2) Let $\mathbf{F}_\Phi$ *be the category described in the following way. The objects are couples* (η, X) *where* X *is an object of* $\mathbf{C}$ *and* ξ *is an object in* $\mathbf{F}(X)$*. An arrow from* (η, X) *to* (ξ, Y) *is given by a map* $f\colon X \to Y$ *of* $\mathbf{C}$ *and a map* $a\colon \eta \to f^*(\xi)$ *of* $\Phi(X)$*. The composition of two arrows*

$$(f,a)\colon (X,\eta) \to (Y,\xi) \quad (g,b)\colon (Y,\xi) \to (Z,\zeta)$$

is defined to be the couple $(gf, b\cdot a)$*, where* $b \cdot a$ *is the composite map*

$$\eta \to f^*\xi \xrightarrow{f^*b} f^*g^*\zeta \xrightarrow{\alpha} (gf)^*\zeta.$$

The forgetful functor $\mathbf{F}_\Phi \to \mathbf{C}$ *that sends* (X,η) *to* X *defines a structure of a fibered category on* $\mathbf{F}_\Phi$*. The choice of Cartesian squares of the following kind*

$$\begin{tikzcd} (X, f^*\xi) \arrow[r] \arrow[d, mapsto] & (Y,\xi) \arrow[d, mapsto] \\ X \arrow[r, "f"] & Y \end{tikzcd}$$

defines a cleavage of $\mathbf{F}_\Phi$ *which is split if and only if* Φ *is a presheaf of categories.*

(3) The two maps described in the first two points define an equivalence between the category of fibered categories over $\mathbf{C}$ *with a cleavage, and the category of pseudo-functors from* $\mathbf{C}$*.*

B.1.7. EXAMPLE. The fibered category associated to the pseudo-functor of B.1.2 is the category $\mathbf{Ar}(\mathbf{C})$ of arrows in $\mathbf{C}$, and the structure functor $\mathbf{Ar}(\mathbf{C}) \to \mathbf{C}$ is the one that associates to a map $f\colon X \to Y$ its target Y. We will denote it simply with $\mathbf{Ar}$, if the context is clear.

B.1.8. DEFINITION. A *split fibered category* is a fibered category that admits a split cleavage. Equivalently, it is a fibered category associated to a pseudo-functor which is a presheaf of categories.

Notice that all fibered categories have a cleavage, so that the two perspectives (pseudo-functors and fibered categories) can really be considered equivalent in some sense. Remark that the definition of a fibered category does not require a choice of a cleavage per se. This is the reason why we will develop the theory using also this perspective, and not only the pseudo-functor perspective. Also, one may wonder where the choice of a cleavage is hidden in a pseudo-functor. The answer lies in the fact that, unlike functors, the very definition of pseudo-functors contains a series of isomorphisms of functors, and these reflect the choice which is made in considering a cleavage of a fibered category.

The previous proposition somehow authorizes our abuse of notation, by which we indicated with $f^*\xi$ both pullbacks in the sense of fibered categories, and images $\Phi(f)(\xi)$ via a pseudo-functor Φ. Also, we can define various properties of fibered categories using their pseudo-functor counterpart.

B.1.9. Definition. Let $p_{\mathbf{F}}\colon \mathbf{F} \to \mathbf{C}$ be a fibered category over $\mathbf{C}$.

(1) We say that $\mathbf{F}$ is a *category fibered in sets* if each fiber $\mathbf{F}(X)$ is a set, i.e. a discrete category.
(2) We say that $\mathbf{F}$ is a *category fibered in grupoids* if each fiber $\mathbf{F}(X)$ is a grupoid.

B.1.10. Example. Consider the Yoneda embedding $\mathbf{C} \to \mathbf{Psh}(\mathbf{C})$, and the embedding $\mathbf{Set} \to \mathbf{Cat}$ by which sets are seen as discrete categories. For each object X of $\mathbf{C}$, there is a fibered category associated to the pseudo-functor h_X. This is the category $\mathbf{C}_{/X}$, together with the forgetful functor $(U \to X) \mapsto U$. We will then denote such fibered category both with h_X and with $\mathbf{C}_{/X}$. It is a split category fibered in sets.

B.1.11. Example. Let now $\mathbf{C}$ be a site, and consider a covering $\mathcal{U} := \{U_i \to X\}$ of an object X. We denote by $h_{\mathcal{U}}$ the subsheaf of h_X that associates to each Y the subset of $\mathrm{Hom}_{\mathbf{C}}(Y, X)$ constituted by those arrows that split over some $U_i \to X$. This defines another split category fibered in sets.

Since fibered categories are categories, they form a 2-category. Puns aside, this means the following fact.

B.1.12. Definition. Let $\mathbf{F}$, $\mathbf{G}$ be fibered categories over $\mathbf{C}$, and let F, G be maps of fibered categories from $\mathbf{F}$ to $\mathbf{G}$. A *natural transformation* α from F to G is a natural transformation of functors $\alpha\colon F \to G$ such that the arrows in $\mathbf{G}$ induced by α lie inside the same fiber, i.e. it is such that for each object ξ of $\mathbf{F}$ that is mapped to X, the map $\alpha_\xi\colon F\xi \to G\xi$ is a map of $\mathbf{G}(X)$. The category whose objects are functors of fibered categories from $\mathbf{F}$ to $\mathbf{G}$, and whose arrows are natural transformation is called $\mathrm{Hom}_{\mathbf{C}}(\mathbf{F}, \mathbf{G})$.

Since the category of fibered categories over $\mathbf{C}$ is a subcategory of $\mathbf{Cat}_{/\mathbf{C}}$, there is a notion of equivalence of fibered categories over $\mathbf{C}$. The following proposition, whose proof is omitted, provides a useful criterion for finding equivalences in this context.

B.1.13. Proposition ([**14**], 3.36). *Let $G\colon \mathbf{F} \to \mathbf{G}$ be a map of fibered categories over $\mathbf{C}$. It is an equivalence if and only if the restriction $G_X\colon \mathbf{F}(X) \to \mathbf{G}(X)$ is an equivalence of categories, for each object X of $\mathbf{C}$.*

Yoneda's lemma has an analogue in the context of fibered category.

B.1.14. Proposition (2-Yoneda's lemma). *Let $\mathbf{F}$ be a fibered category over $\mathbf{C}$, and let X be an object of $\mathbf{C}$. We indicate with h_X the fibered category defined as in B.1.10. The functor*

$$\mathrm{Hom}_{\mathbf{C}}(h_X, \mathbf{F}) \to \mathbf{F}(X)$$

defined by taking the image of id_X, is an equivalence of categories.

PROOF. We can define a functor $\mathbf{F}(X) \to \mathrm{Hom}_{\mathbf{C}}(h_X, \mathbf{F})$ in the following way. Choose a cleavage for $\mathbf{F}$. We associate to an object ξ in $\mathbf{F}(X)$ the functor F_ξ in $\mathrm{Hom}_{\mathbf{C}}(h_X, \mathbf{F})$ that maps an element $\varphi \colon U \to X$ of h_X to the object $\varphi^*\xi \in \mathbf{F}(U)$, and that maps an arrow f of h_X:

$$\begin{tikzcd} U \arrow[rr, "f"] \arrow[dr, "\varphi"'] & & V \arrow[dl, "\psi"] \\ & X & \end{tikzcd}$$

to the arrow $\varphi^*\xi \to \psi^*\xi$ in $\mathbf{F}$ which is induced by the couple of arrows $\varphi^*\xi \to U \to V$ and $\varphi^*\xi \to \xi$. Also, a map $\xi \to \xi'$ in $\mathbf{F}(X)$ induces a natural transformation of the two functors $F_\xi \to F_{\xi'}$ because of the commutativity of the following square, which is induced by the universal property of $\psi^*\xi'$.

$$\begin{tikzcd} \varphi^*\xi \arrow[r] \arrow[d] & \psi^*\xi \arrow[d] \\ \varphi^*\xi' \arrow[r] & \psi^*\xi' \end{tikzcd}$$

Notice that the image of id_X via this functor is the object $id_X^*\xi$, which is canonically isomorphic to ξ. More generally, the composite functor $\mathbf{F}(X) \to \mathrm{Hom}_{\mathbf{C}}(h_X, \mathbf{F}) \to \mathbf{F}(X)$ is isomorphic to the identity functor.

We are left to prove that the image of a functor G in $\mathrm{Hom}_{\mathbf{C}}(h_X, \mathbf{F})$ via the composite functor $\mathrm{Hom}_{\mathbf{C}}(h_X, \mathbf{F}) \to \mathbf{F}(X) \to \mathrm{Hom}_{\mathbf{C}}(h_X, \mathbf{F})$ is isomorphic to G itself. Let $\varphi \colon U \to X$ be in $h_X(U)$. Consider the following Cartesian map of h_X.

$$\begin{tikzcd} \varphi \arrow[r] \arrow[d, mapsto] & id_X \arrow[d, mapsto] \\ U \arrow[r, "\varphi"] & X \end{tikzcd}$$

Since G is a map of fibered categories, also the following square

$$\begin{tikzcd} G(\varphi) \arrow[r] \arrow[d, mapsto] & G(id_X) \arrow[d, mapsto] \\ U \arrow[r, "\varphi"] & X \end{tikzcd}$$

is Cartesian. Hence, there is a canonical isomorphism $G(\varphi) \cong \varphi^*\xi = F_{G(id_X)}(\varphi)$, as wanted. $\square$

We end this section presenting a very important result. It lets us consider only presheaves of categories, or fibered categories with a split cleavage, if you prefer. Since we won't use it in the future, the proof of this fact is omitted.

B.1.15. PROPOSITION ([**14**], 3.45). *Every fibered category is equivalent to a split fibered category.*

B.2. Descent data and stacks

From now on, let $\mathbf{C}$ be a site, i.e. a category with a Grothendieck topology. Since we claimed that fibered categories generalize the notion of a presheaf of categories, we can now give a correct notion of a "lax" sheaf of categories, namely stacks.

We start by describing what happens when substituting h_X with $h_{\mathcal{U}}$ in 2-Yoneda's lemma, where $\mathcal{U}$ is a covering of an object of $\mathbf{C}$ (see B.1.11).

B.2.1. Definition. Let $\mathbf{C}$ be a site, let $\mathcal{U} := \{U_i \to X\}$ be a covering in it and let $\mathbf{F}$ be a fibered category over $\mathbf{C}$. The category $\mathrm{Hom}_{\mathbf{C}}(h_{\mathcal{U}}, \mathbf{F})$ is called the *category of descent data of* $\mathbf{F}$ *on* $\mathcal{U}$, and indicated with $\mathrm{Desc}(\mathcal{U}/X, \mathbf{F})$.

There are alternative explicit description of the category of descent data, which are usually given as its definition. Note that, from now on, we will denote the objects $U_i \times_X U_j$ and $U_i \times_X U_j \times_X U_k$ with U_{ij} and U_{ijk} respectively. Also, we indicate projections with π_α, where α is the (single or double) index that refers to the target object. In particular, for a covering $\{U_i \to X\}$ in a site, we have the following Cartesian cube, for all triples of indices i, j, k.

(4)
$$\begin{tikzcd}
& U_{ijk} \arrow[rr, "\pi_{jk}"] \arrow[dl, "\pi_{ij}"'] \arrow[dd, "\pi_{ik}" near start] & & U_{jk} \arrow[dl, "\pi_j"'] \arrow[dd, "\pi_k"] \\
U_{ij} \arrow[rr, "\pi_j" near start] \arrow[dd, "\pi_i"'] & & U_j \arrow[dd] & \\
& U_{ik} \arrow[rr, "\pi_k" near start] \arrow[dl, "\pi_i"'] & & U_k \arrow[dl] \\
U_i \arrow[rr] & & X &
\end{tikzcd}$$

B.2.2. Proposition. *Let* $\mathbf{C}$ *be a site, and let* $\mathcal{U} := \{U_i \to X\}$ *be a covering in it. Let also* $p_{\mathbf{F}} \colon \mathbf{F} \to \mathbf{C}$ *be a fibered category.*

(1) The category of descent data is equivalent to the category described as follows. Objects are collections $(\{\xi_i\}, \{\xi_{ij}\}, \xi_{ijk}\})$ *together with diagrams*

(5)
$$\begin{tikzcd}
& \xi_{ijk} \arrow[rr] \arrow[dl] \arrow[dd] & & \xi_{jk} \arrow[dl] \arrow[dd] \\
\xi_{ij} \arrow[rr] \arrow[dd] & & \xi_j & \\
& \xi_{ik} \arrow[rr] \arrow[dl] & & \xi_k \\
\xi_i & & &
\end{tikzcd}$$

which are constituted by Cartesian squares and that are mapped to the corresponding part of the cube (4) *via* $p_{\mathbf{F}}$. *Arrows are collections of maps*

$\xi_\alpha \to \xi'_\alpha$ in $\mathbf{F}(U_\alpha)$, defined for each (single, double or triple) index α, such that they induce maps of the corresponding diagrams (5)*.*

(2) Given a choice of pullbacks for $\mathbf{F}$, the category of descent data is equivalent to the category described as follows. Objects are collections $(\{\xi_i\}, \{\sigma_{ij}\})$ of objects ξ_i in $\mathbf{F}(U_i)$, and isomorphisms $\sigma_{ij}\colon \pi_i^*\xi \to \pi_j^*\xi$ in $\mathbf{F}(U_{ij})$ such that the following diagram in $\mathbf{F}(U_{ijk})$ commutes (this is the so called cocycle condition*).*

$$\begin{tikzcd} \pi_i^*\xi_i \arrow[rr, "\pi_{ik}^*\sigma_{ik}"] \arrow[dr, "\pi_{ij}^*\sigma_{ij}"'] & & \pi_k^*\xi_k \\ & \pi_j^*\xi_j \arrow[ur, "\pi_{jk}^*\sigma_{jk}"'] & \end{tikzcd}$$

Arrows from $(\{\xi_i\}, \{\sigma_{ij}\})$ to $(\{\xi'_i\}, \{\sigma'_{ij}\})$ are given by a collection $\{\psi_i\}$ of arrows $\psi_i\colon \xi_i \to \xi'_i$ in $\mathbf{F}(U_i)$ such that all diagrams of the form

$$\begin{tikzcd} \pi_i^*\xi \arrow[r, "\pi_i^*\psi_i"] \arrow[d, "\sigma_{ij}"'] & \pi_i^*\xi' \arrow[d, "\sigma'_{ij}"'] \\ \pi_j^*\xi \arrow[r, "\pi_j^*\psi_j"] & \pi_j^*\xi' \end{tikzcd}$$

commute.

Proof. We only outline the functors that can be defined between the categories we introduced, but we do not prove the fact that they define equivalences. This can be found in [**14**], 4.1.2. Suppose we have chosen pullbacks for $\mathbf{F}$. We define a functor from the category described in 1 to the category described in 2. A diagram of the kind (5) is sent to the collection $(\{\xi_i\}, \{\sigma_{ij}\})$, where the isomorphisms σ_{ij} are defined as the composition of the two isomorphisms $\pi_i^*\xi_i \xrightarrow{\sim} \xi_{ij} \xrightarrow{\sim} \pi_j^*\xi_j$. Arrows are mapped to arrows in the obvious way.

Also, we can define a functor from $\mathrm{Hom}_{\mathbf{C}}(h_{\mathcal{U}}, \mathbf{F})$ to the category described in 1 in the following way. Each projection map $U_\alpha \to U$ obviously splits over some map if the covering, hence we can consider U_α to be objects of the category $h_{\mathcal{U}}$. Hence, to a functor $F\colon h_{\mathcal{U}} \to \mathbf{F}$, we can associate the following diagram.

$$\begin{tikzcd}[row sep=small] & F(U_{ijk}) \arrow[rr] \arrow[dl] \arrow[dd] & & F(U_{jk}) \arrow[dl] \arrow[dd] \\ F(U_{ij}) \arrow[rr, crossing over] \arrow[dd] & & F(U_j) & \\ & F(U_{ik}) \arrow[rr] \arrow[dl] & & F(U_k) \\ F(U_i) & & & \end{tikzcd}$$

Again, the action on arrows is obvious. □

We are now ready to give the definition of a stack.

B.2.3. DEFINITION. Let $\mathbf{F}$ be a fibered category on a site $\mathbf{C}$. For each covering $\mathcal{U}$ of X, the inclusion of categories $h_{\mathcal{U}} \hookrightarrow h_X$ induces a functor

$$\mathbf{F}(X) \xrightarrow{\sim} \mathrm{Hom}_{\mathbf{C}}(h_X, \mathbf{F}) \to \mathrm{Hom}_{\mathbf{C}}(h_{\mathcal{U}}, \mathbf{F}) = \mathrm{Desc}(\mathcal{U}/X, \mathbf{F})$$

where the first arrow is an equivalence of categories by 2-Yoneda's lemma.

(1) We say that $\mathbf{F}$ is a *prestack* if the functor above is fully faithful, for all objects X and all coverings $\mathcal{U}$ of X.
(2) We say that $\mathbf{F}$ is a *stack* if the functor above is an equivalence of categories, for all objects X and all coverings $\mathcal{U}$ of X.

B.2.4. EXAMPLE. Suppose we give to a category $\mathbf{C}$ the trivial topology, where coverings are only identities. In this case $h_X = h_{\mathcal{U}}$, hence any fibered category $\mathbf{F}$ is a stack. More generally, whenever a covering $\mathcal{U}$ contains the identity, then $h_X = h_{\mathcal{U}}$, hence the functor $\mathbf{F}(X) \to \mathrm{Desc}(\mathcal{U}/X, \mathbf{F})$ is an equivalence.

B.2.5. EXAMPLE. Let $\mathbf{F}$ be a split category fibered in sets, i.e. a presheaf of sets over $\mathbf{C}$. It is a stack if and only if it is a sheaf. This comes from the very definition of a sheaf [**1**], 4.3.1, and the fact that a map of sets is a bijection if and only if it is an equivalence, seen as a functor between discrete categories. In particular, if the topology is subcanonical, the fibered categories $\mathbf{C}_{/X}$ defined in B.1.10 are stacks.

It is possible to use the alternative descriptions of $\mathrm{Desc}(\mathcal{U}/X, \mathbf{F})$ given in Proposition B.2.2 in order to rephrase the definition of a stack in more explicit terms.

B.2.6. COROLLARY. *Let $\mathbf{F}$ be a fibered category on a site $\mathbf{C}$. After a choice of pullbacks, for each covering $\mathcal{U} = \{f_i \colon U_i \to X\}$ of X, the functor $\mathbf{F}(X) \to$ $\mathrm{Desc}(\mathcal{U}/X, \mathbf{F})$ can be described in the following way. Using the description of Proposition B.2.2 (2), it associates to an object ξ in $\mathbf{F}(X)$ the collection*

$$(\{f_i^*\xi\}, \{id_{(f_i\pi_i)^*\xi} = id_{(f_j\pi_j)^*\xi}\}).$$

The action on arrows is obvious.

B.2.7. COROLLARY. *Let $\mathbf{F}$ be a fibered category on a site $\mathbf{C}$ with a choice of pullbacks.*

(1) The fibered category $\mathbf{F}$ is a prestack if and only if for any object X of $\mathbf{C}$, and any ξ, ξ' in $\mathbf{F}(X)$, the functor

$$\begin{aligned} \underline{\mathrm{Hom}}_X(\xi, \xi') \colon \mathbf{C}_{/X} &\to \mathbf{Set} \\ (f \colon Y \to X) &\mapsto \mathrm{Hom}_{\mathbf{F}(Y)}(f^*\xi, f^*\xi') \end{aligned}$$

is a sheaf with respect to the comma topology of $\mathbf{C}_{/X}$.
(2) The fibered category $\mathbf{F}$ is a stack if and only if it is a prestack and for any covering $\mathcal{U} = \{U_i \to X\}$ and any collection of diagrams that forms an object of $\mathrm{Desc}(\mathcal{U}, \mathbf{F})$ as described in Proposition B.2.2 (1), there exists an object ξ of $\mathbf{F}(X)$ that completes all diagrams of the collection into Cartesian cubes, as the bottom right vertex.

B.2.8. DEFINITION. Let $\mathbf{F}$ be a fibered category over a site $\mathbf{C}$ together with a choice of pullbacks. Let also $f \colon A \to B$ be an arrow of $\mathbf{C}$, and let $\mathcal{U} = \{U_i \to B\}$ be a covering in $\mathbf{C}$. By abuse of notation, we indicate with f^* the two functors induced by f:

$$\begin{aligned} f^*\colon \mathbf{F}(B) &\to \mathbf{F}(A) \\ \alpha &\mapsto f^*(\alpha) \\ f^*\colon \operatorname{Desc}(\mathcal{U}, \mathbf{F}) &\to \operatorname{Desc}(\mathcal{U} \times_B A, \mathbf{F}) \\ (\alpha_i, \sigma_{ij}) &\mapsto (p_i^* \alpha_i, p_{ij}^* \varphi_{ij}) \end{aligned}$$

where $\mathcal{U} \times_B A := \{p_i\colon U_i \times_B A \to A\}$, and p_{ij} is the composite map

$$p_{ij}\colon (U_i \times_B A) \times_A (U_j \times_B A) \cong (U_i \times_B U_j) \times_B A \to U_i \times_B U_j.$$

Up to now, we only introduced definitions. It is time to display some real theory. In the case of sheaves, one important tool is the sheafification of a presheaf. This construction generalizes also in this context.

B.2.9. Theorem ([**5**], Stacks 8.1). *Let $\mathbf{F}, p_{\mathbf{F}}$ be a fibered category over a site $\mathbf{C}$. There exists a stack $\tilde{\mathbf{F}}, p_{\tilde{\mathbf{F}}}$ and a morphism of fibered categories $F\colon \mathbf{F} \to \tilde{\mathbf{F}}$, which is universal in the sense of 2-categories. Also, $\tilde{\mathbf{F}}$ is such that*

(1) for any object X of $\mathbf{C}$, and any ξ, ξ' in $\mathbf{F}(X)$, the arrow of presheaves

$$\underline{\operatorname{Hom}}_X(\xi, \xi') \to \underline{\operatorname{Hom}}_X(F\xi, F\xi')$$

is a sheafification, and

(2) for any object X of $\mathbf{C}$, and any $\tilde{\xi}$ in $\tilde{\mathbf{F}}(X)$, there exists a covering of X $\{f_i\colon U_i \to X\}$ such that for every i, the object $f_i^*\tilde{\xi}$ lies in the essential image of the functor $F_{U_i}\colon \mathbf{F}(U_i) \to \tilde{\mathbf{F}}(U_i)$.

B.3. Examples of stacks

We still have to prove the theorems which were referred to in Propositions 1.2.12, 2.2.17 and 1.2.14. This is the main aim of this section.

Gluing lemma and subcanonicality can be summarized in the following theorem.

B.3.1. Theorem. *The pseudo-functor*

$$\begin{aligned} \mathbf{Sch} &\to \mathbf{Cat} \\ (X \to Y) &\mapsto (\times_Y X\colon \mathbf{LRS}_{/Y} \to \mathbf{LRS}_{/X}) \end{aligned}$$

defines a stack with respect to the Zariski topology, and its sub-pseudo-functor

$$\begin{aligned} \mathbf{Sch} &\to \mathbf{Cat} \\ (X \to Y) &\mapsto (\times_Y X\colon \mathbf{Sch}_{/Y} \to \mathbf{Sch}_{/X}) \end{aligned}$$

defines a sub-stack.

Proof. This is the fibered category of Example B.1.7, restricted to the full subcategory **Sch** of **LRS**. We show initially that it is a prestack, i.e. that for a given covering $\{U_i \to X\}$ of an object in $\mathbf{C}$ and two elements A, B over it, any arrow between the descent data $\{A_i := A \times_X U_i\}$, $\{B_i := B \times_X X_i\}$ comes from a unique arrow $A \to B$ in $\mathbf{Ar}(\mathbf{C})(X)$.

Because $\{U_i \to U\}$ is a covering, also $\{A_i \to A\}$ and $\{B_i \to B\}$ are coverings. By inspection, one can prove that the composite maps $A_i \to B_i \to B$ coincide at the level of $A_{ij} \cong A_i \times A_j$. Because the topology is subcanonical, h_B is a sheaf. Therefore, the maps $A_i \to B$ induce a unique map $A \to B$. We are left to prove that such a map commutes with the projections to X. This comes from the fact h_X

is a sheaf, and that both maps $A \to Y \to X$, $A \to X$ restrict to the maps $A_i \to X$ with respect to the covering $\{A_i \to A\}$.

We are left to prove that any descent data is isomorphic to the data associated to an object over X. Let $\{A_i \to U_i\}$ be a descent data. The arrow $A_i \times_{U_i} U_{ij} \to A_i$ is an open immersion. Also the arrow $A_i \times_{U_i} U_{ij} \xrightarrow{\sim} A_j \times_{U_j} U_{ij}$ is an open immersion, being the composite of an isomorphism and an open immersion. By the Gluing lemma and by calling A_{ij} the object $A_i \times_{U_i} U_{ij}$, the coequalizer A of the following diagram

$$\coprod A_{ij} \rightrightarrows \coprod A_i \to A$$

lies in **Sch** whenever each A_i does. Also, the arrows $A_i \to U_i \to U$ naturally define a unique map $A \to U$. Hence, we are left to prove that the squares

$$\begin{tikzcd} A_i \arrow[r] \arrow[d] & A \arrow[d] \\ U_i \arrow[r] & U \end{tikzcd}$$

are Cartesian, so that the descent data $\{A_i\}$ is isomorphic to the one induced by the object A. This is clear from the explicit construction of the map $f \colon A \to U$, which is such that $f^{-1}(U_i) = A_i$. □

We remark that in the first part of the previous proof, we only used the fact that the topology is subcanonical.

We make some other remarks about descent criteria which are difficult to find in the literature, but are widely (and sometimes subtly) used in the proofs. The following proposition illustrates some common tricks of stack theory, which enable us to consider only singleton coverings for many purposes.

B.3.2. PROPOSITION. *Let* $\mathbf{C}$ *be a site and* $\mathbf{F}$ *be a fibered category over it that satisfy the following properties.*

- *(i) The category* $\mathbf{C}$ *is distributive, in the sense that fibered products distribute over coproducts.*
- *(ii) The category* $\mathbf{C}$ *is compatible with sums, in the sense that for all collections of objects* $\{U_i\}$*, the collection* $\{U_i \to U := \coprod U_i\}$ *is a covering and* $U_i \times_U U_j = U_i$ *whenever* $i = j$ *and* $\mathbf{F}(U_i \times_U U_j)$ *is the trivial category whenever* $i \neq j$*.*
- *(iii) The fibered category satisfies descent for the coverings of* $\coprod U_i$ *of the kind described in (ii), in the sense that the functor* $\mathbf{F}(\coprod U_i) \to \mathrm{Desc}(\{U_i \to \coprod U_i\}, \mathbf{F})$ *is an equivalence for all collections of objects* $\{U_i\}$*.*

Then the following properties hold.

- *(1) For a fixed covering* $\mathcal{U} = \{U_i \to X\}$*, there is an equivalence of categories*

$$\mathrm{Desc}(\mathcal{U}/X, \mathbf{F}) \xrightarrow{\sim} \mathrm{Desc}\left(\{U \to X\}/\, X, \mathbf{F}\right)$$

where $U = \coprod U_i$ *and the second category is defined as if the single-object set* $\{f \colon U \to X\}$ *formed a covering of* X *in* $\mathbf{C}$*.*

- *(2) The fibered category* $\mathbf{F}$ *is a prestack [resp. stack] if and only if the functor* $f^\star \colon \mathbf{F}(X) \to \mathrm{Desc}(\{U \to X\}, \mathbf{F})$ *is fully faithful [resp. is an equivalence] for all coverings* $\mathcal{U}$*.*

PROOF. Let us prove the first point. We first define a functor from the left hand side to the right hand side. Consider a descent data $\{A_i, \sigma_{ij}\}$ of $\mathrm{Desc}(\mathcal{U}/X, \mathbf{F})$.

Because of the second property, any collection $\{A_i\}$ with $A_i \in \mathbf{F}(U_i)$ defines a descent data in $\mathrm{Desc}(\{U_i \to U\}/U, \mathbf{F})$. Hence, because $\mathbf{F}$ respects descent with respect to the covering $\{U_i \to U\}$, the objects $\{A_i\}$ define an object A in $\mathbf{F}(U)$. Because of the distributivity property, also the collection of natural maps $\{(U_i \times_X U_j) \to \coprod (U_i \times_X U_j)\}$ defines a covering of $\coprod (U_i \times_X U_j) \cong U \times_X U$. In particular, because of descent with respect to this trivial covering, the arrows σ_{ij} glue together to form an isomorphism σ from $\pi_1^* A$ to $\pi_2^* A$. The object (A, σ) is in $\mathrm{Desc}(\{U \to X\}, \mathbf{F})$, as wanted. The uniqueness properties related to descent prove that this functor is an equivalence of categories, hence the first claim. The second claim is an immediate corollary of the first. □

B.3.3. Definition. Under the hypothesis of the previous proposition, we indicate with $\mathrm{Desc}(U \to X, \mathbf{F})$ or just with $\mathrm{Desc}(U \to X)$ the category $\mathrm{Desc}(\{U \to X\}, \mathbf{F})$. Also, we indicate with $\mathrm{Hom}_{U \to X}(\alpha, \beta)$ the set of morphisms between two descent data α, β in the category $\mathrm{Desc}(\{U \to X\}, \mathbf{F})$.

The previous proposition outlines a reason why many authors only consider one-element coverings when proving descent theorems. In most cases, this process is just a restriction to a particular family of coverings. Indeed, for many "geometrical" topologies (say, the étale topology), if $\{U_i \to X\}$ is a covering, then also $\{\coprod U_i \to X\}$ is a (one-element) covering.

B.3.4. Proposition. *Let $\mathbf{C}$ be a site and $\mathbf{F}$ be a fibered category over it. Suppose that for all coverings $\mathcal{U}$, the functor $f^\star \colon \mathbf{F}(X) \to \mathrm{Desc}(\mathcal{U}, \mathbf{F})$ has a right [left] adjoint h, and that by denoting with $\mathcal{U}_i$ the lifted covering $\{U_j \times_X U_i \to U_i\}$, the two composite functors (see B.2.8)*

$$\mathrm{Desc}(\mathcal{U}, \mathbf{F}) \xrightarrow{h} \mathbf{F}(X) \xrightarrow{f_i^*} \mathbf{F}(U_i)$$

$$\mathrm{Desc}(\mathcal{U}, \mathbf{F}) \xrightarrow{f_i^*} \mathrm{Desc}(\mathcal{U}_i, \mathbf{F}) \xrightarrow{h'} \mathbf{F}(U_i)$$

are isomorphic, where h' is the right adjoint of the functor $f_i^\star \colon \mathbf{F}(U_i) \to \mathrm{Desc}(\mathcal{U}_i, \mathbf{F})$. Suppose also that one of the two following hypothesis hold.

(i) For all coverings $\{f_i \colon U_i \to X\}$, the collection $\{f_i^* \colon \mathbf{F}(X) \to \mathbf{F}(U_i)\}$ is conservative (i.e. a map τ in $\mathbf{F}(X)$ is an isomorphism if and only if $f_i^*(\tau)$ is an isomorphism for all i).

(ii) The functor h is a left inverse for $f^\star$, i.e. the natural map $id \to hf^\star$ [$hf^\star \to id$] is an isomorphism.

Then $\mathbf{F}$ is a stack.

Proof. We only discuss the case in which h is a right adjoint, being the other case completely analogous to this one. In case the collection $\{f_i^*\}$ is conservative, then also $f^\star$ is conservative. In case h is a left inverse for $f^\star$, then $f^\star$ is automatically conservative since if $f^\star A \to f^\star B$ is an isomorphism, then $A \cong hf^\star A \to hf^\star B \cong B$ is an isomorphism. Therefore, we are left to prove that the unit of the adjunction $f^\star h \to id$ is an isomorphism. Indeed, if $f^\star hM \cong M$ for all objects M in $\mathbf{F}(X)$, then also $f^\star hf^\star hM \cong f^\star M$. Hence, the map $f^\star M \to f^\star hf^\star hM$ is an isomorphism. Because $f^\star$ is conservative, this implies that $M \to hf^\star M$ is an isomorphism, as wanted.

We remark that descent is easily proven with respect to the coverings of the form $\mathcal{U}_i = \{U_j \times_X U_i \to U_i\}$. Indeed, each map $A \to U_i$ splits over $U_i \times_X U_i$, hence the functor $h_{\mathcal{U}_i}$ equals h_{U_i}, which implies the claim because $\mathbf{F}(U_i) \cong \mathrm{Hom}(h_{U_i}, \mathbf{F})$ and $\mathrm{Desc}(\mathcal{U}_i, \mathbf{F}) = \mathrm{Hom}(h_{\mathcal{U}_i}, \mathbf{F})$.

Consider now a descent data $\alpha := (A_i, \sigma_{ij})$ with respect to $\mathcal{U}$, and the descent data $f^\star h\alpha$. We can pull them back via f_i^*, obtaining two descent data $f_i^*\alpha$ and $f_i^* f^\star h\alpha$ in $\mathrm{Desc}(\mathcal{U}_i, \mathbf{F})$. Descent with respect to $\mathcal{U}_i$ implies that $f_i^*\alpha$ is isomorphic to $f_i^\star h'(f_i^*\alpha)$. Because of the hypothesis, $f_i^* h\alpha$ is isomorphic to $h' f_i^*\alpha$, where h' is the adjoint of $f_i^\star$. Also, from the definition of the pullback f_i^* on descent data (B.2.8), we have that $f_i^* f^\star \cong f_i^\star f_i^*$. We then conclude the following sequence of isomorphisms.

$$f_i^*(f^\star h\alpha) \cong (f_i^\star f_i^*)h\alpha \cong (f_i^\star h')f^*\alpha \cong f_i^*\alpha$$

Hence, $f_i^*\alpha$ and $f_i^* f^\star h\alpha$ are isomorphic, for all i.

In case the first hypothesis holds, then also $f_i^*\colon \mathrm{Desc}(\mathcal{U}, \mathbf{F}) \to \mathrm{Desc}(\mathcal{U}_i, \mathbf{F})$ is conservative, hence that α is isomorphic to $f^\star h\alpha$, as wanted.

We now suppose the second hypothesis holds. Fix an index i, and consider the map $\alpha_i \to f^\star h\alpha_i$ in $\mathbf{F}(U_i)$ induced by the map of the descent data. By what we just proved, this map is sent to an isomorphism via all pullback maps $(U_i \times_X U_j \to U_i)^*$, hence by $f_i^\star$. Because $f_i^\star$ is conservative, we conclude that $\alpha_i \to f^\star h\alpha_i$ is an isomorphism. This holds for all i, hence we conclude the claim. □

We now present another proof of Theorem B.3.1. This translates the final part of the proof in a more abstract language. It is not useful on its own (the previous proof is way clearer in its context), but it lets us gain more familiarity with the standard tricks of descent theorems.

Alternative proof of Theorem B.3.1. We first prove that $\mathbf{Ar}$ is a stack. The three conditions of Proposition B.3.2 are easily checked. For the second one, in particular, we have $U_i \times_U U_j = \emptyset$, and $\mathbf{Ar}(\emptyset)$ is constituted only of the arrow $\emptyset \to \emptyset$. Hence, we can pretend to have a covering $\{f\colon U \to X\}$ constituted by a single element, and check descent only on it. We will use the criterion of Proposition B.3.4. Let $a\colon A \to U$ be an object of $\mathbf{Ar}(U)$ and let $\sigma\colon \pi_1^*A \to \pi_2^*A$ be an isomorphism in $\mathbf{Ar}(U \times_X U)$. In particular, by calling a_1 and a_2 the projections to $U \times_X U$ of π_1^*A and π_2^*A respectively, we have $a_1 = a_2\sigma$. We will call p_1 and p_2 the natural maps from π_1^*A and π_2^*A to A, respectively. We now use the criterion of the last point of the previous proposition, and define a functor h which is a left adjoint of $f^\star$. For an object $A \to U$ associated to a descent data, we define $h(A)$ to be the coequalizer of the following diagram

$$\pi_1^*A \rightrightarrows A \to h(A)$$

where the two arrows are the natural projection p_1, and the composite $p_2\sigma$. The object $h(A)$ lies over X. Indeed, by letting g be the arrow $g = f\pi_1 = f\pi_2\colon U \times_X U \to X$, we have

$$(fa)p_1 = fap_1 = f\pi_1 a_1 = f\pi_2 a_1 = f\pi_2 a_2\sigma = fap_2\sigma = (fa)p_2\sigma.$$

hence a natural arrow $h(A) \to X$. If we prove that $\mathbf{Ar}$ defines a stack via this adjoint couple, then the fact that also the subcategory of schemes is a stack follows. Indeed, from the fact that schemes are closed under open gluing (Gluing Lemma), we conclude that h is still well defined.

We now prove that h is a left adjoint for $f^\star$. Suppose to have a map $b\colon B \to X$ and a map in $\mathrm{Hom}_{U\to X}(A, f^\star B)$. In particular, we have a map $\varphi\colon A \to f^*B$ over U, which induces a map $\psi\colon A \to B$ over X. Analogously, a map $\psi\colon h(A) \to B$ over

X induces a map $\varphi\colon A \to f^*B = B \times_X U$ over U, because of the commutativity of the following diagram

$$\begin{tikzcd} A \arrow[r] \arrow[d] & h(A) \arrow[r] \arrow[d] & B \arrow[dl] \\ U \arrow[r] & X & \end{tikzcd}$$

(we are just using the fact that f^* is the right adjoint of the functor $f_*\colon \mathbf{C}_{/U} \to \mathbf{C}_{/X}$ induced by composition with f). It is easy to see that the compatibility of φ with descent is equivalent to ψ coequalizing the two maps $\pi^*A \rightrightarrows A$, hence we have a natural bijection $\mathrm{Hom}_X(h(A), B) \cong \mathrm{Hom}_{U\to X}(A, f^\star B)$, as claimed.

The fact that $hf^\star$ is an isomorphism on each object comes from the fact that the topology is subcanonical. Indeed, by letting $\bar{A}$ be an object over X, we know that $\bar{A} \times_X U \to \bar{A}$ is a covering and $\bar{A} \times_X U \times_X U \cong (\bar{A} \times_X U) \times_{\bar{A}} (\bar{A} \times_X U)$. Hence, because the topology is subcanonical, we conclude the exactness of all sequences of the following kind

$$\mathrm{Hom}(Y, \bar{A}) \to \prod \mathrm{Hom}(Y, \bar{A} \times_X U) \rightrightarrows \mathrm{Hom}(Y, \bar{A} \times_X U \times_X U)$$

by letting Y vary in $\mathbf{C}$. Hence, by the definition of the colimit, we deduce that also the following sequence is exact

$$\coprod \bar{A} \times_X U \times_X U \rightrightarrows \coprod \bar{A} \times_X U \to \bar{A}$$

as wanted.

Using again subcanonicality, one can prove the compatibility of h with h' (see the notation of B.3.4). Indeed, we can pull back the covering $\{U \times_X U \xrightarrow{\pi_1} U\}$ to $h(A) \times_X U$, obtaining a coequalizing diagram of the following kind

$$h(A) \times_X U \times_X U \times_X U \rightrightarrows h(A) \times_X U \times_X U \to h(A) \times_X U$$

which implies the compatibility condition.

The claim then follows from Proposition B.3.4. $\square$

For a subcanonical site $\mathbf{C}$, Yoneda's lemma implies that the fibered category $\mathbf{Ar}(\mathbf{C})$ is a subcategory of the following one

$$\mathbf{C} \to \mathbf{Cat}$$
$$(f\colon X \to Y) \mapsto (\times_Y X\colon\ \mathbf{Sh}(\mathbf{C})_{/Y} \to \mathbf{Sh}(\mathbf{C})_{/X})$$

where $\mathbf{Sh}(\mathbf{C})_{/X}$ is the category of sheaves over h_X. It is then natural to ask whether this whole fibered category defines a stack. We consider a specific example of this setting.

B.3.5. LEMMA. *Let $\mathbf{C}$ be a closed symmetric monoidal category with all small limits and colimits, and let $\mathbf{Aff}_{\mathbf{C}}$ be the category of affine schemes relative to $\mathbf{C}$ (see 2.2.12), endowed with the Zariski topology (see 2.2.13), and let $\mathbf{Sch}_{\mathbf{C}}$ be the category of schemes over $\mathbf{C}$ (see 2.2.18). Suppose that $\{\mathcal{F}_i \to \mathcal{F}\}$ is a Zariski covering of $\mathcal{F}$ and that $\mathcal{G} \to \mathcal{F}$ is an arrow in $\mathbf{Sch}_{\mathbf{C}}$ such that $\mathcal{G}_i := \mathcal{G} \times_{\mathcal{F}} \mathcal{F}_i$ is a scheme for all i. Then $\mathcal{G}$ is a scheme.*

PROOF. Let's consider affine Zariski coverings $\{h_{\mathrm{Spec}\, A_{ij}} \to \mathcal{G}_i\}$ of each $\mathcal{G}_i$. Because Zariski open immersions are stable under pullbacks, each map $\mathcal{G}_i \to \mathcal{F}$ is open. Hence the composite map $\mathcal{G}_{ij} \to \mathcal{G}_i \to \mathcal{F}$ is open. Also, because of Lemmas 1.2.5 and 1.2.7, the map $\coprod \mathcal{G}_i \to \mathcal{F}$ is an epimorphism. Therefore, the map $\coprod \mathcal{G}_{ij} \to \mathcal{F}$ is an epimorphism. This proves that $\{\mathcal{G}_{ij} \to \mathcal{F}\}$ is a Zariski covering of $\mathcal{F}$, hence the claim. $\square$

B.3.6. Theorem. *Let* $\mathbf{C}$ *be a closed symmetric monoidal category with all small limits and colimits, and let* $\mathbf{Aff}_{\mathbf{C}}$ *be the category of affine schemes relative to* $\mathbf{C}$ *(see 2.2.12), endowed with the Zariski topology (see 2.2.13), and let* $\mathbf{Sh}$ *be the category of sheaves over it. The pseudo-functor*

$$\mathbf{Aff}_{\mathbf{C}} \to \mathbf{Cat}$$
$$(f\colon \operatorname{Spec} B \to \operatorname{Spec} A) \mapsto (f^*\colon \mathbf{Sh}_{/h_{\operatorname{Spec} A}} \to \mathbf{Sh}_{/h_{\operatorname{Spec} B}})$$

defines a stack, and its sub-pseudo-functor

$$\mathbf{Aff}_{\mathbf{C}} \to \mathbf{Cat}$$
$$(f\colon \operatorname{Spec} B \to \operatorname{Spec} A) \mapsto (f^*\colon \mathbf{Sch}_{\mathbf{C}/h_{\operatorname{Spec} A}} \to \mathbf{Sch}_{\mathbf{C}/h_{\operatorname{Spec} B}})$$

defines a sub-stack, where $\mathbf{Sch}_{\mathbf{C}}$ *is the category of schemes over* $\mathbf{C}$ *(see 2.2.18).*

Proof. We start from the first claim. Let $\operatorname{Spec} A$ be an affine scheme, with a covering $\{\operatorname{Spec} A_i \to \operatorname{Spec} A\}$. As always, we indicate with $\operatorname{Spec} A_{ij}$ the fibered products $\operatorname{Spec} A_i \times_{\operatorname{Spec} A} \operatorname{Spec} A_j$. We prove the claim exactly in the same way as Theorem B.3.1, considering $\mathbf{Aff}_{\mathbf{C}}$ as embedded in $\mathbf{Sh}$ via the Yoneda embedding (see Theorem 2.2.15, which is proven independently). The only difference here is that we can't use subcanonicality (we haven't proved subcanonicality of the Zariski topology in the whole of $\mathbf{Sh}$). However, in order to prove that for a sheaf $\mathcal{F}$ over $h_{\operatorname{Spec} A}$ the sequence

$$\coprod (\mathcal{F} \times_{h_{\operatorname{Spec} A}} h_{\operatorname{Spec} A_{ij}}) \rightrightarrows \coprod (\mathcal{F} \times_{h_{\operatorname{Spec} A}} h_{\operatorname{Spec} A_i}) \to \mathcal{F}$$

is exact, we can base-change with $\mathcal{F}$ the epimorphism $\coprod h_{\operatorname{Spec} A_i} \to h_{\operatorname{Spec} A}$, and use Lemmas 1.2.5, 1.2.6, 1.2.7.

We now turn to the second claim. Because the fibered category we are considering is a subcategory of the one of the first point, we are left to prove the following two properties:

(i) If $Y \to X$ is an arrow of $\mathbf{Aff}_{\mathbf{C}}$ and if $\mathcal{F} \to X$ is a scheme over X, then $\mathcal{G} := \mathcal{F} \times_X Y$ is a scheme over Y.
(ii) If $\{X_i \to X\}$ is a covering in $\mathbf{Aff}_{\mathbf{C}}$ and if $\mathcal{F} \to X$ is a sheaf over X such that each $\mathcal{F}_i := \mathcal{F} \times_X X_i$ is a scheme, then also $\mathcal{F}$ is a scheme.

We start by the first point. Suppose that the epimorphism $\{\mathcal{F}_i \to \mathcal{F}\}$ is an affine covering of $\mathcal{F}$. Then each $\mathcal{G}_i := \mathcal{F}_i \times_X Y$ is affine. Because of Lemmas 1.2.5, 1.2.7 and the definition of Zariski open immersions, the collection $\{\mathcal{G}_i \to \mathcal{G}\}$ is a Zariski covering, hence $\mathcal{F}$ is a scheme, as wanted. The second point is a corollary of the previous lemma. $\square$

The previous proposition and Proposition 2.2.19 can hence be summarized in the following result.

B.3.7. Corollary. *Let* $\mathbf{C}$ *be a closed symmetric monoidal category with all small limits and colimits, and let* $\mathbf{Aff}_{\mathbf{C}}$ *be the category of affine schemes relative to* $\mathbf{C}$ *(see 2.2.12), endowed with the Zariski topology (see 2.2.13), and let* $\mathbf{Sch}_{\mathbf{C}}$ *be* *the category of schemes over* $\mathbf{C}$ *(see 2.2.18). The pseudo-functor*

$$\begin{gathered}\mathbf{Sch}_{\mathbf{C}} \to \mathbf{Cat}\\ (f\colon \mathcal{F} \to \mathcal{G}) \mapsto (f^*\colon\ \mathbf{Sch}_{\mathbf{C}/\mathcal{G}} \to \mathbf{Sch}_{\mathbf{C}/\mathcal{F}})\end{gathered}$$

defines a stack.

PROOF. The proof is analogous to the proof of the previous proposition. Notice that the fact that the category $\mathbf{Sch}_{\mathbf{C}}$ is closed under fibered products comes from 2.2.19. □

We also inspect the case of another fibered category related to sheaves over a site. A map $f\colon X \to Y$ in $\mathbf{C}$ induces a functor $\mathbf{C}_{/Y} \to \mathbf{C}_{/X}$, defined in the usual way, taking fibered products. This functor is continue, in the sense that it respects the comma topology induced in the two comma categories. Hence, we have a functor $f_*\colon\ \mathbf{Sh}(\mathbf{C}_{/X}) \to \mathbf{Sh}(\mathbf{C}_{/Y})$ that sends a sheaf $F \in \mathbf{Sh}(\mathbf{C}_{/X})$ to a sheaf f_*F that acts in the following way:

$$f_*F(A \to Y) = F(A \times_Y X \to X).$$

This functor has a left adjoint f^*, which sends a sheaf $F \in \mathbf{Sh}(\mathbf{C}_{/Y})$ to the sheaf f_*F that acts in the following way:

$$f^*F(A \to X) = F(A \to X \to Y).$$

B.3.8. THEOREM. *Let* $\mathbf{C}$ *be a complete category with a Grothendieck topology, and let* $\mathbf{F}$ *be the fibered category associated to the pseudo-functor*

$$\begin{gathered}\mathbf{C} \to \mathbf{Cat}\\ (f\colon X \to Y) \mapsto (f^*\colon\ \mathbf{Sh}(\mathbf{C}_{/Y}) \to \mathbf{Sh}(\mathbf{C}_{/X}))\end{gathered}$$

If $\mathbf{C}$, $\mathbf{F}$ *satisfy the hypothesis of Proposition B.3.2, then* $\mathbf{F}$ *is a stack.*

PROOF. Under these hypothesis, we can check descent only for one-map coverings $f\colon U \to X$ (see B.3.2). We indicate with π_1, π_2 the two projections $U \times_X U \rightrightarrows U$, and with g the composite map $f\pi_1 = f\pi_2$. We also notice that the functor $f^\star$ has a right adjoint h. This maps an object $(\mathcal{F} \in \mathbf{Sh}(\mathbf{C}_{/U}), \sigma\colon\ \pi_1^*\mathcal{F} \to \pi_2^*\mathcal{F})$ to the sheaf that equalizes the following diagram, in the category $\mathbf{Sh}(\mathbf{C}_{/X})$:

$$h(\mathcal{F}) \to f_*\mathcal{F} \rightrightarrows g_*\pi_1^*\mathcal{F}$$

where one map is defined as the composite (we indicate with ε_1 and ε_2 the unit maps of the adjoint couples π_1^*, π_{1*} and π_2^*, π_{2*} respectively)

$$f_*\mathcal{F} \xrightarrow{\varepsilon_1} f_*\pi_{1*}\pi_1^*\mathcal{F} \cong g_*\pi_1^*\mathcal{F}$$

while the other is defined as the composite

$$f_*\mathcal{F} \xrightarrow{\varepsilon_2} f_*\pi_{2*}\pi_2^*\mathcal{F} \xrightarrow{\sigma} g_*\pi_1^*\mathcal{F}.$$

In order to see that h is a right adjoint for $f^\star$, notice that any map $\varphi\colon \mathcal{G} \to f_*\mathcal{F}$ in $\mathbf{F}(X)$ induces a map $\psi\colon f^*\mathcal{G} \to \mathcal{F}$ in $\mathbf{F}(U)$, and vice versa. Also, the cocycle condition for ψ and the equalizing property of φ with respect to the previous diagram boil down to a commutativity check for these two diagrams (we make the identification $\pi_1^* f^* = \pi_2^* f^* = g^*$).

$$\begin{tikzcd} & \pi_1^*\mathcal{F} \arrow[dd, "\sigma"] \\ g^*\mathcal{G} \arrow[ur, "\pi_1^*\psi"] \arrow[dr, "\pi_2^*\psi"'] & \\ & \pi_2^*\mathcal{F} \end{tikzcd} \qquad \begin{tikzcd} & g_*\pi_1^*\mathcal{F} \arrow[dd, "g_*\sigma"] \\ \mathcal{G} \arrow[ur, "\varepsilon_1\varphi"] \arrow[dr, "\sigma\varepsilon_2\varphi"'] & \\ & g_*\pi_2^*\mathcal{F} \end{tikzcd}$$

The commutativity of one of the two diagrams implies the commutativity of the other, applying the functors g^* or g_* and the unit/counit of the adjunction. Therefore, there is a natural bijection between $\mathrm{Hom}_{\mathbf{F}(X)}(\mathcal{G}, h\mathcal{F})$ and $\mathrm{Hom}_{U\to X}(f^\star\mathcal{G}, \mathcal{F})$, as wanted.

We now prove that $hf^\star$ is isomorphic to the identity functor. We have to prove that the equalizer of the diagram

$$f_*(f^*\mathcal{F}) \rightrightarrows g_*\pi_1^*(f^*\mathcal{F}) \cong g_*g^*\mathcal{F}$$

is $\mathcal{F}$ itself. In the category of sheaves, limits can be computed componentwise, hence we are left to prove that for any arrow $Z \to X$, the following diagram is an equalizing diagram.

$$\mathcal{F}(Z \to X) \to f_*f^*(\mathcal{F})(Z \to X) \rightrightarrows (g_*g^*\mathcal{F})(Z \to X).$$

Using the definitions of pullbacks and push-forward, one can see that the previous diagram is equivalent to the diagram

$$\mathcal{F}(Z \to X) \to \mathcal{F}(Z \times_X U \to X) \rightrightarrows \mathcal{F}(Z \times_X U \times_X U \to X)$$

which is an equalizing diagram because $\mathcal{F}$ is a sheaf, $Z \times_X U \to Z$ is a covering for Z, and $(Z \times_X U) \times_Z (Z \times_X U) \cong Z \times_X U \times_X U$. Using similar techniques, one can check compatibility of h with h' (see B.3.4). The claim then follows from Proposition B.3.2. □

We end this chapter with the (arguably) most important result.

B.3.9. LEMMA. *Let* $\mathbf{C}$ *be a closed symmetric monoidal category with all small limits and colimits, and let* $\mathbf{Aff_C}$ *be the category of affine schemes relative to* $\mathbf{C}$ *(see 2.2.12). For* $\operatorname{Spec} A$ *in* $\mathbf{Aff_C}$*, let also* A-$\mathbf{Mod}$ *be the category of* A*-modules (see 2.2.2) and for a map* $f\colon \operatorname{Spec} B \to \operatorname{Spec} A$*, we indicate with* f_**,* f^* *the adjoint couple defined between* A-$\mathbf{Mod}$ *and* B-$\mathbf{Mod}$*. For a Cartesian square in* $\mathbf{Aff_C}$

$$\begin{tikzcd} V \arrow[r, "f'"] \arrow[d, "p'"'] & Y \arrow[d, "p"] \\ U \arrow[r, "f"] & X \end{tikzcd}$$

the natural map of functors $f^*p_* \to p'_*f'^*$ *is an isomorphism.*

PROOF. Considering the category $\mathbf{Mon_C}$, we are left to prove that for a co-cartesian diagram

$$\begin{tikzcd} A \arrow[r] \arrow[d] & B \arrow[d] \\ C \arrow[r] & B \otimes_A C \end{tikzcd}$$

and a B-module M, the natural map of C-modules (we omit forgetful functors)

$$M \otimes_A (B \otimes_A C) \to M \otimes_B (B \otimes_A C)$$

is an isomorphism. This comes from Proposition 2.2.11 and Corollary 2.2.6. $\square$

B.3.10. THEOREM. *Let $\mathbf{C}$ be a closed symmetric monoidal category with all small limits and colimits, and let $\mathbf{Aff_C}$ be the category of affine schemes relative to $\mathbf{C}$, endowed with the Zariski topology. For $\operatorname{Spec} A$ in $\mathbf{Aff_C}$, let also A-$\mathbf{Mod}$ be the category of A-modules. The pseudo-functor*

$$\mathbf{Aff_C} \to \mathbf{Cat}$$
$$(\operatorname{Spec} B \to \operatorname{Spec} A) \mapsto (\otimes_A B\colon A\text{-}\mathbf{Mod} \to B\text{-}\mathbf{Mod})$$

defines a stack with respect to the Zariski topology, where $\otimes_A B$ is defined as in 2.2.3.

PROOF. This theorem is a corollary of Proposition B.3.4. Because the topology is quasi-compact, we can assume that the covering we start with is finite, say $\{f_i\colon \operatorname{Spec} B_i \to \operatorname{Spec} A\}_{i\in I}$. We will also denote with f_{ij} the map $U_i \times_X U_j \to X$. From the definition of the topology, we can also assume that the collection $\{f_i^*\}$ is conservative. We are left to define a right adjoint h of $f^\star$, and check its compatibility with the pullbacks f_i^*.

The functor $f^*\colon A$-$\mathbf{Mod} \to B$-$\mathbf{Mod}$ has a left adjoint (2.2.4), which we will call $f_*\colon B$-$\mathbf{Mod} \to A$-$\mathbf{Mod}$. We can then define a right adjoint functor h of the conservative functor $f^\star$, hugely inspired by the proof of B.3.8. For a descent data $N = (N_i, \sigma_{ij})$, we define $h(N)$ to be the equalizer in the category A-$\mathbf{Mod}$ of the following diagram

$$\prod f_{i*}N_i \rightrightarrows \prod f_{ij*}\pi_i^* N_i$$

where one map is defined as the composite (we indicate with ε_i the units of the adjoint couples π_i^*, π_{i*})

$$\prod f_{i*}N_i \to f_{i*}N_i \xrightarrow{\varepsilon_i} f_{i*}\pi_{i*}\pi_i^* N_i \cong f_{ij*}\pi_i^* N_i$$

while the other is defined as the composite

$$\prod f_{i*}N_i \to f_{j*}N_j \xrightarrow{\varepsilon_j} f_{j*}\pi_{j*}\pi_j^* N_j \xrightarrow{\sigma} f_{ij*}\pi_i^* N_i.$$

Now we prove that h is a right adjoint for $f^\star$. Any collection of maps $\varphi_i\colon M \to f_{i*}N_i$ in $\mathbf{F}(X)$ induces maps $\psi_i\colon f_i^* M \to N_i$ in $\mathbf{F}(U_i)$, and vice versa. Also, the cocycle condition for $\{\psi_i\}$ and the equalizing property of $\prod \varphi_i$ with respect to the equalizing diagram boil down to a commutativity check for these two diagrams (we make the identification $\pi_i^* f_i^* = \pi_j^* f_j^* = f_{ij}^*$).

$$\begin{tikzcd} & \pi_i^* N_i \arrow[dd, "\sigma"] \\ f_{ij}^* M \arrow[ur, "\pi_i^*\psi_i"] \arrow[dr, "\pi_j^*\psi_j"'] & \\ & \pi_j^* N_j \end{tikzcd} \qquad \begin{tikzcd} & f_{ij*}\pi_i^* N_i \arrow[dd, "f_{ij*}\sigma"] \\ M \arrow[ur, "\varepsilon_i\varphi_i"] \arrow[dr, "\sigma\varepsilon_j\varphi_j"'] & \\ & f_{ij*}\pi_j^* N_j \end{tikzcd}$$

The commutativity of one of the two diagrams implies the commutativity of the other, applying the functors f_{ij}^* or f_{ij*} and the unit/counit of the adjunction. Therefore, we conclude that there is a natural bijection between $\mathrm{Hom}_{\mathbf{F}(X)}(M, hN)$ and $\mathrm{Hom}_{\mathrm{Desc}(\mathcal{U})}(f^\star M, N)$, as wanted.

In order to prove the claim, we are then left to prove that, by denoting with $\mathcal{U}_k$ the lifted covering of U_k and with $f_k^\star$, h' the associated adjoint couple from $\mathbf{F}(U_k)$ to $\mathrm{Desc}(\mathcal{U}_k, \mathbf{F})$, we have $h' f_k^* \cong f_k^* h$. Let $N = (N_i, \sigma_{ij})$ be an element of $\mathrm{Desc}(\mathcal{U}, \mathbf{F})$. Using the definition of h and the flatness of f_k^*, we have the following sequence of isomorphisms (all products shown are finite).

$$f_k^* h(N) = f_k^* \lim \left(\prod f_{i*} N_i \rightrightarrows \prod f_{ij*} \pi_i^* N_i \right) \cong \lim \left(\prod f_k^* f_{i*} N_i \rightrightarrows \prod f_k^* f_{ij*} \pi_i^* N_i \right)$$

We now apply Lemma B.3.9 with respect to the Cartesian squares

$$\begin{tikzcd} U_i \times_X U_k \arrow[r, "\pi_i"] \arrow[d, "\pi_k"'] & U_i \arrow[d, "f_i"] \\ U_k \arrow[r, "f_k"] & X \end{tikzcd} \qquad \begin{tikzcd} U_i \times_X U_j \times_X U_k \arrow[r, "\pi_{ij}"] \arrow[d, "\pi_k"'] & U_i \times_X U_j \arrow[d, "f_{ij}"] \\ U_k \arrow[r, "f_k"] & X \end{tikzcd}$$

and we conclude the following isomorphism

$$f_k^* h(N) \cong \lim \left(\prod \pi_{k*}(\pi_i^* N_i) \rightrightarrows \prod \pi_{k*} \pi_{ij}^* (\pi_i^* N_i) \right) = h' f_k^*(N)$$

as wanted. □

We remark that in the previous proof we made no use of the finite presentation property and the epimorphism property which are part of the definition of a Zariski covering. Indeed, one can define a finer topology, just by erasing these two extra conditions. The topology defined in this way is the analogue of the fppf topology of schemes, and the proposition above proves descent for modules over affines, endowed with the fppf topology. Still, we make no use of this stronger fact. Alternative proofs can be found in [**34**], 2.5 (similar to the one we presented here), and in [**14**] 4.2.1 (which yields a similar statement for quasi-coherent sheaves on schemes over $\mathbb{Z}$).

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