NEWSLETTER

OF THE EUROPEAN MATHEMATICAL SOCIETY

Features

An Introduction to Geometric Complexity Theory

Non-Integrable Distributions and the *h*-Principle

Geometry of Polynomial Ordinary Differential Equations

Obituary

Pierre Dolbeault

Society

The London Mathematical Society's 150th Anniversary

European Mathematical Society

March 2016
Issue 99
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Journals published by the
European Mathematical Society

ISSN print 0032-5155
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2016. Vol. 73. 4 issues
Approx. 400 pages. 17.0 x 24.0 cm
Price of subscription:
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236 € print+online

A journal of the Portuguese Mathematical Society

Editor-in-Chief:
Luis Nunes Vicente (Universidade de Coimbra, Portugal)

Aims and Scope
Since its foundation in 1937, *Portugaliae Mathematica* has aimed at publishing high-level research articles in all branches of mathematics. With great efforts by its founders, the journal was able to publish articles by some of the best mathematicians of the time. In 2001 a *New Series of Portugaliae Mathematica* was started, reaffirming the purpose of maintaining a high-level research journal in mathematics with a wide range scope.

ISSN print 0034-5318
ISSN online 1663-4926
2016. Vol. 52. 4 issues
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A journal of the Research Institute for Mathematical Sciences of Kyoto University

Editor-in-Chief:
S. Mochizuki

Aims and Scope
The aim of the *Publications of the Research Institute for Mathematical Sciences* is to publish original research papers in the mathematical sciences. Occasionally surveys are included at the request of the editorial board.

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2016. Vol. 27. 4 issues
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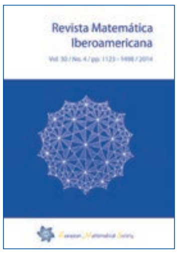
The Accademia dei Lincei (Lynx), founded in 1603, is the oldest academy dedicated to the study of humanities as well as physics, mathematics and the natural sciences in the world. In 1873 the academy began publishing with the *Atti della Reale Accademia dei Lincei, Transunti*. Since 1990 under the present name, the *Rendiconti Lincei* have been one of the best Italian journals. The journal is dedicated to the publication of high-quality peer-reviewed surveys, research papers and preliminary announcements of important results from all fields of mathematics and its applications.



ISSN print 0213-2230
ISSN online 2235-0616
2016. Vol. 32. 4 issues
Approx. 1500 pages. 17.0 x 24.0 cm
Price of subscription:
368 € online only
428 € print+online

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Aims and Scope
Revista Matemática Iberoamericana was founded in 1985 and publishes original research articles on all areas of mathematics. Its distinguished Editorial Board selects papers according to the highest standards.

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2016. Vols. 135 and 136, 1 issue each
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198 € online only
238 € print+online

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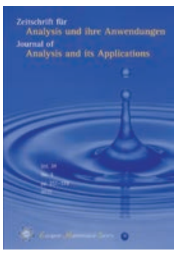
The *Rendiconti del Seminario Matematico della Università di Padova* have been published since 1930. The aim of *Rendiconti* is to publish original high-quality research articles in mathematics. By tradition the journal focuses on algebra, analysis, algebraic and differential geometry, mathematical physics and number theory. The journal is ready to accept long papers of foundational or technical nature, as well as papers containing applications or surveys, provided these are especially well structured and fill a gap in the literature.



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190 € online only
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A periodical edited by the University of Leipzig

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Cover picture: Detail from Hedral 2015, oil on canvas 214 × 274 cm by Mark Francis, commissioned by the London Mathematical Society on the occasion of its 150th anniversary (see p. 43).

EMS Agenda

2016

15–16 April
ERCOM meeting, St. Petersburg, Russia

9–11 May
Meeting of the EMS Committee for Education, Leuven, Belgium

16–17 July
EMS Council, Humboldt University, Berlin, Germany

EMS Scientific Events

2016

29 May–3 June
ESSAM School on Mathematical Modelling, Numerical Analysis and Scientific Computing
Kacov, Czech Republic

27 June–1 July
EMS-EWM Summer School on Geometric and Physical Aspects of Trudinger–Moser Type Inequalities
Institut Mittag-Leffler, Djursholm, Sweden

4–8 July
4th European Summer School in "Modelling, Analysis and Simulation: Crime and Image Processing"
Oxford, UK

4–9 July
Building Bridges: 3rd EU/US Summer School on Automorphic Forms and Related Topics
Sarajevo, Bosnia and Herzegovina

11–15 July
EMS-IAMP Summer School in Mathematical Physics on "Universality, Scaling Limits and Effective Theories"
Roma, Italy
http://www.smp2016.cond-math.it/

18–22 July
7th European Congress of Mathematics, Berlin, Germany
http://www.7ecm.de/

25–29 July
EMS-ESMTB Summer School "Mathematical Modelling of Tissue Mechanics"
Leiden, The Netherlands

17–24 August
European Summer School in Multigraded Algebra and Applications
Constanta, Romania

21–28 August
EMS-ESMTB Summer School "The Helsinki Summer School on Mathematical Ecology and Evolution 2016: Structured Populations"
Linnasmäki Congress Centre, Turku, Finland

25–26 August
Second Caucasian Mathematics Conference (CMC-II)
Lake Van, Turkey

Editorial

Lucia Di Vizio (Editor-in-Chief EMS Newsletter; Université de Versailles-St Quentin, France)

Dear Members of the EMS and Dear Readers,

Here I am with my farewell editorial. Even though I will still be Editor-in-Chief for one more issue, I feel more comfortable saying goodbye before the formal ceremony for the old lady Newsletter starts.

I have a dreadful memory of writing the editorial of my first issue. I remember Vicente Munoz, the former Editor-in-Chief, telling me: "You will get use to it!" Well, I did not get used to it. I've read the editorials of my predecessors to seek inspiration; nonetheless, I have been carefully avoiding the action of actually sitting down and writing. The reason is that I was looking for some formal and public expression of my feelings about this experience, which was neither formal nor public but so private and personal. Being Editor-in-Chief of the Newsletter means dealing with authors, editors, readers, members of the EMS and staff of the publishing house, i.e. people, and actually many more people than you can imagine. For me, the link that one establishes with other people cannot be anything other than personal, even when it is only professional. So let's talk about people.

The Newsletter would not exist without authors; they do the hard work. Although there are a minority of impolite authors, most of them are very nice and interesting – but they are not very good with deadlines. Authors have excellent reasons for that. For instance, I have received a lot (literally a lot) of "I'm on a trek in Nepal. I can hardly find an internet connection in this small village so I will be out of email contact for 2 weeks", or some variant of this sentence. I know for a fact that mountains and deserts are full of mathematicians walking around.

Some authors simply live in another spacetime. I once received an email submitting an article, the author saying with some triumphalism: "I'm two weeks in advance of the deadline!" He was 50 weeks late and he was answering a 14 month old email from me. I'm always grateful to people who make me laugh so everything was forgiven for this author, who actually wrote one of the best papers we have published recently.

Of course, life happens to authors as it does to everybody else and sometimes good and bad events come, as they should, before the Newsletter. I've been happy to renounce an article when I've learnt that a child that was supposed to arrive into this world after their father's article deadline actually came three weeks in advance, but that everybody was well and happy.

Finally, I was touched by one author who wrote me a poem (a sonnet to be precise) of totally professional content!

When I started working on the Editorial Board, I could not even imagine that I would feel the need to say what I'm about to say. Indeed, I have to highlight a coming out that will come as shocking news to some authors: the current Editor-in-Chief of the Newsletter of the European Mathematical Society is a woman. Authors' denial of this fact reveals itself in two ways. I was already quite familiar with the first one: male colleagues addressing me with that "pleasant" linguistic register that those who consider themselves important use to talk to 'small' people, like secretaries and housekeepers. The second one surprised me: I regularly receive emails from colleagues that call me by masculine names, such as Luca, Lucio or Luciano. I'm not a sociologist and I cannot make serious inquiries into this phenomenon but a dear friend of mine, whose name is also Lucia and who is an electronic engineer and works for a Nordic multinational, told me that she also gets a lot of "Dear Lucio". I'm not quite reassured to know that the problem is not limited to mathematics. I also talked to some male colleagues who tried to explain to me that my irritation was unjustified. Indeed, their names were often mismatched with feminine ones when they were not in their countries. I consider that asking yourself whether a name is masculine or feminine and picking the wrong one by mistake is very different to having a subconscious remembering of a masculine name when you have read a feminine one. However, while some are still in denial, it is clear that many women have infiltrated the mathematical community.

Another group of people that work hard for the Newsletter are the Editorial Board. The 60 to 70 pages that you receive every three months cannot be anything but the result of the work of a competent team. I've had the chance of working in a friendly atmosphere for all this time. I want to thank all the former and present members for their enthusiastic work, their inexhaustible quest for new topics and articles, the many pieces of advice and the many discussions.

The most important novelties that we have introduced are the change of the cover, the new section YMCo (Young Mathematicians Column) and the debut of the EMS on social media.[1] All of them have proven to be challenging in different ways. The presence of the EMS on social media started as an activity of the Newsletter and has progressively become an activity of the EMS. Richard Elwes, the publicity officer, is now in charge of this aspect of the communication with members and, more generally, with the mathematical community.

We have tried to make the Newsletter a place of debate and exchange. It is not for me to say if we have succeeded but I can say that the difference between political correctness and a flat interest level of content

[1] Among many useful services that the EMS provides to the mathematical community, I'd like to note the Twitter account @EuroMathJobs, which is exclusively dedicated to European mathematical job announcements.

is sometimes subtle and difficult to discern. One of the very rare aggressive complaints that we have received in the last four years was about an interview. A reader asked the Editorial Board how the interviewee could accept answering such offensive questions. All interviews in the Newsletter receive a final validation from the interviewee. In this specific case, the interviewed mathematician said many politically incorrect but interesting opinions and did not want them to be published, so the interviewer agreed to endorse the statements as his own to keep the interest level of the content. Sometimes, a debate needs a small provocation to ignite and perhaps it is difficult for readers to realise the ease of the relations with contributors to the Newsletter (in 99.9% of cases).

I cannot cite everyone individually but I shall explicitly thank the two former Editors-in-Chief Vicente Muñoz and Martin Raussen, who have been a good source of advice on many occasions.

Backlog has been a good friend over these years, the kind of friend whose presence allows you to sleep as a child and to smile at adversity. He has disappeared twice and I had to go through some tough quarter hours. It is a great pleasure to thank the friends of the Newsletter who have helped me on these occasions, always at quite short notice. Valentin Zagrebnov, who is going to be the next Editor-in-Chief, is one of them; I know that the Newsletter is in good hands. The others are Gert-Martin Greuel and the staff of Imaginary, who have contributed more than once to the Newsletter, and with whom I have to apologise for a regrettable omission. In fact, we are indebted to them since they were at the origin of the publication of "Problems for children from 5 to 15" by V.I. Arnold in the last issue of the Newsletter. The book is actually available in several languages on their website *https://imaginary.org/*, so as many young, enthusiastic mathematicians as possible can profit by reading it.

The Newsletter is published by the EMS Publishing House. Three members of staff contribute to its production: Chris Nunn, our copy-editor, is in charge of transforming the English soup that many foreigners, including myself, speak into the English language; Christoph Eyrich is in charge of TeX-editing the mathematical papers; and Sylvia Fellmann is in charge of the composition. I'm citing them in the order in which they work on the files. Their professionalism has been precious to me, including their advice on more technical editorial matters, copyrights and many other problems. Because of the nature of her work, Sylvia has been the member of the staff with whom I have had more regular contact and I'd really like to thank her for her excellent work and a very pleasant collaboration.

I'd like to thank Thomas Hintermann, Director of the Publishing House, who has supported the changes and the improvements that the Editorial Board has sought to implement. I think we are an even match in the determination of our discussions and the result is the best that we could hope for in improving the Newsletter with the means available.

The Editor-in-Chief of the Newsletter is permanently invited to the meeting of the Executive Committee. I've appreciated these meetings, which have been very interesting and fruitful for the Newsletter but also the occasion for pleasant exchanges. In particular, I'd like to thank the present and former Presidents of the EMS, who have supported me in many ways: Marta Sanz-Solé, who was in office when I started, and Pavel Exner.

I sincerely feel that the Executive Committee has granted me a privilege, giving me the possibility of observing and interacting with the European mathematical community through the loop of the Newsletter, and I'd like to express my deep gratitude for this opportunity.

I apologise in advance for the many I have forgotten to thank here.

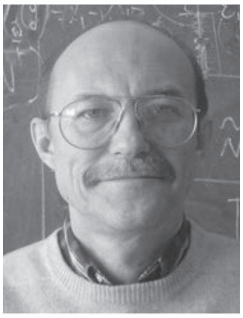
To all our readers, I say that without members, both individual and institutional (but also member-societies), the EMS, with all its committees and the Newsletter, would not exist.

Farewell within the Editorial Board of the EMS Newsletter

In December 2015, the term of office ended for **Eva Maria Feichtner**. Eva Maria has been in charge of the Research Centres column for the last 4 years, introducing us to European and non-European research centres. The Editorial Board express their deep gratitude for the work she has carried out with great competence and thank her for contributing to a friendly and productive atmosphere.

One new member has rejoined the Editorial Board in January 2015. It is a pleasure to welcome **Valentin Zagrebnov**, who has already been appointed by the Executive Committee of the EMS as the next Editor-in-Chief of the Newsletter, starting from the September 2016 issue.

New Editor Appointed



Valentin A. Zagrebnov graduated from Moscow State University and received his PhD and Habilitation in the USSR. He is actually a member of the Analysis-Geometry-Topology group at the *Institut de Mathématiques de Marseille*. His research interests are functional analysis, operator theory and probability theory with applications in mathematical physics and statistical mechanics. He is the author of two books: on the Approximation Hamiltonian Method in quantum statistical mechanics and on the theory of Gibbs semigroups. He is also a member of the editorial boards of several journals in mathematics and mathematical physics and former Editor-in-Chief (2009–2015) of the News Bulletin of the International Association of Mathematical Physics. He has been an emeritus professor at the mathematics department of Aix-Marseille University since September 2015.

Report from the EMS Executive Committee Meeting in Moscow, 27th–29th November 2015

Richard Elwes (EMS Publicity Officer; University of Leeds, UK)

The most recent meeting of the Executive Committee was generously hosted by the Steklov Mathematical Institute of the Russian Academy of Sciences. On Friday it was greeted by Victor Vasiliev, Victor Buchstaber, Sergey Lando (President and Vice-Presidents respectively of the Moscow Mathematical Society), and Anatoly Vershik (former President of St. Petersburg Mathematical Society). On Saturday it was welcomed by Dmitri Orlov, Deputy Director of the Steklov Institute.

Officers' Reports

The President welcomed the Executive Committee and discussed his recent activities (many of which feature as separate items below).

The Treasurer reported on the 2015 EMS accounts, and recorded a surplus. As a consequence, it was agreed to increase the society's 2016 budget for scientific projects. Meanwhile, the Treasurer will seek expert advice on the appropriate balance between budget, assets, and investments. Discussion of financial strategy will continue at the next committee meeting.

With the contribution of the Department of Mathematics at the University of Helsinki now explicitly recorded in the budget, the committee took the opportunity to express its ongoing gratitude to the University of Helsinki for its support of the EMS Office.

Membership

The status of the Armenian Mathematical Union's application for Class 1 Membership will be reviewed, with the intention of inviting them to present their application at the 2016 Council Meeting for approval.

The Executive Committee approved a list of 160 new members, and was pleased to note that this represents an increase on the corresponding period last year (potentially influenced by the forthcoming ECM in Berlin).

Scientific Meetings

The Executive Committee discussed the report from Volker Mehrmann on preparations for the 7th European Congress of Mathematics (ECM) in Berlin (18-22 July 2016). The meeting's finances and the funding of EMS prizes were also considered. The committee expressed confidence in the congress's success and appreciation for the excellent work of the local organisers.

The 8th ECM will be held in 2020, in either Portorož (Slovenia) or Sevilla (Spain). The President presented a summary of recent site visits to the two locations. The committee will encourage both parties to improve their bids, and will invite small delegations to the next EC meeting. The final decision will be taken by the EMS Council in Berlin (July 2016).

Given the healthy state of the EMS finances (above), the committee resolved to sponsor seven EMS Summer Schools in 2016:

- 3rd Barcelona Summer School on Stochastic Analysis, June 27–July 1, 2016
- European Summer School in Modelling, Analysis and Simulation: Crime and Image Processing, Oxford (UK), July 4–8, 2016
- Building Bridges: 3rd EU/US Summer School on Automorphic Forms and Related Topics, Sarajevo (Bosnia & Herzogovina), July 4–9, 2016

- 14th Workshop on Interactions between Dynamical Systems and PDEs (JISD 2016), Barcelona, July 11–15, 2016
- EMS-IAMP Summer School in Mathematical Physics on “Universality, Scaling Limits and Effective Theories”, Rome (Italy), July 11–15, 2016
- European Summer School in Mathematical Modeling of Tissue Mechanics, Lorentz Center, Leiden (The Netherlands), July 25–29, 2016
- European Summer School in Multigraded Algebra and Applications, Moieciu (Romania), August 17–24, 2016

EMS involvement in other upcoming events was also discussed, including the forthcoming UMALCA (Unión Matemática de América Latina y el Caribe) Congress where the EMS will be pleased to support a European plenary speaker.

The committee chose to sponsor *Ernest B. Vinberg* as EMS Distinguished Speaker at the 50th Seminar ‘Sophus Lie’ (25 Sept–1 Oct 2016, Będlewo, Poland).

Society Meetings

The preparations for the 2016 EMS Council (July 2016, Berlin) are underway. One task will be the replenishment of the Executive Committee and its officers. It was agreed to approach member societies for proposals for prospective committee members, to be discussed at the next EC meeting in advance of the elections at Council.

The President then reported back from the Meeting of Presidents of Member Societies (Innsbruck, 28–29 March 2015) where the practice of political lobbying was debated.

Standing Committees

With several EMS committees due for renewal, the Executive Committee had pleasure in making the following appointments: *Giulia Di Nunno* as Chair of the Developing Countries Committee, *Adolfo Quirós* as Chair of the Ethics Committee (each for the term 2016-2017), and *Beatrice Pelloni* as Chair of the Women in Mathematics Committee (for the term 2016–2019).

The committee additionally approved appointments to the Education, Ethics, Publications, and Raising Public Awareness committees (all for the term 2016–2019).

It then considered reports from the Chairs of the Committees on Applied Mathematics, Developing Countries, Education, Electronic Publishing, ERCOM, Ethics, European Solidarity, Raising Public Awareness of Mathematics, and Women in Mathematics.

Projects

The performance of EU-MATHS-IN (European Service Network of Mathematics for Industry and Innovation) was due for evaluation, with two years having passed since its inception as a joint venture between the EMS and ECMI (European Consortium for Mathematics in Industry). The committee concluded that EU-MATHS-IN has played an effective coordinating role linking applied mathematics with industry, and that its lobbying activities in Brussels have yielded valuable results. The committee therefore approved its continuation and congratulated the EU-MATHS-IN board on its success. Maria Esteban was re-appointed as EMS representative to the EU-MATHS-IN board.

Other projects with EMS involvement were discussed, including the European Digital Mathematics Library and Encyclopedia of Mathematics.

Publishing, Publicity, and the Internet

The Executive Committee continued its deliberations about the direction of the EMS Publishing House. It was pleased to hear that the Scientific Advisory Board is functioning well.

With her term coming to an end in 2016, the committee thanked *Lucia Di Vizio* for her excellent work as Editor-in-Chief of the EMS Newsletter, and looked forward to working with the incoming Editor-in-Chief, *Valentin Zagrebnov* from June 2016.

The committee discussed the report of *Gert-Martin Greuel*, Editor-in-Chief of Zentralblatt, and heard that from 1st April 2016, *Klaus Hulek* will succeed him in that role.

Reports from the web-team (led by *Martin Raussen*) and Publicity Officer (*Richard Elwes*) were discussed, along with future plans for EMS Publicity, including its social media, E-news, and website.

Relations with Political and Funding Bodies

The President reported on the new legal status of the Initiative for Science in Europe (ISE). With the EMS’s membership fee set to increase from € 1500 to € 3000, ongoing membership required careful consideration. The Executive Committee agreed to remain a member of the ISE for two further years, at which time the relative costs and benefits of membership will again be re-evaluated.

On behalf of the EMS, the President congratulated *Cédric Villani* on his appointment as member of the High Level Group of Scientific Advisors of the EC Scientific Advice Mechanism. The President also reported on the latest developments for Horizon 2020 and its proposed cuts. The committee’s attention was also drawn to the new COST (European Cooperation in Science and Technology) action call.

Recent developments at the Engineering Research Council were also discussed.

Relations with Other Mathematical Bodies

The committee considered the EMS’s relations with bodies, including ICIAM (the International Council for Industrial and Applied Mathematics) and the International Mathematical Union. It expressed the desire that the 2022 International Congress of Mathematicians take place in Europe. The EMS will therefore support European bids to host this event.

The committee was pleased to note that *Sara van de Geer* will deliver the 2016 EMS-Bernoulli Society Joint Lecture at the Nordic Congress of Mathematics in Stockholm in March. Developments within various research institutes and prize-committees were also noted.

Following his meeting with the President of the Mathematical Society of Japan, the President presented agreements for collaboration and reciprocal membership between our societies. The Executive Committee will be delighted to sign these agreements.

Conclusion

Finally the President expressed the committee's gratitude to the Moscow Mathematical Society and to the Steklov Institute for its warm hospitality, and to committee member *Armen Sergeev* for the faultless organization. The participants especially enjoyed their visit to the Institute's Laboratory for Popularization of Mathematics, headed by *Nikolai Andreev*.

The next Executive Committee meeting will be at Mittag-Leffler Institute in Stockholm, 18th–20th March 2016.

Submission of ERC Grant Proposals

The 2016 calls for ERC grants are approaching. The deadline for advanced grants is 1 Sept 2016. From this year, the funding for the different fields (e.g. PE1 Mathematics) will depend strongly on the number of applications for the field. Unfortunately, in the last calls, the number of applications significantly decreased and this will have a negative effect on the funding for mathematics. If the situation continues like this then a drastic reduction of the budget for funding in mathematics is to be expected in the coming years. So what should be done?

1. Those that have a good idea for a proposal should definitely apply.
2. You should apply for the full grant if the project really has the potential and need for such a large amount.
3. Most research projects in mathematics can get along with much smaller grants.
4. Do not follow the requests of university administrators if they urge you to go for the maximum possible grant sum.

First edition of the Barcelona Dynamical Systems Prize

Amadeu Delshams (Scientific Manager of the BDSP 2015; Universitat Politècnica de Catalunya, Barcelona, Spain)

In July 2014, the first edition of the Barcelona Dynamical Systems Prize was announced. This is an international prize awarded by the Societat Catalana de Matemàtiques, under the patronage of Professor Carles Simó, to the author or authors of a paper, written in English, in the area of dynamical systems.

For this first edition, the paper was required to be published between 1 May 2013 and 30 April 2015 and the authors were invited to send their submission letters no later than 30 May 2015.

The prize is open to all kinds of dynamical systems and the following aspects are rated: global rather than particular descriptions of dynamics, new theoretical approaches, new computational tools and relevant applications. The prize award is 4000 euros, subject to local withholding tax.

The sponsor of this award is Carles Simó, who, after receiving the National Research Award of Catalonia in 2012, decided to share it with researchers of dynamical systems worldwide.

The jury, which was not made public until after the resolution, was formed by Henk Broer, Bob Devaney, Yulij Ilyashenko, Richard Montgomery and Marcelo Viana. The prize received quite a number of submission letters and after long deliberations, in October 2015, the decision of the jury was to award the prize to

Alberto Enciso and Daniel Peralta-Salas,

as the authors of the paper "Existence of knotted vortex tubes in steady Euler flows", *Acta Math*. 214 (2015), no. 1, 61–134,

and *ex aequo* to

Marcel Guardia, Pau Martin and Tere M. Seara,

as the authors of the paper "Oscillatory motions for the restricted planar circular three body problem", *Invent. Math*. 203 (2016), no. 2, 417–492.

The resolution included the following arguments:

In the paper by Alberto Enciso and Daniel Peralta-Salas, the authors construct, for any given knot or link type, a particular type of static solution to the inviscid Euler equation on Euclidean 3-space (a "Beltrami flow") which has among its integral curves one which realises this knot or link type. As a result, they get a single flow which contains all knot types, which is a special form of Lord Kelvin's conjecture.

In the paper by Marcel Guardia, Pau Martin and Tere M. Seara, the authors prove the existence of oscillatory solutions in the restricted planar circular 3-body problem for any value of the mass ratio of the primaries, which had been open since Chazy's 1920s work pointed out oscillatory motion as one possible final behaviour of the motion of the 3-body problem.

The award ceremony took place at the Institut d'Estudis Catalans on 12 November 2015.

Fermat Prize 2015

The laureates of the Fermat Prize 2015 are:

- *Laure Saint-Raymond* (École Normale Supérieure, Paris) for the development of asymptotic theories of partial differential equations, including the fluid limits of rarefied flows, multiscale analysis in plasma physics equations and ocean modelling, and the derivation of the Boltzmann equation from interacting particle systems.
- *Peter Scholze* (Universität Bonn) for his invention of perfectoid spaces and their application to fundamental problems in algebraic geometry and in the theory of automorphic forms.

The prize-giving ceremony took place on Tuesday 22 March 2016 in Toulouse.

The award amounts to a total of 20,000 Euros granted every two years by the Conseil Régional Midi-Pyrénées. The recipients are required to publish, in the mathematical journal *Annales de la Faculté des Sciences de Toulouse*, an article summarising their findings and aiming to explain the significance of the results of their research to professional mathematicians who are not necessarily experts in the subject.

The EMS, as well as the French Académie des Sciences, the Société Mathématique de France and the Société de Mathématiques Appliquées et Industrielles, contribute to the composition of the committee of the Fermat Prize by appointing one of its members.

Barcelona Graduate School of Mathematics BGSMath

Marc Noy (Director of BSGMath; Universitat Politècnica de Catalunya, Barcelona, Spain)

BGSMath
BARCELONA GRADUATE SCHOOL OF MATHEMATICS

The Barcelona Graduate School of Mathematics (BGSMath) is a collaborative initiative of the research groups in mathematics of four main universities located in the Barcelona area and an international research centre: Universitat de Barcelona (UB), Universitat Autònoma de Barcelona (UAB), Universitat Politècnica de Catalunya (UPC), Universitat Pompeu Fabra (UPF) and Centre de Recerca Matemàtica (CRM). Its primary goal is to provide PhDs and postdoctoral training at the highest level in an international environment.

BGSMath is associated to PhD programmes in mathematics and statistics in the universities of Barcelona, as well as several Master's degrees, ranging from core mathematics to statistics, mathematical engineering and financial mathematics. The Barcelona area therefore provides a very attractive environment for carrying out Master's and PhD studies in mathematics and its applications, and for postdoctoral research visits. BGSMath research groups cover most areas of mathematics and its applications.

Recently, BGSMath has been recognised (in 2015) as a María de Maeztu Excellence Unit by the Spanish government. This label carries a grant for the period 2015-2019, which is mainly allocated to international calls for PhDs and postdoctoral positions. We plan to make an open call for positions each year from 2015 to 2019 in all areas of mathematics. Candidates with strong motivation and qualifications for research are invited to apply.

Many international mathematical events take place in the Barcelona area. In addition to these, BGSMath will start running monthly Research Programmes in 2016 (CRM has been running quarterly Intensive Research Programmes for some time). These offer opportunities for researchers to come to Barcelona and collaborate with BGSMath groups.

BGSMath is run by a Governing Board consisting of Marc Noy (UPC, Director), Natàlia Castellana (UAB), Lluís Alsedà (CRM) and Núria Fagella (UB), with the help of an Advisory Scientific Committee consisting of Marta Sanz-Solé (UB, Committee Chair), Tomás Alarcón (CRM), Franco Brezzi (CNR Pavia), Harry Buhrman (CWI Amsterdam), Tere Martínez-Seara (UPC), Joaquim Ortega-Cerdà (UB), Joan Porti (UAB), Pere Puig (UAB), Jean-Michel Roquejoffre (Paul Sabatier University, Toulouse), Víctor Rotger (UPC), Oriol Serra (UPC), Ulrike Tillmann (Oxford University) and Carlos Vázquez (Universidade da Coruña).

An Introduction to Geometric Complexity Theory

J. M. Landsberg (Texas A&M University, College Station, TX, USA)

This article will survey methods from differential geometry, algebraic geometry and representation theory relevant for the permanent vs. determinant problem from computer science, an algebraic analogue of the **P** *vs.* **NP** *problem.*

1 Introduction

The purpose of this article is to introduce mathematicians to uses of geometry in complexity theory. It will focus on a central question: the *Geometric Complexity Theory* version of L. Valiant's conjecture comparing the complexity of the permanent and determinant polynomials, which is an algebraic variant of the **P** ≠ **NP** conjecture. Other problems in complexity such as *matrix rigidity* (see [KLPSMN09, GHIL, Alu15]) and the *complexity of matrix multiplication* (see, for example, [Lan08]) have been treated with similar geometric methods.

2 History

1950s Soviet Union

A travelling saleswoman needs to visit 20 cities: Moscow, Leningrad, Stalingrad, etc. Is there a route that can be taken travelling less than 10,000 km?

Essentially, the only known method to determine the answer is a brute force search through all possible paths. The number of paths to check grows exponentially with the number of cities to visit. Researchers in the Soviet Union asked: *'Is this brute force search avoidable?'* I.e. are there any algorithms that are significantly better than the naïve one?

A possible cause for hope is that if someone proposes a route, it is very easy to check if it is less than 10,000 km (even pre-Google).

1950s Princeton NJ

In a letter to von Neumann (see [Sip92, Appendix]), Gödel attempted to quantify the apparent difference between intuition and systematic problem solving. For example, is it really significantly easier to verify a proof than to write one?

1970s: precise versions of these questions

These ideas evolved to a precise conjecture posed by Cook (preceded by work of Cobham, Edmonds, Levin, Rabin, Yablonski and the question of Gödel mentioned above):

Let **P** denote the class of problems that are "easy" to solve.[1]

Let **NP** denote the class of problems that are "easy" to verify (like the travelling saleswoman problem).[2]

1 Can be solved on a Turing machine in time polynomial with respect to the size of the input data.

Conjecture 1. [Coo71, Kar72] **P** ≠ **NP**.

Late 1970s: L. Valiant, algebraic variant

A *bipartite graph* is a graph with two sets of vertices and edges joining vertices from one set to the other. A *perfect matching* is a subset of the edges such that each vertex shares an edge from the subset with exactly one other vertex.

A standard problem in graph theory, for which the only known algorithms are exponential in the size of the graph, is to count the number of perfect matchings of a bipartite graph.

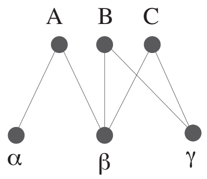


Figure 1. A bipartite graph. Vertex sets are $\{A, B, C\}$ and $\{\alpha, \beta, \gamma\}$.

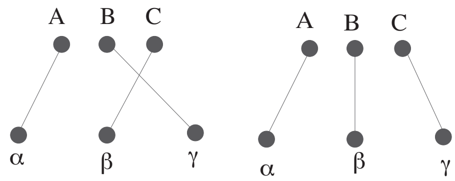


Figure 2. Two perfect matchings of the graph from Figure 1

This count can be computed by evaluating a polynomial as follows. To a bipartite graph Γ, one associates an incidence matrix $X_\Gamma = (x^i_j)$, where $x^i_j = 1$ if an edge joins the vertex i above to the vertex j below and is zero otherwise. For example, the graph of Figure 1 has incidence matrix

$$X_\Gamma = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 1 & 1 \end{pmatrix}.$$

A perfect matching corresponds to a set of entries $\{x^1_{j_1}, \ldots, x^n_{j_n}\}$ with all $x^i_{j_i} = 1$ and $(j_1, \ldots, j_n)$ is a permutation of $(1, \ldots, n)$. Let $\mathfrak{S}_n$ denote the group of permutations of the elements $(1, \ldots, n)$.

Define the *permanent* of an $n \times n$ matrix $X = (x^i_j)$ by

$$perm_n(X) := \sum_{\sigma \in \mathfrak{S}_n} x^1_{\sigma(1)} x^2_{\sigma(2)} \cdots x^n_{\sigma(n)}. \tag{1}$$

Then, $\mathrm{perm}(X_\Gamma)$ equals the number of perfect matchings of Γ.

For example, $\mathrm{perm}_3 \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 1 & 1 \end{pmatrix} = 2.$

2 A proposed solution can be verified in polynomial time.

A fast algorithm to compute the permanent would give a fast algorithm to count the number of perfect matchings of a bipartite graph.

While it may not be easy to evaluate, the polynomial perm_n is relatively easy to write down compared with a random polynomial of degree n in n^2 variables in the following sense.

Let **VNP** be the set of sequences of polynomials that are "easy" to write down.[3] Valiant showed [Val79] that the permanent is *complete* for the class **VNP**, in the sense that **VNP** is the class of all polynomial sequences (p_m), where p_m has degree m and involves a number of variables polynomial in m, such that there is a polynomial $n(m)$ and p_m is an *affine linear projection* of $\mathrm{perm}_{n(m)}$ as defined below. Many problems from graph theory, combinatorics and statistical physics (partition functions) are in **VNP**. A good way to think of **VNP** is as the class of sequences of polynomials that can be written down explicitly.[4]

Let **VP** be the set of sequences of polynomials that are "easy" to compute.[5] For example, one can compute the determinant of an $n \times n$ matrix quickly, e.g. using Gaussian elimination, so the sequence $(\det_n) \in$ **VP**. Most problems from linear algebra (e.g. inverting a matrix, computing its determinant, multiplying matrices) are in **VP**.

The standard formula for the easy to compute determinant polynomial is

$$det_n(X) := \sum_{\sigma \in \mathfrak{S}_n} \mathrm{sgn}(\sigma) x^1_{\sigma(1)} x^2_{\sigma(2)} \cdots x^n_{\sigma(n)}. \qquad (2)$$

Here, $\mathrm{sgn}(\sigma)$ denotes the sign of the permutation σ.

Note that

$$\mathrm{perm}_2 \begin{pmatrix} y^1_1 & y^1_2 \\ y^2_1 & y^2_2 \end{pmatrix} = y^1_1 y^2_2 + y^1_2 y^2_1 = \det_2 \begin{pmatrix} y^1_1 & -y^1_2 \\ y^2_1 & y^2_2 \end{pmatrix}.$$

On the other hand, Marcus and Minc [MM61], building on work of Pólya and Szegö (see [Gat87]), proved that one could not express $\mathrm{perm}_m(Y)$ as a size m determinant of a matrix whose entries are affine linear functions of the variables y^i_j when $m > 2$. This raised the question that perhaps the permanent of an $m \times m$ matrix could be expressed as a slightly larger determinant. More precisely, we say $p(y^1, \ldots, y^M)$ is an *affine linear projection* of $q(x^1, \ldots, x^N)$ if there exist affine linear functions $x^\alpha(Y) = x^\alpha(y^1, \ldots, y^M)$ such that $p(Y) = q(X(Y))$. For example, B. Grenet [Gre14] observed that

$$\mathrm{perm}_3(Y) = \det_7 \begin{pmatrix} 0 & y^1_1 & y^2_1 & y^3_1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & y^3_3 & y^2_3 & 0 \\ 0 & 0 & 1 & 0 & 0 & y^1_3 & y^3_3 \\ 0 & 0 & 0 & 1 & y^1_3 & 0 & y^2_3 \\ y^2_2 & 0 & 0 & 0 & 1 & 0 & 0 \\ y^3_2 & 0 & 0 & 0 & 0 & 1 & 0 \\ y^1_2 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}. \qquad (3)$$

Recently [ABV15], it was shown that perm_3 cannot be realised as an affine linear projection of $\det_n$ for $n \leq 6$, so (3) is optimal.

Valiant showed that if $n(m)$ grows exponentially with respect to m then there exist affine linear functions $x^i_j(y^s_t)$ such that $\det_n(X(Y)) = \mathrm{perm}_m(Y)$. (Grenet strengthened this to show explicit expressions when $n = 2^m - 1$ [Gre14]. See [LR15] for a discussion of the geometry of these algorithms and a proof of their optimality among algorithms with symmetry.) Valiant also conjectured that one cannot do too much better:

Conjecture 2 (Valiant [Val79])**.** Let $n(m)$ be a polynomial of m. Then there exists an m_0 such that, for all $m > m_0$, there do not exist affine linear functions $x^i_j(y^s_t)$ such that $\mathrm{perm}_m(Y) = \det_n(X(Y))$.

Remark 3. The original **P** $\neq$ **NP** is viewed as completely out of reach. Conjecture 2, which would be implied by **P** $\neq$ **NP**, is viewed as a more tractable substitute.

To keep track of progress on the conjecture, for a polynomial $p = p(Y)$, let $dc(p)$ denote the smallest n such that there exists an affine linear map $X(Y)$ satisfying $p(Y) = \det_n(X(Y))$. Then, Conjecture 2 says that $dc(\mathrm{perm}_m)$ grows faster than any polynomial. Since the conjecture is expected to be quite difficult, one could try to prove any lower bound on $dc(\mathrm{perm}_m)$. Several linear bounds on $dc(\mathrm{perm}_m)$ have been shown [MM61, vzG87, Cai90], with the current world record the quadratic bound $dc(\mathrm{perm}_m) \geq \frac{m^2}{2}$ [MR04]. (Over finite fields, one has the same bound by [Cai90]. Over $\mathbb{R}$, one has $dc_{\mathbb{R}}(\mathrm{perm}_m) \geq m^2 - 2m + 2$ [Yab15].) The state of the art was obtained with local differential geometry, as described in Section 3.

Remark 4. There is nothing special about the permanent for this conjecture; it would be sufficient to show that any explicit (in the sense of **VNP** mentioned above) sequence of polynomials p_m has $dc(p_m)$ growing faster than any polynomial. The dimension of the set of affine linear projections of $\det_n$ is roughly n^4 but the dimension of the space of homogeneous polynomials of degree m in m^2 variables grows almost like m^m so a random sequence will have exponential $dc(p_m)$. Problems in computer science to find an explicit object satisfying a property that a random one satisfies are called *trying to find hay in a haystack*.

Coordinate free version

To facilitate the use of geometry, we get rid of coordinates. Let $\mathrm{End}(\mathbb{C}^{n^2})$ denote the space of linear maps $\mathbb{C}^{n^2} \to \mathbb{C}^{n^2}$, which acts on the space of homogeneous polynomials of degree n on $\mathbb{C}^{n^2}$, denoted $S^n\mathbb{C}^{n^2*}$ (where the $*$ is used to indicate the dual vector space to $\mathbb{C}^{n^2}$), as follows: for $g \in \mathrm{End}(\mathbb{C}^{n^2})$ and $P \in S^n\mathbb{C}^{n^2*}$, the polynomial $g \cdot P$ is defined by

$$(g \cdot P)(x) := P(g^T x). \qquad (4)$$

Here, g^T denotes the transpose of g. (One takes the transpose matrix in order that $g_1 \cdot (g_2 \cdot P) = (g_1 g_2) \cdot P$.)

In [MS01], *padding* was introduced, i.e. adding a homogenising variable so all objects live in the same ambient space, in order to deal with linear functions instead of affine linear functions. Let ℓ be a new variable, so $\ell^{n-m}\,\mathrm{perm}_m(y) \in$

3 Such sequences are obtained from sequences in **VP** (defined in the following paragraph) by "projection" or "integration over the fiber", where one averages the polynomial over a subset of its variables specialised to 0 and 1.

4 Here, one must take a narrow view of explicit, e.g. restrict to integer coefficients that are not "too large".

5 Admit a polynomial size arithmetic circuit and polynomially bounded degree (see, for example, [BCS97, §21.1]).

$S^n\mathbb{C}^{(m^2+1)*}$. Then, $\mathrm{perm}_m(y)$ is expressible as an $n \times n$ determinant whose entries are affine linear combinations of the y^i_j if and only if $\ell^{n-m}\,\mathrm{perm}_m$ is expressible as an $n \times n$ determinant whose entries are *linear* combinations of the variables y^i_j, ℓ.

Consider any linear inclusion $\mathbb{C}^{(m^2+1)*} \to \mathbb{C}^{n^2*}$, so, in particular, $\ell^{n-m}\,\mathrm{perm}_m \in S^n\mathbb{C}^{n^2*}$. Then,

$$dc(\mathrm{perm}_m) \le n \Leftrightarrow \ell^{n-m}\,\mathrm{perm}_m \in \mathrm{End}(\mathbb{C}^{n^2}) \cdot \det_n . \qquad (5)$$

Conjecture 2 in this language is:

Conjecture 5 (Valiant [Val79])**.** Let $n(m)$ be a polynomial of m. Then, there exists an m_0 such that, for all $m > m_0$, $\ell^{n-m}\,\mathrm{perm}_m \notin \mathrm{End}(\mathbb{C}^{n^2}) \cdot \det_n$. Equivalently,

$$\overline{\mathrm{End}(\mathbb{C}^{n^2}) \cdot \ell^{n-m}\,\mathrm{perm}_m} \not\subset \overline{\mathrm{End}(\mathbb{C}^{n^2}) \cdot \det_n} .$$

3 Differential geometry and the state of the art regarding Conjecture 2

The best result pertaining to Conjecture 2 comes from local differential geometry: the study of Gauss maps.

Gauss maps

Given a surface in 3-space, form its *Gauss map* by mapping a point of the surface to its unit normal vector on the unit sphere as in Figure 3.

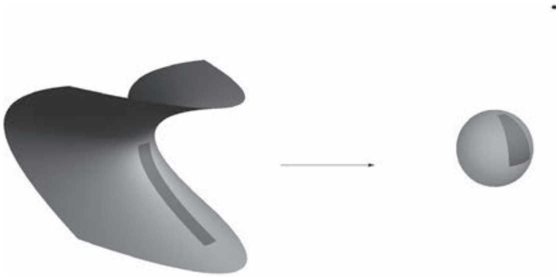


Figure 3. The shaded area of the surface maps to the shaded area of the sphere.

A normal vector to a surface X at x is one perpendicular to the tangent space $T_xX \subset \mathbb{R}^3$. This Gauss image can be defined without the use of an inner product if one instead takes the union of all *conormal lines*, where a conormal vector to $X \subset \mathbb{R}^3$ is one in the dual space $\mathbb{R}^{3*}$ that annihilates the tangent space T_xX. One loses qualitative information. However, one still has the information of the *dimension* of the Gauss image.

This dimension will drop if, through all points of the surface, there is a curve along which the tangent plane is constant. For example, if M is a cylinder, i.e. the union of lines in three space perpendicular to a plane curve, the Gauss image is a curve.

The extreme case is when the surface is a plane. Then, its Gauss image is just a point.

A classical theorem in the geometry of surfaces in three-space classifies surfaces with a degenerate Gauss image. It is stated in the algebraic category for what comes next (for C^∞ versions, see, for example, [Spi79, vol. III, chap. 5]). One may view projective space $\mathbb{P}^3$ as affine space with a plane added at infinity. From this perspective, a cylinder is a cone with its vertex at infinity.

Figure 4. Lines on the cylinder are collapsed to a point.

Theorem 6 (C. Segre [Seg10])**.** If $X^2 \subset \mathbb{P}^3$ is an algebraic surface whose Gauss image is not two-dimensional then X is one of:
- The closure of the union of points on tangent lines to a space curve.
- A generalised cone, i.e. the points on the union of lines connecting a fixed point to a plane curve.

Notice that, in the first picture, the tangent plane along a ray of the curve is constant and, in the second picture, the tangent plane is constant along the lines through the vertex.

One can extend the notion of Gauss maps to hypersurfaces of arbitrary dimension and to hypersurfaces defined over the complex numbers. The union of tangent rays to a curve generalises to the case of *osculating varieties*. One can also take cones with vertices larger than a point.

What does this have to do with complexity theory?

The hypersurface $\{\det_n(X) = 0\} \subset \mathbb{C}^{n^2}$ has a very degenerate Gauss map. To see this, consider the matrix

$$z = \begin{pmatrix} 1 & & & \\ & \ddots & & \\ & & 1 & \\ & & & 0 \end{pmatrix} \in \{\det_n = 0\}.$$

The tangent space to $\{\det_n = 0\}$ at z and the conormal space (in the dual space of matrices) are, respectively,

$$T_z\{\det_n = 0\} = \begin{pmatrix} * & * & * & * \\ \vdots & \ddots & \vdots & \vdots \\ * & * & * & * \\ * & * & * & 0 \end{pmatrix},$$

$$N^*_z\{\det_n = 0\} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & * \end{pmatrix}.$$

But any rank $n-1$ matrix whose non-zero entries all lie in the upper left $(n-1)\times(n-1)$ submatrix will have the same tangent space! Since any smooth point of $\{\det_n = 0\}$ can be moved to z by a change of basis, we conclude that the tangent hyperplanes to $\{\det_n = 0\}$ are parametrised by the rank one matrices, the space of which has dimension $2n-1$ (or $2n-2$ in projective space) because they are obtained by multiplying a column vector by a row vector. In fact, $\{\det_n = 0\}$ may be thought of as an osculating variety of the variety of rank one matrices (e.g. the union of tangent lines to the union of tangent lines … to the variety of rank one matrices).

On the other hand, a direct calculation shows that the permanent hypersurface $\{\mathrm{perm}_m = 0\} \subset \mathbb{P}^{m^2-1}$ has a non-degenerate Gauss map (see Section 5). So, when one includes $\mathbb{C}^{m^2} \subset \mathbb{C}^{n^2}$, the equation $\{\mathrm{perm}_m = 0\}$ becomes an equation in a space of n^2 variables that only uses m^2 of the variables and one gets a cone with vertex $\mathbb{P}^{n^2-m^2-1}$ corresponding to the unused variables. In particular, the Gauss image will have dimension m^2-2.

If one makes an affine linear substitution $X = X(Y)$, the Gauss map of $\{\det(X(Y)) = 0\}$ will be at least as degenerate as the Gauss map of $\{\det(X) = 0\}$. Using this, one obtains:

Theorem 7 (Mignon-Ressayre [MR04])**.** *If $n(m) < \frac{m^2}{2}$ then there do not exist affine linear functions $x^i_j(y^s_t)$ such that $\mathrm{perm}_m(Y) = \det_n(X(Y))$. I.e. $dc(\mathrm{perm}_m) \geq \frac{m^2}{2}$.*

4 Algebraic geometry and Valiant's conjecture

A possible path to show $\mathrm{perm}_m(Y) \neq \det_n(X(Y))$ is to look for a polynomial whose zero set contains all polynomials of the form $\det_n(X(Y))$ and show that perm_m is not in the zero set.

Polynomials

Algebraic geometry is the study of zero sets of polynomials. In our situation, we need polynomials on spaces of polynomials. More precisely, if

$$P(x_1,\dots,x_N) = \sum_{1\leq i_1\leq\cdots\leq i_d\leq N} c_{i_1,\dots,i_d} x_{i_1}\cdots x_{i_d}$$

is a homogeneous polynomial of degree d in N variables, we work with polynomials in the coefficients $c_{i_1,\dots,i_d}$, where these coefficients provide coordinates on the vector space $S^d\mathbb{C}^N$ of all homogeneous polynomials of degree d in N variables.

The starting point of *Geometric Complexity Theory* is the plan to prove Valiant's conjecture by finding a sequence of polynomials P_m vanishing on all affine-linear projections of $\det_{n(m)}$ when n is a polynomial in m such that P_m does not vanish on perm_m.

Disadvantage of algebraic geometry?

The zero set of all polynomials vanishing on

$$S := \{(z,w) \mid w = 0,\ z \neq 0\} \subset \mathbb{C}^2$$

is the line

$$\{(z,w) \mid w = 0\} \subset \mathbb{C}^2.$$

That is, if we want to use polynomials, we may need to prove a more difficult conjecture, in the sense that we will need to prove non-membership in a larger set.

Given a subset Z of a vector space U, the *ideal of* Z, denoted $I(Z)$, is the set of all polynomials vanishing at all points of Z. The *Zariski closure* of Z, denoted $\overline{Z}$, is the set of $u \in U$ such that $P(u) = 0$ for all $P \in I(Z)$. The common zero set of a collection of polynomials (such as $\overline{Z}$) is called an *algebraic variety*.

Conjecture 8 (Mulmuley-Sohoni [MS01])**.** *Let $n(m)$ be a polynomial of m. Then there exists an m_0 such that, for all $m > m_0$, $\ell^{n-m}\,\mathrm{perm}_m \notin \overline{\mathrm{End}(\mathbb{C}^{n^2})\cdot \det_n}$.*

How serious a problem is the issue of Zariski closure? Does it really change Valiant's conjecture?

Mulmuley conjectures [MN] that indeed it does. Namely, he conjectures that there are sequences in the closure of the sequences of spaces $\mathrm{End}(\mathbb{C}^{n^2})\cdot\det_n$ that are *not* in **VP**.

Example 9. Let $P = x_1^3 + x_2^2x_3 + x_2x_4^2 \in S^3\mathbb{C}^4$. Then, $P \notin \mathrm{End}(\mathbb{C}^{n^2})\cdot\det_n$ for $n < 5$ [ABV15]. However, $\overline{\mathrm{End}(\mathbb{C}^9)\cdot\det_3} \supset S^3\mathbb{C}^4$, i.e. *every* homogeneous polynomial of degree 3 in 4 variables is in the Zariski closure of degenerations of $\det_3$ (see, for example, [Bea00]).

However, Mulmuley also conjectures [MN] that any path to resolving Valiant's conjecture will have to address "wild" sequences in the closure, so that the stronger conjecture is the more natural one. Moreover, Grochow makes the case [**?**] that essentially all lower bounds in algebraic complexity theory have come from algebraic geometry.

Advantage of the stronger conjecture: representation theory

Representation theory is the systematic study of symmetry in linear algebra.

The variety $\overline{\mathrm{End}(\mathbb{C}^{n^2})\cdot\det_n}$ may be realised as an *orbit closure* as follows. Let GL_N denote the group of invertible $N\times N$ matrices. It acts on the space of polynomials $S^d\mathbb{C}^N$ by (4). Any element of $\mathrm{End}(\mathbb{C}^N)$ may be described as a limit of elements of GL_N, so the Euclidean closure of $\mathrm{End}(\mathbb{C}^N)\cdot\det_n$ equals the Euclidean closure of $GL_N\cdot\det_n$. In general, Euclidean and Zariski closure can be quite different (e.g. the Zariski closure of $\{(z,w) \mid z = 0,\ w \in \mathbb{Z}\} \subset \mathbb{C}^2$ is the line but this set is already Euclidean closed). However, in this situation, Euclidean closure equals Zariski closure (see [Mum95, Thm. 2.33]), so we have the following equality of Zariski closures:

$$\overline{GL_{n^2}\cdot\det_n} = \overline{\mathrm{End}(\mathbb{C}^{n^2})\cdot\det_n}.$$

Substantial techniques have been developed to study orbits and their closures.

Let

$$\hat{\mathcal{D}et}_n := \overline{GL_{n^2}\cdot\det_n}$$

and let

$$\hat{\mathcal{P}erm}^m_n := \overline{GL_{n^2}\cdot\ell^{n-m}\,\mathrm{perm}_m}.$$

Conjecture 10. [MS01] *Let $n = m^c$ for any constant c. Then, for all sufficiently large m,*

$$\hat{\mathcal{P}erm}^m_n \not\subset \hat{\mathcal{D}et}_n.$$

Conjecture 10 would imply Conjecture 2. The programme to use representation theory to prove Conjecture 10 is described in Section 7.

5 State of the art for Conjecture 10: Classical algebraic geometry

Classical algebraic geometry detour: B. Segre's dimension formula

In algebraic geometry, it is more convenient to work in projective space. (From a complexity perspective, it is also natural, as changing a function by a scalar will not change its complexity.) If W is a vector space then $\mathbb{P}W$ is the associated projective space of lines through the origin: $\mathbb{P}W = (W\backslash 0)/\sim$ where $w_1 \sim w_2$ if $w_1 = \lambda w_2$ for some nonzero complex number λ. Write $[w] \in \mathbb{P}W$ for the equivalence class of $w \in W\backslash 0$ and if $X \subset \mathbb{P}W$, let $\hat{X} \subset W$ denote the corresponding cone in W. Define $\overline{X} = \pi(\overline{\hat{X}})$, the Zariski closure of X.

If $X \subset \mathbb{P}W$ is a hypersurface, let $X^\vee \subset \mathbb{P}W^*$ denote its Gauss image, which is called its *dual variety*. If X is an irreducible algebraic variety, $X^\vee$ will be too. More precisely, $X^\vee$ is the Zariski closure of the set of *conormal lines* to smooth points of X. Here, if $\hat{T}_x X \subset W$ denotes the tangent space to the cone over X, the conormal space is $N^*_x X = (\hat{T}_x X)^\perp \subset W^*$.

Proposition 11 (B. Segre [Seg51])**.** Let $P \in S^d W^*$ be irreducible and let $d \geq 2$. Then, for a Zariski open subset of points $[x] \in \mathrm{Zeros}(P)$,

$$\dim \mathrm{Zeros}(P)^\vee = \mathrm{rank}(Hess(P)(x^{d-2})) - 2.$$

Here, $(Hess(P)(x^{d-2})) \in S^2 W^*$ is the Hessian matrix of second partial derivatives of P evaluated at x. Note that the right side involves second derivative information and the left side involves the dual variety (which is first derivative information from $\mathrm{Zeros}(P)$) and its dimension, which is a first derivative computation on the dual variety and therefore a second derivative computation on $\mathrm{Zeros}(P)$.

Proof. For a homogeneous polynomial $P \in S^d W^*$, write $\overline{P}$ when we consider P as a d-multi-linear form. Let $x \in \hat{Zeros}(P) \subset W$ be a smooth point, so $P(x) = \overline{P}(x, \ldots, x) = 0$ and $dP_x = \overline{P}(x, \ldots, x, \cdot) \neq 0$. Take $h = dP_x \in W^*$, so $[h] \in \mathrm{Zeros}(P)^\vee$. Consider a curve $h_t \subset \hat{Zeros}(P)^\vee$ with $h_0 = h$. There must be a corresponding curve $x_t \in \hat{Zeros}(P)$ such that $h_t = \overline{P}(x_t, \ldots, x_t, \cdot)$ and thus its derivative is $h'_0 = \overline{P}(x^{d-2}, x'_0, \cdot)$. The dimension of $\hat{T}_h \mathrm{Zeros}(P)^\vee$ is then the rank of $Hess(P)(x^{d-2}) = \overline{P}(x^{d-2}, \cdot, \cdot)$ minus one (we subtract one because $x'_0 = x$ is in the kernel of $Hess(P)(x^{d-2})$). Finally, $\dim X = \dim \hat{T}_x X - 1$. □

First steps towards equations

Segre's formula implies, for $P \in S^d W^*$, that $\dim \mathrm{Zeros}(P)^\vee \leq k$ if and only if, for all $w \in W$, letting $G(q, W)$ denote the Grassmannian of q-planes through the origin in W,

$$P(w) = 0 \;\Rightarrow\; \det{}_{k+3}(Hess(P)(w^{d-2})|_F) = 0 \;\forall F \in G(k+3, W).$$

Equivalently (assuming P is irreducible), for any $F \in G(k+3, W)$, the polynomial P must divide $\det_{k+3}(Hess(P)|_F) \in S^{(k+3)(d-2)} W^*$.

Thus, to find polynomials on $S^d W^*$ characterising hypersurfaces with degenerate duals, we need polynomials that detect if a polynomial $P \in S^d W^*$ divides a polynomial $Q \in S^e W^*$. Now, P divides Q if and only if $Q \in P \cdot S^{e-d} W^*$, i.e. letting x^{I_j} be a basis of $S^{e-d} W^*$ and $\wedge$ denote exterior (wedge) product,

$$x^{I_1} P \wedge \cdots \wedge x^{I_D} P \wedge Q = 0. \tag{6}$$

Let $\dim W = N$ and let $\mathcal{D}_{k,d,N} \subset \mathbb{P}S^d W^*$ denote the zero set of the equations (6) in the coefficients of P taking $Q = \det_{k+3}(Hess(P)|_F)$. By our previous discussion, $[\det_n] \in \mathcal{D}_{2n-2,n,n^2}$.

The lower bound on $\overline{dc}(\mathrm{perm}_m)$

When

$$x = \begin{pmatrix} 1-m & 1 & \cdots & 1 \\ 1 & 1 & \cdots & 1 \\ \vdots & \vdots & \cdots & \vdots \\ 1 & 1 & \cdots & 1 \end{pmatrix}, \tag{7}$$

a short calculation shows that $Hess(\mathrm{perm}_m)(x^{m-2})$ is of maximal rank. This fills in the missing step of the proof of Theorem 7. Moreover, if one works over $\mathbb{R}$ then the Hessian has a signature. For $\det_n$, this signature is $(n-1, n-1)$ but, for the permanent, the signature on an open subset is at least $(m^2 - 2m + 1, 2m - 3)$, thus:

Theorem 12 (Yabe [Yab15])**.** $dc_{\mathbb{R}}(\mathrm{perm}_m) \geq m^2 - 2m + 2$.

Were we to just consider perm_m as a polynomial in more variables, the rank of the Hessian would not change. However, we are also adding padding, which could *a priori* have a negative effect on the rank of the Hessian. Fortunately, as was shown in [LMR13], it does not and we conclude:

Theorem 13. [LMR13] $\mathcal{P}erm^m_n \not\subset \mathcal{D}_{2n-2,n,n^2}$ when $m < \frac{n^2}{2}$. In particular, when $m < \frac{n^2}{2}$, $\mathcal{P}erm^m_n \not\subset \mathcal{D}et_n$.

On the other hand, since cones have degenerate duals, $\ell^{n-m}\,\mathrm{perm}_m \in \mathcal{D}_{2n-2,n,n^2}$ whenever $m \geq \frac{n^2}{2}$.

In [LMR13], it was also shown that $\mathcal{D}_{k,d,N}$ intersected with the set of irreducible hypersurfaces is exactly the set (in $\mathbb{P}S^d W^*$) of irreducible hypersurfaces of degree d in $\mathbb{P}W$ with dual varieties of dimension k, which solved a classical question in algebraic geometry.

6 Necessary conditions for modules of polynomials to be useful for GCT

Fixing a linear inclusion $\mathbb{C}^{m^2+1} \subset \mathbb{C}^{n^2}$, the polynomial $\ell^{n-m}\,\mathrm{perm}_m \in S^n \mathbb{C}^{n^2}$ has evident pathologies: it is *padded*, that is, divisible by a large power of a linear form, and its zero set is a *cone* with an $n^2 - m^2 - 1$ dimensional vertex, that is, it only uses $m^2 + 1$ of the n^2 variables in an expression in good coordinates. To separate $\ell^{n-m}\,\mathrm{perm}_m$ from $\det_n$, one must look for modules in $I(\mathcal{D}et_n)$ that do not vanish automatically on equations of hypersurfaces with these pathologies. It is easy to determine such modules with representation theory. Before doing so, the irreducible representations of the general linear group will be reviewed below.

$GL(V)$-modules

Let V be a complex vector space of dimension $\mathbf{v}$. The irreducible representations of $GL(V)$ are indexed by sequences of integers $\pi = (p_1, \dots, p_{\mathbf{v}})$ with $p_1 \geq \cdots \geq p_{\mathbf{v}}$ and the corresponding module is denoted $S_\pi V$. The representations occurring in the tensor algebra of V are those with $p_{\mathbf{v}} \geq 0$, i.e. by partitions. For a partition π, let $\ell(\pi)$ denote its *length*, the smallest s such that $p_{s+1} = 0$. In particular, $S_{(d)}V = S^dV$, and $S_{(1,\dots,1)} =: S_{(1^d)}V = \Lambda^d V \subset V^{\otimes d}$, the skew-symmetric tensors.

One way to construct $S_\pi V$, where $\pi = (p_1, \dots, p_{\mathbf{v}})$ and its conjugate partition is $\pi' = (q_1, \dots, q_{p_1})$, is to form a projection operator from $V^{\otimes|\pi|}$ by first projecting to $\Lambda^{q_1}V\otimes\cdots\otimes\Lambda^{q_{p_1}}V$ by skew-symmetrising and then re-ordering and projecting the image to $S^{p_1}V\otimes\cdots\otimes S^{p_{\mathbf{v}}}V$. In particular, if an element of $V^{\otimes|\pi|}$ lies in some $W^{\otimes|\pi|}$ for some $W \subset V$ with $\dim W < q_1$ then it will map to zero.

Polynomials useful for GCT

To be useful for GCT, a module of polynomials should not vanish identically on cones or on polynomials that are divisible by a large power of a linear form. The equations for the variety of polynomials whose zero sets are cones are well known – they are all modules where the length of the partition is longer than the number of variables needed to define the polynomial.

Proposition 14. [KL14] Necessary conditions for a module $S_\pi\mathbb{C}^{n^2} \subset I_d(\mathcal{D}et_n)$ to not vanish identically on polynomials in m^2 variables padded by ℓ^{n-m} are:
1. $\ell(\pi) \leq m^2 + 1$.
2. If $\pi = (p_1, \dots, p_t)$ then $p_1 \geq d(n-m)$.

Moreover, if $p_1 \geq \min\{d(n-1), dn-m\}$ then the necessary conditions are also sufficient. In particular, for p_1 sufficiently large, these conditions depend only on the partition π and not on how the module $S_\pi\mathbb{C}^{n^2}$ is realised as a space of polynomials.

7 The programme to find modules in $I[\mathcal{D}et_n]$ via representation theory

The programme initiated in [MS01] and developed in [BLMW11, MS08] to find modules in the ideal of $\mathcal{D}et_n$ will be presented in this section.

Preliminaries

Let $W = \mathbb{C}^{n^2}$ and consider $\det_n \in S^nW^*$. Define $\mathbb{C}[\mathcal{D}et_n] := Sym(S^nW)/I(\mathcal{D}et_n)$, the homogeneous coordinate ring of $\hat{\mathcal{D}}et_n$. This is the space of polynomial functions on $\hat{\mathcal{D}}et_n$ inherited from polynomials on the ambient space S^nW.

Since $Sym(S^nW)$ and $I(\mathcal{D}et_n)$ are $GL(W)$-modules, so is $\mathbb{C}[\mathcal{D}et_n]$ and, since $GL(W)$ is reductive (a complex algebraic group G is reductive if $U \subset V$ is a G-submodule of a G-module V and there exists a complementary G-submodule U^c such that $V = U \oplus U^c$), we obtain the splitting as a $GL(W)$-module:

$$Sym(S^nW) = I(\mathcal{D}et_n) \oplus \mathbb{C}[\mathcal{D}et_n].$$

In particular, if a module $S_\pi W$ appears in $Sym(S^nW)$ and it does not appear in $\mathbb{C}[\mathcal{D}et_n]$, it must appear in $I(\mathcal{D}et_n)$.

For those not familiar with the ring of regular functions on an affine algebraic variety, consider $GL(W) \subset \mathbb{C}^{n^2+1}$ as the subvariety of $\mathbb{C}^{n^2+1}$, with coordinates (x^i_j, t) given by the equation $t\det(x) = 1$ and $\mathbb{C}[GL(W)]$ defined to be the restriction of polynomial functions on $\mathbb{C}^{n^2+1}$ to this subvariety. Then, $\mathbb{C}[GL(W)\cdot\det_n] = \mathbb{C}[GL(W)/G_{\det_n}]$ can be defined as the subring of $G_{\det_n}$-invariant functions $\mathbb{C}[GL(W)]^{G_{\det_n}}$. Here, $G_{\det_n} := \{g \in GL(W) \mid g\cdot\det_n = \det_n\} \simeq SL_n \times SL_n \ltimes \mathbb{Z}_2$. A nice proof of this result (originally due to Frobenius [Fro97]) is due to Dieudonné [Die49] (see [Lan15] for an exposition). It relies on the fact that, in analogy with a smooth quadric hypersurface, there are two families of maximal linear spaces on the Grassmannian $G(n^2-n, \mathbb{C}^n\otimes\mathbb{C}^n)$ with prescribed dimensions of their intersections. One then uses that the group action must preserve these intersection properties.

There is an injective map

$$\mathbb{C}[\mathcal{D}et_n] \to \mathbb{C}[GL(W)\cdot\det_n]$$

given by restriction of functions. The map is an injection because any function identically zero on a Zariski open subset of an irreducible variety is identically zero on the variety. The algebraic Peter-Weyl theorem below gives a description of the G-module structure of $\mathbb{C}[G/H]$ when G is a reductive algebraic group and H is a subgroup.

Plan of [MS01, MS08]: *Find a module $S_\pi W$ not appearing in $\mathbb{C}[GL(W)/G_{\det_n}]$ that does appear in $Sym(S^nW)$.*

By the above discussion, such a module must appear in $I(\mathcal{D}et_n)$.

One might object that the coordinate rings of different orbits could coincide, or at least be very close. Indeed, this is the case for generic polynomials but, in GCT, one generally restricts to polynomials whose symmetry groups are not only "large" but they *characterise* the orbit as follows:

Definition 15. Let V be a G-module. A point $P \in V$ is *characterised by* its stabiliser G_P if any $Q \in V$ with $G_Q \supseteq G_P$ is of the form $Q = cP$ for some constant c.

One can think of polynomial sequences that are complete for their complexity classes and are characterised by their stabilisers as "best" representatives of their class. Corollary 17 will imply that if $P \in S^dV$ is characterised by its stabiliser, the coordinate ring of its G-orbit is unique as a module among orbits of points in V.

The algebraic Peter-Weyl theorem

Let G be a complex reductive algebraic group (e.g. $G = GL(W)$) and let V be an irreducible G-module. Given $v \in V$ and $\alpha \in V^*$, define a function $f_{v,\alpha} : G \to \mathbb{C}$ by $f_{v,\alpha}(g) = \alpha(g\cdot v)$. These are regular functions and it is not hard to see that one obtains an inclusion $V\otimes V^* \subset \mathbb{C}[G]$. Such functions are called *matrix coefficients* because, if one takes bases, these functions are spanned by the elements of the matrix $\rho(g)$, where $\rho : G \to GL(V)$ is the representation. In fact, the matrix coefficients span $\mathbb{C}[G]$:

Theorem 16. [Algebraic Peter-Weyl theorem] Let G be a reductive algebraic group. Then, there are only countably many non-isomorphic, irreducible, finite dimensional G-modules. Let Λ^+_G denote a set indexing the irreducible G-modules and let V_λ denote the irreducible module associated to $\lambda \in \Lambda^+_G$. Then, as a $G\times G$-module,

$$\mathbb{C}[G] = \bigoplus_{\lambda\in\Lambda^+_G} V_\lambda\otimes V^*_\lambda.$$

For a proof and discussion, see, for example, [Pro07].

Corollary 17. Let $H \subset G$ be a closed subgroup. Then, as a G-module,

$$\mathbb{C}[G/H] = \mathbb{C}[G]^H = \bigoplus_{\lambda\in\Lambda_G^+} V_\lambda \otimes (V_\lambda^*)^H = \bigoplus_{\lambda\in\Lambda_G^+} V_\lambda^{\oplus \dim(V_\lambda^*)^H}.$$

Here, G acts on the V_λ and $(V_\lambda^*)^H$ is just a vector space whose dimension records the multiplicity of V_λ in $\mathbb{C}[G/H]$.

Corollary 17 motivates the study of polynomials characterised by their stabilisers: if $P \in V$ is characterised by its stabiliser then $G \cdot P$ is the unique orbit in V with coordinate ring isomorphic to $\mathbb{C}[G \cdot P]$ as a G-module. Moreover, for any $Q \in V$ that is not a multiple of P, $\mathbb{C}[G \cdot Q] \not\subset \mathbb{C}[G \cdot P]$.

Schur–Weyl duality

The space $V^{\otimes d}$ is acted on by $GL(V)$ and $\mathfrak{S}_d$ (permuting the factors) and these actions commute so we may decompose it as $GL(V) \times \mathfrak{S}_d$-module. The decomposition is

$$V^{\otimes d} = \bigoplus_{\pi \mid |\pi|=d} S_\pi V \otimes [\pi],$$

where $[\pi]$ is the irreducible $\mathfrak{S}_d$-module associated to the partition π (see, for example, [Mac95]). This gives us a second definition of $S_\pi V$ when π is a partition: $S_\pi V = \mathrm{Hom}_{\mathfrak{S}_d}([\pi], V^{\otimes d})$.

The coordinate ring of $GL(W) \cdot \det_n$

Let $E, F \simeq \mathbb{C}^n$. We first compute the $SL(E)\times SL(F)$-invariants in $S_\pi(E \otimes F)$, where $|\pi| = d$. As a $GL(E) \times GL(F)$-module, since $(E\otimes F)^{\otimes d} = E^{\otimes d} \otimes F^{\otimes d}$,

$$\begin{aligned} S_\pi(E\otimes F) &= \mathrm{Hom}_{\mathfrak{S}_d}\left([\pi], E^{\otimes d}\otimes F^{\otimes d}\right) \\ &= \mathrm{Hom}_{\mathfrak{S}_d}\left([\pi], (\bigoplus_{|\mu|=d}[\mu]\otimes S_\mu E)\otimes(\bigoplus_{|\nu|=d}[\nu]\otimes S_\nu F)\right) \\ &= \bigoplus_{|\mu|=|\nu|=d} \mathrm{Hom}_{\mathfrak{S}_d}([\pi], [\mu]\otimes[\nu]) \otimes S_\mu E \otimes S_\nu F \end{aligned}$$

The vector space $\mathrm{Hom}_{\mathfrak{S}_d}([\pi], [\mu]\otimes[\nu])$ simply records the multiplicity of $S_\mu E \otimes S_\nu F$ in $S_\pi(E\otimes F)$. The integers $k_{\pi\mu\nu} = \dim \mathrm{Hom}_{\mathfrak{S}_d}([\pi], [\mu]\otimes[\nu])$ are called *Kronecker coefficients*.

Now, $S_\mu E$ is a trivial $SL(E)$ module if and only if $\mu = (\delta^n)$ for some $\delta \in \mathbb{Z}$. Thus, so far, we are reduced to studying the Kronecker coefficients $k_{\pi\delta^n\delta^n}$. Now, take the $\mathbb{Z}_2$ action given by exchanging E and F into account. Write $[\mu]\otimes[\mu] = S^2[\mu] \oplus \Lambda^2[\mu]$. The first module will be invariant under $\mathbb{Z}_2 = \mathfrak{S}_2$ and the second will transform its sign under the transposition. So, define the *symmetric Kronecker coefficients* $sk^\pi_{\mu\mu} := \dim(\mathrm{Hom}_{\mathfrak{S}_d}([\pi], S^2[\mu]))$. For a $GL(V)$-module M, write M_{poly} for the submodule consisting of isotypic components of modules $S_\pi V$, where π is a partition.

We conclude with:

Proposition 18. [BLMW11] Let $W = \mathbb{C}^{n^2}$. The polynomial part of the coordinate ring of the $GL(W)$-orbit of $\det_n \in S^n W$ is

$$\mathbb{C}[GL(W)\cdot \det_n]_{poly} = \bigoplus_{d\in\mathbb{Z}_+} \bigoplus_{\pi \mid |\pi|=nd} (S_\pi W^*)^{\oplus sk^\pi_{d^n d^n}}.$$

8 Asymptotics of plethysm and Kronecker coefficients via geometry

The above discussion can be summarised as:

Goal: Find partitions π which have $\mathrm{mult}(S_\pi W, S^d(S^n W)) \neq 0$, $sk^\pi_{d^n d^n} = 0$ and have few parts, with a first part large.

Kronecker coefficients and the plethysm coefficients $\mathrm{mult}(S_\pi W, S^d(S^n W))$ have been well-studied in both geometry and combinatorics literature. Here, a geometric method will be discussed of L. Manivel and J. Wahl [Wah91, Man97, Man98, Man15], based on the *Borel-Weil theorem* that realises modules as spaces of sections of vector bundles on homogeneous varieties. Advantages of the method are: (i) the vector bundles come with filtrations that allow one to organise information, (ii) the sections of the associated graded bundles can be computed explicitly, giving one upper bounds for the coefficients, and (iii) Serre's theorem on the vanishing of sheaf cohomology tells one that the upper bounds are achieved asymptotically.

A basic, if not *the* basic problem in representation theory is: given a group G, an irreducible G-module U and a subgroup $H \subset G$, decompose U as an H-module. The determination of Kronecker coefficients can be phrased this way with $G = GL(V\otimes W)$, $U = S_\lambda(V\otimes W)$ and $H = GL(V) \times GL(W)$. The determination of plethysm coefficients may be phrased as the case $G = GL(S^n V)$, $U = S^d(S^n V)$ and $H = GL(V)$.

Focusing on plethysm coefficients, we want to decompose $S^d(S^n V)$ as a $GL(V)$-module or, more precisely, to obtain qualitative asymptotic information about this decomposition. Note that $S^{dn} V \subset S^d(S^n V)$ with multiplicity one. Let $x_1, \ldots, x_{\mathbf{v}}$ be a basis of V, so $((x_1)^n)^d$ is the highest weight vector in $S^d(S^n V)$. (A vector $v \in V$ is a highest weight vector for $GL(W)$ if $B[v] = [v]$, where $B \subset GL(W)$ is the subgroup of upper triangular matrices. There is a partial order on the set of highest weights.) Say $S_\pi V \subset S^d(S^n V)$ is realised with highest weight vector

$$\sum_I c^I (x_{i_{11}} \cdots x_{i_{1n}}) \cdots (x_{i_{d1}} \cdots x_{i_{dn}})$$

for some coefficients c^I, where $I = \{i_{s,\alpha}\}$. Then,

$$\sum_I c^I (x_1)^n (x_{i_{11}} \cdots x_{i_{1n}}) \cdots (x_{i_{d1}} \cdots x_{i_{dn}}) \in S^{d+1}(S^n V)$$

is a vector of weight $(n) + \pi$ and is a highest weight vector. Similarly,

$$\sum_I c^I (x_1 x_{i_{11}} \cdots x_{i_{1n}}) \cdots (x_1 x_{i_{d1}} \cdots x_{i_{dn}}) \in S^d(S^{n+1} V)$$

is a vector of weight $(d) + \pi$ and is a highest weight vector. This already shows qualitative behaviour if we allow the first part of a partition to grow:

Proposition 19. [Man97] Let μ be a fixed partition. Then, $\mathrm{mult}(S_{(dn-|\mu|,\mu)}, S^d(S^n V))$ is a non-decreasing function of both d and n.

One way to view what has just been done is to write $V = x_1 \oplus T$, so

$$S^n(x_1 \oplus T) = \bigoplus_{j=0}^{n} x_1^{n-j} \otimes S^j T. \tag{8}$$

Then, decompose the d-th symmetric power of $S^n(x_1 \oplus T)$ and examine the stable behaviour as we increase d and n. One could think of the decomposition (8) as the osculating sequence of the n-th Veronese embedding of $\mathbb{P}V$ at $[x_1^n]$ and the further decomposition as the osculating sequence of the d-th Veronese re-embedding of the ambient space refined by (8).

For Kronecker coefficients and more general decomposition problems, the situation is more complicated in that the ambient space is no longer a projective space but a homogeneous variety and, instead of an osculating sequence, one examines jets of sections of a vector bundle. As mentioned above, in this situation, one gets the bonus of vanishing theorems. For example, with the use of vector bundles, Proposition 19 can be strengthened to say that the multiplicity is eventually constant and state for which d, n this constant multiplicity is achieved.

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Note
While this article was in press, the Goal stated on p. 17 was proven to be unachievable in [IP15]. Nevertheless there continues to be substantial work towards Conjecture 8 using other geometric approaches.

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Non-Integrable Distributions and the *h*-Principle

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Topologically stable distributions were locally classified by E. Cartan. He showed that they corresponded to four classes of "non-integrable" distributions, giving rise to four different geometries. This article reviews their history. Special care is devoted to the latest developments of Engel geometry.

1 Introduction

Fix a smooth n-dimensional manifold M. An r-dimensional distribution over M is a rank r vector bundle $\mathcal{D} \to M$ together with a monomorphism of vector bundles

$$\begin{tikzcd} \mathcal{D} \arrow[r] \arrow[d] & TM \arrow[d] \\ M \arrow[r, "id"] & M. \end{tikzcd}$$

It is simple to give a topological classification of these objects. Two distributions $\mathcal{D}_0$ and $\mathcal{D}_1$ are equivalent if they represent the same abstract vector bundle $\mathcal{D}_0 \simeq \mathcal{D}_1 \simeq \mathcal{D}$ and there is a homotopy of monomorphisms $i_t : \mathcal{D} \to TM$ connecting the two monomorphisms. The existence and classification of distributions is an obstruction theory problem, lying in the realm of algebraic topology. Let us briefly explain it.

Realise that distributions are in 1-to-1 correspondence with sections of a fiber bundle $\mathcal{G} \to M$ with fiber over a point p equal to $\mathcal{G}_p = Gr(r, T_pM) \simeq O(n)/(O(r) \times O(n-r))$. Therefore, classical obstruction theory tells us that the obstruction for a section to exist is measured by a sequence of cohomology classes in the groups

$$H^{j+1}(M, \pi_j(\mathcal{G}_p)), \qquad j = 0, \ldots, n-1.$$

Each class controls the extension of the section from the $(j-1)$–skeleton to the j–skeleton of the manifold with respect to some fixed CW–decomposition. Here is an example. Assume that $r = n-1$ and that orientations have been fixed both in the manifold and in the vector bundles. The fiber $\mathcal{G}_p$ is diffeomorphic to the sphere $\mathbb{S}^{n-1}$. This implies that all the cohomology groups are trivial, except the last one, so the section can always be extended to the $(n-1)$–skeleton; the only obstruction lies in $H^n(M, \pi_{n-1}(\mathbb{S}^{n-1})) = H^n(M, \mathbb{Z})$ and it is easily identified as the Euler class $e(M)$ of the tangent bundle of the manifold M. This is just telling us that the necessary and sufficient condition for an oriented manifold to admit a codimension 1 distribution is to have vanishing Euler characteristic.

After the existence question has been answered, we study how many homotopy classes of k–dimensional distributions there are on M. In this case, the obstruction for a pair of sections of $\mathcal{G} \to M$ to be homotopic is controlled by a sequence of cohomology classes in

$$o_j \in H^j(M, \pi_j(\mathcal{G}_p)), \quad j = 0, \ldots, n.$$

We compute the obstructions for $r = n - 1$ in the oriented setting. In that case, there are two of them. The first one appears with $j = n - 1$: the class $o_{n-1} \in H^{n-1}(M, \pi_{n-1}(\mathbb{S}^{n-1})) = H^{n-1}(M, \mathbb{Z})$. For instance, for $n = 3$, $o_2 = e(\mathcal{D}_0) - e(\mathcal{D}_1)$. That is, the condition for the two distributions to be deformable over the 2-skeleton is that they possess the same Euler class as abstract vector bundles. The last obstruction lies in $H^n(M, \pi_n(\mathbb{S}^{n-1}))$ and it is an Hopf invariant.

We have solved a purely topological problem. However, we can impose "geometrical" conditions to the distributions. The most classical one is the integrability condition. We say that a distribution $\mathcal{D}$ is integrable at a point $p \in M$ if

$$[\mathcal{D}_p, \mathcal{D}_p] \subset \mathcal{D}_p. \tag{1}$$

The reason for the name is the following result.

Theorem 1 (Frobenius)**.** A distribution defines a foliation if and only if it is everywhere integrable.

We need to introduce what a foliation is. Abusing language, a distribution $\mathcal{D}$ defines a foliation if for any point $p \in M$ it admits a foliated chart in a neighbourhood of the point. A chart $\phi_p : U_p \to \mathbb{V}_p \subset \mathbb{R}^n$ is called foliated if $(\phi_p)_*\mathcal{D} = \mathcal{D}_0$, where $\mathcal{D}_0 = \mathbb{R}^k \oplus \{0\} \subset \mathbb{R}^k \oplus \mathbb{R}^{n-k} = \mathbb{R}^n$, using the usual identification between $\mathbb{R}^n$ and the fiber of its tangent bundle at each point.

Define the space

$$\mathcal{D}\text{ist}(k, M) = \{k\text{–dimensional distributions}\},$$

endowed with the compact–open topology. In the previous discussion, we understood some of its global topology: the number of connected components it has. Similar arguments provide information about its higher homotopy groups. On the other hand, we can define

$$\mathcal{F}\text{ol}(k, M) = \{k\text{–dimensional foliations}\},$$

which is a more geometric entity, since its elements are defined by the partial differential equation (1). One of the goals of the theory of foliations is to understand the topology of the natural inclusion map

$$\iota : \mathcal{F}\text{ol}(k, M) \to \mathcal{D}\text{ist}(k, M). \tag{2}$$

The initial dream was to show that the map was a homotopy equivalence and hence that the classification of foliations up to deformation was purely topological. In codimension greater than one, it was fairly simple to produce connected components of the distribution space that did not possess a foliation representing them [8]. However, the codimension 1 case was completely different. First of all, there is the following theorem.

Theorem 2 ([41])**.** For codimension 1 distributions, the map $\pi_0(\iota)$ is surjective in any manifold.

On the other hand, the injectivity of $\pi_0(\iota)$ has remained an important open problem for many years. Quite recently (2014), the following result has been proven by Hélène Eynard-Bontemps.

Theorem 3 ([20])**.** For codimension 1 distributions, the map $\pi_0(\iota)$ is injective in any 3–manifold.

There is a chance that the result may be true in all dimensions. These results give a topological flavour to codimension 1 foliation theory.

The goal of this note is to discuss some other classes of distributions possessing a local model and also defined by a *geometric* condition. We will show that we have to go to the opposite end of the spectrum: maximally non-integrable distributions.

2 Local theory

Definition 4. An open subset $\mathcal{O} \subset Dist(k, M)$ is called topologically stable if, $\forall p \in M$, the germs of distributions of $\mathcal{O}$ at p act transitively by the group of germs of diffeomorphisms at p.

An obvious corollary is that such a class has to be Diff–invariant. A way of rephrasing it is to say that any $\mathcal{D} \in \mathcal{O}$ has a local canonical model.

Let us study the case $k = 1$ (line fields). The germs of diffeomorphisms act locally transitively in $Dist(1, M)$. Thus, if there is a topologically stable class, it has to be $\mathcal{O} = Dist(1, M)$. This is indeed the case.

Lemma 5 (Flow-box Theorem)**.** Let l be a germ of a line field at the origin of $\mathbb{R}^n$. There exists a chart at the origin $\varphi : U \subset \mathbb{R}^n \to V \subset \mathbb{R}^n$ such that $\varphi_* l = \langle \partial x_1 \rangle$.

The informal classification of topologically stable classes of distributions goes back to Cartan and is summarised in the following.

Theorem 6. ([43]) Any class of topologically stable distributions belongs to one of the following families:
1. n arbitrary, $k = 1$; $\mathcal{O} = Dist(1, M)$.
2. n odd, $k = n-1$; $\mathcal{O} \subsetneq Dist(n-1, M)$. $\mathcal{O}$ even-contact class.
3. n even, $k = n-1$; $\mathcal{O} \subsetneq Dist(n-1, M)$. $\mathcal{O}$ contact class.
4. $n = 4$, $k = 2$; $\mathcal{O} \subsetneq Dist(2, M)$. $\mathcal{O}$ Engel class.

Sketch of the proof.
The space of k–dimensional distributions in $\mathbb{R}^n$ is defined as the space of maps $\varphi : \mathbb{R}^n \to Gr(k, \mathbb{R}^n)$.

Fixing a distribution at the origin $\mathcal{D}(0) \in Gr(k, \mathbb{R}^n)$, we can take a chart of the Grassmannian around it $\psi : \mathcal{U}_{\mathcal{D}_0} \to \mathbb{R}^{k(n-k)}$. Therefore, a germ of distribution is given by $\psi \circ \varphi : \mathbb{R}^n \to \mathbb{R}^{k(n-k)}$ (in other words, by $k(n-k)$ local functions).

Now, a germ of diffeomorphism is given by a map $F : \mathbb{R}^n \to \mathbb{R}^n$, i.e. $F = (f_1, \cdots, f_n)$ (n local functions). If the diffeomorphisms act transitively on the distribution, we need this "functional dimension" to be bigger than or equal to the one of the space of distributions. We deduce that

$$n \geq k(n-k).$$

Solving this equation in the natural numbers provides a necessary condition for the existence of a topologically stable class. Note that the solution pairs (k, n) correspond to the four cases in the statement.

We need to find sufficient conditions. As has already been stated, Lemma 5 concludes the result for $k = 1$, n arbitrary.

For $k = n - 1$, the generic situation for a generic point $p \in M$ is that $[\mathcal{D}_p, \mathcal{D}_p] = T_pM$. Moreover, we have, generically, that the morphism

$$\phi_p := [\cdot, \cdot] : \mathcal{D}_p \times \mathcal{D}_p \to T_pM/\mathcal{D}_p \qquad (3)$$

is a bilinear skew-symmetric form of maximal rank.

For n odd, maximal rank implies that ϕ_p is a non-degenerate bilinear skew-symmetric form. If we impose this non-degeneracy condition at a given point, we say that $\mathcal{D}$ is a contact distribution at p. The contact condition is dense and open. Moreover, it is a Diff–invariant condition. This implies that any candidate class $\mathcal{O}$ has to contain the set of contact distributions. The following lemma is well known.

Lemma 7 (Darboux)**.** Let (M, ξ^{2n}) be a contact manifold. Then, for any $p \in M$,

$$\exists \varphi_p : \mathcal{U}_p \to \mathbb{R}^{2n+1}(z, x_1, y_1, \cdots x_n, y_n)$$

such that $(\varphi_p)_* \xi = \ker(dz + \sum x_i dy_i)$.

This concludes the proof of the second case.

For n even, maximal rank implies that ϕ_p is a bilinear skew-symmetric form with 1–dimensional kernel $\mathcal{W}_p \subset \mathcal{D}_p$. Under this maximal rank hypothesis we say that the distribution is even-contact. Being open, dense and Diff–invariant implies that any topologically stable class needs to contain the even-contact distributions. To conclude, we use another lemma.

Lemma 8 (Darboux)**.** Let (M, ξ^{2n-1}) be an even-contact manifold. Then, for any $p \in M$,

$$\exists \varphi_p : \mathcal{U}_p \to \mathbb{R}^{2n}(z, x_1, y_1, \cdots x_{n-1}, y_{n-1}, t)$$

such that $(\varphi_p)_* \xi = \ker(dz + \sum x_i dy_i)$.

We are left with the case $(k = 2, n = 4)$. Again, a generic distribution at a generic point is not integrable, i.e. $[\mathcal{D}_p, \mathcal{D}_p] = \mathcal{E}_p$, where $\mathcal{E}_p$ is a 3–dimensional vector space. But now, we can further apply the Lie bracket and again, generically, $[\mathcal{E}_p, \mathcal{E}_p] = T_pM$. We say that if the two previous conditions are met at p then $\mathcal{D}$ is Engel at p.

Any sensible choice of topologically stable class needs to contain the Engel distributions. They owe their name to F. Engel, who proved the canonical local model theorem.

Lemma 9 ([18])**.** Let $(M, \mathcal{D}^2)$ be a distribution that is Engel everywhere. Then, for any $p \in M$,

$$\exists \varphi_p : \mathcal{U}_p \to \mathbb{R}^4(z, x, y, t)$$

such that $(\varphi_p)_* \mathcal{D} = \ker(dz + x dy) \cap \ker(dx + t dy)$.

This concludes the proof. □

We have isolated the four classes of distributions that do not possess local geometry. Therefore, any non–trivial phenomenon they might display must be of a global nature. The reader is surely familiar with the first case: dynamical systems – the study of the global properties of line fields. The second case relates to contact topology, a rapidly growing area within differential topology. We will discuss some of its history. Finally, we will show what can be expected from global even-contact geometry and global Engel geometry.

3 The h-principle

The next aim is to understand inclusions like the one given in (2). There will be a set of objects whose nature is *topological* (e.g., distributions) and, within it, a subset of objects for which a certain *geometric* condition holds (e.g., foliations, as in the example above). This is the mantra behind Gromov's h-principle.

Jet spaces

Let $f : \mathbb{R}^n \to \mathbb{R}^q$ be a smooth map. The r–jet of f at $v \in \mathbb{R}^n$ is $J^r_f(v) = (v, f(v), f^{(1)}(v), \ldots, f^{(r)}(v)) \in \mathbb{R}^{n+N}$, where $N = q\frac{(n+r)!}{n!r!}$.

The r-jet space of maps from $\mathbb{R}^n$ to $\mathbb{R}^q$ is defined to be $\mathbb{R}^n \times \mathbb{R}^N$.

Fix a locally trivial smooth fibration $X \to V$ with $\dim(V) = n$ and $\dim(X) = n + q$. For a given section $s : V \to X$, fix a fibration chart at $(v, x_v) \in X$, i.e. a couple of neighbourhoods $\mathcal{U}_x$ of x_v and $\mathcal{U}_v$ of v and diffeomorphisms $\varphi_x : \mathcal{U}_x \to \mathbb{R}^{n+q}$, $\varphi_v : \mathcal{U}_v \to \mathbb{R}^n$, making the following diagram commute:

$$\begin{array}{ccc} X \supset \mathcal{U}_x & \to & \mathbb{R}^{n+q} \\ \downarrow & & \downarrow \\ V \supset \mathcal{U}_v & \to & \mathbb{R}^n. \end{array}$$

We say that $s, \sigma : V \to X$ are r-tangent at (v, x_v) if the r-jets associated to $(\varphi_x)_* s$ and $(\varphi_x)_* \sigma$ at $(\varphi_v(v), \varphi_x(x_v))$ are the same. The relation is well-defined independently of the choice of charts. We define $X^{(r)}$ as the set of r-tangency classes. There is a natural chain of projections:

$$\begin{tikzcd} X^{(r)} \arrow[r] \arrow[rrrd] & X^{(r-1)} \arrow[r] \arrow[rrd] & \cdots \arrow[r] & X^{(0)} = X \arrow[d] \\ & & & V \end{tikzcd}$$

Given a section $\sigma : V \to X$, we can build a section $j_r\sigma : V \to X^{(r)}$ that is called the associated r-jet. A section $F : V \to X^{(r)}$ is called holonomous if $\exists f : V \to X$ such that $j_r f = F$.

Classical h-principle

A subset $\mathcal{R} \subset X^{(r)}$ is called a partial differential relation. Denote by $\mathcal{S}\mathrm{ec}(\mathcal{R})$ the space of sections of the fibration $X^{(r)} \to V$ such that $s(v) \in \mathcal{R}$, $\forall v \in V$. Denote

$$\mathcal{H}\mathrm{ol}(\mathcal{R}) = \{s \in \mathcal{S}\mathrm{ec}(\mathcal{R}) \text{ such that } s \text{ is holonomous}\}.$$

Examples:

1. V arbitrary, $X = V \times W$. A section is just a pair $(id, f) : V \to V \times W$, where f is a differentiable map. $\mathcal{R}_{imm} \subset X^{(1)}$ is defined as the set $(v, w, A) \in X^{(1)}$, where $A \in End(T_vV, T_wW)$ has rank n. A holonomous section $j_1\sigma : V \to \mathcal{R}_{imm} \subset X^{(1)}$ is an immersion and $\mathcal{H}ol(\mathcal{R})$ is the space of immersions.
2. V arbitrary, $X \to V$ the fibration with fiber $X_v = Gr(n-1, T_vV)$ at $v \in V$. A section of X is nothing but an $(n-1)$-dimensional distribution. Being contact (resp. even-contact) is a relation $\mathcal{R}_{cont} \subset X^{(1)}$ (resp. $\mathcal{R}_{even-ct}$) and a contact (resp. even-contact) distribution is an element of $\mathcal{H}\mathrm{ol}(\mathcal{R}_{cont})$ (resp. $\mathcal{H}\mathrm{ol}(\mathcal{R}_{even-ct})$).
3. V a 4-dimensional manifold, $X \to V$ the fibration with fiber $X_v = Gr(2, T_vV)$. A section of X is a 2-distribution.

The Engel condition is a relation $\mathcal{R}_{Eng} \subset X^{(2)}$ and an Engel distribution is an element of $\mathcal{H}\mathrm{ol}(\mathcal{R}_{Eng})$.

A fibration $X \to V$ is said to be natural if there is a lift of the group $\mathrm{Diff}(V)$ to X. A relation $\mathcal{R}$ is $\mathrm{Diff}(V)$-invariant if it is invariant under the action of this lift. The three previous examples are both natural and $\mathrm{Diff}(V)$-invariant.

Theorem 10 (Holonomous Lemma [30, 17])**.** If $X \to V$ is natural, V is open and $\mathcal{R} \subset X^{(r)}$ is open and $\mathrm{Diff}(V)$-invariant then the inclusion

$$\mathcal{H}\mathrm{ol}(\mathcal{R}) \to Sec(\mathcal{R})$$

is a weak homotopy equivalence.

In other words, under the hypothesis, topology controls geometry.

Applications

Immersion theory

If $n \leq q$, $\mathcal{R}_{imm}$ satisfies the holonomous lemma. This says that the topology of the space of immersions between two differentiable manifolds (the source being open) can be derived from the space of *formal* immersions and, therefore, it is an obstruction theory problem. In particular, one immediately recovers Smale's eversion of the sphere theorem.

Contact geometry

We did not explicitly describe the relation $\mathcal{R}_{cont}$. It suffices to say that if $(v, \xi_v, A_v) \in \mathcal{R}_{cont}$ then the 1-jet component equips ξ_v with a bilinear skew-symmetric non-degenerate form[1] (a symplectic form). So any formal solution induces a "symplectic distribution" by (3). From a topological point of view, since $\mathrm{Sp}(2n)$ retracts to $\mathrm{U}(n)$, this is completely equivalent to the distribution being complex.

Therefore, a more elegant way of stating the result is to say that the natural inclusion

$$\mathrm{Cont}\,(M^{2n+1}) \to \mathcal{D}\mathrm{ist}_{\mathbb{C}}\,(2n, M^{2n+1}) \tag{4}$$

of the space of contact distributions into the space of complex distributions is a weak homotopy equivalence as long as M is an open manifold. A complex distribution will be called a formal contact structure and their classification is fully understood in terms of obstruction theory.

Even-contact geometry

A similar discussion in the even-contact case leads to the following definition. A formal even-contact structure in M^{2n} is a flag of distributions $\mathcal{W}^1 \subset \xi^{2n-1}$ and a fixed complex structure in $\xi/\mathcal{W}$. There is a natural inclusion map

$$\mathcal{E}\mathrm{Cont}(M) \to \mathcal{F}\,\mathcal{E}\mathrm{Cont}(M) \tag{5}$$

from the space of even-contact structures into the space of formal even-contact structures and Theorem 10 implies that this is a weak homotopy equivalence whenever M is open.

Engel geometry

Fix an Engel distribution $\mathcal{D}^2 \subset TM^4$. Recall that $[\mathcal{D}, \mathcal{D}] = \mathcal{E}^3$ is a rank 3 vector bundle and so we have an isomorphism of line bundles

$$[\cdot,\cdot] : \bigwedge^2 \mathcal{D} \to \mathcal{E}/\mathcal{D}. \tag{6}$$

The second non-integrability condition states that $[\mathcal{E}, \mathcal{E}] = TM$ and thus it induces a bilinear skew-symmetric morphism

$$[\cdot,\cdot] : \mathcal{E} \times \mathcal{E} \to TM/\mathcal{E}.$$

Recall that its kernel is a line bundle $\mathcal{W}$ and so we obtain a second isomorphism of line bundles

$$[\cdot,\cdot] : \bigwedge^2 (\mathcal{E}/\mathcal{W}) \to TM/\mathcal{E}. \tag{7}$$

Moreover, a linear algebra computation shows that $\mathcal{W} \subset \mathcal{D}$.

As in the contact and even-contact cases, $\mathcal{R}_{Eng}$ does not intersect all the connected components of the space of sections $X^{(2)} \to M^4$, only a few components are touched by $\mathcal{R}_{Eng}$ and their elements are called formal Engel structures. More explicitly, a formal Engel structure is a complete flag $\mathcal{W}^1 \subset \mathcal{D}^2 \subset \mathcal{E}^3 \subset TM^4$ together with a fixed pair of isomorphisms as in (6) and (7). Again, we have the natural inclusion

$$\mathcal{E}\mathrm{ng}(M^4) \to \mathcal{F}\,\mathcal{E}\mathrm{ng}(M^4) \tag{8}$$

of the space of Engel structures into the space of formal Engel structures. Theorem 10 implies that the inclusion is a weak homotopy equivalence whenever M is open. Recall that the elements in $\mathcal{E}\mathrm{ng}(M)$ are given by a *geometric* condition, whereas the ones in $\mathcal{F}\,\mathcal{E}\mathrm{ng}$ can be fully understood in terms of algebraic topology.

Closed manifolds

The previous discussion shows that the study of maximally non-integrable distributions is not very exciting when the manifold is open. Still, even though Theorem 10 does not apply to closed manifolds, another method of proof for h-principles called convex integration (also introduced by Gromov) does. It goes back to methods applied in PDEs and analysis.

Convex integration deals with a class of first order partial differential relations called ample. Gromov's work shows that the h-principle holds for ample relations in closed manifolds and, unfortunately for even contact geometry, $\mathcal{R}_{even}$ is one of them. Therefore, we have the following theorem.

Theorem 11 ([32])**.** For any even dimensional closed manifold, the inclusion (5) is a weak homotopy equivalence.

Thus, there is no global even-contact geometry, since all meaningful questions about even-contact structures reduce to studying their formal counterparts.

Fortunately for us, contact and Engel structures are not defined by an ample partial differential relation. In fact, the last 30 years of contact geometry have been a *tour de force* to show that the h-principle simply breaks down for those distributions. The beauty of the theory lies in the interplay between the failure of the h-principle (rigidity in contact geometry) and the testing of the limits of this failure (flexibility in contact geometry). This will be explained in Section 4.

Engel structures are still quite mysterious. They will be discussed in Section 5.

1 Being precise up to a positive scalar multiple, it therefore defines a conformal class.

4 Flexibility in contact geometry

Gromov proved in the late 1960s that the map (4) is a weak homotopy equivalence whenever the ambient manifold is open. For many years, the closed case was studied mainly in three dimensions. In the 1970s, the existence of a contact structure on any closed 3-manifold [34] was proven followed by the surjectivity of the map at π_0 level [31]. The birth of modern contact geometry can be considered to be the paper in which Bennequin proves that the map (4) fails to be injective in π_0 for some closed 3-manifolds [4].

The introduction of pseudo-holomorphic curves by Gromov [29] allowed for a deeper understanding of the failure of the h-principle in contact geometry, since they form the basis for many of the constructions of global invariants for contact distributions [16]. Pseudo-holomorphic curves have brought us examples of the lack of injectivity of the homotopy maps induced by (4): one particular instance of this is the fact that some formal classes admit infinitely many contact representatives [42]. Furthermore, [28] and [7] give examples of manifolds in which (4) is not injective at π_1 level.

However, some flexibility remains. In dimension 3, Eliashberg [14] proved that the map (4) induces surjections in homotopy. In fact, he proved much more: he was able to define a special subclass of contact distributions that he called overtwisted, such that the inclusion

$$\mathrm{Cont}_{ot}(M^3) \hookrightarrow \mathrm{Cont}(M^3) \to \mathcal{D}\mathrm{ist}(2, M^3) \tag{9}$$

induces a weak homotopy equivalence. The overtwisted condition was given by the existence of a particular local model of contact distribution in a ball: the overtwisted disc. The overtwisted class became a test for exotic behaviours in contact structures. Remarkably enough, all the classical constructions of contact distributions arising from physics (e.g., the space of contact elements over a differentiable manifold [2], the contact connection associated to a prequantum bundle [5] and a regular level of a plurisubharmonic function in a Stein domain [15]) are not overtwisted. The main reason is that overtwisted contact manifolds cannot be the boundary of a symplectic manifold.

Definition 12. A contact manifold (M, ξ) is said to be fillable if there exists a symplectic manifold (W, ω) satisfying:
1. $\partial W = M$.
2. There is a vector field X in a collar around M, positively transverse to M, such that the 1-form
$$i_X\omega = \lambda$$
satisfies $\ker \lambda|_M = \xi$ and $\omega = d\lambda$.

A direct consequence of [29] is the following proposition.

Proposition 13. Overtwisted contact manifolds are not fillable.

The classification of tight (i.e., non-overtwisted) contact 3-manifolds became a hot topic in the 1990s. This has produced a rich literature (see [25, 26, 12, 13] and the references within).

Higher dimensional contact geometry

In the last 25 years, there have been some attempts to generalise the overtwisted class to higher dimensions. One task was to study the map (4) in the 5 dimensional case. There is a sequence of articles that solve this question: simply connected 5-folds [21], with some specific finite fundamental group [24], etc. The most advanced result in this *prehistory* is the following theorem.

Theorem 14 ([10])**.** For any 5-fold, the map (4) induces a surjection at π_0 level.

Another goal of this *prehistory* was the attempt to generalise the notion of overtwisted. K. Niederkrüger introduced a definition built over the fillability property. Using Proposition 13 as the property to be generalised, he introduced a class of contact structures showing the following.

Proposition 15 ([36])**.** PS-overtwisted contact manifolds are not fillable.

Examples of such structures were immediately constructed.

Theorem 16 (see [40, 38, 19])**.** For any contact manifold (M, ξ), there exists a PS-overtwisted contact structure $\tilde{\xi}$ on M representing the same formal class as ξ. $(M, \tilde{\xi})$ is hence non-fillable.

More attempts of capturing the notion of overtwisted in higher dimension were made, in particular through the notion of adapted open book [27] and by measuring the size of the normal neighbourhoods of overtwisted contact submanifolds [37]. However, none of those *exotic* classes were shown to satisfy an h-principle.

Building upon flexibility phenomena detected for higher dimensional Legendrian embeddings, Borman, Eliashberg and Murphy have found the right notion of high dimensional overtwisted contact structure [6]. Again, it is done by imposing a particular contact distribution in a ball of the manifold: the overtwisted disc model. They prove that the inclusion

$$\mathrm{Cont}_{ot}\left(M^{2m+1}\right) \hookrightarrow \mathrm{Cont}\left(M^{2m+1}\right) \to \mathcal{D}\mathrm{ist}_{\mathbb{C}}\left(2m, M^{2m+1}\right) \tag{10}$$

induces a weak homotopy equivalence. Moreover, most of the prehistoric notions of overtwistedness (PS-overtwisted, open book based definition and size of neighbourhoods) has been shown to be equivalent [9]. This links the prehistory and the future of the area in a very satisfying way.

Now, the interplay that occurred between the flexible side and the rigid side of contact geometry in dimension 3 has begun to be replicated in higher dimension because, finally, we know what a *flexible* contact structure is.

5 Engel geometry

We are left with the most intriguing class. We have explained that even-contact structures are boring and contact structures are much deeper and very much explored. However, Engel structures have remained a mystery till very recently. Let us summarise what is known.

There is a classical construction of Engel distributions due to Cartan: the prolongation of a contact structure. For many years, this was the only known construction of Engel distributions for a reasonably ample class of closed 4-manifolds. Fix a contact 3-manifold (N^3, ξ). Construct the circle fiber bundle $M = \mathbb{P}(\xi) \xrightarrow{\pi} N$. Define the prolongation $\mathcal{D}(\xi)$ of ξ as follows: a vector $v \in T_pM$ belongs to $\mathcal{D}(\xi)_p$ if and only if $d\pi_p(v) \in \langle v \rangle \subset T_{\pi(p)}N$. A simple computation shows that

$$\mathcal{E} = [\mathcal{D}(\xi), \mathcal{D}(\xi)] = \pi^*\xi,$$

where $\pi^*\xi$ is a rank 3-bundle, the inverse image of ξ, defined as

$$\pi^*\xi(p) = \{v \in T_pM : \pi_* v \in \xi_{\pi(p)}\}.$$

The contactness of ξ immediately implies that $[\mathcal{E}, \mathcal{E}] = TM$ and so $\mathcal{D}(\xi)$ is an Engel distribution. Observe that $\mathcal{W} = \ker(d\pi)$. This construction was quite well understood and was an early example of developing Engel geometry [35].

The next existence result was largely unnoticed: H. Geiges was able to produce an Engel structure in a mapping torus of a 3-manifold [22]. It was the first modern construction that implicitly related the geometry of families of convex curves in the 2-sphere with the study of Engel structures (see Lemma 26 below). We say modern because there was another construction, going back to E. Cartan, that had been forgotten (see Corollary 28 below).

Still, the topological properties of the map (8) remained unknown.

Vogel's result

For a simpler description, let us assume that our formal Engel structure over the 4-manifold, that is, the flag, is orientable and oriented. Then, after a choice of metric, it induces a global orthonormal framing and we deduce that the 4-manifold is parallelisable. Then, the non-emptiness of the space $\mathcal{FE}ng(M^4)$ is equivalent to the parallelisability of M. Now, the meaning of the following result is clear.

Theorem 17 ([44]). Any parallelisable 4-manifold admits an Engel structure.

Stated in the language of the map (8), the result just says that whenever there is a formal Engel structure, there is an Engel structure. However, there is no control on the formal class of the produced Engel structure.

Vogel's proof is based on the notion of Engel cobordism. Let $(M, \mathcal{D})$ be an Engel structure and $(\mathcal{W} \subset \mathcal{D} \subset \mathcal{E} \subset TM)$ its associated flag. Assume that H is a 3-dimensional embedded submanifold of M such that $\mathcal{W}$ is transverse to it. Then, the 2-distribution $TH \bigcap \mathcal{E} \subset TH$ is contact over H.

Definition 18. A 4-dimensional manifold M is an Engel cobordism if it admits an Engel structure $\mathcal{D}$ such that $\mathcal{W}$ is transverse to the boundary of M.

An Engel cobordism has a canonical contact boundary.

Vogel starts by using D. Asimov's theorem [3] on the existence of a round handle decomposition for manifolds with Euler characteristic zero. Using the extra assumption on parallelisability, the round handles are formally Engel with $\mathcal{W}$ transverse to their boundaries. Vogel then defines model Engel structures in each round handle. Finally, he is left with the task of gluing the contact boundaries. For this, he uses many of the known flexibility properties for contact structures in dimension 3.

Existence *h*-principle

The first classification result for Engel structures is the following.

Theorem 19 ([11]). The map (8) induces surjections in homotopy groups π_j for all $j \geq 0$.

This, in particular, proves that given a full flag in the 4-manifold, there is a deformation through full flags such that the final flag is the flag associated to an Engel structure. The argument does not extend to prove injectivity. In fact, an *overtwisted* class might exist. The discovery of such a class would officially begin Engel topology as a sensible area of differential topology. On the other hand, if injectivity holds then the theory will be completed. I do not dare state a conjecture about what will be the answer.

Let us briefly explain the *state of the art* for a related problem in which the picture is somewhat clearer. The inspiration comes from contact geometry: there is a well developed branch that classifies n-dimensional embedded submanifolds tangent to a fixed rank $2n$ contact distribution. They are called Legendrian knots. As with contact distributions, their study can be formulated in terms of an h-principle. Again, they do not quite satisfy it. But there is a sub-class of them that does: the loose Legendrian knots [33]. There is a deep relationship between Legendrian knot theory and contact geometry. What is the equivalent notion in Engel geometry?

Definition 20. Fix an Engel distribution $(M, \mathcal{D})$. An Engel knot[2] is an embedding $\gamma : \mathbb{S}^1 \to M$ satisfying $\gamma'(t) \in \mathcal{D}_{\gamma(t)}$.

Realise that the non-integrability of the Engel condition allows us to C^0-approximate any smooth knot by an Engel one and therefore this class of objects is non–trivial. Moreover, we have an invariant by deformation. In order to define it, recall that the bundle $\mathcal{D}$ is canonically trivialised by $\mathcal{W}$ (oriented case).

Definition 21. The rotation number $r(\gamma)$ associated to an Engel knot γ is the degree of the map

$$r(\gamma) : \mathbb{S}^1 \to \mathbb{S}^1 \subset \mathbb{R}^2 \simeq \gamma^*\mathcal{D},$$
$$t \to \frac{\gamma'(t)}{\|\gamma'(t)\|}.$$

Definition 22. A formal Engel knot is a pair of maps $\gamma : \mathbb{S}^1 \to M$ and $F_s : \mathbb{S}^1 \times [0,1] \to T_{\gamma(t)}M$ such that $F_0(t) = \gamma'(t)$, $F_1(t) \in \mathcal{D}_{\gamma(t)}$ and $F_s(t) \neq 0$ for all (t,s).

The map F_1 is called the formal derivative of γ and we are just asking the formal derivative to be tangent to the Engel distribution in the spirit of the h-principle. There is an analogous definition of rotation invariant for formal Engel knots. As usual, we have a natural inclusion

$$\iota_{\mathcal{K}} : \mathcal{K}(M, \mathcal{D}) \to \mathcal{FK}(M, \mathcal{D}) \tag{11}$$

from the space of Engel knots into the space of formal Engel knots. Realise that the connected components of the space of formal knots are completely characterised by the class of γ as an element of $\pi_1(M)$ and the rotation number $r(F_1)$.

Theorem 23 ([1, 23]). The homotopy classes of Engel knots in $(\mathbb{R}^4, \mathcal{D}_{std})$ are determined by the rotation number.

2 They are usually called in the literature horizontal knots but I do not particularly like the name.

In other words, the map (11) is an isomorphism at π_0-level for standard $\mathbb{R}^4$. Two remarks are in order:
1. This sharply contrasts with the theory of Legendran knots in $(\mathbb{R}^{2m+1}, \xi_{std})$, where the injective h-principle strongly fails.
2. The proof does not completely adapt to the case of a general Engel manifold. The problem has to do with the estimate of the size of the Engel charts. My intuition is that it should be true but the proof needs to be detailed.

Out of Geiges' proof, one can easily guess the following conjecture.

Conjecture 24. There is a 1-parametric family of Engel knots on $\mathbb{R}^4$ standard, γ_θ, $\theta \in \mathbb{S}^1$, such that it defines a non–trivial element of $\pi_1(\mathcal{K}(\mathbb{R}^4, \mathcal{D}_{std}))$ and $[\gamma_\theta] \in \ker \iota_{\mathcal{K}}$.

This would provide the first example of rigidity in Engel topology.

6 Convex curves and Engel geometry

Let us explain the key geometric idea behind Theorem 19. Instead of sketching its proof, we will prove a related result that showcases the mantra behind the whole theory.

Theorem 25 ([39])**.** Let ξ_0 and ξ_1 be two co–oriented contact distributions representing the same formal contact class on M^3. Then, its Cartan extensions $(X = \mathbb{P}(\xi_0), \mathcal{D}_0)$ and $(X = \mathbb{P}(\xi_1), \mathcal{D}_1)$ are in the same connected component of the space of Engel structures on X.

We are just showing that the lack of injectivity of the map (4) partially disappears when we lift it to Cartan extensions. Realise that this is still far away from proving injectivity of the map (8), since Cartan extensions are a very special type of Engel structures and, even in the manifold X, Theorem 19 provides Engel structures very different from a Cartan extension.

We want to understand an Engel distribution $\mathcal{D}$ in a small 4-ball. Choose coordinates $(p, t) \in D^3 \times (0, 1)$ such that $\partial_t \in \mathcal{D}$; this can be done without loss of generality thanks to Lemma 5. Thus, the 2–distribution $\mathcal{D}$ is $\langle \partial_t, X\rangle$ for some vector field X. The behaviour of X determines whether $\mathcal{D}$ is Engel.

Without loss of generality, the vector field X can be chosen to lie in $\mathbb{S}^2 \times \{t\} \subset D^3 \times \{t\}$. The first non-integrability condition reads as $\mathcal{D}$ not being involutive, i.e.

$$[\partial_t, X] = \frac{\partial X}{\partial t} = \dot{X} \notin \mathcal{D}.$$

Then, $\mathcal{E} = \langle \partial_t, X, \dot{X}\rangle$. Geometrically, the condition says that for each fixed $p_0 \in D^3$ the curve $X(p_0, t) \in \mathbb{S}^2$ is *immersed*.

The second non-integrability condition reads as $[\mathcal{E}, \mathcal{E}] = TM$. There are three vectors generating $\mathcal{E}$ and we may choose any pair of them to try to escape $\mathcal{E}$. The possible choices are
1. $[\partial_t, X] = \dot{X} \in \mathcal{E}$.
2. $[\partial_t, \dot{X}] = \ddot{X}$.
3. $[X, \dot{X}]$.

Thus, for the second non-integrability condition to hold, either the curve X is *convex* at the given time t (choice 2) or the span $\langle X, \dot{X}\rangle$, for t fixed, is a *contact* distribution in a neighbourhood of p (choice 3) (See Figure 1). We summarise with the following lemma.

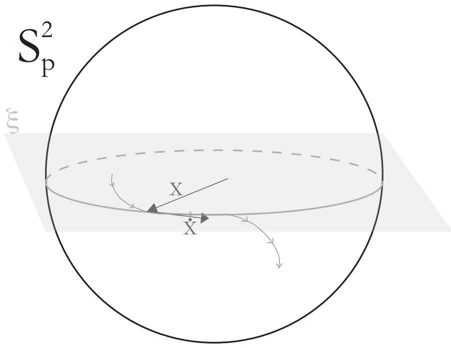

Figure 1. A curve $X(p_0, t)$ generating an Engel structure

Lemma 26. A sufficient condition for a 2-distribution $\mathcal{D} = \langle \partial_t, X(p, t)\rangle$ in $D^3 \times [0, 1](p, t)$ to be Engel at (p_0, t_0) is that $X(p_0, t)$ is an immersed curve at t_0 and either $X(p_0, t)$ is convex at t_0 or the 2-plane $\langle X, \dot{X}\rangle$ is contact around p_0 for t_0 fixed.

Corollary 27. The prolongation of a contact structure is Engel.

Proof. Fix $\partial_t \in \ker \pi$. Then, the curve $X(p_0, t)$ follows the equator defined by the intersection of the contact plane ξ_p and $\mathbb{S}^2_p \subset T_pM$. The second hypothesis of Lemma 26 applies everywhere. □

Fix a Lorentzian $(1, 2)$-metric h in a 3-fold N. Denote its light cone by $C \to N$. Construct the circle fiber bundle $M = \mathbb{P}(C) \xrightarrow{\pi} N$. Define the extension $\mathcal{D}(h)$ of h on M as follows: a vector $v \in T_pM$ belongs to $\mathcal{D}(h)_p$ if and only if $\pi_* v \in v \subset T_{\pi(p)}N$. This is called the prolongation of a Lorentzian manifold (its definition goes back to E. Cartan).

Corollary 28. The prolongation of a Lorentzian structure is Engel.

Proof. Fix $\partial_t \in \ker \pi$. Then, the curve $X(p_0, t)$ follows the non–maximal circle defined by the intersection of the light cone and $\mathbb{S}^2_{p_0} \subset T_{p_0}M$. The curve is everywhere convex and so the first hypothesis of Lemma 26 applies (see Figure 3). □

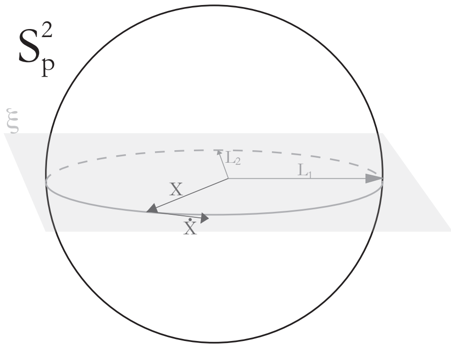

Figure 2. A contact prolongation

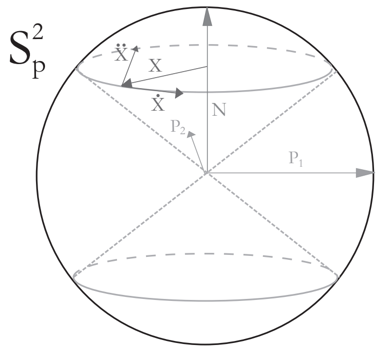

Figure 3. A Lorentzian prolongation

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Proof of Theorem 25.

Since ξ_0 and ξ_1 are in the same formal class, there is a family of co–oriented 2-planes ξ_t connecting them. Fix an auxiliary Riemannian metric g. Construct a bi–parametric family of Lorentzian metrics $h_{t,s}$, $(t,s) \in [0,1] \times (0,1]$. We declare $h_{t,s}(v_1, v_2) = s \cdot g(v_1, v_2)$ for all $v_1, v_2 \in \xi_t$. Define R_t to be the unitary, positively oriented (with respect to the co–orientation) vector field orthogonal to ξ_t with respect to g. Define $h_{t,s}(R_t, R_t) = -1$ and $h_{t,s}(R_t, v) = 0$ for all $v \in \xi_t$. Check that $h_{t,s}$ degenerates for $s = 0$ and its light cone converges to ξ_t. We have that the family of Lorentzian prolongations

$$\mathcal{D}_u = \begin{cases} \mathcal{D}(h_{0,3u}) & u \in (0, 1/3], \\ \mathcal{D}(h_{3u-1,1}) & u \in [1/3, 2/3], \\ \mathcal{D}(h_{1,3-3u}) & u \in [2/3, 1), \end{cases}$$

connects $\mathcal{D}(\xi_0)$ and $\mathcal{D}(\xi_1)$. Figure 4 pictorially presents the construction. □

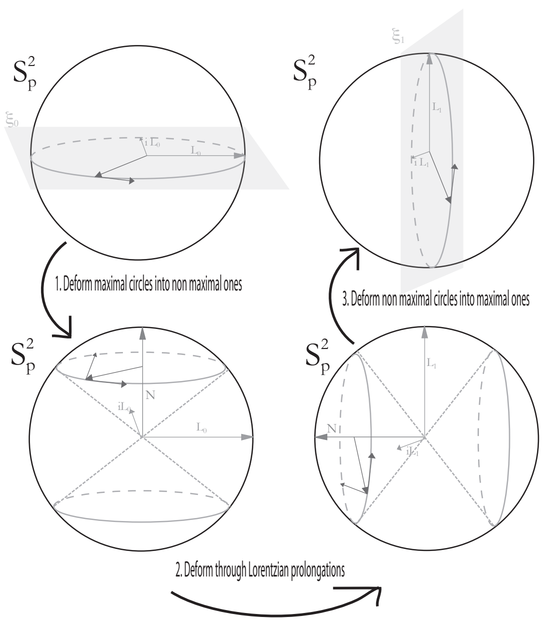

Figure 4. The deformation from $\mathcal{D}(\xi_0)$ to $\mathcal{D}(\xi_1)$

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Geometry of Polynomial Ordinary Differential Equations

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Algebraic varieties defined by polynomial equations over real or complex numbers admit explicit upper bounds for their topological complexity (e.g. Betti numbers). These bounds depend on the dimension and the degree of the equations. In this survey, we identify a few cases where similar bounds can be established if the polynomials are replaced by solutions of differential equations (ordinary or Pfaffian) with polynomial right sides.

1 Introduction and motivation: Hilbert's 16th problem

In 1900, David Hilbert presented his famous list of problems (parts of them very concrete, other parts rather loosely formulated), which played a prominent role in the development of mathematics in the 20th century. Almost all problems from this list have been solved in one way or another, with only a couple of die hards left open, one of them the Riemann hypothesis.

The problem listed as number 16 consisted of two parts. The first part was the question about the number and relative position of real algebraic ovals, i.e. compact connected components of the set defined by the equation $\{H(x, y) = 0\} \subset \mathbb{R}^2$, $H \in \mathbb{R}[x, y]$. The second part asked "the same" question about limit cycles, i.e. isolated closed compact solutions of the ordinary differential equation $P(x, y)\,\mathrm{d}x + Q(x, y)\,\mathrm{d}y = 0$ (in Pfaffian form) defined by two real polynomials $P, Q \in \mathbb{R}[x, y]$.

The algebraic part of the problem had already seen substantial progress by the time it was formulated. A. Harnack obtained a sharp bound for the *maximal* possible number of algebraic ovals; the question remained about their mutual position and the way these ovals can be nested into each other. In the specific case of sextics (curves of degree 6), two configurations, one constructed by Harnack, the other by Hilbert himself, were discovered and Hilbert conjectured that there were no more possibilities. It took almost 70 years until a third configuration was found by D. Gudkov, who showed that this new list was indeed exhaustive. Since then, the emphasis has shifted to the general study of the interplay between real algebraic geometry and its proper complexification. A paper by O. Viro [21] describes the spectacular achievements of this programme.

All the way through, the progress with the transcendental part of the question has been formally rather unimpressive. The only general result is the finiteness theorem due (independently) to Yu. Ilyashenko [12] and J. Ecalle [7], which asserts that any polynomial differential equation has at most finitely many limit cycles. It is not even known whether this finite number is *uniformly bounded* over, say, differential equations of degree 2 (equations of degree 1 cannot have limit cycles at all). The historical survey [11] paints a dramatic story of discoveries and overturns in this area.

One can speculate about general reasons why the two parts differ so strikingly. One clear distinction is that the transcendental part lacks appropriate complexification. While the notion of a planar polynomial differential equation can be naturally complexified to that of a singular holomorphic foliation on $\mathbb{C}P^2$, the natural complex analogues of limit cycles generically coexist in infinite numbers.

Arguing more broadly, one could question the mere possibility of generalising the counting problems, which is so natural for algebraic objects (real or complex) and their characteristics, for similarly looking transcendental objects defined by differential equations. In a nutshell, tallying the roots of a polynomial is a meaningful problem while counting the roots of the sine, a transcendental solution of a second order differential equation, is not.

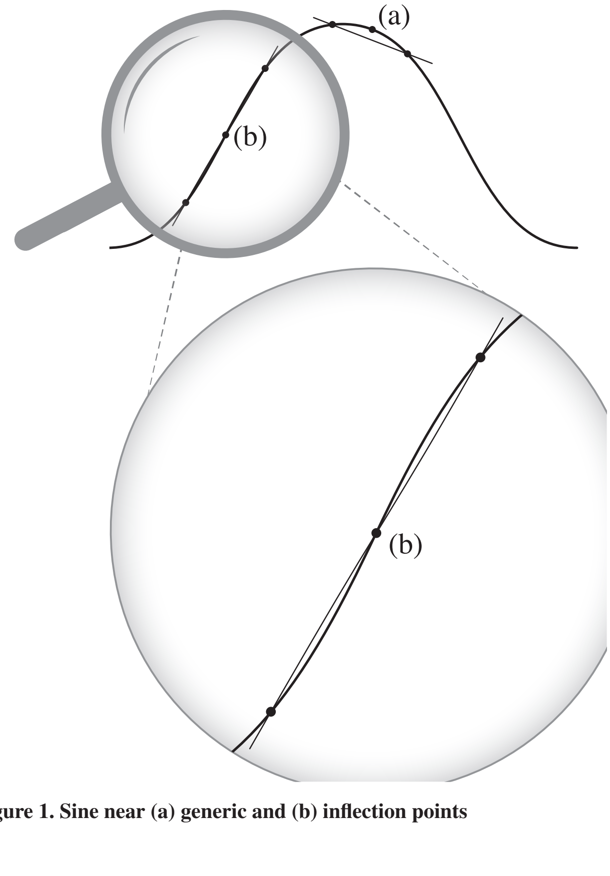

Figure 1. Sine near (a) generic and (b) inflection points

The goal of this text is to describe several results showing that counting problems sometimes admit finite and explicit solutions, if the problem is considered locally. In other words, objects defined by polynomial differential equations *locally* behave as if they are algebraic (compare with Example 6 below) and their "geometric complexity" can be explicitly estimated in terms of the dimension and the degree of the equations.

By the geometric complexity of (singular) analytic varieties, one can understand various numeric characteristics (e.g. Betti numbers) but most of these characteristics can be estimated from above by the number of isolated intersection points of certain auxiliary varieties in a standard way, e.g. using Morse theory (see the foundational work in [15]). To simplify our task, in the text below, we will focus exclusively on the isolated intersections between invariant subvarieties of polynomial differential equations (e.g. the integral curves of polynomial vector fields) and algebraic subvarieties in the ambient space (most often affine subspaces of complimentary dimension). We will show that sometimes an explicit upper bound for the number of such intersections is possible.

2 Oscillation of linear equations and trajectories of polynomial vector fields

The sine as a paradigm

The simplest manifestation of the feature above is the function $y = \sin x$ on the real line. Having infinitely many real isolated roots, it is obviously non-algebraic. Moreover, adding an extra parameter $\lambda \gg 0$, we may squeeze any number of real roots of the function $\sin(\lambda x)$ on any finite open interval $(-r, r)$. However, for any finite λ and any point $a \in \mathbb{R}$, the graph of sine restricted to a sufficiently small neighbourhood of a is indistinguishable from the graph of a polynomial of degree $\leqslant 3$. More precisely, there exists $r > 0$ (depending on a, λ) such that the equation $\sin(\lambda x) = px + q$, describing the intersection of the graph $y = \sin x$ with an arbitrary line $y = px + q$, has no more than 3 solutions on $(a - r, a + r)$. This result can be improved: if a is not an inflection point on the graph then the number of intersections is no greater than 2. Both bounds follow from the Rolle theorem on interlacing between roots of a function and its derivative and the fact that the second derivative of $\sin x$ vanishes only at the inflection points where the third derivative is nonzero.

Solutions of linear ordinary differential equations

This example is a manifestation of the general fact about solutions of arbitrary homogeneous linear ordinary differential equations with bounded coefficients. Consider the equation on the real interval $[0, r]$ with, say, real analytic coefficients,

$$a_0(t)\, y^{(n)} + a_1(t)\, y^{(n-1)} + \cdots + a_{n-1}(t)\, y' + a_n(t)\, y = 0. \quad (1)$$

If the leading coefficient $a_0(t)$ is nonvanishing on $[0, r]$, one can divide by it and assume that $a_0 \equiv 1$. This case is nonsingular and zeros of nontrivial solutions to (1) are subject to the de la Vallée Poussin theorem (see [22] for the short proof via Rolle theorem).

Theorem 1. If the length r of the interval is sufficiently small compared to the magnitude of the coefficients then any solution of the nonsingular equation (1) may have no more than $n-1$ isolated roots on the interval.

Remark 2. The bound $n-1$ is sharp: an equation of order n always has a nontrivial solution with roots at arbitrarily placed $n-1$ points of any interval.

The smallness condition above is completely explicit:

$$\sum_{i=1}^{n} A_i \frac{r^i}{i!} < 1, \quad \text{where} \quad A_i = \sup_{t\in[0,r]} \frac{|a_i(t)|}{|a_0(t)|} < +\infty.$$

The assertion of Theorem 1 remains true if, instead of real equations on the interval, we consider equations with complex coefficients $a_1, \dots, a_n \in \mathscr{O}(U)$ holomorphic in a convex domain $U \subset \mathbb{C}$ of diameter $\leqslant r$.

Trajectories of polynomial vector fields

Given a smooth real analytic parametrised curve $\gamma : (-r, r) \to \mathbb{R}^n$, $\gamma(0) = 0$, the question about its intersection with hyperplanes passing through the origin can be reduced to that about roots of solutions of a linear nth order homogeneous differential equation. Indeed, each such intersection corresponds to zeros of a linear combination $\langle c, \gamma(t)\rangle = 0$ for some $c = (c_1, \dots, c_n) \in \mathbb{R}^n$. All such combinations form a linear n-dimensional subspace in the space of real analytic functions on $(-r, r)$. By the standard arguments using Wronskians, such a subspace coincides with the space of solutions of a certain linear differential equation with real analytic coefficients. Yet this equation may well be singular, with the leading coefficient vanishing at some points.

One can, however, construct a *nonsingular* linear differential equation satisfied by an arbitrary affine function $f = \langle c, x\rangle + c_0$ restricted on γ, assuming that γ is a nonsingular integral trajectory of a *polynomial* vector field P in $\mathbb{R}^n$. Suppose that P is defined in $\mathbb{R}^n$ by a system of polynomial differential equations

$$\dot{x}_i = P_i(x_1, \dots, x_n), \quad i = 1, \dots, n, \ \deg P_i \leqslant \delta. \tag{2}$$

Denote by $f_k \in \mathbb{R}[x_1, \dots, x_n]$ the sequence of iterated Lie derivatives,

$$f_{k+1} = \sum_{i=1}^{n} \frac{\partial f_k}{\partial x_i} P_i, \qquad k = 0, 1, 2 \dots, f_0 = f, \tag{3}$$

and the associated ascending chain of ideals in the ring of polynomials $\mathbb{R}[x_1, \dots, x_n]$,

$$I_0 \subset I_1 \subset I_2 \subset \cdots, \qquad I_k = \langle f_0, \dots, f_k\rangle. \tag{4}$$

Because of the Noetherianity of the ring, the chain must stabilise at a certain moment $N < +\infty$, meaning that the last Lie derivative f_N is a combination of the preceding derivatives $f_N = \sum_{k=1}^{N-1} a_{N-k} f_k$. Restricting this identity on the integral curve γ, we obtain a linear differential equation of order N with real analytic coefficients. This allows for the application of Theorem 1. The following result was proved in [19]. Let $a \in \mathbb{R}^n$ be a *nonsingular* point, $P(a) \neq 0$.

Theorem 3. Let a be a nonsingular point of the vector field (2). Then, there exists $r > 0$ such that any integral curve γ of (2) intersects any affine hyperplane $\Pi = \{\sum_1^n p_i x_i = q\}$ inside the small ball $B_r = \{|x-a| < r\}$ by no more than N points: $\#(\gamma \cap \Pi \cap B_r) \leqslant N$.

Here, $N = N(n, \delta)$ is an explicit bound, polynomial in δ and growing no faster than $\delta^{2^{O(n^2 \ln n)}}$ (double exponentially) as $n \to \infty$. The radius $r > 0$ depends on the field (2) and the choice of the point a and is also explicitly bounded from below.

In other words, small pieces of integral curves of a polynomial vector field of degree δ from the point of view of intersection theory behave as algebraic curves defined by polynomial equations of the same degree δ, yet in a much higher-dimensional fictitious space $\mathbb{R}^\nu$, $\nu = \nu(n) \leqslant 2^{O(n^2 \ln n)}$.

This result also holds (with the same estimates) for integral curves of polynomial vector fields in $\mathbb{C}^n$.

Remark 4. It is important to note that the size r of the ball B_r in which the bound established in Theorem 3 holds is explicit. In addition to the parameters n, d, it depends polynomially on the norm $|a|$ and the magnitude of the coefficients of the polynomials P_i defining the field (2).

This is, very roughly, where the "genuine" counting of solutions reaches its current frontier. For instance, the following statement is only conjectural. Consider two polynomial *commuting* vector fields P, Q in $\mathbb{R}^n$ of degree δ, $n \geqslant 4$ and assume that a point $a \in \mathbb{R}^n$ is nonsingular: $(P \wedge Q)(a) \neq 0$.

Conjecture 5. There exists $r > 0$ depending on P, Q, a such that any intersection between any integral 2-dimensional manifold Γ and any affine subspace Π of codimension 2 inside the ball $B_r = \{|x-a| < r\}$, if isolated, consists of no more than M points, $\#(\Gamma \cap \Pi \cap B_r) \leqslant M$, where $M = M(n, d)$ is an explicit bound, polynomial in δ.

3 Multiplicity estimates

Let $f_1, \dots, f_n$ be n germs at the origin $x = 0$ of holomorphic functions of n variables $x = (x_1, \dots, x_n)$ and $I = \langle f_1, \dots, f_n\rangle$ the ideal generated by them in the ring $\mathscr{O}(\mathbb{C}^n, 0)$ of such functions. One of the basic results of singularity theory is as follows: the ideal I has finite codimension

$$\mu = \mu_f = \dim_{\mathbb{C}} \mathscr{O}(\mathbb{C}^n, 0)/I < +\infty \tag{5}$$

if and only if the intersection $X_0 = \bigcap_{i=1}^n \{f_i = 0\}$ is an isolated point (at the origin) in $\mathbb{C}^n$ as a germ of an analytic variety. This number μ is called the *multiplicity of intersection* (and sometimes the Milnor number of the tuple f_i). If μ_f is finite, the number of solutions of the system $X_c = \{f_i(x) = c_i,\ i = 1, \dots, n\} \subseteq \mathbb{C}^n$ is equal to μ (if counted with multiplicities) for all sufficiently small values of $c_1, \dots, c_n \in \mathbb{C}$ (we abuse the notation, identifying the germs f_i with their representatives defined in a small ball in $\mathbb{C}^n$). However, when the multiplicity is infinite, no conclusion on the number of points in X_c can be made.

Example 6. The classical Bézout theorem implies that, for polynomial germs $f_i \in \mathbb{C}[x_1, \dots, x_n]$, the multiplicity of the intersection, if finite, is less than or equal to d^n, where $d = \max_i \deg_x f_i$. The bound is polynomial in d and exponential in n.

Isolated intersections

Instead of counting the number of intersections between a subvariety defined by a differential equation and an affine subspace, one can attempt to estimate the maximal multiplicity of the intersection, *assuming that the latter is finite*. This task turns out to be much more amenable. For instance, the most direct counterpart of Theorem 3, achieved by A. Gabrielov in [9], improves[1] the double exponential bound to a single exponential.

Theorem 7. The multiplicity of the intersection between the trajectory of a polynomial vector field (2) and an affine hyperplane Π at a *nonsingular* point a, if finite, is no greater than $2^{3n}d^{2n}$.

Noetherian functions

In fact, a much more general fact can be proved. Assume that a tuple of germs of complex analytic functions $\mathscr{F} = \{f_1(x), \dots, f_\nu(x)\}$ in $(\mathbb{C}^n, 0)$ satisfies a system of polynomial Pfaffian equations of the form

$$\mathrm{d}z_i = \sum_{j=1}^{n} P_{ij}(z, x)\,\mathrm{d}x_j, \qquad i = 1, \dots, \nu, \quad z_i = f_i(x). \tag{6}$$

Here, $P_{ij} \in \mathbb{C}[z, x]$ are polynomials of degree $\leqslant \delta \in \mathbb{N}$. Then, one can define the ring of *$\mathscr{F}$-Noetherian germs* as the subring $\mathbb{C}[x, f(x)]$ of functions in $\mathscr{O}(\mathbb{C}^n, x)$ that are polynomials in x and $z = f(x)$. This ring is filtered by the degree $d = \deg_{x,z}$. The dimension $\nu = \dim z$ and the degree $\delta = \max_{i,j} \deg P_{ij}$ are parameters of the ring.

It turns out that the Noetherian germs behave similarly to polynomials. The following result was achieved by A. Gabrielov and A. Khovanskii [10]. It can be considered to be an infinitesimal version of the Bézout theorem for Noetherian functions (see Example 6).

Theorem 8. If $q_1, \dots, q_n \in \mathbb{C}[x, f]$ are Noetherian germs,

$$q_i = Q_i(x, f(x)), \qquad \deg Q_i \leqslant d,$$

then the multiplicity of the intersection is either infinite or explicitly bounded,

$$\mu = \dim_{\mathbb{C}} \mathscr{O}(\mathbb{C}^n, 0) / \langle q_1, \dots, q_n \rangle \leqslant C_{n,\nu,\delta}\, d^{2(n+\nu)},$$

where $C_{n,\nu,\delta}$ is some explicitly given expression, polynomial in δ and exponential in n, ν.

Sharpening the bounds

The Gabrielov–Khovanskii bound is polynomial in d, δ and single exponential in n, ν. A simple comparison with the Bézout bound d^n suggests that it has a right asymptotic behaviour. However, being a function of several arguments, it allows a certain measure of trade-off.

For instance, for applications in transcendental number theory, it is very important to study and sharpen asymptotics of the bound of contact between a fixed trajectory of a polynomial vector field (2) of degree δ in $\mathbb{C}^n$ and an arbitrary algebraic hypersurface of degree d as a function of d with the remaining parameters (n, δ) being less important. In [9], it is shown that the multiplicity of an isolated intersection in this case does not exceed

$$2^{2n-1} \sum_{i=1}^{n} \left[d + (i-1)(\delta - 1)\right]^{2n}. \tag{7}$$

This bound, however, is insufficient for the purposes of proving transcendence results in analytic number theory (see [18]): one needs a bound of the form $C(n, \delta)\, d^n$. Such a bound was achieved very recently by G. Binyamini [3]: by refinement of the Gabrielov–Khovanskii methods, he proved that the multiplicity does not exceed $2^{n+1}[d + (n-1)(\delta - 1)]^n$, thus closing the gap asymptotically as $d \to \infty$. In fact, in [3], he gives the answer in terms of the *Newton polytope* of the polynomial that is to be restricted on the integral curve.

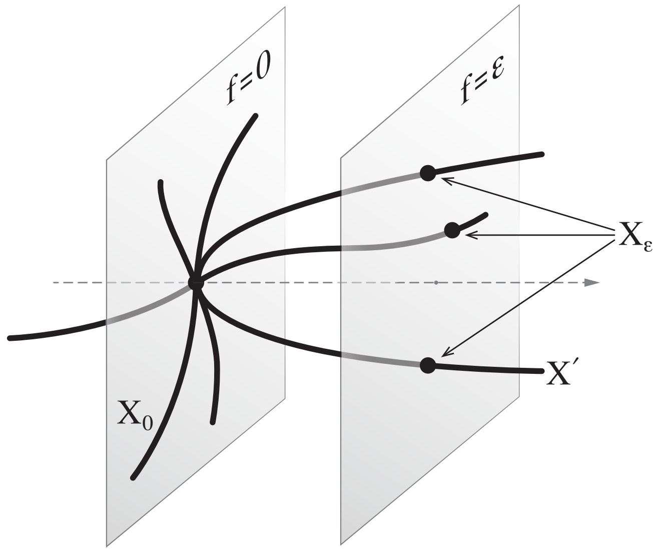

Figure 2. Multiplicity of non-isolated intersections

Nonisolated intersections

In the case of a nonisolated intersection, one can also find interesting points to count. In the simplest case, the problem is as follows.

Let $X \subset (\mathbb{C}^n, 0)$ be an analytic Noetherian variety (defined by Noetherian functions of known complexity). Assume that another Noetherian function f does not vanish identically on X. Then, it may well happen that, for all sufficiently small $\varepsilon \neq 0$, the intersection $X_\varepsilon = X \cap \{f = \varepsilon\}$ is zero-dimensional, that is, it consists of a finite number of isolated points that converge to the origin as $\varepsilon \to 0$. Geometrically, this means that the closure of the difference $X' = \overline{X \smallsetminus X_0}$ is a (singular) curve whose intersection with the hypersurface $\{f = 0\} \subset (\mathbb{C}^n, 0)$ is isolated.

The question about the number of solutions of the system

$$x \in X, \quad f(x) = \varepsilon, \tag{8}$$

that converge to $x = 0$ as $\varepsilon \to 0$ has been addressed in the works of G. Binyamini and D. Novikov [1, 2].

Theorem 9. If $f_1, \dots, f_n$ are Noetherian functions then the number of confluent solutions of (8) can be explicitly bounded. The bound is polynomial in $d + \delta$ and *double exponential* in the dimension $n + \nu$.

Clearly, this result implies an explicit upper bound for solutions of any one-parametric system of equations $f_i(x, \varepsilon) =$

1 In fact, both Theorems 7 and 8 have been proved with slightly better estimates than stated but the precise expressions are much more involved while being of the same growth rate.

0, $i = 1, \ldots, n$, provided that the functions f_i depend on the one-dimensional parameter ε in a Noetherian way.

The main difficulty of the proof consists of the estimate of the intersection complexity of the curve $X' = \overline{X \setminus X_0}$. For algebraic varieties, there exists a well developed technique of decomposition into irreducible components that gives an explicit bound on the degree $\deg_{\mathbb{C}} X'$ and hence on the multiplicity of the isolated intersection $X' \cap \{f_n = 0\}$. For Noetherian functions and the Noetherian varieties (null sets of the corresponding functions), there is no similar technique, hence one has to develop ad hoc tools.

4 Singular case

All previous results on multiplicity explicitly or implicitly assume nonsingularity of the system of differential equations defining the "transcendental variables", and for good reason.

Example 10. Consider a planar linear vector field $\dot{x} = ax+by$, $\dot{x} = cx + dy$, with the real matrix $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$ having two non-real eigenvalues. The integral curves of this field are spirals that intersect any line passing through the origin infinitely many times.

Of course, integral curves of this field are not analytic at the origin.

Example 11. The planar vector field associated with the system of equations $\dot{x} = x$, $\dot{y} = py$, $p \in \mathbb{N}$ ("the resonant node"), apart from the coordinate axes $x = 0$ and $y = 0$, has a one-parametric family of invariant curves of the form $y = cx^p$. These curves intersect the line $y = 0$ with multiplicity p, which can be arbitrarily large, whereas the degree of all equations $\delta = d = 1$ and the dimension $\nu = 2$ stay constant (and rather small).

In the other case, the need to address the singular case is rather pressing. Indeed, consider the situation described in Conjecture 5, where one is required to count intersections between a common 2-dimensional integral surface S of two commuting vector fields P, Q and an algebraic subvariety X defined by polynomial equations $\{f = g = 0\}$ in $\mathbb{R}^n$ or $\mathbb{C}^n$. One might hope to reduce the problem to the case of one vector field and one hypersurface as follows. The vector field $R = uP + vQ$, where $u = Qf$, $v = -Pf$ are polynomials (the Lie derivatives of f along the fields P, Q), is tangent to all level surfaces $\{f = \text{const}\}$ and hence the intersection $S \cap X$ coincides with the intersection of trajectories of the field R with the hypersurface $\{g = 0\}$. Unfortunately, the polynomial vector field R is usually singular (i.e. vanishes at some points), which renders Theorem 3 inapplicable.

Multiplicity estimates (singular case)

It turns out that if γ is a smooth analytic *separatrix* (analytic curve through the origin) of a polynomial vector field and it is "sufficiently non-algebraic" then one can still estimate its order of contact with an algebraic hypersurface of degree d even when the vector field is singular.

Assume first that the curve γ is totally transcendental, that is, does not belong to any proper algebraic subvariety of $(\mathbb{C}^\nu, 0)$. Then, one should expect that a polynomial hypersurface of degree d will have the multiplicity of isolated intersection with γ at least $c_\gamma d^\nu$. Indeed, the monomials of degree $\leqslant d$ after restriction on γ are linearly independent (because of the transcendence), hence there must exist a linear combination with vanishing order at least $\dim \mathscr{P}(\nu, d) - 1$, where $\mathscr{P}(\nu, d)$ is the linear $\mathbb{C}$-space of polynomials in ν variables of degree $\leqslant d$.

The corresponding *upper* bound for the multiplicity of the isolated intersection between γ and a polynomial hypersurface was achieved by Yu. Nesterenko in [17] in exactly the same exponential form. For that purpose, the condition of transcendence imposed on γ should be reinforced: Nesterenko introduced the so-called D-property in [16]. The curve γ is said to satisfy the D-property with an exponent $\varkappa \in \mathbb{N}$ if its order of tangency with any *algebraic* invariant subvariety of the vector field through the singular point is less than or equal to $\varkappa$. The D-exponent depends only on the curve and enters explicitly into the multiplicity bounds. For curves satisfying the D-property, the multiplicity of the isolated intersection with hypersurface of degree d can be at most $c'_\gamma d^\nu$ (with a different constant c'_γ, double exponential in the dimension ν). G. Binyamini improved these results in [4] by reducing this constant to single-exponential size.

Fuchsian singularities

Let us return to the study of roots of solutions of linear differential equations (see §2). This time, we assume that the origin $t = 0$ is a singular point of equation (1), that is, one of the ratios $a_i(t)/a_0(t)$ has a pole at the origin.

The simplest examples show, in this case, that Theorem 1 no longer holds and solutions may have an infinite number of isolated roots accumulating at the singular point. However, under certain rather natural extra assumptions, one can still guarantee existence of finite and explicit bounds.

The first assumption requires the singularity at the origin to be relatively mild so that solutions of equation (1) grow at most polynomially in $|t|^{-1}$ as $t \to 0$. This condition, called *regularity*, is equivalent to the requirement that equation (1) is *Fuchsian*. The latter condition means that it can be put in the form

$$\epsilon^n y + b_1(t)\,\epsilon^{n-1} y + \cdots + b_{n-1}(t)\,\epsilon y + b_n(t)\,y = 0, \qquad (9)$$

with real analytic coefficients $b_1, \ldots, b_n \in \mathscr{O}(\mathbb{C}, 0)$, written with respect to the *Euler derivation* $\epsilon = t\frac{\mathrm{d}}{\mathrm{d}t}$. In other words, the equation is Fuchsian if it becomes nonsingular after expansion in the iterated Euler derivatives $\epsilon^k y$ rather than the usual derivatives $y^{(k)} = \left(\frac{\mathrm{d}}{\mathrm{d}t}\right)^k y$.

Another condition refers to the roots of the characteristic equation

$$z^n + \beta_1 z^{n-1} + \cdots + \beta_{n-1} z + \beta_n = 0, \qquad \beta_i = b_i(0). \qquad (10)$$

If all the roots of equation (10) are real then one can produce an explicit upper bound for the number of isolated roots of any solution of (9) (see [20] for the real case and [5] for the complex version).

Theorem 12. If all roots $\lambda_1, \ldots, \lambda_n$ of equation (10) are real then there exists $r > 0$ such that any solution of the Fuchsian equation (9) has no more than $(2n + 1)(2L + 1)$ isolated roots on the interval $(0, r)$, where $L = \max_i |\lambda_i|$.

As in the nonsingular case, the size r of the interval is explicit and depends on the magnitude of the coefficients

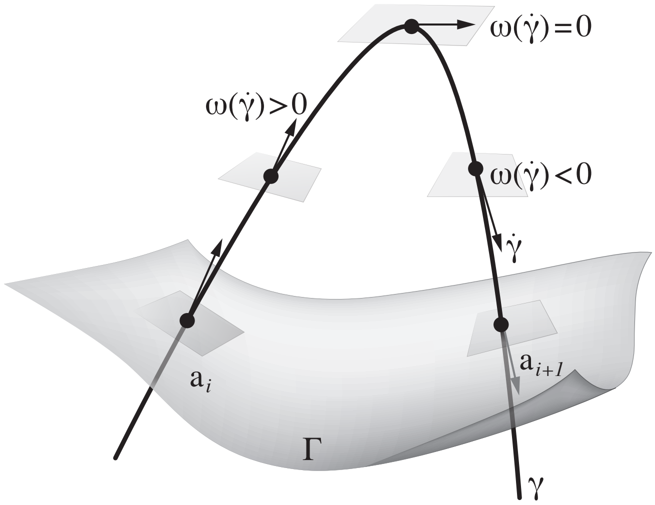

Figure 3. Geometric Rolle theorem in $\mathbb{R}^n$

$B = \max_i \max_{t\in[0,r]} |b_i(t)|$. The claim remains valid for Fuchsian equations with complex analytic coefficients if the interval $(0, r)$ is replaced by a sector $\{0 < |z| < r,\ |\arg t| < \pi\}$. Note, in this case, that the bound on the number of roots (even very close to the singularity) depends not only on the order of the equation but also on the magnitude of its coefficients. This is a new but unavoidable phenomenon.

5 Global bounds

Local complexity estimates under certain assumptions can be aggregated into global bounds, provided the necessary local conditions are satisfied "at infinity" (recall that polynomial differential equations can be naturally extended onto projective spaces that are compact).

Example 13. The equation $y'' + y = 0$ has a non-Fuchsian singular point at infinity: written with respect to the variable $s = \frac{1}{t}$, it takes the form $s^4 y'' + 2s^2 y' + y = 0$ (the prime now stands for $\frac{d}{ds}$). Thus, Theorem 12 is not applicable and cannot prevent the sine from having infinitely many isolated zeros accumulating at the infinite point $s = 0$.

On the other hand, if a linear equation with rational coefficients has only Fuchsian singular points on the Riemann sphere $\mathbb{C}P^1 = \mathbb{C}\cup\{\infty\}$ and all characteristic roots for all points are real then the number of isolated roots of any solution is finite, and one may hope to obtain an explicit bound.

Pfaffian intersections

One approach to counting intersections globally was discovered by A. Khovanskii by generalising the Descartes rule and is known by the name *Fewnomials theory* [13]. It applies to submanifolds in $\mathbb{R}^n$, defined by systems of Pfaffian equations, rather than to trajectories of vector fields.

The simplest example (actually, a version of the Rolle theorem) is as follows. Assume that Γ is an integral hypersurface of a polynomial Pfaffian 1-form $\omega = \sum_{i=1}^n P_i(x)\,dx_i$ in $\mathbb{R}^n$, such that $\mathbb{R}^n \setminus \Gamma$ consists of two connected components (we say that Γ is a separating solution of the Pfaffian equation $\omega = 0$).

Consider a real algebraic curve $\gamma \subset \mathbb{R}^n$, assuming for simplicity that it is a smooth complete intersection (i.e. defined by $n-1$ polynomial equations $Q_1(x) = 0, \dots, Q_{n-1}(x) = 0$). This curve consists of several compact components (ovals) and several non-compact components diffeomorphic to the real line $\mathbb{R}^1$. Each component can be oriented, which means that the points of its intersection with Γ are ordered (possibly cyclically). Assume (again only for simplicity) that all intersections are transversal, that is, the 1-form ω takes nonzero value on the tangent vector $\dot\gamma(a_i)$ to the curve at all intersection points $a_i \in \Gamma \cap \gamma$.

Then, the topological conditions imposed on Γ imply that the signs $\omega(\dot\gamma(a_i))$ are alternating along each component of γ: the curve must enter and then leave the respective half-spaces into which Γ separates $\mathbb{R}^n$. Looking at the continuous function $\omega(\dot\gamma(a))$ along each component, we conclude that, between any two consecutive intersections $\gamma \cap \Gamma$, there must be at least one *point of contact* where the value $\omega(\dot\gamma)$ vanishes. The vanishing condition is *polynomial*:

$$\omega(\dot\gamma(a)) = 0 \iff (dQ_1 \wedge \cdots \wedge dQ_{n-1} \wedge \omega)(a) = 0. \tag{11}$$

By the Rolle theorem, this implies that the number of isolated intersections $\#\gamma \cap \Gamma$ does not exceed the number of solutions of (11) plus the number of non-compact components of γ. The first number does not exceed the product of degrees of $\deg dQ_i$ and $\deg \omega$. To estimate the second number, note that each non-compact component must twice intersect any sufficiently big sphere $\sum_1^n x_i^2 = R$, $R \gg 1$, so the number is at most $\frac{1}{2} \cdot 2 \prod_{i=1}^{n-1} \deg Q_i$. These estimates yield an explicit Bézout-type bound for the number of intersections between $\gamma \cap \Gamma$. The simplifying assumptions can actually be replaced by deformation-type arguments which admit generalisation for the complex case [8].

Example 14. Consider the common integral surface Γ of two commuting polynomial vector fields u, v of degree d in $\mathbb{R}^3$ (compare with Conjecture 5). This surface locally separates a neighbourhood of a nonsingular point and can be defined by the Pfaffian form $\omega = dV(u, v, \cdot)$, where dV is the Euclidean volume 3-form. The 1-form ω is polynomial of degree $\leqslant d^2$ and, by the Pfaffian elimination above, it can intersect any line (of degree 1) in $\mathbb{R}^3$ by no more than $d^2 + 1$ isolated points.

The above construction allows one to reduce the study of a "mixed" system of $n-1$ polynomial equations and one Pfaffian equation to that of two systems of n polynomial equations. Under certain assumptions, it can be iterated.

Let $U \subseteq \mathbb{R}^n$ be a domain. An (ordered) tuple of real analytic in U functions $\mathscr{P}(U) = \{f_1, \dots, f_\nu\}$ is called a *Pfaffian chain* if they satisfy a system of polynomial Pfaffian equations (6) with an additional property: the polynomials $P_{ij}(z, x)$ on the right side *do not depend* on the variables z_k for $k > i$:

$$\begin{aligned} dz_1 &= \sum_{j=1}^n P_{1j}(z_1, x)\,dx_j, \\ dz_2 &= \sum_{j=1}^n P_{2j}(z_1, z_2, x)\,dx_j, \\ &\cdots \\ dz_\nu &= \sum_{j=1}^n P_{\nu j}(z_1, \dots, z_\nu, x)\,dx_j. \end{aligned} \tag{12}$$

The triangular structure of the equations (12) allows one to apply Pfaffian elimination inductively as explained in [13] and produce explicit bounds for the number of solutions for systems of equations of the form

$$Q_1(x, f) = 0, \dots, Q_n(x, f) = 0, \quad x \in U,\ f = (f_1, \dots, f_\nu).$$

Unfortunately, the Fewnomials theory can only partially be complexified, the main obstruction being the absence of a suitable analogue of the Rolle theorem (see [8, 14]).

Fuchsian equations on $\mathbb{R}P^1$ and $\mathbb{C}P^1$

One context in which the problem of counting complex intersections can be fully globalised is that of Fuchsian equations on $\mathbb{C}P^1$, i.e. linear equations with rational coefficients and with only Fuchsian singular points (compare with §4). Because of the homogeneity, any linear equation with rational coefficients can be put in the form where the coefficients are polynomial:

$$\begin{gathered} a_0(t)\, y^{(n)} + \cdots + a_1(t)\, y' + a_n(t)\, y = 0, \\ a_0, \dots, a_n \in \mathbb{C}[t], \quad \gcd\{a_0, \dots, a_n\} = 1. \end{gathered} \tag{13}$$

The roots $t_1, \dots, t_\mu$ of the leading coefficient a_0 and possibly the point $t = \infty$ are singular for (13). The assumption that all these points are Fuchsian imposes additional constraints on the polynomials a_i.

Theorem 12 implies (by compactness of the real projective line) that if equation (13) has real coefficients $a_i \in \mathbb{R}[t]$ and all its real singular points $t_i \in \mathbb{R}$ have only real characteristic roots then the number of isolated zeros of any real solution of this equation on any real interval free from singularities is finite. To find an explicit bound for this number, one has to identify parameters on which the answer might depend. Apart from the order n of the equation and the degree d of the coefficients, the answer must necessarily depend on the "magnitude of the coefficients", i.e. the numeric measure of how large non-principal coefficients $a_1(t), \dots, a_n(t) \in \mathbb{C}[t]$ are compared to the principal one $a_0(t)$.

The string of polynomials $\{a_0, a_1, \dots, a_n\}$ in (13) is defined modulo a common scalar factor. To normalise it, let $\|\cdot\|$ be an l_1-norm on the $\mathbb{C}$-space of polynomials $\mathbb{C}[t]$ (the sum of absolute values of all coefficients). Then, the expression $S = \max_{i=1,\dots,n} \frac{\|a_i\|}{\|a_0\|}$ may be considered as the natural measure of the "magnitude of the coefficients" of equation (13), (compare with the parameters $A = \max_{i=1,\dots,n} A_i$ from Theorem 1 and B from Theorem 12). We call the value S the *slope* of the differential equation (13). Clearly, any bound on the number of isolated roots of solutions of (13), besides the order of the equation and the degree of its coefficients $\max_i \deg a_i$, must depend on S, explicitly or implicitly.

It turns out that the order, degree and slope of the equation alone are insufficient to explicitly majorise the number of roots of its solutions. The "hidden" parameter is the configuration of the singularities. In [5], one can find a discussion of what may happen with roots of solutions in a parametric family of equations with *confluent singularities*: without knowing a lower bound on the distance $|t_i - t_k|$ between different singular points, one cannot achieve any bound on the number of roots of solutions, even in the case where each singularity by itself has a real spectrum and is hence covered by Theorem 12. On the other hand, the collection of the discrete (n and $d = \max \deg a_i$) and continuous data (the slope S and $\rho = \min_{i,j} |t_i - t_j| > 0$) are sufficient to produce an explicit global upper bound for the number of real roots of solutions of (13) (see [5]).

The above phenomenon does not exclude the possibility that there are certain families of equations of the form (13) that admit explicit upper bounds for the number of zeros, uniform over all configurations of singularities. It turns out that if the family is *isomonodromic* in the complex domain[2] then one can obtain *uniform* bounds for the number of isolated *complex* roots of solutions. Such families are most conveniently represented by systems of linear Pfaffian equations with rational coefficients.

Remark 15. If instead of the real roots of solutions to (13) we decide to count complex isolated roots then it is necessary to address the ramification of solutions over the singular locus $\Sigma \subset \mathbb{C}P^1$. Indeed, after analytic continuation of a given solution $y = f_1(t)$, one obtains another solution $y = f_2(t)$ and so on; the total number of different branches obtained by analytic continuation is in general infinite. Even in the "good" case where each branch has only a limited number of roots in a simply connected domain, all branches together in general would have infinitely many such roots. Thus, the problem of counting of roots should be restricted to all simply connected subsets of $\mathbb{C}P^1 \setminus \Sigma$. Yet, if such a domain spirals around a singular point, the above phenomenon may still be possible. The correct method is to consider only subdomains of the simplest form, e.g. triangles (in the affine chart t). For a given linear equation $Ly = 0$, as in (13) with the singular locus $\Sigma(L)$, denote by $\mathscr{N}(L)$ the supremum (finite or not):

$$\mathscr{N}(L) = \sup_{T \subset \mathbb{C} \setminus \Sigma} \ \sup_{f\colon Lf = 0} \#\{t \in T : f(t) = 0\},$$

taken over all open triangles $T \subset \mathbb{C}$ free from singular points (including those with one vertex at infinity) and all solutions of the equation.

Integrable linear systems on $\mathbb{C}P^m$

Consider a matrix-valued rational 1-form Ω on the complex projective m-space $\mathbb{C}P^m$: such a form is defined by n^2 rational 1-forms ω_{ij}, $i, j = 1, \dots, n$. Denote by d the maximal degree $\max_{i,j} \deg \omega_{ij}$. This form defines a vector Pfaffian equation $\mathrm{d}x = \Omega x$ or an equivalent system $\mathrm{d}x_i - \sum_{j=1}^n x_j \omega_{ij} = 0$, $i = 1, \dots, n$, of Pfaffian equations linear with respect to n dependent variables $x_1, \dots, x_n$.

Denote by Σ the polar divisor of the form Ω (the union of polar divisors of the entries ω_{ij}). We consider only *integrable* Pfaffian systems which admit holomorphic solutions in $\mathbb{C}P^m \setminus \Sigma$ ramified over the polar divisor Σ. The condition $\mathrm{d}\Omega = \Omega \wedge \Omega$ is necessary and sufficient for integrability. The system is called *regular* if its solutions grow at most polynomially near Σ (compare with §4).

For any oriented closed loop $\gamma : [0, 1] \to \mathbb{C}P^m \setminus \Sigma$, $\gamma(0) = \gamma(1) = a$, the result of analytic continuation along γ defines an automorphism M_γ of the n-space of solutions of the Pfaffian

2 A parametric family of linear differential equations or their systems is called isomonodromic if the result of analytic continuation of any fundamental system of its solutions along a loop avoiding singular points locally does not depend on the parameters, in particular when singularities inside the loop collide.

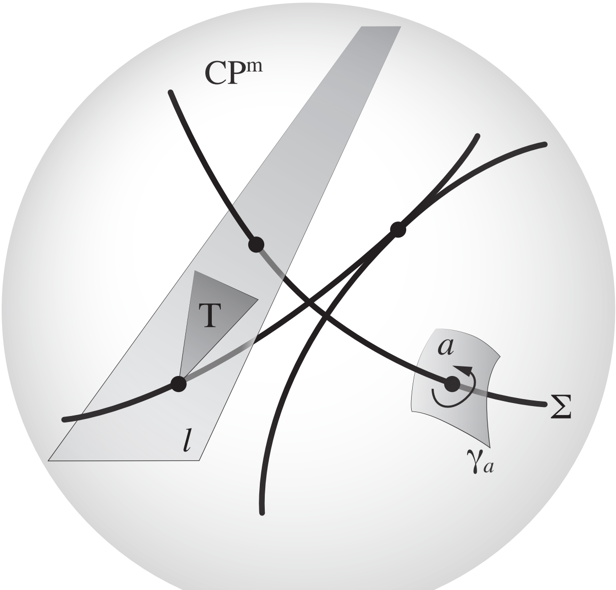

Figure 4. Singular Pfaffian system on $\mathbb{C}P^m$ with singular locus Σ and its restriction on a line ℓ

system, as any solution $x(t)$ is locally uniquely determined by its value $x(a) \in \mathbb{C}^n$. The correspondence $\gamma \mapsto M_\gamma$ is called the *monodromy* of the system; it is a linear representation of the fundamental group $\pi_1(\mathbb{C}P^m \smallsetminus \Sigma, a)$.

Definition 16. A *small loop* around a point $a \in \Sigma$ is the image of a sufficiently small circle $\{|z| = r\}$ by a map (holomorphic curve) $\chi\colon (\mathbb{C}, 0) \to (\mathbb{C}P^m, a)$ such that $\chi(z) \notin \Sigma$ for $z \neq 0$. This loop is well defined as a free homotopy class. If a is a smooth point on the polar divisor Σ then the small loop also does not depend (again as a free homotopy class) on the choice of χ. Consequently, the monodromy operator M_γ along a small loop γ is well defined modulo conjugacy in $\mathrm{GL}_n(\mathbb{C})$.

For any 1-dimensional line $\ell \subset \mathbb{C}P^m$, not entirely lying in Σ and equipped with an affine chart t, the system $\mathrm{d}X = \Omega X$ can be restricted on ℓ and reduced to a system of ordinary linear differential equations $\dot{x} = A(t)x$ with some rational matrix function $A(t)$. If we consider a pencil of lines passing through a fixed point in $\mathbb{C}P^m$, the corresponding restrictions can be considered as a parametric family of linear systems, analytically depending on an $(m-1)$-dimensional parameter. If the initial system was integrable then the corresponding family will be isomonodromic: the result of analytic continuation of solutions along a path avoiding the singular locus is independent of the parameter.

Solutions of this system are ramified over the locus $\Sigma_\ell = \Sigma \cap \ell$ and one can attempt to count zeros of arbitrary linear combinations $y_c(t) = \sum c_i x_i(t)$ "globally" on ℓ exactly as described in Remark 15. Define the number $\mathscr{N}(\Omega)$ as the supremum of the number of isolated zeros of all linear combinations in all open triangles:

$$\mathscr{N}(\Omega) = \sup_{T \subset \mathbb{C}P^m \smallsetminus \Sigma} \ \sup_{x\colon \mathrm{d}x = \Omega x} \ \sup_{c \in \mathbb{C}^n} \#\{t \in T\colon \textstyle\sum c_i x_i(t) = 0\}. \tag{14}$$

We will specifically consider the linear systems of the form $\mathrm{d}x = \Omega x$, defined over $\mathbb{Q}$: this means that in some affine chart $(t_1, \ldots, t_m)$ on $\mathbb{C}P^m$, each of the rational forms ω_{ij} has coefficients from the field $\mathbb{Q}(t_1, \ldots, t_m)$. In other words, the matrix 1-form Ω can be explicitly written using the variables t_i, their differentials and natural numbers. This allows one to introduce the "magnitude-like" parameter of the linear system: its *size* $\mathscr{S}(\Omega)$, as the maximal natural number required to write Ω down explicitly.

Quasialgebraic systems

The principal result can be formulated as follows: an integrable regular Pfaffian system admits a uniform upper bound on the number of roots of its solutions if the monodromy operators along all small loops only have modulus one eigenvalues. For systems defined over $\mathbb{Q}$, this bound is explicitly double exponential. A more precise formulation follows (see [6]).

Theorem 17. Consider an integrable Pfaffian n-dimensional system $\mathrm{d}x = \Omega x$ with rational coefficients of degree $\leqslant d$ on the projective space $\mathbb{C}P^m$. Assume that:
1. The system is regular, i.e. all its solutions grow at most polynomially when approaching the singular locus Σ.
2. Each monodromy operator M_γ along any small loop γ has its spectrum on the unit circle $|\lambda| = 1$.

Then, the supremum defined in (14) is finite: $\mathscr{N}(\Omega) < \infty$. If, in addition,
3. Ω is defined over $\mathbb{Q}$ and has known size $\mathscr{S}(\Omega) \leqslant s$

then the supremum (14) is explicitly bounded:

$$\mathscr{N}(\Omega) \leqslant s^{2^{\mathrm{Poly}(n,m,d)}}, \tag{15}$$

where $\mathrm{Poly}(n, m, d)$ is some explicit polynomial of degree $\leqslant$ 20 in the parameters n, m, d.

Applications

In practice, Theorem 17 is applied in the cases where a tuple of solutions is given initially and the Pfaffian system appears only a posteriori as a differential identity satisfied by these solutions (see [6, Appendix A]). The principal application concerns *periods* or (complete) abelian integrals. These are integrals of rational 1-forms over cycles on algebraic curves, and vanishing of periods is a condition describing appearance of limit cycles in the perturbation of planar integrable polynomial vector fields.

A general planar algebraic curve C_t of degree $\leqslant k+1$ on the (X, Y)-plane is defined by the equation $\sum_{i+j=0}^{k+1} t_{ij} X^i Y^j = 0$. The nonzero string of coefficients $\{t_{ij}\}_{i+j=0}^{k+1}$ defined modulo a common factor parametrises the space of all such curves by points of the projective space $\mathbb{C}P^m$, $m = \frac{1}{2}(l+1)(k+2)$. Generically, the monomial forms $\alpha_{ij} = X^i Y^j\,\mathrm{d}X$, $0 \leqslant i, j \leqslant k$, generate the cohomology of C_t and the tuple of periods $x_{ij}(t) = \oint_{C_t} \alpha_{ij}$ satisfies a system of Pfaffian equations of dimension $n = k^2$, known as the Picard–Fuchs system or, more geometrically, the Gauss–Manin connection. The polar locus Σ of this system consists of several components but one can easily see that integrals grow at most polynomially as the parameters t approach Σ. The monodromy condition follows from the Picard–Lefschetz formulas and the Kashiwara theorem. Thus, the only nontrivial things left to do are to verify the fact that the Picard–Fuchs system is defined over $\mathbb{Q}$ and then to estimate its size. Application of Theorem 17 allows one to prove the following result.

Corollary 18. The integral of a polynomial 1-form of degree $\leqslant k$ over ovals (compact closed components of level curves) of a polynomial of degree $k + 1$ may vanish no more than $2^{2^{P(k)}}$ times unless it vanishes identically. Here, P is an explicit polynomial of degree no greater than 61.

By the classical Poincaré–Pontryagin criterion, this implies that a perturbation of the polynomial Hamiltonian system $\dot{X} = \frac{\partial H}{\partial Y}$, $\dot{Y} = -\frac{\partial H}{\partial X}$, with a polynomial Hamiltonian $H \in \mathbb{R}[X, Y]$ of degree $k + 1$ by polynomial *non-conservative* perturbation of degree k, may produce no more than double exponential in a k number of limit cycles. This brings us back to Hilbert's 16th problem discussed in the introduction.

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On the Mathematical Works of Pierre Dolbeault

Christine Laurent-Thiébaut (CNRS/Université Grenoble Alpes, France)

Pierre Dolbeault was born on 10 October 1924 and died on 12 June 2015. After his secondary education in Paris, he was accepted as a student at the Ecole Normale Supérieure for three years starting in 1944, following which he obtained a position as a researcher (attaché de recherche) with the CNRS (Centre National de la Recherche Scientifique) from 1947 to 1953. During that time, he was given a grant to spend the academic year 1949–1950 at Princeton University, where he met Kodaira and Spencer and started learning the techniques and tools used in research in complex analysis. On his return to Paris, he attended Henri Cartan's Ecole Normale Supérieure seminar and began preparing his French "Thèse d'Etat" under his supervision. He defended his thesis, entitled "Formes différentielles et cohomologie sur les variétés analytiques complexes", in Paris in 1955, and the results were published in the Annals of Mathematics ([3, 4]) in 1956 and 1957. From 1953 onwards, Pierre Dolbeault held positions at several French universities, such as Montpellier and Bordeaux; in 1960, he became a professor at the University of Poitiers and finally moved to Paris 6 (Université Pierre et Marie Curie) in 1972, where he stayed until his retirement in 1992. His intense mathematical activity continued after retirement, both through participation at international conferences and workshops and through the publication of papers. He submitted his last paper for publication in January 2015, only six months before his death.

During his stay in Poitiers, he developed a school of complex analysis, supervising the French Thèses d'Etats of Jean-Louis Cathelineau, Joseph Le Potier and Jean Poly. Arriving in Paris in 1972, he joined Pierre Lelong and Paul Malliavin in organising the Seminar of Complex Analysis, founded by Pierre Lelong in the 1960s, which took place every Tuesday at the Institut Henri Poincaré in Paris. He also created a working group in Paris 6 where residue theory, the complex Plateau problem and many other questions were studied.

Alongside Pierre Lelong and Paul Malliavin, Pierre Dolbeault was one of the founders of the Institute of Complex Analysis and Geometry in Paris, which was at the origin of the current Institute of Mathematics of Jussieu. He was its director from the creation of the institute in 1974 until 1982.

Pierre Dolbeault was my advisor in Paris from 1975 until the defence of my French Thèse d'Etat in June 1985. Initially, he gave me some pointers but I quickly escaped and went by my own road. He always respected that and I am very grateful for the autonomy he gave me. I will never forget that he was always there to answer my mathematical questions when I needed him.

At the beginning of the 1990s, the European Commission created the European Human Capital and Mobility networks to promote the mobility of researchers through the different countries of the European community. European mathematicians working in the field of complex analysis and geometry decided to submit an application and Pierre Dolbeault was naturally the leader of this project. The project was successful for the first time in 1994 and was renewed in 1998 by Henri Skoda. The network connected centres in France (Paris and Grenoble), Germany (Wuppertal and Berlin), Italy (Firenze) and Sweden (Göteborg) and the administration by Pierre Dolbeault was very efficient.

Pierre Dolbeault was a lovely man. The mathematicians he met all over the world will remember him as a very discrete and kind colleague, always ready to help and to discuss mathematics. He had a deep understanding of the fields to which he contributed and a very wide mathematical culture around them, as is obvious from the fine survey papers he wrote throughout his life ([9, 11, 15, 17, 19, 24]). Let us now continue to develop the work of Pierre Dolbeault, which is involved with the origins of modern complex analysis.

1 The Dolbeault–Grothendieck lemma and the Dolbeault isomorphism

The main results

The very important results that are today called the *Dolbeault–Grothendieck lemma* and the *Dolbeault isomorphism* are part of Pierre Dolbeault's thesis that he prepared under the supervision of Henri Cartan and defended in 1955. These results were announced in 1953 in two Notes aux Comptes Rendus de l'Académie des Sciences de Paris [1] and [2], presented by Jacques Hadamard and published in the Annals of Mathematics in 1956 [3].

Let V be a complex manifold of complex dimension n. Then, for $p, q \geq 0$, we denote by Ω^p the sheaf of holomorphic p-forms on V, $\mathcal{E}^{p,q}$ the sheaf of germs of smooth differential forms of bidegree (p, q) and $(\mathcal{D}')^{p,q}$ the sheaf of germs of currents of bidegree (p, q). For a family Φ of supports in V (the most common families of supports are the family of all closed subsets of V and the family of all compact subsets of V), we consider, on one hand, the Čech cohomology groups $H^q_\Phi(V, \Omega^p)$, $q \geq 0$, of the sheaf Ω^p with support in the family Φ and, on the other hand, the cohomology groups, denoted respectively by $H^{p,q}_{\Phi,\infty}(V)$ and $H^{p,q}_{\Phi,cur}(V)$, of the differential complexes of sections with support in Φ, $(\Gamma_\Phi(V, \mathcal{E}^{p,\bullet}), \overline{\partial})$ or $(\Gamma_\Phi(V, (\mathcal{D}')^{p,\bullet}), \overline{\partial})$, where $\overline{\partial}$ is the Cauchy–Riemann operator.

The main result of the first part of Pierre Dolbeault's thesis is the following.

Theorem 1. Let V be a complex manifold. For any $p, q \geq 0$ and any family of support in V, the cohomology group $H^q_\Phi(V, \Omega^p)$ is canonically isomorphic to $H^{p,q}_{\Phi,\infty}(V)$ and $H^{p,q}_{\Phi,cur}(V)$.

This isomorphism is called the *Dolbeault isomorphism*. The Dolbeault isomorphism says that a global problem in complex analytic geometry involving the cohomology of the sheaf of holomorphic functions can be solved using global $\bar{\partial}$-equations on the complex manifold. It makes a link between complex analytic geometry and the theory of partial differential equations.

The first application is a vanishing result for the cohomology of the sheaf Ω^p on V in large degrees.

Corollary 2. Let V be a complex manifold. If q is an integer strictly greater than the complex dimension of V then $H^q_\Phi(V, \Omega^p) = 0$.

The proof of Theorem 1 is based on the following result of sheaf theory and cohomology from the early 1950s.

Proposition 3. Let X be a topological space, Φ a family of supports in X and $\mathcal{F}$ a sheaf on X. If $\mathcal{F}$ admits a resolution $\mathcal{R}$

$$0 \to \mathcal{F} \to \mathcal{R}^0 \to \mathcal{R}^1 \to \cdots \to \mathcal{R}^p \to \dots,$$

such that $H^k_\Phi(X, \mathcal{R}^p) = 0$, for all $k > 0$ and all $p \geq 0$, the canonical homomorphism

$$\delta^q \; : \; H^q(\Gamma_\Phi(X, \mathcal{R})) \to H^q_\Phi(X, \mathcal{F}^p)$$

is an isomorphism.

The idea of Pierre Dolbeault was to introduce a new tool, $\bar{\partial}$-cohomology, today called *Dolbeault cohomology* of a complex manifold, to describe the cohomology of the sheaf Ω^p on V via the cohomological result from Proposition 3. The $\bar{\partial}$-cohomology is the cohomology of the complexes $(\Gamma_\Phi(V, \mathcal{E}^{p,\bullet}), \bar{\partial})$ or $(\Gamma_\Phi(V, (\mathcal{D}')^{p,\bullet}), \bar{\partial})$; it is the holomorphic analogue of the de Rahm cohomology in real differentiable manifolds.

To derive Theorem 1 from Proposition 3, it is sufficient to prove that the sequence of sheaves $\mathcal{E}^{p,\bullet}$ or $(\mathcal{D}')^{p,\bullet}$, with the Cauchy–Riemann operator $\bar{\partial}$, defines a resolution of fine sheaves of the sheaf Ω^p.

The sheaves $\mathcal{E}^{p,q}$ or $(\mathcal{D}')^{p,q}$ are sheaves of $\mathcal{E}$-modules, so they are fine and the hypothesis of Proposition 3 on the vanishing of the cohomology with support in a family of support Φ is satisfied as soon as a resolution of Ω^p is defined. Therefore, it remains to prove that the sequences

$$0 \to \Omega^p \to (\mathcal{D}')^{p,0} \to (\mathcal{D}')^{p,1}$$

and

$$(\mathcal{D}')^{p,q-1} \to (\mathcal{D}')^{p,q} \to (\mathcal{D}')^{p,q+1}$$

are exact sequences of sheaves.

The first sequence is exact since the restriction of any $\bar{\partial}$-closed $(p, 0)$-current to a coordinates chart is harmonic, so it is a smooth, $\bar{\partial}$-closed $(p, 0)$-form, hence an holomorphic p-form. The exactness of the second sequence follows from the Dolbeault–Grothendieck lemma.

Lemma 4. Let T be a germ of $\bar{\partial}$-closed (p, q)-current (resp. smooth differential form), $q > 0$, on a complex manifold V. Then, there exists a germ of $(p, q-1)$-current (resp. smooth differential form) S such that $T = \bar{\partial}S$.

This lemma was proved independently by Pierre Dolbeault and Alexander Grothendieck. Grothendieck's proof is an inductive process on the dimension based on the non-homogeneous Cauchy formula in complex dimension 1. In Pierre Dolbeault's thesis, an argument of potential theory due to Henri Cartan reduces the proof of the lemma to the real analytic case.

Lemma 5. Let φ be a germ of $\bar{\partial}$-closed, real analytic (p, q)-form, $q > 0$, on a complex manifold V. Then, there exists a germ of real analytic $(p, q-1)$-form ψ such that $\varphi = \bar{\partial}\psi$.

Lemma 5 was proved by homotopy, by Pierre Dolbeault, in the spirit of the proof of the Poincaré lemma for the operator d.

Some applications

Let us describe some applications of Theorem 1 given by Pierre Dolbeault in his thesis.

First, he considered the classes, modulo isomorphism, of holomorphic principal bundles over a complex manifold V, with fiber a complex abelian Lie group G. Let $\mathcal{G}$ be the sheaf of germs of holomorphic functions on V with values in G. He proved that there exists a one-to-one correspondence between these classes and the elements of the cohomology group $H^1(V, \mathcal{G})$.

In the special case when $G = \mathbb{C}$, $\mathcal{G} = \Omega^0$ is nothing other than the sheaf of germs of holomorphic functions on V and, by Theorem 1, the cohomology group $H^1(V, \mathcal{G})$ is isomorphic to the group $H^{0,1}(V)$. In particular, for such a bundle to have an holomorphic section, it is necessary and sufficient that the associated cohomology class vanishes. Therefore, if the manifold V satisfies $H^{0,1}(V) = 0$ then all holomorphic principal bundles over V with fiber $\mathbb{C}$ are trivial.

For $G = \mathbb{C}^*$, he analysed the topological and analytical obstructions for a principal bundle to be trivial. Once again, they are given in terms of differential forms.

Another application is the first Cousin problem. Let V be a complex manifold and M^p be the sheaf of germs of meromorphic p-forms on V, $p \geq 0$. We get the exact sequence of sheaves

$$0 \to \Omega^p \to M^p \to M^p/\Omega^p \to 0,$$

from which we can derive the long exact sequence of cohomology

$$\cdots \to H^0(V, M^p) \to H^0(V, M^p/\Omega^p) \to H^1(V, \Omega^p) \to \dots .$$

An element t of $H^0(V, M^p/\Omega^p)$ is called Cousin data of degree p. Solving the first Cousin problem means, given Cousin data t, finding a meromorphic function on V whose image in $H^0(V, M^p/\Omega^p)$ is t. If there is a solution, the data t is called solvable. It follows from Theorem 1 that the obstruction to the solvability of the first Cousin problem is an element of $H^{p,1}(V)$. In the case when V is such that $H^{p,1}(V) = 0$, all Cousin data are solvable. Thanks to the Dolbeault isomorphism, solving the first Cousin problem can be reduced to solving a $\bar{\partial}$-equation.

2 The residue theory

The residue theory in several complex variables was founded by Poincaré in 1887 and continued by Picard at the very beginning of the 20th century. In the 1930s, de Rham's currents where introduced in residue theory and then used more systematically through the Cauchy principal value in the 1950s. Cohomological residue theory takes its origin in the seminal work of Leray from 1959 and was completed by Norguet the same year.

The one-dimensional case

Let X be a Riemann surface and g a meromorphic function defined on an open subset U in X, such that g has only one pole at a point P in U. If z is an holomorphic coordinate on U such that $z(P) = 0$, the coefficient a of $\frac{1}{z}$ in the Laurent series is called the *Cauchy residue* of g at P. In fact, a is an invariant of the closed meromorphic $(1,0)$-form $\omega = g(z)\,dz$. The smooth differential form ω defines a current on $U \setminus P$, which can be extended to U as a current, denoted $\mathrm{vp}(\omega)$ – the *principal Cauchy value* of ω – and defined by

$$< \mathrm{vp}(\omega), \psi >= \lim_{\varepsilon\to 0} \int_{|z|\geq\varepsilon} \omega \wedge \psi,$$

where $\psi \in \mathcal{D}^{0,1}(U)$ is a smooth $(0,1)$-form with compact support in U. Moreover, since ω is $\overline{\partial}$-closed in $U \setminus P$, the current $\overline{\partial}\,\mathrm{vp}(\omega)$ is supported by $\{P\}$ and satisfies

$$\overline{\partial}\,\mathrm{vp}(\omega) = 2i\pi a\delta_P + \partial B,$$

where δ_P is the Dirac measure at P and B a current supported by $\{P\}$. In particular, $B = 0$ if the pole P is simple.

Consider now a general meromorphic form ω and denote by $S = \{P_l \mid l \in I\}$ the set of poles of ω (it is a discrete subset of points in X). Let $\mathrm{res}_{P_j}(\omega)$ be the Cauchy residue of ω at P_j. Let $J \subset I$ be a finite subset of I and, for each $j \in J$, let D_j be a disc centred at P_j such that $\overline{D}_j \cap S$ is reduced to P_j and let γ_j be the boundary of D_j. If $(n_j)_{j\in J}$ is a family of elements in $\mathbb{Z}$, $\mathbb{R}$ or $\mathbb{C}$ then we have the *residue formula*

$$\int_{\sum_{j\in J} n_j\gamma_j} \omega = \sum_{j\in J} 2i\pi\, n_j \mathrm{res}_{P_j}(\omega).$$

Moreover, if X is compact and connected, the *residue theorem* holds: for any discrete (hence finite) set $S = \{P_l \mid l \in I\}$ of X and any meromorphic 1-form ω whose set of poles is S, we have

$$\sum_{l\in I} \mathrm{res}_{P_l}(\omega) = 0.$$

Conversely, for any such subset S of X and any subset $(\alpha_j)_{j\in I}$ of complex numbers such that $\sum_{j\in I}\alpha_j = 0$, there exists a meromorphic 1-form ω on X having simple poles exactly on S and such that, for any $j \in I$, $\alpha_j = \mathrm{res}_{P_j}(\omega)$.

Cauchy principal values and residue currents

In general, if X is a complex manifold of arbitrary complex dimension n, a differential form ω on X is called *semi-meromorphic* if any point x in X admits a neighbourhood U such that $\omega_{|U} = \frac{\alpha}{f}$, where α is a smooth form and f an holomorphic function that does not vanish identically on U. The set $S = \{x \in U \mid f(x) = 0\}$ is the *polar set* of ω on U.

If ω is a closed form ($d\omega = 0$), the idea is to associate to ω a current supported by S corresponding to the current $a\delta_P$ when $n = 1$, which will be called the *residue current* of ω.

We say that a differential operator D is *semi-holomorphic* if, for any $x \in X$, there exists a neighbourhood U of x and holomorphic coordinates $(z_1, \ldots, z_n)$ such that, on U,

$$D = \sum_{i_1,\ldots,i_n} \alpha_{i_1,\ldots,i_n}(z) \frac{\partial^{i_1}}{\partial z_1^{i_1}} \cdots \frac{\partial^{i_n}}{\partial z_n^{i_n}},$$

where the $\alpha_{i_1,\ldots,i_n}$ are smooth functions. This definition is independent of the choice of coordinates. The set of semi-holomorphic differential operators in X is a ring denoted by $\Delta(X)$. Schwartz proved in 1953 that the space of semi-meromorphic forms on X is a $\Delta(X)$-module.

Following the one dimensional case, to define the residue current associated to a semi-meromorphic form ω, whose polar set is contained in the analytic subset S of X, the idea is first to extend ω as a current T (corresponding to the principal Cauchy value) to X, which satisfies $T(D\omega) = DT(\omega)$ for any $D \in \Delta(X)$.

When the set S is a complex manifold, this was done by Schwartz in 1953. In the second part of his thesis [4], Pierre Dolbeault considered the case of semi-meromorphic p-forms, whose polar set S admits special singularities; in particular, the singular set S' of S has to be a manifold and some other regularity conditions are added. Under these conditions on S, he proved that there exists a current $\mathrm{vp}(\omega)$, which canonically extends ω such that $\mathrm{vp}(\omega) = \omega$, when ω is a smooth form on X and $D\mathrm{vp}(\omega) = \mathrm{vp}(D\omega)$ for any $D \in \Delta(X)$. If ω is closed, he defined the residue current of ω by $\overline{\partial}\,\mathrm{vp}(\omega)$. He also proved that if $S = \cup_k S_k$, with S_k the irreducible components of S in a neighbourhood of a point $x \in S$, then

$$\overline{\partial}\,\mathrm{vp}(\omega) = 2i\pi \sum_k \mathrm{vp}_{S_k} u_k + \partial B, \tag{1}$$

where the u_k are closed meromorphic $(p-1)$-forms on S_k with polar set $S_k \cap S'$ and B a $(p-1,1)$-current supported by S and $\overline{\partial}$-closed on $S \setminus S'$. He also gave necessary and sufficient conditions for a closed current of the form (1) with $B = 0$ to be the residue of a meromorphic form on X with polar set S of multiplicity 1.

At the end of the 1960s (see [5] and [7]), Pierre Dolbeault considered polar sets with normal crossings and, when X is an irreducible, projective algebraic variety, using Hironaka's resolution of singularities reducing general singularities to normal crossings, he solved the problem in the general case.

Later in 1972 [8], using the Cauchy principal value defined by Herrera and Lieberman in reduced complex spaces, he obtained an axiomatic definition of the canonical extension of a semi-meromorphic p-form defined on a reduced complex space proving the uniqueness of the Cauchy principal values.

Theorem 6. Let X be a reduced complex space, $\mathcal{S}^\bullet(X)$ be the space of semi-meromorphic differential forms and $\Delta(X)$ be the ring of semi-holomorphic differential operators in X. Let T_X be a map from $\mathcal{S}^\bullet(X)$ into the space $\mathcal{D}'_\bullet(X)$ of currents on X such that:

(1) If X is a manifold and ω a semi-meromorphic form on X with polar set with normal crossings and with locally integrable coefficients then $T_X(\omega)$ coincides with the current defined by the L^1_{loc}-form ω.
(2) If X is a manifold, the restriction of T_X to the subspace of $\mathcal{S}^\bullet(X)$ of semi-meromorphic forms with given polar set with normal crossings is $\Delta(X)$-linear.
(3) Let X' be another reduced complex space. For any morphism $\Phi : X' \to X$, there exists a map

$$^*\Phi : \mathrm{Im}\,T_X \to \mathrm{Im}\,T_{X'},$$

with the following property: for any $\omega \in \mathcal{S}^\bullet(X)$, we have

$$^*\Phi T_X(\omega) = T_{X'}(\Phi^*\omega).$$

(4) For any $\omega \in \mathcal{S}^\bullet(X)$ whose polar set S contains the singular set of X and for any resolution of singularities π such that X' is a manifold and $S' = \pi^{-1}(S)$ has normal crossings, we have
$$T_X(\omega) = \pi_* {}^*\pi\, T_X(\omega).$$
The T_X is unique and coincides with the Cauchy principal value.

Moving to differential forms with more general singularities, in 1977, he considered, with Jean Poly [10], the case of subanalytic singularities.

In 1978, Coleff and Herrera generalised the notion of Cauchy principal value and residue current in relation to composed residues. These new currents are called *residual currents*. Later, in the middle of the 1980s, a new definition of residual currents was given by Passare. In 1988-89, Pierre Dolbeault decided to study the structure of residual currents in analogy with the structure theorem for holomorphic chains by Harvey and Shiffman. After considering the normal crossings [13], in a collaboration with Letellier around 1990 [16], he described the local structure of the first residual current in the general case in terms of holomorpic differential operators and Cauchy principal values on the irreducible components of the polar set.

At last, in 2009, Pierre Dolbeault proved a generalisation of a theorem by Picard, characterising residues of closed meromorphic p-forms [23]. For $p = 1$, the theorem is as follows.

Theorem 7. Let X be a Stein manifold or a compact Kähler manifold. Then, the following conditions are equivalent:
(i) The locally rectifiable closed current T of bidimension $(n-1, n-1)$ is exact.
(ii) The current T is the residue current of a closed meromorphic 1-form on X having the support of T as a polar set with multiplicity 1.

Residue theory and homology

Consider the case when S is a polar set of a closed semi-meromorphic p-form ω with multiplicity k. Then, locally, S is the zero set of an holomorphic function s on an open subset U of X such that $ds \neq 0$ and $\omega = \frac{ds}{s}\wedge\psi + d(\frac{\theta}{s^{k-1}})$ on U, where ψ and θ are smooth forms on U. In fact, $\psi_{|S}$ is a smooth, globally defined, closed $(p-1)$-form on S, called the *residue form* of ω.

From the cobord homomorphism δ^* of the exact cohomology sequence with complex coefficients and compact supports associated to the closed subset S of X
$$\cdots \to H^r_c(X) \to H^r_c(S) \xrightarrow{\delta^*} H^{r+1}_c(X \setminus S) \to H^{r+1}_c(X) \to \ldots, \tag{2}$$
the duality isomorphism of Poincaré defines the homology cobord
$$\delta : H^c_{2n-r-2}(S) \to H^c_{2n-r-1}(X \setminus S).$$
Then, since the vector space of cohomology of degree q with closed support is the dual of the vector space of homology of degree q with compact support, Leray could define the *residue homomorphism*
$$\mathrm{Res} \,:\, H^{2n-r-1}(X \setminus S) \to H^{2n-r-2}(S)$$
as the transposed homomorphism of δ. This allows one to associate to each p-class of cohomology $\overline{\omega}$ of $X \setminus S$ a cohomology class $\mathrm{Res}[\overline{\omega}]$ of $H^{p-1}(S)$ (in the case when ω is a closed semi-meromorphic form, the image of the cohomology class of ω by the residue homomorphism is nothing other than the cohomology class of the residue form of ω) and the pairing of these two morphisms gives the *residue formula*
$$\int_\gamma \omega = 2i\pi \int_h \mathrm{Res}[\overline{\omega}],$$
with $\overline{h} \in H_{p-1}(S)$, $\overline{\omega} \in H^p(X \setminus S)$ and $\gamma \in \delta(\overline{h})$.

Leray also proved that, given a class $c \in H^p(X \setminus S)$, there exists a semi-meromorphic closed p-form with polar set S of multiplicity 1, whose restriction to $X \setminus S$ belongs to c and whose residue form is $\mathrm{Res}[c]$.

In [6], Pierre Dolbeault considered the Borel-Moore homology, whose groups are canonically isomorphic to the homology groups of singular locally finite chains, and the associated exact sequence
$$\cdots \leftarrow H_r(X) \leftarrow H_r(S) \leftarrow H_{r+1}(X \setminus S) \leftarrow H_{r+1}(X) \leftarrow \ldots \tag{3}$$
is the dual of the exact sequence of cohomology with compact support (2). Its connection morphism $\delta_* : H_{r+1}(X\setminus S) \to H_r(S)$ comes from the residue homomorphism Res by composition with the Poincaré isomorphisms for homology and cohomology with closed supports of S and $X \setminus S$. It is called the *homological residue homomorphism* and is defined in much more general situations than Res, in particular when S is no longer a complex manifold but a complex analytic subset of X of codimension 1.

3 Boundary problems

Let X be a complex manifold of complex dimension n. The boundary problems are geometric extension problems. Given an odd real-dimensional compact submanifold M of X with negligible singularities in the sense of the Hausdorff measure $H_{\dim M}$ and satisfying the necessary conditions to be the boundary of a complex manifold in X, can we extend M as a complex manifold whose boundary is M?

Another natural question is: when is a 2-codimensional real compact submanifold S of X the boundary of a compact real hypersurface M such that $M \setminus S$ is Levi-flat?

In both cases, these problems can also be considered as Plateau problems, i.e. the search of manifolds with prescribed boundary and minimal volume, since, if X is a Kähler manifold, complex varieties and Levi-flat hypersurfaces minimise the volume among even dimensional manifolds or foliated manifolds.

Pierre Dolbeault began to be interested in these problems after hearing Harvey's talk on the joint work with Lawson on boundaries of complex varieties during the conference on complex analysis organised in Williamstown in 1975. Back in France, he decided that the working group he used to organise each year in Paris 6 would study Harvey's and Lawson's papers to form a complete understanding of their works.

The complex Plateau problem

Let X be a complex Hermitian manifold of complex dimension n and M a smooth oriented closed real manifold of real dimension $2p-1$, $0 < p \leq n$. We denote by $[M]$ the integration current on the manifold M. Then, $[M]$ has locally finite mass and $d[M] = 0$. The complex Plateau problem consists of looking for necessary and sufficient conditions on M such that there exist holomorphic p-chains T of $X \setminus M$ of locally finite mass in the neighbourhood of M with $dT = [M]$.

For example, consider the case when $X = V \times \mathbb{C}$, with V a Stein manifold of complex dimension $p \geq 2$. Let D be a relatively compact domain in V with smooth boundary and f be a smooth CR function on ∂D (i.e. the differential of f, restricted to the complex subspace to the tangent space to ∂D, at each point, is $\mathbb{C}$-linear). Then, the graph M of f defines a smooth closed real $(2p-1)$-dimensional submanifold of X. By the Hartogs-Bochner theorem, the function f extends as an holomorphic function F to D. Then, taking the graph of F in X, we get a complex hypersurface of X whose boundary is M, so it solves the complex Plateau problem for M.

The study of the complex Plateau problem was initiated by Wermer in 1958 for $X = \mathbb{C}^n$, $p = 1$ and M an holomorphic image of the unit circle of $\mathbb{C}$. For $X = \mathbb{C}^n$, the complex Plateau problem has been solved by Harvey and Lawson in 1975 with M an oriented C^1 compact manifold with negligible singularities. The necessary and sufficient condition on M for the existence of a solution is: M is maximally complex for $p \geq 2$ and the moment condition $\int_M \varphi = 0$ for every holomorphic 1-form φ on $\mathbb{C}^n$ for $p = 1$. For $p = n-1$, the method, inspired from the above example with good choices of projections, consists of building T as a graph, with multiplicities on the irreducible components, of an holomorphic function with a finite number of determinations. For the other p, it is reduced to the previous case using projections. Two years later, in 1977, they considered the case when $X = \mathbb{C}P^n \setminus \mathbb{C}P^{n-r}$ for compact M. In both cases the solution is unique.

The main contributions of Pierre Dolbeault in this field are the joint works with Henkin on the complex Plateau problem for $X = \mathbb{C}P^n$ and, more generally, a q-linearly concave domain X of $\mathbb{C}P^n$, i.e. a union of projective subspaces of dimension q (for $p = 1$ in [18] and for general p in [20]). The boundary problem in $\mathbb{C}P^n$ was set up for the first time by King in 1979. In that case, uniqueness no longer holds, since two solutions differ by an algebraic p-chain. The necessary and sufficient condition for the existence of a solution is an extension of the moment condition: it uses a Cauchy residue formula in one variable and a nonlinear differential condition that appears in many questions of geometry and mathematical physics. In the simplest case, $p = 1$ and $n = 2$, this is the shock wave equation for a local holomorphic function of 2 variables ξ and η: $f\frac{\partial f}{\partial \xi} = \frac{\partial f}{\partial \eta}$.

Levi flat hypersurfaces with prescribed boundaries

Let S be a smooth 2-codimensional real submanifold of $\mathbb{C}^n$, $n \geq 2$. The problem of finding a Levi-flat hypersurface $M \subset \mathbb{C}^n$ with boundary S has been extensively studied for $n = 2$ (during the 1980s and the 1990s), when S is contained in the boundary of a bounded, strictly pseudoconvex domain. In two joint works with Tomassini and Zaitsev ([21] and [25]), Pierre Dolbeault proved some results for $n > 2$. The situation is quite different from how it is in $\mathbb{C}^2$, since a submanifold of real codimension 2 in general position is no longer totally real if $n > 2$. They first studied the necessary local conditions to ensure S bounds a Levi-flat hypersurface at least locally. They observed that, near a CR point, S has to be nowhere minimal, i.e. all local CR orbits must be of positive codimension, and some flatness condition has to occur at complex points. In [25], the following theorem is proved.

Theorem 8. Let $S \subset \mathbb{C}^n$, $n > 2$, be a compact connected smooth real 2-codimensional submanifold such that the following holds:
(i) S is nonminimal at every CR point.
(ii) Every complex point of S is flat and elliptic and there is at least one such point.
(iii) S does not contain complex submanifolds of dimension $n-2$.

Then, S is a topological sphere with two complex points and there exists a smooth submanifold $\widetilde{S}$ and a Levi-flat $(2n-1)$-subvariety $\widetilde{M}$ in $\mathbb{R} \times \mathbb{C}^n$ (i.e. $\widetilde{M}$ is Levi-flat in $\mathbb{C} \times \mathbb{C}^n$), both contained in $[0,1] \times \mathbb{C}^n$ such that $\widetilde{S} = d\widetilde{M}$ in the sense of currents, and the natural projection $\pi : [0,1] \times \mathbb{C}^n \to \mathbb{C}^n$ restricts to a diffeomorphism between $\widetilde{S}$ and S.

They first proved the existence of a foliation of class C^∞ with 1-codimensional CR orbits as compact leaves and then reduced the problem to a complex Plateau problem with parameter to get the result. To complete the proof, they extended to the smooth case the analytic solution of the boundary problem for an analytic Levi-flat subvariety in a real hyperplane of $\mathbb{C}^n$ studied by Pierre Dolbeault in [12].

When S is a smooth graph over the boundary of a strictly convex domain Ω in $\mathbb{C}^n \times \mathbb{R}$, M is the graph of a Lipschitz function defined on the closure of Ω [26].

In a later paper [22], Pierre Dolbeault studied the case when S admits one hyperbolic point and then, in [27], he gave examples of 2-codimensional submanifolds bounding a Levi-flat hypersurface with one special 1-hyperbolic point. He first considered a sphere with two horns that has one special hyperbolic point and three special elliptic points, then a torus with two special hyperbolic points and two special elliptic points and then other elementary models and their gluing to get more complicated examples.

4 Quaternionic analysis

Beside classical complex analysis, Pierre Dolbeault was also interested in the study of quaternionic analysis, mostly after 2000 and until the end of his life. On the set $\mathbb{H} \simeq \mathbb{C}^2$ of quaternionic numbers, he considered the modified Cauchy-Fueter operator, which was introduced in 2007 by a group of Italian and Mexican mathematicians: Colombo, Luna-Elizarraras, Sabadini, Shapiro and Struppa. The elements of the kernel of this operator, inside the space of smooth $\mathbb{H}$-valued functions, are called *hyperholomorphic functions*. In contrast to the classical Cauchy-Fueter operator, this kernel contains all holomorphic maps from an open subset of $\mathbb{C}^2$ into $\mathbb{C}^2$ and in particular all $\mathbb{C}$-valued holomorphic functions of two variables.

In a first paper [28], Pierre Dolbeault characterised the quaternionic functions that are almost everywhere hyperholomorphic and whose inverses, with respect to right side multiplication, are also hyperholomorphic almost everywhere on an open subset U of $\mathbb{H}$, as the solutions of a system of two nonlinear partial differential equations. These functions are called *weak-hypermeromorphic functions*. The subspace of the set of weak-hypermeromorphic functions that is stable by sum and product is also characterised by nonlinear partial differential equations. It contains the space of meromorphic functions and is called the space of *hypermeromorphic functions*. In this setting, he extended the notion of Cauchy principal value and of residue to quaternionic 1-forms.

In a second paper he submitted for publication six month before his death [29], he defined *Hamilton 4-manifolds* as the analogue to Riemann surfaces of complex analysis. To do this, he followed the lines of construction of Riemann surfaces developed in his book [14].

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A former student of the Ecole Normale Supérieure de Fontenay aux Roses, Christine Laurent-Thiébaut prepared her Thèse d'Etat under the supervision of Pierre Dolbeault. After 12 years as an assistant professor in Paris at the University Pierre et Marie Curie, she became a full professor in Grenoble at the University Joseph Fourier in 1989. She was Director of the Institut Fourier between 1999 and 2002. She retired as a professor at the end of October 2015.

Mum and Postdoc at SISSA

Virginia Agostiniani (SISSA, Trieste, Italy)

The fact that most of my female colleagues of a similar age to me (around 30) and typically holding non-permanent positions have made the choice of not having a baby or planning to postpone the event has never weakened or delayed in me the idea of becoming a mum. I have always optimistically thought that being a mum and a postdoc couldn't be, in the end, so difficult. At least, that's what I thought before my child was born… The aim of this short article is to report my experience and that it was, indeed, difficult for me, even if my institution (SISSA, Trieste) gave support in helping me to survive.

In October 2014, I started my postdoc at SISSA, whose scope is contributing to the mathematical modelling of some active materials in the framework of the ERC grant "Micromotility" held by Professor A. DeSimone. Just a few months later, my maternity leave began, which in Italy covers a period of five months, one month or two before the birth of the baby plus, respectively, four or three after. Among the adversities that an Italian mum-postdoc encounters, the financial one is probably at the top. In this respect, the economic treatment of SISSA is 100% satisfying. And not only satisfying but also quite singular in the Italian panorama. To explain this, let me first recall that Italian maternity leave for postdocs is meant to be an interruption of the period of the contract, to which five months are then added at the end. At the same time, INPS (the Italian Social Security Service) pays 80% of the salary during the interruption (usually many months later and with many bureaucratic disruptions). Well, SISSA covers the part of the salary that is not covered by INPS so that, during the months of leave, one can count on a full salary. In my case, SISSA's contribution to maternity leave was even more relevant because the INPS regulations stipulate that a postdoc is entitled to 80% of the salary during maternity leave only if they have paid contributions for at least two months in the two years before the leave starts. Otherwise, nothing! Well, I didn't fulfil the requirement since I had been a postdoc for two years in Oxford before my Italian contract started and SISSA on their own covered my entire salary during the leave. Note that if I didn't work at SISSA but, for example, in an arbitrary Maths Department of some Italian University, the state wouldn't have given any economic support to me. And all this in a period where the need for a salary is more urgent than ever.

Exactly four months after my baby's birth, I am, at 9 am, in my office. From that point on, a very very hard period begins for me, in family as well as at work. Usually, in order to maximise the working time, I hardly find a moment to breathe during my office hours. This is due to many facts coming together: (1) I am working on a very hot topic and I need to proceed quickly, also to recover some of the "time lost" during the leave, this not only because I am required to produce some results during my present post but also in view of my overall career, since my position is not permanent; (2) My husband works and lives for most of the time in another town, situated four to five hours by train away from Trieste; and (3) Neither my family nor my husband's family live in Trieste or close to it. Here, again, SISSA's regulation helps but in this respect its help is less incisive. Here, we deal with a KINDERGARTEN. I feel the need to write this word in capital letters because it represents a dream for me (not for a few weeks yet, though). The fact is that SISSA has a very nice kindergarten situated in the beautiful park surrounding the main building. It is also highly ranked as a kindergarten for the activities undertaken and the overall organisation and concept; it is far from being a simple "baby parking". Moreover, SISSA provides economic support for childcare, so that the monthly rate is approximately 300 euros, which is a very good rate. The drawback is that the kindergarten only accepts babies who are older than 13 months. But this is a problem because most of the people working at SISSA, excluding admin people, are people whose families live far away and are therefore unable to offer their help when a baby arrives. So, I am sure that when my baby is finally accepted at the kindergarten, he will be superhappy and the teachers will prove to be supergood and trained, but arriving at that moment has been so difficult, for me as well as for my little son!! Just to give an idea, we spent the last nine months – between the end of the leave and now – finding and changing temporary and private (and bad) kindergartens, together with carefully scheduling grandparents' humanitarian visits.

Apart from SISSA's regulations concerning maternity leave and childcare, I have to say that the first year of a baby's life requires specific attention towards them and, even if I have had to renounce some conferences and some occasions to make progress in my job, I feel happy for not having deprived my baby of all my attention and support. We cannot have everything at the same time (at least, we mums) and the first year of a son happens once in a life. Also, carrying him to some conferences with me so as not to interrupt the breastfeeding – with the key aid of my husband – has been doubly stressing in terms of preparation for the conference but also doubly rewarding: my little son, in his official role of little mascot, has

been many times the subject of amusement for the conference participants, making social dinners and coffee breaks less conventional!

Virginia Agostiniani, after obtaining a BSc and a Master's degree in mathematics at the University of Florence, obtained her PhD at SISSA under the supervision of Gianni Dal Maso and Antonio DeSimone. She has been a postdoctoral research associate at the Oxford Centre for nonlinear PDE for two years and has had a postdoc position at SISSA since October 2014.

The London Mathematical Society's 150th Anniversary

Stephen Huggett (University of Plymouth, UK)

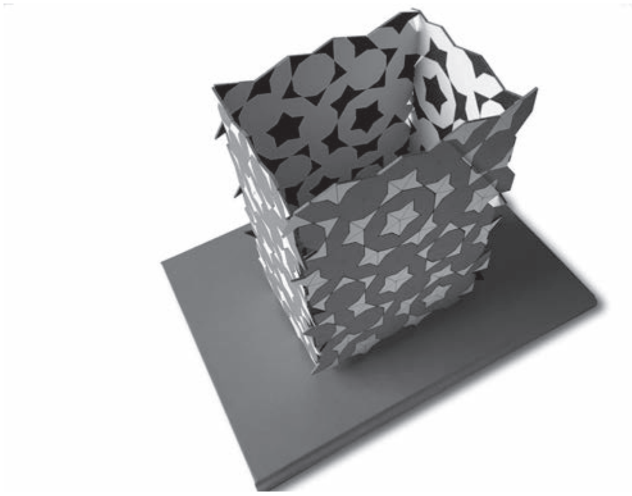
As this is a personal account of the LMS's 150th birthday I will start with the thing which most pleased me and, I admit, surprised me. Among all the many and varied things which we did to celebrate the anniversary, it was our work with the arts which stands out for me as the most intriguing. We commissioned a painter, Mark Francis (see a detail of his abstract painting on the cover of this Newsletter) and an architect, George Legendre, to be our artists-in-residence, and we commissioned the artist Heidi Morstang to make a film called *Thinking Space*.

Our interactions with these three people have been curious and thought-provoking for us, and will bring mathematical thinking to new audiences. Of course we are not the only people working at the arts/mathematics interface, and indeed it is almost part of the zeitgeist, but it was a new experience for me, anyway. All three artists have engaged very deeply with the project and produced beautiful work. The Society is very grateful to them.

Why does one celebrate an anniversary such as this? Most important is to reinforce the sense of community which holds us together and inspires us to work for the Society. That is not all, of course: it is also a great opportunity to try new things, especially things which increase the appreciation of mathematics among new audiences. Here, it begins to feel that we are pushing at an open door: in contrast to a decade or two ago, mathematicians are now acknowledged in polite society.

Exactly 150 years after the first meeting of the Society, we launched our celebrations on the 16th of January 2015 at Goldsmiths Hall, London, with an afternoon of short talks including several describing how mathematics is used in various ways, such as the making of special effects in films. There was a substantial online audience (and the talks are still available online) but only those physically present were able to enjoy the reception afterwards.

Of course we also continued with our usual activities during 2015, but tried to make them special somehow. One of these was the Mary Cartwright Lecture, given on the 27th of February in De Morgan House, which was certainly enhanced by the lecturer, Maria Esteban, who spoke on *Eigenvalue problems in relativistic Quantum Mechanics, theory and applications.*



Penrose Lantern, by George Legendre.

Similarly, the LMS meeting on the 1st of April at the joint British Mathematical Colloquium/British Applied Mathematics Colloquium in Cambridge was made very special by lectures from Robert Calderbank on *The art of measurement* and Andrew Wiles *On the arithmetic of ideal class groups*, followed by an excellent reception and dinner.

The Women in Mathematics Committee of the LMS has been holding extremely successful annual meetings designed to encourage and inspire young female mathematicians to stay in the subject, but this year they really went overboard and held a spectacular event from the 14th to the 17th of April in Oxford, called "It all adds up: celebrating women across the mathematical sciences". The first two days were devoted to mathematically inclined schoolgirls, while the second two days were for female mathematicians at all stages of their university careers. There were talks, discussions, workshops, and poster displays. It was a huge success, and the Society is very grateful to the Mathematical Institute at Oxford for its big contribution.

On the 9th of May we held a joint meeting with the British Society for the History of Mathematics in De Morgan House, and on the subject of De Morgan. It included a talk by John Heard on *Augustus De Morgan* *and the early history of the London Mathematical Society*, about which I will say more later.

The artists-in-residence scheme was officially launched at our Anniversary Dinner, which we held on the 18th of June, again in Goldsmiths Hall. This is also where we had held the anniversary dinner on our 100th birthday, but even more significant for us was that Michael Atiyah was present on both occasions, and it was a very great pleasure to hear him give us a short speech. The main after-dinner speaker, though, was Jim Simons, who was both fascinating and very entertaining.

In a normal year we invite two mathematicians to give our Popular Lectures in London and to repeat them in Birmingham. This year we added Glasgow and Leeds to the venues, and invited six mathematicians to speak (two on each occasion): Hannah Fry on *Patterns in human behaviour*, Ben Green on *A good new millennium for prime numbers*, Martin Hairer on *The mathematics of randomness*, Ruth King on *Epidemics and viruses: the mathematics of disease*, Joan Lasenby on *The mathematics of processing digital images*, and Colva Roney-Dougal on *Party hard! The mathematics of connections*. So the whole programme was significantly bigger. As with many of our events this year, we wish we could continue at this higher level.

Somewhat to my embarrassment the first European Mathematical Society Joint Mathematical Weekend ever to be held in the UK was the one held in Birmingham with the LMS on the 18th, 19th, and 20th of September. But it was of such high quality that I hope we are now forgiven for the delay! There was a very impressive list of speakers from many countries on algebra, analysis, and combinatorics, together with a special lecture from Niccolò Guicciardini on *Reading the Principia with the help of Newton*. The weekend was a fitting way simultaneously to celebrate the LMS anniversary and the 25th anniversary of the EMS.

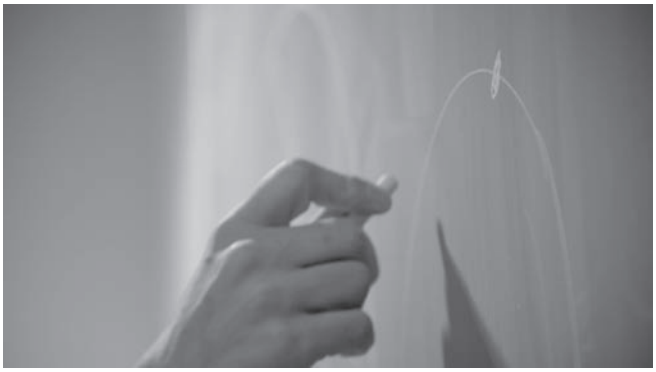
Another highly significant birthday was the centenary of Einstein's field equations for general relativity. Our joint meeting with the Institute of Physics and the Royal Astronomical Society on the 28th and 29th of November at Queen Mary College, University of London, was one of many such celebrations across Europe of Einstein's beautiful work. Our final joint meeting of the year was with the Edinburgh Mathematical Society, on the 10th and 11th of December at the ICMS in Edinburgh, on mathematical aspects of big data. This was yet another highly successful meeting, and we are very grateful to our hosts in Edinburgh for their warm welcome.

These are just the highlights: there were many more events. Our regional meetings, our research schools, and our Hardy Lectures were all enhanced or made special in some way. There were departmental celebrations, a display of LMS archival material in the Library at University College London, a Computer Science colloquium at the Royal Society, and several "local heroes" events at museums across Britain.

We started work on a new web page of case studies of the impact of mathematics, we prepared a new Handbook and "who's where" for our members, we designed an "infographic" on mathematics in Britain over the last 150 years, and we facilitated the publication (by CUP) of a new book by John Heard on the history and role of the LMS from its foundation up until just after the First World War.

In addition to our usual portfolio of research grants, we for the first time awarded a series of postdoctoral mobility grants, and we started a series of undergraduate summer schools. We awarded new prizes, for original and innovative work in the history of mathematics, and for excellence in communication of mathematical ideas. We also elected six new honorary members (instead of two, as has become the norm).

I would like to end this account by describing an event we held at the Science Museum in London in November. It consisted of a four-day festival, in which several research groups interacted with the public, mediated by the interactive theatre company non zero one. Also, on the 25th, there was a special evening in which Roger Penrose gave a talk on *Einstein's amazing theory of gravity: black holes and novel ideas in cosmology*, followed by the premiere screening of *Thinking Space*, the film directed and produced by Heidi Morstang.



Still image from the film *Thinking Space*, by Heidi Morstang.

Even by his standards Roger's talk was a superb combination of brilliant exposition and beautiful mathematical ideas. Heidi's utterly captivating film featured nine UK-based mathematicians offering insights into their mathematical thinking across a broad range of mathematical research fields. Together, the talk and the film summed up the entire year for me.

In order not to try your patience, dear reader, I have of course omitted a great deal of detail here, but the LMS web site has more information. In spite of this being a personal account, I am sure that my fellow LMS Trustees would want to join with me in thanking everybody, LMS members and staff, who worked so hard for our celebrations, and all our friends across the world who wished us such a happy birthday.

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How Mercator Did It in 1569: From Tables of Rhumbs to a Cartographic Projection

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In 1569, the Flemish cartographer Gerardus Mercator (1512–1594), his name being a Latinised version of Gerard de Kremer, presented to the world a large printed planisphere with the title *Nova et aucta orbis terrae descriptio ad usum navigantium emendate accomodate*, that is, 'New and enlarged description of the world properly adapted for use in navigation' (Figure 1). On this map, meridians are equally spaced and form, with the parallels, a rectangular mesh, in which the spacing between adjacent parallels increases with latitude in such a way that the proportion between the lengths of the parallel and meridian segments is everywhere equal to the one on the spherical surface of the Earth. This property implies that linear scale at every point does not vary with direction and that angles are conserved, making the projection *conformal*. An additional feature of extraordinary importance to marine navigation is that all rhumb lines – the curved tracks of constant course followed by ships at sea – are represented by straight segments making true angles with the meridians, a property that allows rhumb line courses to be directly read, traced and transported on a chart using a simple protractor and a ruler. Mercator's projection was a major achievement destined to change the history of cartography and navigation.

In the present day, deriving an expression for the spacing of parallels in the Mercator projection is straightforward. Consider a map projection in which meridians are represented by equidistant segments north-south oriented and aligned with the y-direction of a Cartesian coordinate system, and parallels perpendicular to the meridians. The problem consists of

Figure 1. World map of Mercator (1569), *Nova et aucta orbis terrae descriptio ad usum navigantium emendate accomodate*. The map depicts the world as it was known in the middle of the 16th century. Notice the *Organum Directorium* abacus in the bottom left corner of the map.
Bibliothèque nationale de France, GE A-1064 (RES).

finding a general expression for the ordinate $y = y(\varphi)$ of a parallel of latitude φ, measured from the equator, so that the proportion between the lengths of a parallel and a meridian arc on the spherical surface of the Earth is conserved in the projection. It can be shown that such an expression is the solution of the equation ([1], 49–51):

$$\frac{dy}{d\varphi} = \mu \sec\varphi, \tag{1}$$

where μ is an arbitrary linear scale. That is,

$$y(\varphi) = \mu\psi(\varphi), \tag{2a}$$

$$\psi(\varphi) = \ln\left[\tan\left(\frac{\pi}{4} + \frac{\varphi}{2}\right)\right]. \tag{2b}$$

In one of the Latin legends of the map, Mercator shows that he is fully aware of the mathematical nature of the problem by stating that he has 'increased the length of the degrees of latitude in proportion with the lengthening of the parallels relative to the equator', which is exactly what is expressed by equation (1). However, he could do no better than resorting to some kind of empirical solution in order to accomplish such requirements because both calculus and logarithms had not been introduced at that time. But how did he actually do it? This is a question that had been open since Mercator's achievement until very recently. In two articles published in the prestigious journal *Imago Mundi* ([3, 4]), we believe we have finally solved the enigma. In these papers, we demonstrate that the method most certainly used by Mercator in 1569 is not only deceptively simple – as a matter of fact, the simplest possible at the time – but also the one that would come naturally to the mind of a cosmographer aware of the contributions of his contemporaries.

No explanations were given by Mercator about how the projection was calculated, aside from his statement about the proportion between the lengths of meridians and parallels. Some 30 years later, the young English mathematician Edward Wright (1561–1615) presented a table of *meridional parts* (that is, a list of latitude ordinates) from which a Mercator graticule could be accurately drawn. This table was calculated by iteratively summing the secants of the latitudes in one-minute intervals, from the equator to 89º 50', an empirical approach that solved, with remarkable accuracy, the theoretical formulation in equation (1). Although Wright admitted that his work was inspired by Mercator's, he clarified in the preface of his book that the 'way how this should be done, I learned neither from Mercator nor anyone else' [10]. The relevant point to recall here is that the method used by Mercator to construct his projection was unknown to his contemporaries and remained unknown up to modern day.

Two types of method have been proposed in the literature over the last 125 years (starting with the one by Finish geographer Nils Adolf Erik Nordenskjöld in 1889): (i) those based on a formula and (ii) those based on a graphical construction (a review of the methods proposed in the literature, from 1889 to 2003, has been made by Raymond D'Hollander [2], 85–106). Most of the methods in the first group were iterative solutions of the formula

$$y(\varphi) = \Delta\varphi \sum \sec\varphi, \tag{3}$$

derived directly from condition (1), where the value of the latitude increment $\Delta\varphi$ and the argument of the secant can vary. For example, Nordenskjöld used $\Delta\varphi = 10°$ and the alternative formulation $y(\varphi) = \Delta\varphi \sum \sec(\varphi + \Delta\varphi/2)$, where the secant in each interval $[\varphi, \varphi + \Delta\varphi]$ is calculated for the middle latitude value. A few methods used non-iterative expressions based on some kind of *ad hoc* formulation. One type of graphical solution [9] consisted of transferring the coordinates of rhumb lines previously traced on a globe (specifically the globe made by Mercator in 1541, which depicts a dense mesh of rhumb lines) to the plane of the projection.

1 Error assessment

The first step in our study was to assess the accuracy of the cartographic projection in Mercator's map, that is, the accuracy of its mesh of meridians and parallels. Two unrelated and independent types of errors affect the graticule of the map: (i) those associated with the *method* used to calculate the ordinates of the parallels, as measured from the equator; and (ii) those related to the *physical distortion* suffered by the sheets after printing. In order to correctly identify the construction method used by Mercator, it was absolutely necessary to separate these two components. The obvious way of assessing physical distortion is to compare some simple forms depicted on the map (circles, squares, etc.) with their theoretical shapes and then use the differences to correct all coordinate values. However, no adequate forms are depicted in the area of the map that can be used effectively to assess distortion of the main latitude scale (for a detailed description of this part of the research, see [3], 2–8).

In the bottom right corner of the map is an abacus entitled *Organum Directorium*, composed of a graduated

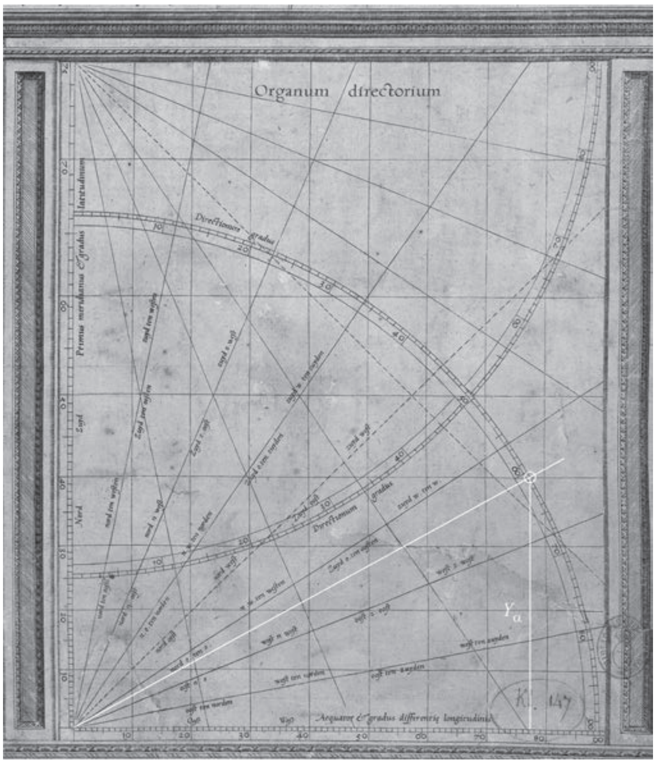

Figure 2. The *Organum Directorium* of Mercator's world map of 1569. The original ordinate of the 40º parallel (Y_α) is estimated from the polar coordinates of its intersection with the lower quarter circle (reproduced from [3], 3).

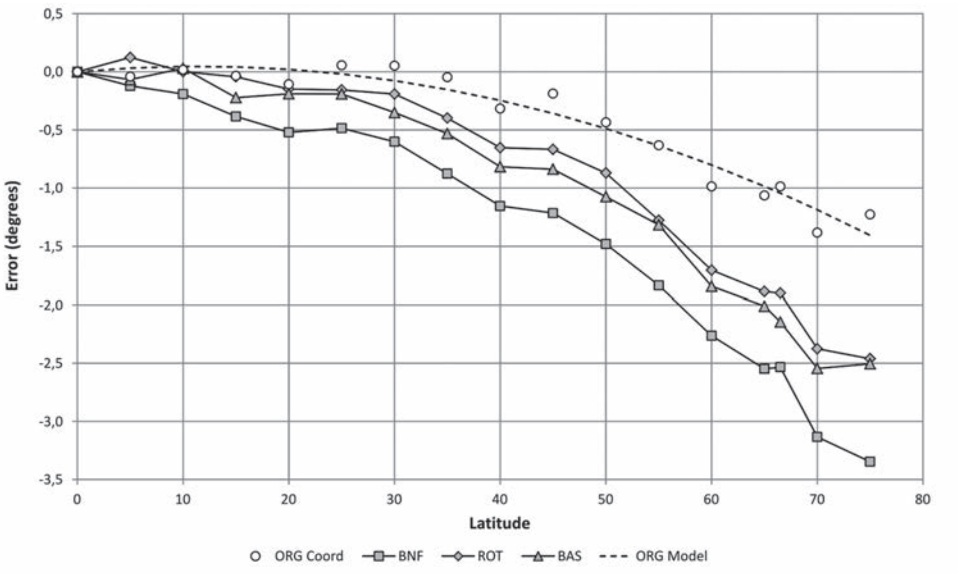

Figure 3. The error signature derived from the *Organum Directorium* (dashed line) compared to the errors derived from direct ordinate measurements made in three different exemplars of Mercator's world map: Bibliothèque nationale de France (BNF), University Library of Basel (BAS) and Maritime Museum of Rotterdam (ROT). The dashed line is a smooth quadratic interpolation through the measured error values, represented by the open circles. Notice how physical distortion affected the three copies differently (reproduced from [3], 5).

graticule of meridians and parallels for the Northern Hemisphere, which matches the one of the map (Figure 2). According to one of the Latin legends, the *Organum* was intended to help in solving simple problems of navigation, such as finding the course and distance between two points or determining the coordinates of the point of arrival given the course and distance from some point of departure. Seven straight lines, representing the seven classical rhumbs (counted from north and south to east and west): 11¼º; 22½º, 33¾º, etc., radiate from the bottom left and top left corners of the abacus. Two graduated quarter circles centred on the same two points are shown, each with a radius of a little less than 90 equatorial units. The important point to note is that it is possible to retrieve the original ordinates used to draw the parallels on the abacus by reading the polar coordinates of their intersections with the circles. This fact, which remained hitherto unnoticed by all historians studying the map, was a critical landmark in our study. For example (Figure 2), the polar coordinates of the intersection between the lower quarter circle and the 40º parallel are $\alpha = 60.8^{\circ}$ and $\varrho = 88.95$ (the radius of the circle in equatorial units). Thus, the corresponding ordinate will be $Y_{\alpha}(40) = \rho \cos \alpha = 43.4$. It is important to stress that this value is independent of any physical distortions affecting the sheets of the *Organum* after they were printed and is the same for all extant copies of the map. Alternatively, if one directly measures the ordinate of $\varphi = 40^{\circ}$ in the latitude scales of each *Organum*, different values are obtained for each of them.

The next step in our analysis involved determining the original ordinates Y of the parallels in the *Organum Directorium* and comparing them with the theoretical ordinates of the Mercator projection ψ, as given by equation (2). We called the distribution of the differences $\varepsilon(\varphi) = Y - \psi$ with latitude, illustrated in Figure 3 (open circles and dashed line), the *error signature* of the chart. Notice how different this distribution is from the ones derived from the direct ordinate measurements made in the three extant copies of the map.

This error signature was then compared to the errors produced by the various empirical procedures that Mercator may have used to calculate the projection, which have been proposed in the literature since the end of the 19th century (Figure 4). The one closest to the error signature is the iterative solution described by Müller-Reinhardt (1914), which is a direct application of equation (3) for $\Delta\varphi = 1^{\circ}$ [5]. The maximum deviation is one third of a degree, at latitude 60º, which corresponds to about 0.7 millimetres measured on the *Organum*. However, this was a purely conjectural hypothesis that did not take into consideration any measurements made on the map or any assessment of distortion. Thus, we were forced to conclude that none of the methods proposed in the literature reproduced Mercator's result and we were back to the initial question: how did Mercator calculate his projection? The only possible way to shed some light on this issue was to look at it *historically*, trying to better understand how his achievement related to the contemporary know-how on the subject.

2 The pre-history of the Mercator projection

When Mercator started working on his map, mathematicians and cosmographers had been addressing the mathematical problems underlying navigation for some three decades. This historical background is an indispensable prerequisite to the understanding of how he came to his solution (for a detailed description of the historical background underlying the construction of Mercator projection in 1569, see [4]).

In 1537, the Portuguese mathematician Pedro Nunes (1502–1578) published a collection of works, two of them discussing the mathematical problems related to navigation and nautical cartography: *Treatise on certain doubts of navigation* and *Treatise in defence of the nautical chart* ([6], 105–119; 120–141). In those two works, Nunes introduced, for the first time, the concept of a *rhumb line,* that is, the line on the surface of the sphere that intersects all meridians with a con-

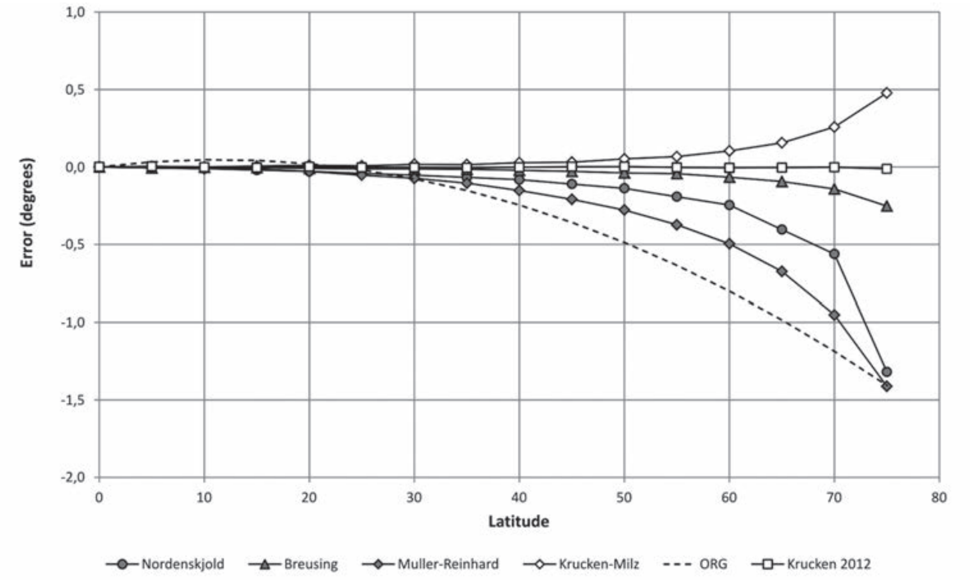

Figure 4. Distribution of the errors associated with five methods proposed in the literature, compared with the error signature of the chart (ORG, dashed line): Nordenskjöld (1889), Breusing (1892), Müller-Reinhardt (1914), Krücken & Milz (1994) and Krücken (2012). Each degree of error translates to a length of about 2 millimetres on the *Organum* (reproduced from [3], 9).

stant angle. This curve was later called a *loxodromic curve* or *loxodrome* (terms that are still in use today). Nunes carefully distinguished between navigation with a constant rhumb (that is, along a rhumb line) and navigation along the shortest distance (that is, along a great circle). While only the basic concepts and minimal remarks about the properties of the two curves were given in the two treatises, it is obvious that what he had in mind was the drawing of rhumb lines on a globe.

Immediately after Nunes published his works, a scholar (today unidentified) countered with a refutation, to which Nunes replied, first with a lengthy defence of his initial ideas and later (before 1541) with a treatise on how to draw rhumb lines on a globe. Pedro Nunes' reply, which may have been written in the late 1530s, is a manuscript now kept in Florence (BNCF, Cod. Palatino 825). The treatise on how to draw rhumb lines on a globe is referred to in a later work of Pedro Nunes but was probably never printed. The texts of this polemic circulated only in manuscript and most of them are lost today. From what has reached us, it is clear that one of the issues at stake was the correct way to mathematically define a rhumb line and how to construct what was called a 'table of rhumbs': a set of coordinates (latitude vs. longitude) defining a rhumb line for a specific course on a sphere. These tables were typically made for each of the seven classical rhumbs (in 11¼º intervals), starting at the equator and progressing towards the pole. News about Nunes' original works – and possibly also about the polemic that ensued – travelled fast. Several authors in Europe became familiar with the new concept in the subsequent years. In 1541, Mercator drew rhumb lines on his well known globe; in 1545, in the Low Countries, Gemma Frisius (1508–1555) referred to rhumb lines and revealed that he had already represented them on his world map of 1540 (now lost). In the mid 1550s, the English cosmographer and mathematician John Dee (1527–1608/9) was deeply interested in problems of navigation and was corresponding with Pedro Nunes, having calculated a table of rhumbs that he called *Canon Gubernauticus* [8]. Nunes continued to work on the problem and finally, in his *Opera* of 1566, published a detailed study of the mathematical properties of rhumb lines together with a complete set of instructions on how to calculate tables of rhumbs. He also clarified the asymptotic behaviour of rhumb lines near the pole ([7], 214–224). The simplest form of calculating a table of rhumbs for a given rhumb R is to iteratively solve a series of plane triangles along the corresponding line on the sphere. Departing from a point on the equator, one possible way is to choose some constant latitude interval $\Delta\varphi$, trace a segment making an angle R with the meridian and successively find the coordinates of the next point on the line. This is done by solving the right-angled triangle whose hypotenuse is the segment with direction R and whose catheti are the arc of meridian with length $\Delta\varphi$ and the arc of parallel with length $\Delta s = \Delta\varphi \tan R$, where $\Delta\lambda = \Delta s \sec \varphi$. Variants of this method consist of adopting constant intervals of longitude (instead of latitude) and in using the middle latitude $\varphi + \Delta\varphi/2$ as the argument for the secant function. At the time that Mercator engraved his world map, at least three different tables had been calculated in Europe: one by the unknown Portuguese scholar around 1540, one by John Dee – who used the middle latitude as argument for the secant – (the *Canon Gubernauticus*) around 1558, and one formally described by Pedro Nunes in his *Opera* of 1566, who adopted a more complex solution involving arcs of great circle [7].

3 The solution

Constructing a Mercator's graticule with a table of rhumbs is an intuitive and straightforward process. For a table containing the coordinates (φ, λ) of a series of points located on a loxodrome of rhumb R, separated by equal intervals of latitude, the procedure is (Figure 5):

- On squared paper draw: a horizontal line (the equator) and graduate this line in degrees of longitude; a series of equally spaced meridians, perpendicular to the equator; and

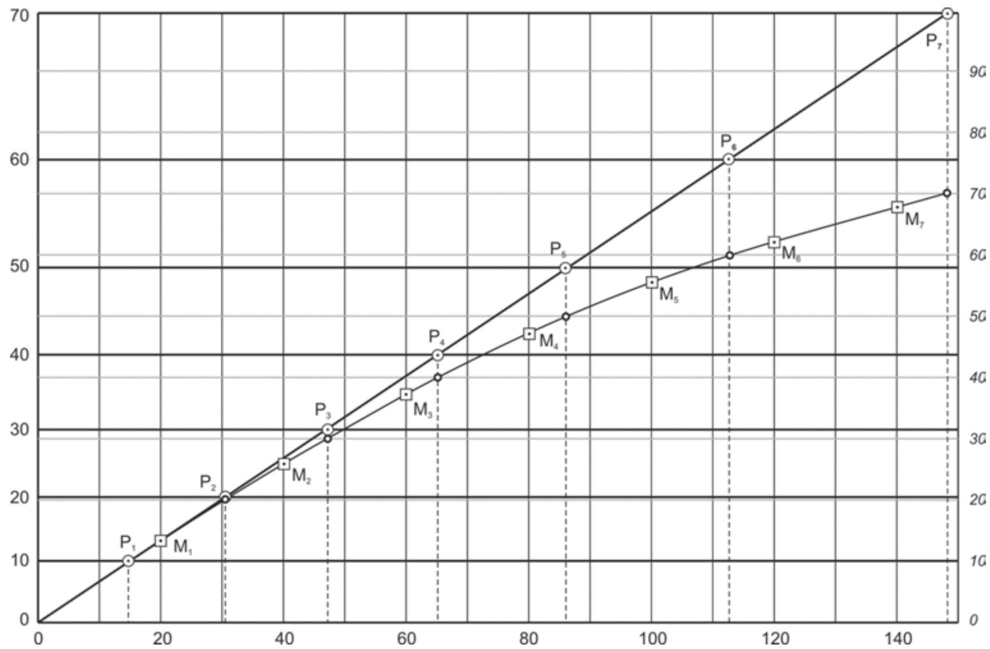

Figure 5. Constructing a Mercator graticule with a table of rhumbs, taking the fifth rhumb (56¼º) as an example. Two methods are illustrated according to the type of table: tables with constant intervals of latitude (points P_1, P_2, etc.) and tables with constant intervals of longitude (points M_1, M_2, etc.). The ordinate of a parallel of latitude φ is found at an intersection of the meridian of longitude λ (dashed lines) with the oblique rhumb, where φ and λ are the coordinates of a point taken from the table of rhumbs (reproduced from [3], 12).

a series of equally spaced parallels perpendicular to the meridians, forming a square grid with them.

- Choose one of the traditional rhumbs (R) and represent it as a straight segment passing through the bottom left corner of the graticule and making an angle R with the meridians.
- From the table of rhumbs, take a point P_1 with coordinates (φ_1, λ_1) and mark it on the segment representing the rhumb, according to its longitude. The horizontal line containing this point will be the parallel of latitude φ of the projection. Do the same with all points P_2, P_3, etc., in the table of rhumbs.

When all latitudes have been dealt with, the graticule is complete. For a table with constant intervals of longitude, the procedure will be slightly different owing to the necessity of interpolating, as illustrated in the figure. Assuming that the table of rhumbs is exact, the choice of a particular rhumb is a question of practical convenience because all of them will produce the same result. Instead of finding the ordinates of the parallels graphically, as illustrated in the figure, a more convenient and accurate way would be to use the analytical equivalent $y(\varphi) = \lambda \cot(R)$.

Tables for several different rhumbs and intervals of latitude and longitude were tested in our study, with the purpose of finding the one whose associated errors were closest to the error signature of the chart, as determined in the previous part (for a complete description of this phase of the study, see [3], 10–14). The best match was found to be a table for the second rhumb (22½º), using constant intervals of one degree of longitude, whose error curve has a maximum difference of one fifth of a degree to the error signature (Figure 6). This is, by far, the method producing the results closest to Mercator's original ordinates, as determined from the *Organum Directorium*. An argument of a different nature makes this solution even more historically plausible, which is the fact that tables with constant increments of longitude produce rhumb lines that never reach the pole – a feature previously explained by Pedro Nunes and to which Mercator was certainly sensitive.

4 Final remarks

Our study has shown that Mercator, both for *historical* and *numerical* reasons, most certainly used a table of rhumbs for calculating his projection in 1569. This settles a century-old debate about the issue. Mathematicians may be pleased to know that, in the conception of the most important cartographic projection of all time, the best mathematicians and mathematics of the time were involved.

Finally, it is interesting to note that Mercator, according to his biographer Walter Ghim, considered his invention to 'correspond to the squaring of the circle in a way that nothing seemed to be lacking save a proof'. The reference to the old puzzle (unsolved at the time) is either an analogy between the two problems or, as is often suggested, a metaphor emphasising the difficulty of the task. But the most important point here is the statement about the completeness of the solution ('nothing seemed to the lacking ...') and the lack of a proof ('... save a proof'). In our view, this further confirms that Mercator used a table of rhumbs to construct his projection. Suppose that the latitude scale was calculated in such a way that a certain rhumb R was represented as a straight segment making an angle R with the meridians. In that case, the *completeness* of the solution would consist: firstly, in all other rhumbs (and not only the one used in the construction) also being represented by straight segments; and secondly, in the proportion between the lengths of meridians and parallels being conserved as well, as mentioned by Mercator in one of the map's legends. One important point to stress is that he was doubtless aware that the observance of the first require-

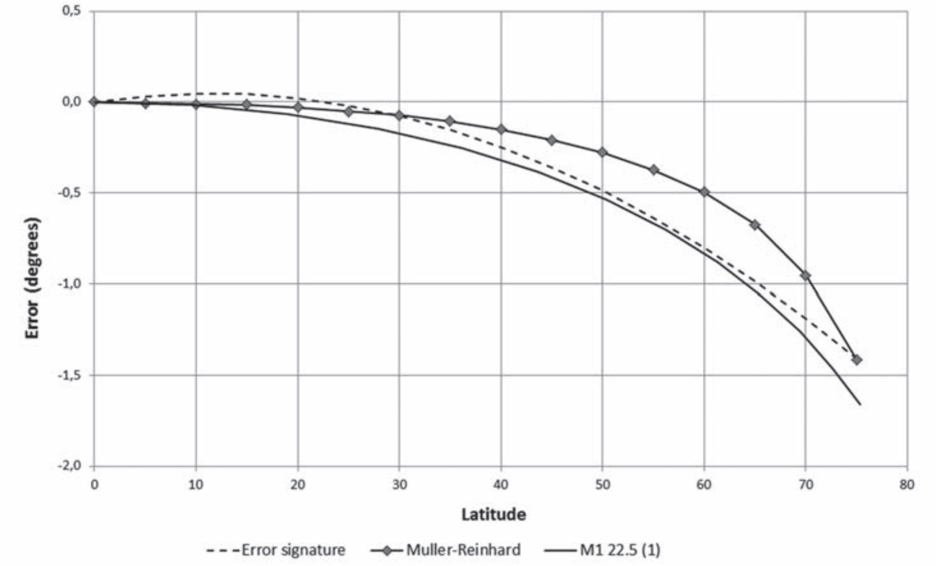

Figure 6. Error signature of the chart (dashed line) compared with those associated with the use of a table of rhumbs for the second rhumb and increments of one degree of longitude (solid line). The maximum difference between the two error curves is one fifth of a degree (at 75 degrees of latitude), which corresponds to a length of about 0.4 millimetres measured on the *Organum*. The upper line, reproduced from Figure 4, refers to the method proposed by Müller-Reinhardt (1914) [9] based on a secant formula.

ment would lead automatically to the observance of the other. Thus, the proof to which he was likely referring was that 'in a projection depicting rhumb lines as correctly oriented straight segments, the proportion between the lengths of meridians and parallels is also conserved, and vice versa'. The formal demonstration of this property was beyond the reach of mathematics in Mercator's time.

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ICMI Column

Jean-Luc Dorier (University of Geneva, Switzerland)

The 2015 ICMI Awards Felix Klein and Hans Freudenthal Medals

The ICMI is proud to announce the seventh awardees of the Klein and Freudenthal medals.

The *Felix Klein Medal* for 2015 is awarded to **Alan J. Bishop**, Emeritus Professor of Education, Monash University, Australia.
The *Hans Freudenthal Medal* for 2015 is awarded to **Jill Adler**, University of the Witwatersrand, Johannesburg, South Africa.

The ICMI Awards, given in each of the odd-numbered years since 2003, are the two prizes created by the ICMI to recognise outstanding achievement in mathematics education research. They respectively honour a lifetime achievement (*Felix Klein Award*, named after the first president of the ICMI: 1908–1920) and a major cumulative programme of research (*Hans Freudenthal Award*, named after the eighth president of the ICMI: 1967–1970). By paying tribute to outstanding scholarship in mathematics education, the ICMI Awards serve not only to encourage the efforts of others but also to contribute to the development of high standards in the field through the public recognition of exemplars. The awards consist of a medal and a certificate, accompanied by a citation.

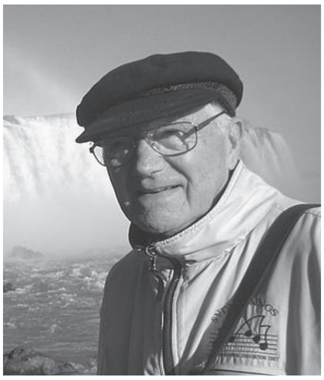
The ICMI Awards represent the judgement of an anonymous jury of distinguished scholars of international stature. The jury for the 2015 awards was chaired by Professor Carolyn Kieran, Université du Québec à Montréal, Canada.

We give some key biographical elements below whilst full citations of the work of the two 2015 medallists can be found at http://www.mathunion.org/icmi/activities/awards/introduction/.

Presentation of the medals and invited addresses of the 2013 and 2015 medallists and of the 2014 Emma Castelnuovo award (attributed for the first time to Hugh Burkhardt and Malcolm Swan, University of Nottingham, UK) will occur at the ICME-13 opening ceremony, to be held in Hamburg, 25 July 2016.

The following table gives a list of all the previous awardees since the creation of the medals in 2003:

	Felix Klein Medal	*Hans Freudenthal Medal*
2003	Guy Brousseau	Celia Hoyles
2005	Ubiratan D'Ambrosio	Paul Cobb
2007	Jeremy Kilpatrick	Anna Sfard
2009	Gilah Leder	Yves Chevallard
2011	Alan Schoenfeld	Luis Radford
2013	Michèle Artigue	Frederick Leung



Alan Bishop's early research on spatial abilities and visualisation was transformed during sabbatical leave in 1977 to Papua New Guinea, where he began to think about the process of mathematical enculturation and how it is carried out in different countries. His subsequent book, *Mathematical Enculturation: A Cultural Perspective on Mathematics Education*, published in 1988, was groundbreaking in that it developed a new concept of mathematics – the notion of mathematics as a cultural product and the cultural values that mathematics embodies. Further evolution of this notion occurred as a result of co-organising a special day-long event during the 1988 Sixth International Congress on Mathematical Education devoted to "Mathematics, Education and Society", which eventually led to successive conferences on the political and social dimensions of mathematics education. Alan Bishop has been instrumental in bringing the political, social and cultural dimensions of mathematics education to the attention of the field. Alan Bishop succeeded Hans Freudenthal, founding editor of *Educational Studies in Mathematics*, with Volume 10 in 1979, ending with Volume 20 in 1989. In 1980, he founded and became the series editor for Kluwer's (now Springer's) *Mathematics Education Library*, which currently comprises 63 volumes. He was the chief editor of the *International Handbook of Mathematics Education* (1996) and the *Second International Handbook of Mathematics Education* (2003) and he continued as an editor for the *Third International Handbook* (2013). Through his tireless and scholarly work in the area of publication, Alan Bishop has enabled research in mathematics education to become an established field. He is therefore an eminently worthy recipient of the Felix Klein Medal for 2015.

Professor Jill Adler led an outstanding research programme dedicated to improving the teaching and learning of mathematics in South Africa – from her 1990s groundbreaking, sociocultural research on the inherent dilemmas of teaching mathematics in multilingual classrooms through to her subsequent focus on problems related to mathematical knowledge for teaching and mathematics teacher professional development. Her research of multilingual classrooms during a period of change in South Africa puts into stark relief the tensions involved in teaching and learning mathematics in classrooms where the language of instruction is different from the language of teachers' and students' everyday lives. In her 2001 book, *Teaching Mathematics in Multilingual Classrooms*, she displays the strong theoretical grounding that has served to advance the field's understanding of the relationship between language and mathematics in the classroom. From 1996 onward, Jill has spearheaded several large-scale teacher development projects. The most recent one, begun in 2009, called the *Wits Maths Connect Secondary project*, aims to further develop mathematics teaching practice at the secondary level so as to enable more learners from disadvantaged communities to qualify for the study of mathematics-related courses at university. This ongoing research and development project is a further testament to Jill's unstinting efforts to face head-on the challenges of improving mathematics teaching in post-apartheid South Africa – efforts that have been recognised by several awards over the years, including the *University of the Witwatersrand Vice Chancellor's Research Award* for 2003, the FRF Chair of Mathematics Education in 2009, the Gold Medal for Science in the Service of Society from the *Academy of Science of South Africa* in 2012, and the *Svend Pedersen Lecture Award* in Mathematics Education from Stockholm University in 2015. For the inspiring, persistent and scholarly leadership that Jill Adler has provided to the field of mathematics education research and practice in South Africa and beyond, she is truly deserving of the Hans Freudenthal medal for 2015.

ERME Column

Cristina Sabena (University of Torino, Italy) and Susanne Schnell (University of Cologne, Germany)

ERME Session at ICME-13

ICME-13 (24–31 July 2016 in Hamburg, Germany) is an outstanding opportunity to promote and strengthen communication, cooperation and collaboration between researchers in mathematics education from Europe and around the world. In this spirit, ERME invites attendants in two parallel sessions to gain firsthand insights into the work and philosophy of European research and ways of promoting young researchers.

The session "Voices from a CERME Working Group experience – From a study of mathematics teaching to issues in teacher education and professional development" is coordinated by *Stefan Zehetmeier* (Austria), *Despina Potari* (Greece) and *Miguel Ribeiro* (Norway). It will give insights into the academic work conducted within and between Thematic Working Groups at the bi-annual ERME conference (CERME). The study of mathematics teacher education and professional development is prominent in Europe and has been addressed in three different Thematic Working Groups, which will collaborate for the ICME session: "Mathematics teacher education and professional development", "Mathematics teacher and classroom practices" and "Mathematics teacher knowledge, beliefs and identity". The session will not only allow insights into the topics that have been raised in previous CERMEs and activities that have taken place but also highlight recent developments in European research. It will also provide invited presentations and potential for small group work and discussions. The session aims to enable collaborative work and further elaboration of emerging issues in the field with researchers from all over the world.

The session "Young Researchers in Mathematics Education – Building bridges between Europe and the world" will be hosted by YERME (Young Researchers in the European Society for Research in Mathematics Education) with support from the German Young Researchers Group (*Nachwuchsvertretung der GDM*). It will be coordinated by *Susanne Schnell* (Germany), *Jason Cooper* (Israel), *Cristina Sabena* (Italy), *Raja Herold* (Germany) and *Angel Mizzi* (Germany). This session will focus on introducing the work and structure of the YERME (Young Researchers in ERME) and also highlight ERME offers for young researchers (YERME day, YESS Summer School). Furthermore, presentations and discussions will give insights into the situation of young researchers in mathematics education in Germany and Europe and look at implications for the future of the young scientific community. All participants will be invited to share and discuss their experiences.

Both sessions are scheduled for Saturday 30 July, 16:30–18:00, and will include a short presentation about the society, held by the President of ERME (*Viviane Durand-Guerrier*) and the Vice-President (*Susanne Prediger*).

YESS-8

The 8th YERME Summer School (YESS-8) will take place 13–20 August 2016 in Poděbrady, Czech Republic, with the support of the local committee chaired by Nad'a Vondrová. Young Researchers in ERME (YERME), especially PhD students and postdocs in mathematics education, are invited to participate in a rich programme of workshop sessions, lectures and Thematic Working Groups. The latter represent the heart of the summer school, allowing approximately 20 hours of work with peers from similar fields of research, guided by internationally renowned experts. In accordance with the spirit of ERME, YESS-8 promotes collaboration, cooperation and communication among young scholars but also enables exchanges with more experienced researchers. Past experience shows that the working groups not only provide profound opportunities to discuss research related issues in depth but also provide the foundation for work collaborations and friendships among the participants which continue long after the summer schools are over.

Proposed topics for the Thematic Working Groups at YESS-8 are:

- Teacher knowledge and practice; teacher education and professional development.
- Teaching and learning mathematics at primary level.
- Teaching and learning mathematics at secondary and advanced level.
- Information technologies in mathematics teaching and learning (software, internet, etc.).
- Cognitive and affective factors in learning and teaching mathematics.
- Theoretical perspectives, linguistic and representational aspects of teaching and learning mathematics.

Participants are asked to prepare and present a paper concerning the current status of their research, e.g. preliminary results, work in progress or comprehensive information about graduate studies and/or future research plans. This paper will be the basis to compare, integrate and define their own research within the group of other young researchers, as well as in the broad field of mathematics education research.

YESS is a well-established ERME institution, which has taken place in Klagenfurt (Austria, 2002), Poděbrady (Czech Republic, 2004), Jyväskylä (Finland, 2006), Trabzon (Turkey, 2008), Palermo (Italy, 2010), Faro (Portugal, 2012) and Kassel (Germany, 2014).

The international programme committee is formed of Paolo Boero, Jason Cooper, João Pedro da Ponte, Susanne Schnell, Konstantinos Tatsis and Nad'a Vondrová. Paolo Boero is the scientific coordinator.

More details can be found on the YESS-8 website: http://ocs.pedf.cuni.cz/index.php/YESS/YESS8, and the YERME website: http://www.mathematik.uni-dortmund.de/~erme/index.php?slab=yerme.

The deadline for applications for admission was 20 January 2016.

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Practices for Identifying, Supporting and Developing Mathematical Giftedness in Schoolchildren: The Opener for a Series of Country Reports

EMS Education Committee

The EMS Education Committee has begun publishing a series of reports about practices for identifying, supporting and developing mathematical giftedness in school children who reside in countries represented within the EMS. The idea to report on the respective scenarios within different countries in the pages of the newsletter was suggested by the President of the EMS Pavel Exner at the meeting of the committee in Prague in February 2015. The committee further discussed this idea at the meeting in Sienna in October 2015 and the discussion converged to the opinion that the idea is timely and important. It is timely because a topic of nurturing the mathematically gifted is aligned to the well-recognised need to increase the number of STEM[1] students in Europe. Additionally, this topic is of importance to many members of the EMS community, who themselves have had experience of being "the gifted" in particular education systems before later becoming involved in gifted education.

The committee recognises that there is not a unanimous approach to defining what it means to be *mathematically gifted*, as well as the related concepts of *high*

[1] STEM stands for Science, Technology, Engineering and Mathematics.

mathematical ability, *mathematical talent* and *mathematical promise*. Mönks and Pflüger's (2005) inventory of gifted education in Europe refers to *giftedness* as "the individual potential for high or outstanding achievements in one or more areas of ability" (p. 3). Accordingly, mathematical giftedness can be referred to as the individual's potential for high or outstanding achievements in mathematics.

It is relatively easy to characterise the mathematically gifted a posteriori, if and when they produce results that are acknowledged by the mathematics community as outstanding. The main challenge of gifted education is to recognise and nurture the gifted *before* they have a chance to demonstrate outstanding achievements (i.e. while they in kindergarten or in school).

Different education systems deal with this challenge in different ways and, as a rule, these are driven by the *equity principle.* This principle states that each student should be provided with equal opportunities to fulfil their potential in accordance with their special characteristics and needs, and that excellence in mathematics requires high expectations and worthwhile opportunities for all. The principle applies, among others, to those students who show signs of being insufficiently challenged by opportunities provided by a country's regular mathematics curriculum or show interest in mathematics beyond the regular curriculum.

Practices of identifying and supporting such students, some of whom will constitute the next generation of mathematicians, will be presented in the current series of country specific reports. The reports are intended to be of different formats but, generally speaking, each report will provide: (1) a brief historical account of a country's approach to gifted education; (2) background information on regular mathematics education; (3) an overview of the range of the country's educational activities that aim to enrich or deepen the regular mathematics curriculum (this is the main part); and (4) trends. We begin by presenting the scene in Israel.

References

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Practices for Identifying, Supporting and Developing Mathematical Giftedness in Schoolchildren: The Scene in Israel[1]

Boris Koichu (Technion, Haifa, Israel, on behalf of the EMS Education Committee)

Guiding principles

The education of gifted students in Israel relies on the following positions (e.g. Leikin & Berman, 2016): the *equity principle* (refers to equal opportunities for students with different needs), the *diversity principle* (refers to the diversity of fields in which human talent can be manifested), *dynamic perspective* (acknowledges that cultivating human talents requires designing unique learning environments, distinct study tracks, and appropriate teachers and curricula) and *holistic approach* (acknowledges the range of instructional approaches for promoting a range of abilities and skills). These positions reflect and stipulate the Israeli scene, where various in-school programmes and out-of-school activities for mathematically promising students are conducted in different forms and formats.[2]

Practices for elementary school students (Grades 1–6)

Israeli first graders are enrolled in elementary school by their place of residence and study mathematics in heterogeneous classes. In Grades 2 or 3, all Israeli children take an examination for determining eligibility for acceptance onto two special governmental programmes. About 1% of the children are enrolled in special classes for the gifted that operate in selected elementary and secondary schools (Grades 3–12) around the country. The top 5% are eligible for acceptance onto a *weekly enrichment day programme*, which operates at 52 regional centres (Grades 3–6). The students leave their regular school to go to one of these centres for one day a week in order to study a variety of scientific topics, from medicine to mathematics.

In addition, interested parents can easily find after-school mathematics clubs, forums and circles for their children, operated by universities or public associations. Recently, the Math-by-mail programme for students of

[1] This is an abridged version of the article published on the website of the EMS Education Committee: http://www.euro-math-soc.eu/ems_education/education_homepage.html.

[2] See http://cms.education.gov.il/EducationCMS/Units/Gifted/English.

Grades 3–9 has received a good response all over the country.[3]

Practices for junior high school students (Grades 7–9)

There is a one-level mathematics curriculum for junior high school but many schools split their students into A-stream and B-stream classes in mathematics lessons. Additionally, the Ministry of Education, in cooperation with the Technion and the Hebrew University of Jerusalem, conducts a programme that gives an equal number of extra hours to low-achieving and high-achieving mathematics students. There also exist special mathematics classes, at different levels of inclusiveness, in which mathematics is studied 7–9 hours a week, compared to the regular 5 hours a week.

The most prestigious programme is the four-year *Future Scientists and Inventors* programme[4] for exceptionally gifted mathematics students (beginning in Grade 9). The participants of the programme attend one of the participating universities two days a week in order to take academic courses, listen to lectures by distinguished professors and experience laboratory work.

Practices for senior high school students (Grades 10–12)

Israeli 10th graders are enrolled in mathematics study at one of three levels: low, regular or high. Currently, 8% of senior high school students study mathematics at the high level. Needless to say, there are students who need more advanced opportunities to study mathematics than even the high level curriculum can provide. Some of the above mentioned programmes begin in elementary and junior high school and continue up to senior high school. Those not yet mentioned include integration of school students in university courses. The participants in these programmes pass the matriculation exams in Grades 10 or 11 and begin studying academic courses toward a BSc degree.

In addition, in-school and out-of-school programmes aimed at engaging students in research are in the mainstream of Israeli education for the gifted. For example, each summer the Technion organises a two-week *Number Theory Camp* for 9th–12th grade students. At the camp, the students cope with challenging problems requiring exploration. The Technion also runs an international summer research programme for high school students called SciTech.

Mathematics competitions

The oldest mathematics competitions in Israel are the Grossman Olympics (which started in 1960) and the Gillis Olympics (which started in 1968). The Gillis Olympics begins with a test sent to all interested schools and continues with a competition conducted at the Davidson Institute of Science Education. This competition serves as a tool for choosing an Israeli team for the IMO. It is worth mentioning that Israel does not make the shortlists at the IMOs; the country's best results have been 11th place in 2000 and 13th place in 2013. The main mathematics competition for junior high school students is the *Zuta Olympiad* (*Mini Olympics*). There are also mathematics competitions "for all", e.g. the Open Competition (since 1989) and the Kangaroo (since 2013).

Thus, the main types of mathematics competition are represented in Israel but the mathematics competition movement is less extensive than in some other countries.

Trends

Mathematics education is high on society's agenda nowadays. Preparation of sufficient numbers of teachers who can teach the high level mathematics curriculum and work on the special programmes for the gifted is the bottleneck of the system. The emerging trend is a greater involvement of the public sector and IT industry. Obtaining an additional BSc degree in STEM education has become popular among middle-aged IT specialists looking for a second career or a way to contribute to the educational system while continuing to work in industry.[5]

Another trend is integrating distance learning into regular school timetables. The Virtual High School[6] has recently become a reality for hundreds of students in peripheral schools. The best teachers reach these students by means of an interactive digital platform.

Finally, efforts are made to address the needs of exceptionally gifted students within the regular school system. In 2015, a new high school curriculum containing four levels of mathematics study was launched. The new advanced high level is for about 1% of students who aspire to become mathematicians, whereas the high level remains for those who strive for careers in the IT industry (currently 8% of high school students, while the Ministry's goal is for it to reach 15% in 2020). It is planned that the Virtual High School will also operate at the new advanced high level of mathematics study.

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3 See http://davidson.weizmann.ac.il/en/mbm.

4 See http://www.rashi.org.il/#!future-scientists-and-inventor/ca5x.

5 An example is the Technion VIEWS programme. See http://www.focus.technion.ac.il/Jan15/education_story2.asp.

6 See http://jtec.macam.ac.il/portal/ArticlePage.aspx?id=1431.

Will All Mathematics Be on the arXiv (Soon)?

Fabian Müller and Olaf Teschke (both FIZ Karlsruhe, Berlin, Germany)

Introduction

For about 25 years, the arXiv has become established as an efficient tool for rapid research dissemination. Initially a service to cater for the needs of high-energy physics, it was soon adapted in related research areas (e.g. in mathematics on a broad scale by algebraic geometers since 1998). Perhaps the most common use is still as a notification tool about recent developments, though it increasingly serves other functions (e.g. archival/repository).

In order to fulfil these functions, a natural question is that of completeness. What share of recent research can one expect to be covered in a framework based on voluntary contributions by individuals? Personal experience gives very different answers, depending on the subject of research, from "almost everything I need" to "only a small part". Some years ago, a study analysing a sample of several thousand articles[1] by hand obtained very different ratios for several areas in physics. Given the possibility of bias for certain samples, it would certainly be desirable to have results on a larger scale; but it is obviously non-trivial to obtain reliable data due to inherent effects like the delay of publications, different area definitions, etc.

A first rough approximation – relating 34,797 submissions in 2015 in arXiv:math/math-ph (including cross-lists)[2] to about 120,000 new entries indexed in MathSciNet[3] or zbMATH during the same period – would lead one to deduce that almost 30% of the recent mathematical literature is freely available on the arXiv, with a still growing share. However, it is not clear at all to what extent these collections overlap.

As mentioned earlier in this column,[4] a recently introduced feature of zbMATH, in the effort to connect the various digital resources in mathematics, is the integration of links to arXiv versions of zbMATH documents via matching heuristics. Besides the obvious benefits of providing an alternative free source, this also allows for a more detailed analysis of the questions mentioned above, though the inherent inaccuracies due to heuristics, submission behaviour and indexing policies need to be taken into account.

[1] T. Ingoldsby, "Physics Journals and the arXiv: What is Myth and What is Reality?", *Technical Report*, American Institute of Physics (2009).
[2] http://arxiv.org/year/math/15.
[3] See also E. Dunne, *MathSciNet: Digital Guide to the Mathematical Literature*, 2015 SIAM-OPSA, Gaithersburg.
[4] O. Teschke, zbMATH as a Hub Connecting Digital Sources, *Eur. Math. Soc. Newsl.* 98, 61–62, 2015.

Matching the arXiv and zbMATH

During the course of 2015, zbMATH released its new citation matching interface.[5] Here, users can enter a citation string, which will be matched to a zbMATH database entry if possible.[6] Due to several factors, such as the wealth of different citation formats and norms, the possibility of misspellings and the existence of different competing transcription systems for non-English author names, this cannot be done in a completely rigorous fashion. Instead, the algorithm must allow for a certain degree of variation.

In the general case, where a user just enters a citation string without any formatting information, the first step is to determine which parts of that string constitute an author's name, a title, a page number and so on. Methods from Natural Language Processing – specially adapted to the problem domain – are used to accomplish this. Once this information is obtained, a specialised search index is used to make a fuzzy search, which tries to match this data as accurately as possible before returning a list of results, each result having a matching score that measures the degree of agreement with the search query.

The question then becomes at what point the matcher should be trusted, i.e. what should be the minimum value for the matching score in order to accept the result as a valid match. To determine this, a test dataset (gold standard) was collected from about 4,800,000 references matched via DOI (Digital Object Identifier), which implies a large degree of confidence, as well as overlap with articles from the computer science bibliography dblp.[7]

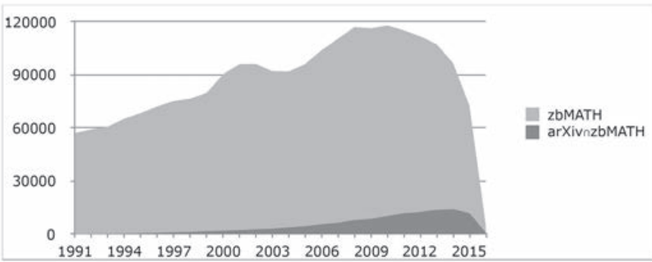
The same algorithm can be used to match zbMATH entries to arXiv articles, whose metadata can be harvested using the OAI (Open Archives Initiative) standard, which is supported by the arXiv and several other large institutional repository providers.[8] While the bibliographic information in this case is already split into author and title, the matching is made more difficult due to the common lack of journal source and pagination data. There are about 60,000 articles in the arXiv maths set with a DOI supplied (of which 45,000 have corresponding entries in zbMATH), from which another reliable test dataset was created and used to again compute a minimum matching score appropriate to this use case. This made it possible to match more than 75,000 further

[5] Available at https://zbmath.org/citationmatching/; at times affectionately called zbMATcH.
[6] An interface for processing BibTeX files and an API for use in scripts are available as well.
[7] http://dblp.uni-trier.de/.
[8] See https://arxiv.org/help/bulk_data and links therein.

zbMATH articles to their corresponding arXiv preprints with a precision of 97.0%, bringing the total number of articles with an arXiv link to over 120,000.

Exploring some general figures on submission and publication behaviour

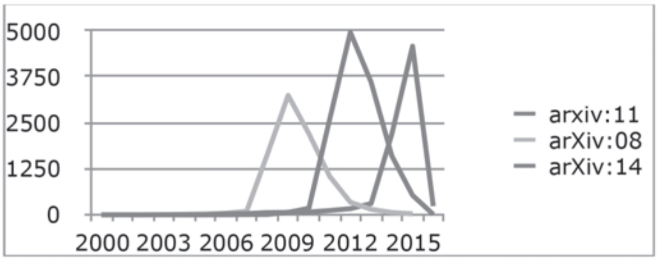
On this basis, we can employ zbMATH rather easily to create some interesting statistics. The reader is invited to try out some examples of their own – simply adding arxiv* before a query will restrict the search to articles with an arXiv link.[9] As an overview, searching for arXiv* alone currently provides about 125,000 results, which makes up 3.5% of mathematics literature since 1868. Perhaps more interesting is the share of recent publications. These figures can be directly read from the zbMATH filter (which also offers a breakdown by publication year). Starting with an almost negligible ratio of 0.05% in 1991, the overall share of mathematics publications available via the arXiv has recently approached almost 20% – a quite impressive development. For a number of subject areas, the ratio is much higher still (see the last section of this article).



arXiv versions available for zbMATH documents in relation to their publication year.

In principle, these figures could be increased further by retroarchiving efforts. However, a second look indicates that this option is rarely used. There are certainly prominent examples, including the arXiv version of SGA1.[10] However, refining the zbMATH search to arXiv submission years (e.g. "arXiv:11*" in the new enumeration or "arXiv:math/06*" in the old pattern) shows that the dominating use case is still the preprint function of the arXiv. For less than 5%,[11] the publication year predates the submission. On the other hand, the distribution also provides some figures about the publication delay (counted as the difference between the submission date and the publication date; naturally, this only makes sense for non-retroarchived papers).

Overall, the figures show an approximately Poisson distribution, with a clear peak in the year after submission, followed by the year after that and then the sub-



Publication delay and retroarchiving figures for three arXiv submission years.

mission year. On average, the publication delay is about 1.5 years. It should be noted that there is a significant bias here due to the Journal of High Energy Physics – its considerable bulk of papers (basically all of which are on the arXiv) in some years accounts for almost 10% of the arXiv/zbMATH overlap and, for a large proportion of them, the time gap between submission and publication is only about two months. This accounts for a large proportion of the same-year publications and decreases the total average by almost two months overall. It should also be kept in mind that this is an ongoing process and some research might be published with a huge delay that is not yet visible.[12]

Growth rates and missing papers

Whilst the amount of new mathematical research available via various platforms is growing all the time, it does so at various rates. Over the last decade, the number of mathematical articles added to zbMATH every year has grown by (only) about 4%, whilst yearly submissions to arXiv:math/math-ph have increased by 11% (the overlap of the arXiv with zbMATH has grown even a bit faster). If this momentum is maintained, one could expect that, within 20 years, recent mathematical research (at least) would be almost completely available via the arXiv; on the other hand, the extremely low rate of retroarchiving submissions makes it highly unlikely that a complete corpus will be achieved (in the sense of the envisioned Global Digital Mathematics Library[13]) in this way. However, there is no reason for pessimism in overcoming this obstacle – it just shows that retroarchiving by mathematicians may not be an effective approach. Frameworks like the European Digital Mathematics Library,[14] relying on suitable moving wall policies, seem to be more adequate solutions.

Whilst the amount of research available in zbMATH but not on the arXiv may be explained by subject specifics (see below), the question remains about arXiv submissions that do not seem to make it to publication.

[9] *Cum grano salis*, since the phrase may also appear in the search index via the review text – but this concerns less than 5000 documents, or 0.15% of the database.

[10] A. Grothendieck, M. Raynaud, Revêtements étales et groupe fundamental. *Lecture Notes in Mathematics* 224 (1971; Zbl 0234.14002); arXiv:math/0206203; currently the math paper on arXiv with the earliest publication year.

[11] Precise figures can not be given reliably since matching inaccuracy may influence this significantly.

[12] The most extreme case so far seems to be Alfredo Poirier, Hubbard forests. *Ergodic Theory Dyn. Syst.* 33, No. 1, 303–317 (2013), arXiv:math/9208204, with a delay of no less than 21 years.

[13] Th. Bouche, Update on recent Global Digital Mathematics Library (GDML) developments. *Eur. Math. Soc. Newsl.* 94, 41–42 (2014).

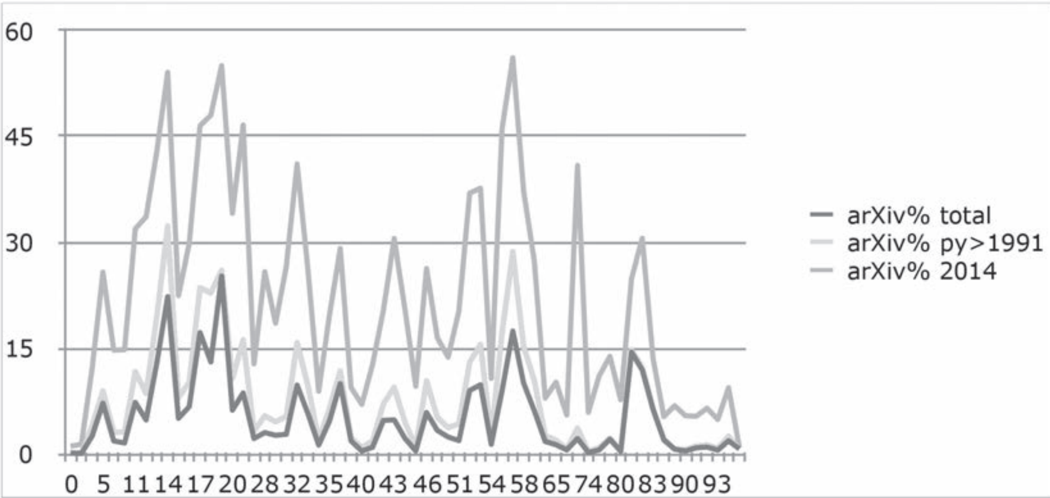
[14] Th. Bouche, Introducing EuDML, the European Digital Mathematics Library. *Eur. Math. Soc. Newsl.* 76, 11–16 (2010); J. Rákosník, O. Teschke, EuDML: the prototype and further development. *Eur. Math. Soc. Newsl.* 85, 57–58 (2012).

There are a surprisingly large number of them: overall, there are more than 280,000 arXiv:math/math-ph submissions since 1991 compared to only about 125,000 published in a source indexed in zbMATH. There is no easy answer to explain this gap, since the difference between published mathematics and arXiv submissions includes such diverse contributions as Perelman's work on the Geometrisation Conjecture alongside frequent submissions of elementary proofs of Fermat's Last Theorem or the Riemann Hypothesis. Whilst the latter may even succeed in getting published occasionally, journals containing them will usually not be indexed in zbMATH.[15] On the other hand, the scope differs (certain math-ph submissions may very well be beyond the scope of zbMATH) and there is also a possible contribution by an indexing delay in zbMATH.[16] These two effects can be roughly estimated by comparing the DOI figures above – since 60,000 arXiv submissions with DOI match to 45,000 in zbMATH, one may expect that about 166,000 arXiv:math/math-ph articles have been published. A further effect comes from the publication delay – from the figures above, one may reasonably expect that about 40,000 recent arXiv submissions are currently still being processed. That leaves about 70,000 arXiv submissions, or 25%, which do not make it to publication at all.

Different cultures in different areas

zbMATH also allows for a further analysis according to subject area, as encoded in the Mathematics Subject Classification. By employing the filters again, one sees immediately that the proportion of publications with arXiv versions available varies dramatically with field: from almost a third of the publications in algebraic geometry since 1991 or almost 30% in algebraic topology and K-Theory, to only about 2% in numerical mathematics and less than 0.1% in mathematics history. This pattern still prevails for recent publications, though some changes are certainly visible, e.g. for the publication year 2014, about 55% of research in algebraic geometry, algebraic topology and K-Theory is available through the arXiv but only about 10% in numerical mathematics and 1% in mathematics history or mathematics education.

These figures make it a bit less likely that one may reach reasonably complete coverage of recent research via preprint repositories soon, since the areas very active on the arXiv seem to be approaching a level of saturation, while for others its use is still rather uncommon. But, as can be seen from the diagrams, the behaviour can also change quickly.



Percentage of arXiv coverage for main MSC subject classes.

Overall, the analysis shows that there are promising approaches to preserve the mathematical research in the public domain – for more recent publications this is more and more the arXiv and for mathematical heritage this is digital libraries like EuDML[17]. While there is no hope that a general solution will work for the whole corpus, the joint forces of different approaches may eventually prevail, and the main task will be to ensure a sustainable framework to connect the different services and make them work well together.

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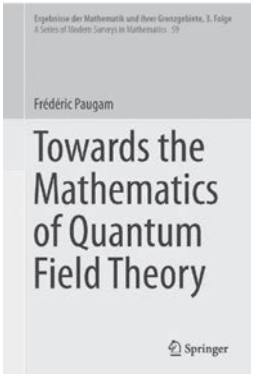
Olaf Teschke [olaf.teschke@fiz-karlsruhe.de] is a member of the Editorial Board of the EMS Newsletter, responsible for the Zentralblatt Column. Currently, he also serves as a member of the GDML working group and as acting Editor-in-Chief of zbMATH.

[15] See also G.-M. Greuel, D. Werner, in: Open access – four opinions. *Eur. Math. Soc. Newsl.* 91, 39–43 (2014).

[16] Caused, e.g., by the necessary selection process for sources which contain not only mathematics, or by print-only publications, like several conference proceedings.

[17] Both services – almost 125,000 publications available via arXiv and more than 240,000 via EuDML, with an overlap (according to zbMATcH) of about 4000 – already ensure free access to more than 10% of the math corpus.

Book Reviews



Frédéric Paugam

Towards the Mathematics of Quantum Field Theory

Springer, 2014, xvi + 487 pp.
ISBN print: 978-3-319-04563-4
ISBN e-book: 978-3-319-04564-1

Reviewer: Hirokazu Nishimura (Tsukuba, Japan)

Quantum Field Theory (or QFT) is a splendid hybrid of three major themes of modern physics: quantum theory, the field concept and special relativity. Although many mathematicians are quite afraid of it, the central concepts and technicalities of QFT are easy to grasp. In particular, many daily tools of working quantum field theorists, such as Feynman diagrams and renormalisation group flows, are even picturesque. It is pleasing to live in a universe where low energy physics is decoupled from high energy physics, the former surely being described by quantum field theories and the latter being depicted, possibly, by string theories or similar. Contemporary mathematicians are no longer allowed to boast of the bliss of their ignorance of QFT. Recall, by way of example, how Witten's supersymmetric QFT of $U(1)$ gauge symmetry (leading to Seiberg-Witten theory) replaced Donaldson's topological QFT of $SU(2)$ gauge symmetry, which could be called a revolution.

This book is by no means a textbook on QFT. Technical results are usually presented without proofs, for which the reader is referred to appropriate references. Even technical terms are often introduced without palpable definitions. It is very difficult to imagine a novice in category theory not being swallowed up by the flood of jargon of higher category theory. The book should be regarded as a rough design of the author's grandiose approach to QFT. For a good textbook on QFT for contemporary mathematicians, Zeidler's ongoing, all-inclusive, six volume (when complete) textbook [Quantum Field Theory, Springer] is recommended. The first three volumes [27], [28] and [29] are already available, the fourth is supposed to deal with quantum mathematics in general and the sixth is intended to cover quantum gravity and string theory. All mathematicians and physicists are awaiting the remaining three volumes with a lot of enthusiasm. If the author had wanted to write a textbook on QFT along these lines, it would have been at least as voluminous as Zeidler's.

The main bulk of this book is divided into three parts: Part 1 (Mathematical Preliminaries) from Chapter 2 to Chapter 13 over about 280 pages, Part 2 (Classical Trajectories and Fields) from Chapter 14 to Chapter 16 over about 40 pages, and Part 3 (Quantum Trajectories and Fields) from Chapter 17 to Chapter 24 over about 100 pages. The first chapter is an introduction of about 30 pages. The remarkably unique contents of Part 1 (covering mathematical tools found in books oriented towards QFT) are Chapter 2 (A Categorical Toolbox), Chapter 3 (Parametrized and Functional Differential Geometry), Chapter 9 (Homotopical Algebra), Chapter 10 (A Glimpse at Homotopical Geometry) and Chapter 13 (Gauge Theories and Their Homotopical Poisson Reduction).

Chapter 2 consists of five sections, with a sequel in Chapter 9. The first section § 2.1 (Higher Categories, Doctrines and Theories) introduces higher categories, doctrines such as $(n+1)$-categories, theories and models such as objects of the same doctrine. Adjunctions, (left and right) Kan extensions, limits and colimits (such as special cases of right and left Kan extensions) are dealt with from the standpoint of higher categories. The Yoneda dual n-categories $C^\vee$ and $C^\wedge$ of an n-category C are defined and the higher Yoneda lemma is stated as an hypothesis. Even Ehresmann's sketches, as a natural generalisation of algebraic theories in the sense of Lawvere [14], are touched upon from the standpoint of higher categories. The book uses a presentation based upon the setting of doctrines and higher categorical logic. With the advantage of a direct homotopical generalisation, § 2.2 (Monoidal Categories) gives monoidal categories in a doctrinal guise, while symmetric monoidal categories are introduced in § 2.3 (Symmetric Monoidal Categories) equivalently to how the traditional one is introduced by generalising Segal's theory of Γ-spaces. The next section § 2.4 (Grothendieck Topologies) is a succinct review. The final section § 2.5 (Categorical Infinitesimal Calculus) gives Quillen's tangent category [20] for a genuinely categorical treatment of differential calculus, with the advantage of allowing one to deal simultaneously with all geometric structures in the book as well as generalising directly to the higher categorical setting of doctrines and theories. Grothendieck connections introduced in this section subsequently play a unifying role in Chapter 7 (Connections and Curvature).

Chapter 3, consisting of three sections, gives appropriate tools for differential geometry on spaces of fields, which are directly susceptible to homotopical generalisations in Chapter 10. Roughly speaking, there are two approaches: parametrised and functional. The parametrised approach is appropriate for treating differential forms, while the functional one is good for dealing with vector fields. The parametrised approach can be seen in [4] and [1] in the arena of algebraic geometry and in [8] in the terrain of differential geometry. Synthetic differential geometry (see [13]) is an adequate mixture of both approaches. After § 3.1 (Parametrized Geometry) and § 3.2 (Functional Geometry), § 3.3 is devoted to showing how the categorical infinitesimal calculus depicted in § 2.5 will be applied to differential geometry.

Most problems one encounters in physics and mathematics carry obstructions with them and Chapter 9, consisting of

11 sections, gives the main tools for general obstruction theory. All the obstructions considered in the book can be defined as some kind of higher Kan extension of models of theories within a given doctrine. Nobody would expect a full treatment of homotopical algebra in only 40 pages or so and one is referred to [5], [2], [11], [3] and [7] for a full treatment of axiomatic homotopy theory and other intimately related topics. The principal objective in homotopical algebra is the localisation of categories. While § 9.6 (∞-Categories) gives an elementary presentation of ∞-categories, § 9.10 (Higher Categories) gives their advanced presentation based upon Rezk (see [21] and [22]). The next section § 9.11 (Theories up to Homotopy and the Doctrine Machine) is concerned with homotopical doctrines and homotopical theories based upon the homotopical higher categories discussed in § 9.10.

Homotopical geometry, a rigorous device intended for studying obstruction theory problems in geometry from a genuinely geometric standpoint, started with [23], [9], [10], [20] and others in the latter half of the previous century. Roughly speaking, it is obtained from the parametrised and functional geometry discussed in Chapter 3 by simply replacing the category **Sets** of sets and maps by the category **sSets** of simplicial sets and simplicial maps endowed with the standard model category structure or the homotopical ∞-category ${}^{\infty}\mathbf{GRPD}$ of ∞-groupoids, in the author's terminology (Theorem 9.5.1 and Definition 9.5.5). We have to produce a kind of differential calculus up to homotopy, carrying a well-behaved notion of higher stack in the formalisation of quotients and moduli spaces in covariant gauge theory. In Yang-Mills theory, we have to deal with a variable principal G-bundle P over a given spacetime M, which can be reformulated as a map $P : M \to BG$, with BG being the smooth classifying space for principal G-bundles. It is the derived geometry that provides the proper setting for differential calculus on spaces like BG. It is interesting to note that, as is often the case, physicists have encountered similar mathematical structures, independently of mathematicians, in endeavours related to BRST-BV formalism. Quantisation can be regarded as a kind of deformation and the modern theory of deformation has a great deal to do with homotopical geometry.

Homotopical algebraic geometry is now a well-established branch of mathematics, for which the reader is referred to [15], [25] and [26]. Homotopical differential geometry is by no means settled (in fact, it is currently in a mess). For the first attempts at homotopical differential geometry, the reader is referred to [24] and [12]. This stark contrast between homotopical algebraic geometry and homotopical differential geometry comes simply from the fact that algebraic geometry, as such, is completely axiomatised or conceptually purified due to Grothendieck's revolution in this arena in the middle of the previous century, while differential geometry has not yet been addressed in this way. For the burgeoning axiomatic approach to differential geometry inspired by synthetic differential geometry, the reader is referred to [16], [17] and [18]. In Chapter 10, the author gives a systematic construction of homotopical spaces based upon the doctoral approach to categorical logic depicted in Chapter 2. Homotopical or derived geometry gives a natural setting for non-abelian cohomology. Following § 10.3 (Non-abelian Cohomology) and § 10.4 (Differential Cohomology), § 10.5 (Geometric Stacks) gives a short account of geometric higher stacks as in [6]. The next section § 10.6 (Homotopical Infinitesimal Calculus) explains how to adapt the methods in § 2.5 to the ∞-categorical setting. Then, § 10.7 (Derived Symplectic Structures) sketches the notion of a closed form and a derived symplectic form on a derived stack. A striking difference between the classical and derived settings is that a differential form being closed is not an intrinsic property but an exotic structure in the latter setting. The final section § 10.8 (Deformation Theory and Formal Geometry) is devoted to the derived deformation theory programme due to Deligne, Drinfeld and Kontsevich.

Chapter 13 is an expansion of [19], defining gauge theories and investigating their classical aspects. The problems considered in this chapter are called local variational problems, and their equations of motion, expressed by Euler-Lagrange equations, are studied in the use of the non-linear algebraic analysis depicted in Chapter 12 (Algebraic Analysis of Non-Linear Partial Differential Equations).

Note

A longer version of this review is available at http://tinyurl.com/zja7f3d.

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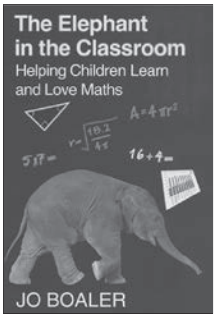
For information on Hirokazu Nishimura please refer to issue 94 of the Newsletter (December 1994), page 56.



Jo Boaler

The Elephant in the Classroom: Helping Children Learn and Love Maths

Souvenir Press, 2nd ed. 2015
288 p.
ISBN 978-0285-63847-1

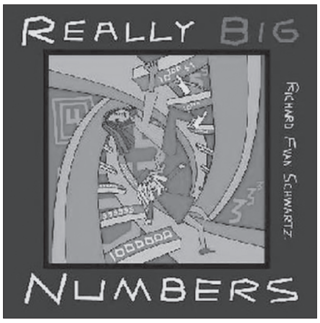
Reviewer: Jean-Paul Allouche

The title of this book is illuminated by its subtitle "Helping children learn and love mathematics".

This looks like a doubly ambitious project but, after reading the book, one sees that it seems quite doable. The author, inspired by what he has seen and learned in Great Britain and in the US, presents several chapters explaining his point of view. The introduction's title "Understanding the urgences" and the first sentence "Far too many students hate mathematics" give the general tone.

After the first chapter "What is maths, and why do we *all* need it?", the author tries to establish what goes wrong in classrooms: in particular, labelling children by assigning them a level, transforming teaching into a can-or-can't-do exercise, prioritising immediately applicable formulas and hints rather than thought, and learning without talking and without reality. Then, the author tries to give a "vision for a better future": keeping children interested with projects and teaching by communicating rather than dumping one-way lectures on them, replacing the present form of assessments with "assessments for learning" and avoiding the classification of children very early into good-at-maths and not-good-at-maths. The author then explains how girls and women are kept out of maths and science, in relation to the difference between trick-learning and deep understanding. The last chapters are devoted to strategies and ways of working and advice for parents (e.g. giving children a good mathematical start in life with, for example, games and puzzles, avoiding praising children with sentences like "you are smart" but praising them about *what* they have done, never sharing personal stories of maths failures or dislike, encouraging children to work on challenging problems and to learn from their mistakes, avoiding leading them through work step-by-step and encouraging drawing and figures). The book ends with an appendix giving solutions to the delicious exercises suggested in all the chapters.

I do not know whether every teacher, every parent and – most importantly – every pupil or student will quickly know how to learn mathematics, or whether they will all love mathematics, but, without doubt, this book can really help to form a "different" and better vision of mathematics; it might even be inspiring for the teaching (and research activities) of professional mathematicians.



Richard Evan Schwartz

Really Big Numbers

AMS, 2014
192 p.
ISBN 978-1-4704-1425-2

Reviewer: Jean-Paul Allouche

As explained on the back cover of this delicious book, "This book talks about really big numbers in terms of everyday things, such as the number of basketballs needed to cover New York City…". But this book also goes far beyond this. Written in a language quite accessible to children, it also shows how notation permits one to *name* huge numbers. Remember when you were a child: you were first extraordinarily proud to count from 1 to 16, say, and then from 1 to 99… At some point you essentially understood that you could count from 1 to virtually *any* number. But was it really true? Did you know, for example, that a 1 followed by 78 zeros is called a "quinquavigintillion"?

So it is possible to name really big numbers without even really conceiving of how big they are and without being quite sure of how to count from 1 to say 10^{78}…

The next idea is to build up towers of symbols (the simplest being $a^{b^{c^{\cdot^{\cdot^{\cdot}}}}}$, to invent an abbreviation for such a tower with a given number of floors and then to consider towers of towers, etc., in other words using *recursion* (in the sense of starting from a simple rule and applying it again and again).

Children (and also less young people) will love this book, especially the attractive colour pictures accompanying the short sentences on every single page. I cannot resist quoting some sentences from the end of the book: "Each new addition to the language is a chariot moving so quickly it makes all the previous ones seem to stand still", "We skip from chariot to chariot, impatient with them almost as soon as they are created", "Unhindered by any ties to experience, giddy with language, we race ever faster through the number system", "Now and then we pluck numbers from the blur … numbers which have no names except the one we might now give them … souvenirs from alien, unknowable worlds", and lastly "INFINITY is farther away than you thought".

The European Mathematical Society hosts a book review page on its website. A large number of book reviews have been collected since 2011 and they can be browsed by subject (see http://www.euro-math-soc.eu/book-reviews).

Solved and Unsolved Problems

Themistocles M. Rassias (National Technical University, Athens, Greece)

Do not worry about your difficulties in mathematics.
I can assure you mine are still greater.
Albert Einstein (1879–1955)

I Six new problems – solutions solicited

Solutions will appear in a subsequent issue.

155. Let $f : I \subset \mathbb{R} \to \mathbb{R}$ be a convex function on the interval I, with $a, b \in \mathring{I}$ (interior of I), $a < b$ and $\nu \in [0,1]$. Show that

$$\begin{aligned}(0 \leq)\nu(1-\nu)(b-a)\left[f'_+((1-\nu)a+\nu b) - f'_-((1-\nu)a+\nu b)\right] \quad (1)\\ \leq (1-\nu)f(a) + \nu f(b) - f((1-\nu)a+\nu b)\\ \leq \nu(1-\nu)(b-a)\left[f'_-(b) - f'_+(a)\right],\end{aligned}$$

where $f'_\pm$ are the lateral derivatives of the convex function f.
In particular, for any $a, b > 0$ and $\nu \in [0,1]$, show that the following reverses of Young's inequality are valid:

$$(0 \leq)(1-\nu)a + \nu b - a^{1-\nu}b^\nu \leq \nu(1-\nu)(a-b)(\ln a - \ln b) \quad (2)$$

and

$$(1 \leq) \frac{(1-\nu)a+\nu b}{a^{1-\nu}b^\nu} \leq \exp\left[4\nu(1-\nu)\left(K\left(\frac{a}{b}\right)-1\right)\right], \quad (3)$$

where K is Kantorovich's constant *defined by*

$$K(h) := \frac{(h+1)^2}{4h}, \; h > 0. \quad (4)$$

(*Sever S. Dragomir, Victoria University, Melbourne City, Australia*)

156. Evaluate

$$\lim_{n\to\infty}\left[\frac{(1+\frac{1}{n^2})(1+\frac{2}{n^2})\cdots(1+\frac{n}{n^2})}{\sqrt{e}}\right]^n.$$

(*Dorin Andrica, Babeş-Bolyai University of Cluj-Napoca, Romania*)

157. Let X be a compact space and $f : X \to X$ be continuous and expansive, that is,

$$d(f(x), f(y)) \geq d(x, y) \quad \forall\, x, y \in X.$$

What can be said about the function f?

(*W. S. Cheung, University of Hong Kong, Pokfulam, Hong Kong*)

158. Find all differentiable functions $f : \mathbb{R} \to \mathbb{R}$ which satisfy the equation

$$xf'(x) + kf(-x) = x^2 \quad \forall\, x \in \mathbb{R},$$

where $k > 0$ is an integer.

(*Ovidiu Furdui, Technical University of Cluj-Napoca, Cluj-Napoca, Romania*)

159. Let $f : I \subset \mathbb{R} \to \mathbb{R}$ be a twice differentiable function on the interval $\mathring{I}$ (interior of I). If there exist the constants d, D such that

$$d \leq f''(t) \leq D \text{ for any } t \in \mathring{I}, \quad (5)$$

show that

$$\begin{aligned}\frac{1}{2}\nu(1-\nu)\,d\,(b-a)^2 &\leq (1-\nu)\,f(a) + \nu f(b) - f((1-\nu)a+\nu b)\\ &\leq \frac{1}{2}\nu(1-\nu)\,D\,(b-a)^2 \quad (6)\end{aligned}$$

for any $a, b \in \mathring{I}$ and $\nu \in [0,1]$.
In particular, for any $a, b > 0$ and $\nu \in [0,1]$, show that the following refinements and reverses of Young's inequality are valid:

$$\begin{aligned}&\frac{1}{2}\nu(1-\nu)(\ln a - \ln b)^2 \min\{a,b\}\\ &\leq (1-\nu)\,a + \nu b - a^{1-\nu}b^\nu \quad (7)\\ &\leq \frac{1}{2}\nu(1-\nu)(\ln a - \ln b)^2 \max\{a,b\}\end{aligned}$$

and

$$\begin{aligned}\exp\left[\frac{1}{2}\nu(1-\nu)\left(1-\frac{\min\{a,b\}}{\max\{a,b\}}\right)^2\right] &\leq \frac{(1-\nu)\,a+\nu b}{a^{1-\nu}b^\nu} \quad (8)\\ \leq \exp\left[\frac{1}{2}\nu(1-\nu)\left(\frac{\max\{a,b\}}{\min\{a,b\}}-1\right)^2\right].\end{aligned}$$

(*Sever S. Dragomir, Victoria University, Melbourne City, Australia*)

160. Let p be the partition function (counting the ways to write n as a sum of positive integers), extended so that $p(0) = 1$ and $p(n) = 0$ for $n < 0$. Prove that, for $n \geqslant 0$,

$$1 \leqslant \frac{2p(n+2) - p(n+3)}{p(n)} \leqslant \frac{3}{2}.$$

(*Mircea Merca, University of Craiova, Romania*)

II Two new open problems

161*. For $c > 0$, $n \in \mathbb{N}$, $x \geq 0$, if we define the m-th order moment as

$$T^c_{n,m}(x) = \sum_{k=0}^{\infty}\left(\frac{c}{1+c}\right)^{ncx}\frac{(ncx)_k}{k!(1+c)^k}\left(\frac{k}{n}\right)^m,$$

where $(a)_k = a(a+1)\ldots(a+k-1)$, examine whether one can find a recurrence relation between $T^c_{n,m+1}(x)$ and $T^c_{n,m}(x)$?

(*Vijay Gupta, Netaji Subhas Institute of Technology, New Delhi, India*)

162*. Does there exist an harmonic homeomorphism of the open unit ball B in $\mathbb{R}^4$ onto $\mathbb{R}^4$?

(*Themistocles M. Rassias, National Technical University of Athens, Greece*)

III Solutions

147. Prove or disprove the following. If $f : \mathbb{R} \to \mathbb{R}$ has both a left limit and right limit at every point then f is continuous, except perhaps on a countable set.

(*W. S. Cheung, The University of Hong Kong, Pokfulam, Hong Kong*)

Solution by the proposer. True!

Define $\varphi : \mathbb{R} \to \mathbb{R}$ by

$$\varphi(x) := \max\left\{|f(x) - f(x^+)|\,,\, |f(x) - f(x^-)|\right\}.$$

Observe that $\varphi \geq 0$ and $\varphi(x) > 0$ exactly when x is a point at which f is discontinuous.

For any $n \in \mathbb{N}$, define

$$S_n := \left\{x \in \mathbb{R} : \varphi(x) \geq \frac{1}{n}\right\}.$$

Then,

$$\begin{aligned}&\{x \in \mathbb{R} : f \text{ is discontinuous at } x\}\\ &= \{x \in \mathbb{R} : f(x) > 0\}\\ &= \bigcup_{n=1}^{\infty} S_n\,.\end{aligned}$$

For any $n \in \mathbb{N}$ and any $t \in S_n$, since $f(t^+) = \lim_{x \to t^+} f(x)$, $\exists\, \delta > 0$ such that

$$\left|f(x) - f(t^+)\right| < \frac{1}{4n} \quad \forall\, x \in (t, t+\delta)\,.$$

Hence, for any $x \in (t, t+\delta)$,

$$\begin{aligned}\varphi(x) &\leq \left|f(x) - f(x^+)\right|\\ &\leq \left|f(x) - f(t^+)\right| + \left|f(x^+) - f(t^+)\right|\\ &< \frac{1}{4n} + \Big|\lim_{\substack{y \to x^+\\ y \in (t,t+\delta)}} f(y) - f(t^+)\Big|\\ &\leq \frac{1}{4n} + \frac{1}{4n}\\ &= \frac{1}{2n}\end{aligned}$$

and so $x \notin S_n$. Similarly, $\exists\, \delta > 0$ such that no point in $(t-\delta, t)$ lies in S_n. Therefore, points in S_n are isolated points and so it is at most countable. Hence, $\bigcup_{n=1}^{\infty} S_n$ is also countable.

Also solved by Vincenzo Basco (Universita degli Studi di Roma "Tor Vergata", Italy), Mihaly Bencze (Brasov, Romania), Soon-Mo Jung (Hongik University, Chochiwon, Korea), Socratis Varelogiannis (National Technical University of Athens, Greece)

148. Let $(a_n)_{n\geq 1}$ and $(b_n)_{n\geq 1}$ be two sequences of positive real numbers. If

1. $(a_n)_{n\geq 1}$ and $(b_n)_{n\geq 1}$ are both unbounded; and
2. $\limsup_{n\to\infty} \frac{a_{n+1}}{a_n} = 1$,

prove that the set $M = \left\{\frac{a_n}{b_m} : m, n \geq 1\right\}$ is everywhere dense in the interval $[0, \infty)$.

(*Dorin Andrica, Babeş-Bolyai University of Cluj-Napoca, Romania*)

Solution by the proposer. Let $z \in [0, \infty)$ and let V be a neighbourhood of z in $\mathbb{R}$. There exists an interval $[x, y) \subset V$, with $x \leq z < y$.

When $z = 0$, we have $x = z = 0$ and, from hypothesis 1, it follows that there is a positive integer m with $b_m > a_1/y$ such that $a_1/b_m \in M \cap V$.

When $z > 0$, we suppose that $0 < x < z$. Let $\varepsilon = (x - y)/x > 0$. From hypothesis 2, it follows that there is a positive integer n_0 such that, for every $n > n_0$,

$$\frac{a_n}{a_{n-1}} < 1 + \varepsilon. \tag{9}$$

From hypothesis 1, it follows that there is a positive integer m with $b_m \geq a_{n_0}/x$. Define the positive integer n by $n = \min\{k : k > n_0, a_k > xb_m\}$. Then, it satisfies

$$\frac{a_{n-1}}{x} \leq b_m < \frac{a_n}{x}. \tag{10}$$

Therefore, for $n > n_0$, by (1), we get

$$x\frac{a_n}{a_{n-1}} < x + \varepsilon x = y. \tag{11}$$

From (10) and (11), we have

$$x < \frac{a_n}{b_m} \leq x\frac{a_n}{a_{n-1}} < y,$$

hence $a_n/b_m \in M \cap (x, y) \subset M \cap V$.

In both cases, we have obtained $z \in \bar{M}$, the closure of M, and we are done.

Remark. Taking $a_n = p_n, n \geq 1$, where p_n is the n^{th} prime, and using the well known result $\lim_{n\to\infty} \frac{p_{n+1}}{p_n} = 1$, from the property proved above, it follows that for every unbounded sequence $(b_n)_{n\geq 1}$ of positive real numbers the set

$$M = \left\{\frac{p_n}{b_m} : m, n \geq 1\right\}$$

is everywhere dense in the interval $[0, \infty)$.

Also solved by Vincenzo Basco (Universita degli Studi di Roma "Tor Vergata", Italy), Mihaly Bencze (Brasov, Romania), Soon-Mo Jung (Hongik University, Chochiwon, Korea), Socratis Varelogiannis (National Technical University of Athens, Greece)

149. (a) Prove that

$$\lim_{n\to\infty}\left(2\zeta(3) + 3\zeta(4) + \cdots + n\zeta(n+1) - \frac{n(n+1)}{2}\right) = 0.$$

(b) *An Apéry's constant series.* Calculate

$$\sum_{n=2}^{\infty}\left(\frac{n(n+1)}{2} - 2\zeta(3) - 3\zeta(4) - \cdots - n\zeta(n+1)\right),$$

where ζ denotes the Riemann zeta function.

(*Ovidiu Furdui, Technical University of Cluj-Napoca, Cluj-Napoca, Romania*)

Solution by the proposer.

(a) We have

$$x_n = 2\zeta(3)+3\zeta(4)+\cdots+n\zeta(n+1)-\frac{n(n+1)}{2} = \sum_{k=2}^{n} k\,(\zeta(k+1)-1)-1$$

and this implies that

$$\begin{aligned}\lim_{n\to\infty} x_n &= \sum_{k=2}^{\infty} k\,(\zeta(k+1)-1)-1\\ &= \sum_{k=2}^{\infty}\sum_{i=2}^{\infty}\frac{1}{i^{k+1}}-1\\ &= \sum_{i=2}^{\infty}\frac{1}{i}\sum_{k=2}^{\infty}k\left(\frac{1}{i}\right)^k-1\\ &= \sum_{i=2}^{\infty}\frac{1}{i}\left(\frac{i}{(i-1)^2}-\frac{1}{i}\right)-1\\ &= \sum_{i=2}^{\infty}\left(\frac{1}{(i-1)^2}-\frac{1}{i^2}\right)-1\\ &= 0.\end{aligned}$$

(b) The series equals $2\zeta(3)-1$. We need in our analysis Abel's summation by parts formula [1, p. 258], which states that if $(a_n)_{n\geq 1}$ and $(b_n)_{n\geq 1}$ are two sequences of real numbers and $A_n = \sum_{k=1}^{n} a_k$ then $\sum_{k=1}^{n} a_k b_k = A_n b_{n+1} + \sum_{k=1}^{n} A_k\,(b_k - b_{k+1})$. The equality can be proved by elementary calculations. Also, we will be using, in our calculations, the infinite version of the formula above:

$$\sum_{k=1}^{\infty} a_k b_k = \lim_{n\to\infty}(A_n b_{n+1}) + \sum_{k=1}^{\infty} A_k\,(b_k - b_{k+1}). \tag{12}$$

Now, we are ready to calculate our series. We apply formula (12), with $a_k = 1$ and

$$b_k = \frac{(k+1)(k+2)}{2} - 2\zeta(3) - 3\zeta(4) - \cdots - (k+1)\zeta(k+2),$$

and we get that

$$\begin{aligned}&\sum_{n=2}^{\infty}\left(\frac{n(n+1)}{2} - 2\zeta(3) - 3\zeta(4) - \cdots - n\zeta(n+1)\right)\\ &= \sum_{n=1}^{\infty}\left(\frac{(n+1)(n+2)}{2} - 2\zeta(3) - 3\zeta(4) - \cdots - (n+1)\zeta(n+2)\right)\\ &= \lim_{n\to\infty} n\left(\frac{(n+2)(n+3)}{2} - 2\zeta(3) - 3\zeta(4) - \cdots - (n+2)\zeta(n+3)\right)\\ &\qquad + \sum_{n=1}^{\infty} n(n+2)\,(\zeta(n+3)-1)\\ &= \sum_{n=1}^{\infty} n(n+2)\,(\zeta(n+3)-1)\\ &= \sum_{n=1}^{\infty} n(n+2)\sum_{i=2}^{\infty}\frac{1}{i^{n+3}}\\ &= \sum_{i=2}^{\infty}\sum_{n=1}^{\infty} n(n+2)\left(\frac{1}{i}\right)^{n+3}\\ &= \sum_{i=2}^{\infty}\frac{3i-1}{i^2(i-1)^3}\\ &= 2\zeta(3)-1,\end{aligned}$$

since

$$\lim_{n\to\infty} n\left(\frac{(n+2)(n+3)}{2} - 2\zeta(3) - 3\zeta(4) - \cdots - (n+2)\zeta(n+3)\right) = 0.$$

We also used in this calculation the power series formula

$$\sum_{n=1}^{\infty} n(n+2)x^{n+3} = \frac{x^4(3-x)}{(1-x)^3} \quad \text{for } |x| < 1.$$

The preceding limit can be proved using the 0/0 case of Cesaro–Stolz's lemma ([1, p. 265]). The problem is solved. □

Remark. It appears that this series is new in the mathematical literature.

[1] Ovidiu Furdui, *Limits, Series and Fractional Part Integrals. Problems in Mathematical Analysis*, Springer, London, 2013.

Also solved by Mihaly Bencze (Brasov, Romania), Soon-Mo Jung (Hongik University, Chochiwon, Korea), Sotirios E. Louridas (Athens, Greece), Socratis Varelogiannis (National Technical University of Athens, Greece)

150. We say that the function $f : I \subset \mathbb{R}\setminus\{0\} \to \mathbb{R}$ is *HA-convex* if

$$f\left(\frac{xy}{tx+(1-t)\,y}\right) \leq (1-t)\,f(x) + t\,f(y) \tag{13}$$

for all $x, y \in I$ and $t \in [0,1]$.

Let $f, h : [a,b] \subset (0,\infty) \to \mathbb{R}$ be such that $h(t) = tf(t)$ for $t \in [a,b]$. Show that f is HA-convex on the interval $[a,b]$ if and only if h is convex on $[a,b]$.

(*Sever S. Dragomir, Victoria University, Melbourne, Australia*)

Solution by the proposer. Assume that f is *HA*-convex on the interval $[a,b]$. Then, the function $g : \left[\frac{1}{b},\frac{1}{a}\right] \to \mathbb{R}$, $g(t) = f\left(\frac{1}{t}\right)$ is convex on $\left[\frac{1}{b},\frac{1}{a}\right]$. By replacing t with $\frac{1}{t}$, we have $f(t) = g\left(\frac{1}{t}\right)$.

If $\lambda \in [0,1]$ and $x, y \in [a,b]$ then, by the convexity of g on $\left[\frac{1}{b},\frac{1}{a}\right]$, we have

$$\begin{aligned}h((1-\lambda)x+\lambda y) &= [(1-\lambda)x+\lambda y]\,f((1-\lambda)x+\lambda y)\\ &= [(1-\lambda)x+\lambda y]\,g\Big(\frac{1}{(1-\lambda)x+\lambda y}\Big)\\ &= [(1-\lambda)x+\lambda y]\,g\Big(\frac{(1-\lambda)x\frac{1}{x}+\lambda y\frac{1}{y}}{(1-\lambda)x+\lambda y}\Big)\\ &\leq [(1-\lambda)x+\lambda y]\,\frac{(1-\lambda)xg(\frac{1}{x})+\lambda yg(\frac{1}{y})}{(1-\lambda)x+\lambda y}\\ &= (1-\lambda)xg\Big(\frac{1}{x}\Big)+\lambda yg\Big(\frac{1}{y}\Big)\\ &= (1-\lambda)xf(x)+\lambda yf(y) = (1-\lambda)h(x)+\lambda h(y),\end{aligned}$$

which shows that h is convex on $[a,b]$.

We have $f(t) = \frac{h(t)}{t}$ for $t \in [a,b]$. If $\lambda \in [0,1]$ and $x, y \in [a,b]$ then, by the convexity of h on $[a,b]$, we have

$$\begin{aligned}f\left(\frac{xy}{\lambda x+(1-\lambda)\,y}\right) &= \frac{h\left(\frac{xy}{\lambda x+(1-\lambda)y}\right)}{\frac{xy}{\lambda x+(1-\lambda)y}}\\ &= \frac{\lambda x+(1-\lambda)\,y}{xy}h\left(\frac{xy}{\lambda x+(1-\lambda)\,y}\right)\\ &= \frac{\lambda x+(1-\lambda)\,y}{xy}h\left(\frac{1}{(1-\lambda)\,\frac{1}{x}+\lambda\frac{1}{y}}\right)\\ &= \frac{\lambda x+(1-\lambda)\,y}{xy}h\left(\frac{(1-\lambda)\,\frac{1}{x}x+\lambda\frac{1}{y}y}{(1-\lambda)\,\frac{1}{x}+\lambda\frac{1}{y}}\right)\end{aligned}$$

$$\leq \frac{\lambda x + (1-\lambda) y}{xy} \frac{(1-\lambda)\frac{1}{x}h(x) + \lambda\frac{1}{y}h(y)}{(1-\lambda)\frac{1}{x} + \lambda\frac{1}{y}}$$
$$= (1-\lambda)\frac{1}{x}h(x) + \lambda\frac{1}{y}h(y)$$
$$= (1-\lambda) f(x) + \lambda f(y),$$

which shows that f is HA-convex on the interval $[a,b]$.

Also solved by Soon-Mo Jung (Hongik University, Chochiwon, Korea), Panagiotis T. Krasopoulos (Athens, Greece), John N. Lillington (Wareham, UK), Socratis Varelogiannis (National Technical University of Athens, Greece)

151. Let $f : [a,b] \subset (0,\infty) \to \mathbb{R}$ be an HA-convex function on the interval $[a,b]$. Show that we have

$$f\left(\frac{a+b}{2}\right) \leq \frac{1}{\ln b - \ln a}\int_a^b \frac{f(t)}{a+b-t}dt$$
$$\leq \frac{af(a) + bf(b)}{a+b}. \tag{14}$$

(*Sever S. Dragomir, Victoria University, Melbourne, Australia*)

Solution by the proposer. Since the function $h(t) = tf(t)$ is convex, we have

$$\frac{x+y}{2} f\left(\frac{x+y}{2}\right) \leq \frac{xf(x) + yf(y)}{2}$$

for any $x, y \in [a,b]$.

If we divide this inequality by $xy > 0$, we get

$$\frac{1}{2}\left(\frac{1}{x} + \frac{1}{y}\right) f\left(\frac{x+y}{2}\right) \leq \frac{1}{2}\left(\frac{f(x)}{y} + \frac{f(y)}{x}\right), \tag{15}$$

for any $x, y \in [a,b]$.

If we replace x by $(1-t)a + tb$ and y by $ta + (1-t)b$ in (15), we get

$$\frac{1}{2}\left(\frac{1}{(1-t)a+tb} + \frac{1}{ta+(1-t)b}\right) f\left(\frac{a+b}{2}\right)$$
$$\leq \frac{1}{2}\left(\frac{f((1-t)a+tb)}{ta+(1-t)b} + \frac{f(ta+(1-t)b)}{(1-t)a+tb}\right) \tag{16}$$

for any $t \in [0,1]$.

Integrating (16) on $[0,1]$ over t, we get

$$\frac{1}{2}\left(\int_0^1 \frac{1}{(1-t)a+tb}dt + \int_0^1 \frac{1}{ta+(1-t)b}dt\right) f\left(\frac{a+b}{2}\right)$$
$$\leq \frac{1}{2}\left(\int_0^1 \frac{f((1-t)a+tb)}{ta+(1-t)b}dt + \int_0^1 \frac{f(ta+(1-t)b)}{(1-t)a+tb}dt\right). \tag{17}$$

Observe that, by an appropriate change of variable,

$$\int_0^1 \frac{1}{(1-t)a+tb}dt = \int_0^1 \frac{1}{ta+(1-t)b}dt$$
$$= \frac{1}{b-a}\int_a^b \frac{du}{u}$$
$$= \frac{\ln b - \ln a}{b-a}$$

and

$$\int_0^1 \frac{f((1-t)a+tb)}{ta+(1-t)b}dt = \int_0^1 \frac{f(ta+(1-t)b)}{(1-t)a+tb}$$
$$= \frac{1}{b-a}\int_a^b \frac{f(u)}{a+b-u}du$$

and, by (17), we get the first inequality in (14).

From the convexity of h, we also have

$$((1-t)a+tb)f((1-t)a+tb) \leq (1-t)af(a) + tbf(b)$$

and

$$(ta+(1-t)b)f(ta+(1-t)b) \leq taf(a) + (1-t)bf(b)$$

for any $t \in [0,1]$.

Add these inequalities to get

$$((1-t)a+tb)f((1-t)a+tb) + (ta+(1-t)b)f(ta+(1-t)b)$$
$$\leq af(a) + bf(b)$$

for any $t \in [0,1]$.

If we divide this inequality by $((1-t)a+tb)(ta+(1-t)b)$, we get

$$\frac{f((1-t)a+tb)}{ta+(1-t)b} + \frac{f(ta+(1-t)b)}{(1-t)a+tb} \leq \frac{af(a)+bf(b)}{((1-t)a+tb)(ta+(1-t)b)} \tag{18}$$

for any $t \in [0,1]$.

If we integrate the inequality (18) over t on $[0,1]$, we obtain

$$\int_0^1 \frac{f((1-t)a+tb)}{ta+(1-t)b}dt + \int_0^1 \frac{f(ta+(1-t)b)}{(1-t)a+tb}dt \tag{19}$$
$$\leq [af(a)+bf(b)]\int_0^1 \frac{dt}{((1-t)a+tb)(ta+(1-t)b)}.$$

Since

$$\int_0^1 \frac{dt}{((1-t)a+tb)(ta+(1-t)b)} = \frac{1}{b-a}\int_a^b \frac{du}{u(a+b-u)}$$

and

$$\frac{1}{u(a+b-u)} = \frac{1}{a+b}(\frac{1}{u} + \frac{1}{a+b-u}),$$

we have

$$\int_a^b \frac{du}{u(a+b-u)} = \frac{1}{a+b}\int_a^b (\frac{1}{u} + \frac{1}{a+b-u})du$$
$$= \frac{2}{a+b}(\ln b - \ln a).$$

By (19), we then have

$$\frac{2}{b-a}\int_a^b \frac{f(u)}{a+b-u}du \leq 2\left[\frac{af(a)+bf(b)}{a+b}\right]\frac{\ln b - \ln a}{b-a},$$

which proves the second inequality in (14).

Also solved by Soon-Mo Jung (Hongik University, Chochiwon, Korea), Panagiotis T. Krasopoulos (Athens, Greece), John N. Lillington (Wareham, UK).

Notes

1. John N. Lillington (Wareham, UK) also solved problems 143 and 144.
2. G. C. Greubel (Newport News , Virginia, USA) also solved problems 139, 140 and 153*.
3. Ovidiu Furdui (Technical Univeristy of Cluj-Napoca, Romania) solved problem 154*.

We wait to receive your solutions to the proposed problems and ideas on the open problems. Send your solutions both by ordinary mail to Themistocles M. Rassias, Department of Mathematics, National Technical University of Athens, Zografou Campus, GR-15780, Athens, Greece, and by email to trassias@math.ntua.gr.
We also solicit your new problems with their solutions for the next "Solved and Unsolved Problems" column, which will be devoted to *real analysis*.

Letters to the Editor

Abdus Salam Shield of Honour

Speedy publication of relatively trivial research articles in mathematics in journals with so-called high impact factors has set a rather dangerous trend of research in most of the world's developing nations. These journals usually publish quite superficial research and this is a trend that may, in the long run, be a threat to the global research community, substituting "good research" for "popular research". However, this is not, at present, the most dangerous part of the system. The real threat is something more fundamental, associated with this kind of research publication.

In an effort to encourage research in mathematics, governments in numerous developing countries are giving awards, prizes and various financial incentives to their researchers. But not knowing how to evaluate the quality of research, the governmental bodies in these developing countries have found a very easy way out. They just add up the impact factors of the publications of the researchers applying for some national award or prize. The persons with the highest sums of impact factors are declared to be the winners. This process provides strong encouragement for publishing large number of trivial papers in journals with positive impact factors. Research performance in such countries has taken on a very different meaning, a meaning that honours triviality and mediocrity.

If the Pakistani criteria for honouring research performance had been applied globally then most of the Fields medallists and Abel laureates would never have received any award.

In order to bring the Pakistani mathematical community back on the track of quality research, the Abdus Salam Shield of Honour (ASSH) was created in April 2015. This is an initiative of the National Mathematical Society of Pakistan.

The first Shield of Honour goes to

Professor Hassan Azad

a Pakistani national. The evaluation committee, chaired by Professor *Cédric Villani*, had the following members: Professor *Juergen Herzog,* Professor *Stefano Luzzatto* and Professor *Ioan Tomescu.*

The main research interest of the first recipient of ASSH, Professor Hassan Azad, is Lie groups and algebraic groups, and algorithms related to these fields. Currently, he is working on real algebraic groups and constructive procedures for computing their invariants, with a view toward their applications in symmetry methods in differential equations.

Professor Cédric Villani wrote in the final report of the committee: "It was our unanimous vote, independently of each other, that Hassan Azad is the most deserving candidate. He did not sacrifice the quality for the quantity. This is exactly the kind of example that we wish to promote."

The award ceremony will take place on 16 March in Lahore, Pakistan.

Alla Ditta Raza Choudary,
Government College University, Lahore, Pakistan

"On jugera"

Equal standards are a necessity for fair evaluation and for excellence in research. Publication in journals with an editorial board or committee review is a principal element in the development of a professional career, for hiring, promotions, salary increases, invitations, prize awards, etc., in particular for young researchers. Unfortunately, the standards for publication of some supposedly reputed editorial boards are not equal for everyone. The chances of an article being accepted by some journals depends crucially on who the author is and, more precisely, to which "group" they belong (in the wide sense research group and national group but unfortunately sometimes also religious or ethnic group) and their connections with the editors. If you are not "connected", it is very likely that you will get the typical dry and brief two line rejection letter without any congruent reason, for example stating that your paper is not interesting enough, that it is too long or too short, that it is not general enough (or too general!), etc., without going any deeper into the mathematics of the paper that probably nobody has read. Throughout time, fundamental results have been rejected without serious grounds, starting with Galois' fundamental paper rejected by Poisson… "On jugera" (E. Galois).

But it is the perception of the author that this problem is becoming much worse in our time. I have recently experienced the arrogant behaviour of an editorial board when an article I originally submitted in 1997 to the *Annals of Mathematics* was rejected without any reason 18 (eighteen) years later in 2015, after going though the hands of two editors, Professor Fefferman (10 years) and Professor Sarnak (8 years). It is the same reputed journal that publishes fundamentally flawed articles, such as "The dynamics of the Hénon map" paper from 1991, just to cite one example from the same field where the review standards were not the same.[1] This happens in all fields. We should pay a special tribute to the late Abbas Bahri who courageously wrote a paper "Five gaps in mathematics".[2]

Unfortunately, this is not limited to the publication system. It is amusing to see how the same families of people award each other prizes and distinctions in turn, and invitations to international congresses, etc., how some editors get invitations because of their powerful positions to accept or reject papers, and how favours are exchanged and bargained. It is a shame to see this happening and allowed by a silent majority. There is no mafia without omerta.

Because some of us decided to be mathematicians and remain faithful to mathematics, and because we value most our freedom to voice our mathematical opinions without compromise, we believe it is our duty to break and not collaborate with a corrupt publication system. A centralised authoritarian review system only makes sense if it functions in an honest, transparent, fair and equal way. Otherwise, we learned from Thoreau that we have a duty of "journal publication disobedience".

An open publication system, such as the preprint server repository arXiv, and an open review system, such as the one used in other disciplines on the site pubpeer, should be the way to go. An open decentralised version that guarantees continuity over time will render the traditional publication system obsolete.

And never forget what we learned from Ramanujan … we only need a notebook …

Ricardo Perez-Marco,
Université Paris 13, Villetaneuse, France

[1] As is recognised by subsequent literature that tries to fix the problem, the parameter selection has a hole since, in the inductive procedure, only the measure and not the geometry of the "good parameters" are considered. Indeed, at each stage of the induction, one needs connected components of good parameters of minimal size, depending on the rate of expansion of the iterate to mimic the one dimensional argument as the authors pretend. That this rate of expansion overcomes the combinatorial explosion of the bad parameters is a mathematical miracle for which we can't find a proof in the literature (one would need to compare the "constants" in some of the subsequent papers ... but these constants are never estimated).

[2] Incidentally, we can find an excellent review of his article on the site pubpeer.com. From Peer 1: "It may come as a surprise to non-mathematicians, but the level of proof required for the acceptance of a mathematical theorem has a strong social element. Contributing factors include how surprising the result is, how it fits within the reigning beliefs of the field, the status of the author, as well as the general culture of rigour within the community. Mathematics, as a human activity, is as open to distortion and fraud as any area of science.
This uncomfortable fact is largely ignored by working mathematicians, although most have anecdotal evidence of incorrect results being waved through peer-review by 'friendly' editors."

Daniel Kastler (1926–2015) – Memories of a friend

Memories of my first collaboration with Daniel, in the Autumn of 1965, are intertwined with the colours of Provence and with the dark blue of its sea. We would work in Daniel's living room in Bandol, writing with chalk on the window panes so that we were staring at our C*-algebras as they floated over a landscape spanned by the Mistral.

Lisl, Daniel's wife, would not complain; rather, she would find our dusty traces decorative and would keep us at dinner after each session.

But another Mistral blows through those memories of 50 years ago and continued to blow along the decades: Daniel's inextinguishable enthusiasm, whose frequent bursts kept everybody's spirits up. Between those bursts, he would move to the piano to try out the opening of Beethoven's third concerto, interrupting the nap of one of his many cats sleeping on the scores.

After those few unforgettable years I spent in Bandol, the passing of time has seen many things change, except one: Daniel's incredible enthusiasm, which always seemed more fresh and lively.

As a child wondering on each little pretty stone he collects on the seashore – as Newton said about himself, a comparison once extended to Daniel by our late friend Gert Pedersen – Daniel would light up with joy at each further idea of new question or path, no matter how realistic or promising. And that was contagious in a marvellous way.

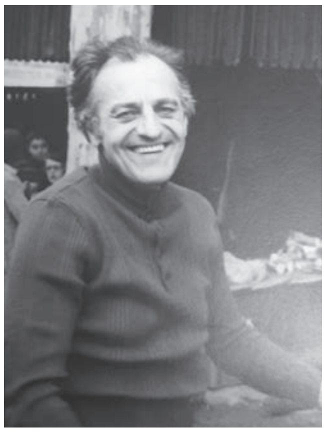
In the collaboration I mentioned, we introduced covariance algebras (later called crossed products); we were fascinated by our new concept in operator algebras but motivated by the historical paper that Daniel had written with Rudolf Haag a couple of years earlier, which opened the way to a new science, initiating the algebraic approach to quantum field theory – an alternative (or complementary) to the Wightman formulation – and equally inspired to a mathematical rigour otherwise mostly unknown in those times.

The motivation of our paper was the hope of controlling the spectrum condition in the algebraic approach to quantum field theory.

Derek Robinson contributed with a rather uncorrelated final section, where, independently of the parallel work by David Ruelle at the time, the notion of asymptotic abelianness was introduced.

The possibility of applications to statistical mechanics lit Daniel's enthusiasm as a spark in a haystack. This resulted in a series of papers he wrote in various combinations with Derek Robinson, Dick Kadison, Erling Stormer and me.

Daniel and Lisl Kastler, probably early 50ies.
Courtesy of Nora Kastler.



Daniel Kastler, probably early 60ies.
Courtesy of Nora Kastler.

In those years at the end of the 1960s, Bandol was a natural place to visit for scientists working on operator algebras and on mathematical problems of quantum field theory and statistical mechanics; the "Bandol Institute", of course, did not exist as such but many talks took place, sometimes even on the tiny benches of the elementary school.

The first account I heard of the Atiyah-Singer Index Theorem, at that point unpublished (except for Palais' lecture notes), was expounded by Isy Singer in Daniel's living room, in the Summer of 1966, again writing with chalk on the window glass.

At the start of the 1970s, during a memorable visit to Bandol, Alain Connes exposed the essentials of his Classifications of AFD Factors to Daniel, Dick and me, again in Daniel's living room sitting with teacups in our hands but this time without writing anything, just with words, slowed down by the re-expression of each sentence in crystal clear terms by Dick (or at least a slower rephrasing, giving us more time to grasp the meaning). An unforgettable session!

After the late 1960s, I still had frequent scientific contact with Daniel and, of course, the friendship could only grow. And it did continue to grow all life long but our collaborations relaxed. While I got engaged in a collaboration on algebraic quantum field theory with Rudolf Haag, starting in the Autumn of 1967 and joined in the Summer of 1968 by John Roberts, Daniel was more and more engaged in basic questions of quantum statistical mechanics.

Marvellous results emerged: the mathematical basis of the structure of states on funnels of type I factors in a paper by Daniel with R. Haag and R. V. Kadison; the stability of properties of von Neumann algebras under perturbations, studied with R. V. Kadison; the basis of the KMS conditions in the stability of equilibrium states, discovered with R. Haag and E. B. Trych-Pohlmeyer; and the algebraic foundations of the notion of chemical potential, developed with H. Araki, R. Haag and M. Takesaki.

Soon, the enthusiasm of Daniel was lit by another fire, which thereafter kept burning all his life. This was Alain Connes' noncommutative geometry and the related foundations of the Standard Model, proposed by Connes and Lott and developed to include the gravitational forces in the classical form of the Action by Chamseddine and Connes. Daniel gave relevant contributions to the field with many expositions of noncommutative geometry, cyclic cohomology and the theory of the Standard Model, with several research papers, alone and in collaborations (notably, with Thomas Schucker, Bruno Iochum and Robert Coquereaux).

In 1984, Daniel was awarded the Prix Ampère de l'Académie des Sciences of France and was a corresponding member of the Göttingen Academy of Sciences and of the Austrian Academy of Sciences, and a member of the German National Academy of Sciences Leopoldina.

But many other aspects of Daniel's personality have to be mentioned. First of all, there was his passionate dedication to developing an important scientific centre in Marseille-Luminy, to which he devoted an enormous proportion of his time and energy, with unselfish sacrifice of his own research activity, starting with the famous May 1968 in which he was involved hand-in-hand with the students.

But his thoughts were not only devoted to mathematics and mathematical physics; he strongly desired a modern school of music and musical research, the central personality of which he saw in Jean-Claude Risset, whose appointment was obtained as a professor at the University of Aix-Marseille in 1979.

At the same time, he devoted much energy into the promotion of a painter living in Bandol, behind the hill in a place from where the sea was not visible, a place not visible to the flaw of Summer tourists. That painter was Roger van Rogger, who unfortunately died in 1983, probably poisoned by the colours he would prepare himself and use to paint with bare hands for many years. But he remained all those years working in his country house, largely built with his own hands (as was the case for his atelier in the middle of his lands). It was Daniel's activity in avoiding the realisation of destructive projects, like the choice of route threatened for the construction of a highway, as well as the new location of Bandol's cemetery, that prevented the destruction of Roger's atelier.

Instead, in that location, a "Fondation van Rogger" exists and can be visited, with the astonishing collection of Roger's paintings, and this is certainly due, to a large extent, to the efforts of Daniel; time will eventually place that unfortunate painter at the high rank that he deserved.

Daniel's conversations were intertwined with splendid sentences that one could often not understand, whether they were his own thoughts or quotes. Maybe, if asked, he would not always have been able to explain himself, since his culture really became his flesh and blood. Quotes, for instance, went from Blaise Pascal to Denis Diderot to Paul Valery to Albert Einstein.

And he loved to tell stories about famous scientists – living or of the past – but also lovely anecdotes about himself and about his father, the physicist Nobel Laureate Alfred Kastler. He was a delightful person, as was Daniel's father-in-law, a professor in mineralogy and petrography called Bruno Sander from Innsbruck. About him, too, Daniel had many interesting stories, besides those I had heard from Sander himself.

Many of his personal and original thoughts are collected in a little book, which is half joking and half serious, "Éphémérides de Kashtlerus", Société des Écrivains, 2005. We like to remember him lit by his enthusiasm and his love for science, his marvellous interest for scientific intelligence as well as for the human richness of other people, his immense generosity, yet always half joking, with ideas, science and culture, and the way he liked to smile in the company of friends and his beloved family, his children Nora, Danielle (Poppi) and Bruno, and his adored wife Lisl especially (to whom many of his elegant, delicate and deeply heartfelt unpublished poems were dedicated).

Dear Daniel. Nothing can ever fill the huge empty space you have left; we will have to live with that but also with the vivid memory of your precious friendship.

Sergio Doplicher,
Università di Roma "La Sapienza", Italy

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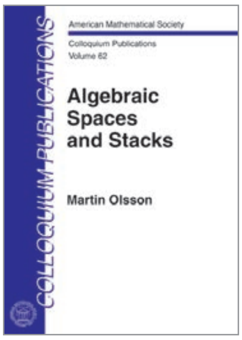

ALGEBRAIC SPACES AND STACKS

Martin Olsson, University of California

An introduction to the theory of algebraic spaces and stacks intended for graduate students and researchers familiar with algebraic geometry at the level of a first-year graduate course. The first several chapters are devoted to background material including chapters on Grothendieck topologies, descent, and fibered categories. Following this, the theory of algebraic spaces and stacks is developed. The last three chapters discuss more advanced topics including the Keel-Mori theorem on the existence of coarse moduli spaces, gerbes and Brauer groups, and various moduli stacks of curves.

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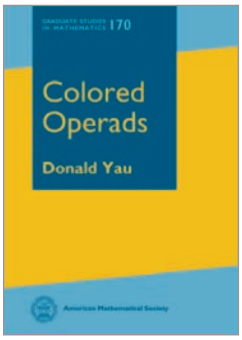

COLORED OPERADS

Donald Yau, The Ohio State University at Newark

The subject of this book is the theory of operads and colored operads, sometimes called symmetric multicategories. A (colored) operad is an abstract object which encodes operations with multiple inputs and one output and relations between such operations. The theory originated in the early 1970s in homotopy theory and quickly became very important in algebraic topology, algebra, algebraic geometry, and even theoretical physics (string theory). Topics covered include basic graph theory, basic category theory, colored operads, and algebras over colored operads. Free colored operads are discussed in complete detail and in full generality.

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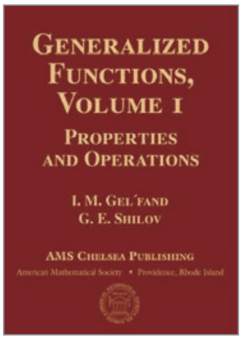

GENERALIZED FUNCTIONS

I.M. Gel'fand et al

The first systematic theory of generalized functions (also known as distributions) was created in the early 1950s, although some aspects were developed much earlier, most notably in the definition of the Green's function in mathematics and in the work of Paul Dirac on quantum electrodynamics in physics. The six volume collection, *Generalized Functions*, written by I. M. Gel'fand and co-authors and published in Russian between 1958 and 1966, gives an introduction to generalized functions and presents various applications to analysis, PDE, stochastic processes, and representation theory.

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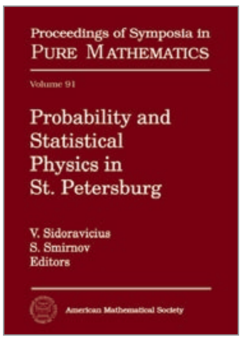

PROBABILITY AND STATISTICAL PHYSICS IN ST. PETERSBURG

Edited by V. Sidoravicius, Courant Institute and NYU–Shanghai & S. Smirnov, University of Geneva and St. Petersburg State University

Brings the reader to the cutting edge of several important directions of the contemporary probability theory, which in many cases are strongly motivated by problems in statistical physics. The authors of these articles are leading experts in the field and the reader will get an exceptional panorama of the field from the point of view of scientists who played, and continue to play, a pivotal role in the development of the new methods and ideas, interlinking it with geometry, complex analysis, conformal field theory, etc., making modern probability one of the most vibrant areas in mathematics.

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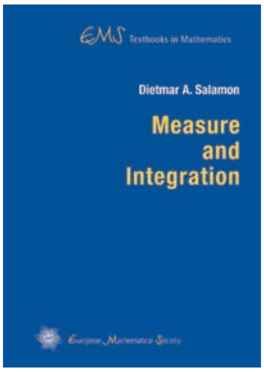
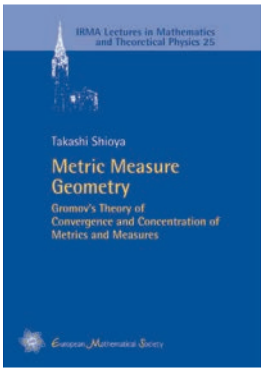


Dietmar A. Salamon (ETH Zürich, Switzerland)
Measure and Integration (EMS Textbooks in Mathematics)

ISBN 978-3-03719-159-0. 2016. 363 pages. Hardcover. 16.5 x 23.5 cm. 48.00 Euro

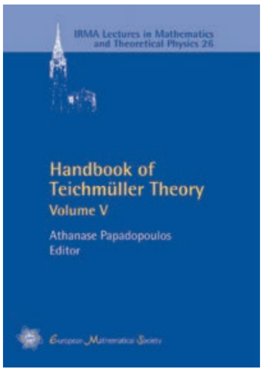
The book is intended as a companion to a one semester introductory lecture course on measure and integration. After an introduction to abstract measure theory it proceeds to the construction of the Lebesgue measure and of Borel measures on locally compact Hausdorff spaces, L^p spaces and their dual spaces and elementary Hilbert space theory. Special features include the formulation of the Riesz Representation Theorem in terms of both inner and outer regularity, the proofs of the Marcinkiewicz Interpolation Theorem and the Calderon–Zygmund inequality as applications of Fubini's theorem and Lebesgue differentiation, the treatment of the generalized Radon–Nikodym theorem due to Fremlin, and the existence proof for Haar measures. Three appendices deal with Urysohn's Lemma, product topologies, and the inverse function theorem.
The book assumes familiarity with first year analysis and linear algebra. It is suitable for second year undergraduate students of mathematics or anyone desiring an introduction to the concepts of measure and integration.



Takashi Shioya (Tohoku University, Sendai, Japan)
Metric Measure Geometry. Gromov's Theory of Convergence and Concentration of Metrics and Measures
(IRMA Lectures in Mathematics and Theoretical Physics, Vol. 25)

ISBN 978-3-03719-158-3. 2016. 194 pages. Hardcover. 17 x 24 cm. 42.00 Euro

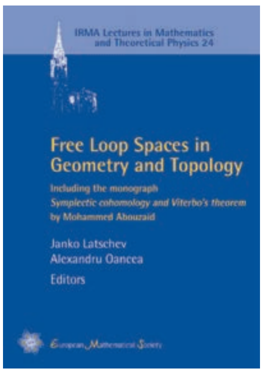
This book studies a new theory of metric geometry on metric measure spaces, originally developed by M. Gromov in his book "Metric Structures for Riemannian and Non-Riemannian Spaces" and based on the idea of the concentration of measure phenomenon due to Lévy and Milman. A central theme in this text is the study of the observable distance between metric measure spaces, defined by the difference between 1-Lipschitz functions on one space and those on the other. The topology on the set of metric measure spaces induced by the observable distance function is weaker than the measured Gromov–Hausdorff topology and allows to investigate a sequence of Riemannian manifolds with unbounded dimensions. One of the main parts of this presentation is the discussion of a natural compactification of the completion of the space of metric measure spaces. The stability of the curvature-dimension condition is also discussed.
This book makes advanced material accessible to researchers and graduate students interested in metric measure spaces.



Handbook of Teichmüller Theory, Volume V (IRMA Lectures in Mathematics and Theoretical Physics, Vol. 26)
Athanase Papadopoulos (Université de Strasbourg, France), Editor

ISBN 978-3-03719-160-6. 2016. 596 pages. Hardcover. 17 x 24 cm. 88.00 Euro

This volume is the fifth in a series dedicated to Teichmüller theory in a broad sense, including the study of various deformation spaces and of mapping class group actions. It is divided into four parts: Part A: The metric and the analytic theory; Part B: The group theory; Part C: Representation theory and generalized structures; Part D: Sources.
The topics that are covered include identities for the hyperbolic geodesic length spectrum, Thurston's metric, the cohomology of moduli space and mapping class groups, the Johnson homomorphisms, Higgs bundles, dynamics on character varieties, and there are many others.
Besides surveying important parts of the theory, several chapters contain conjectures and open problems. The last part contains two fundamental papers by Teichmüller, translated into English and accompanied by mathematical commentaries.
The chapters, like those of the other volumes in this collection, are written by experts who have a broad view on the subject. They have an expository character (which fits with the original purpose of this handbook), but some of them also contain original and new material.
The Handbook is addressed to researchers and to graduate students.



Free Loop Spaces in Geometry and Topology. Including the monograph *Symplectic cohomology and Viterbo's theorem* by Mohammed Abouzaid (IRMA Lectures in Mathematics and Theoretical Physics, Vol. 24)
Janko Latschev (University of Hamburg, Germany) and Alexandru Oancea (Université Paris 6, France), Editors

ISBN 978-3-03719-153-8. 2015. 500 pages. Hardcover. 17 x 24 cm. 78.00 Euro

One of the main purposes of this book is to facilitate communication between topologists and symplectic geometers thinking about free loop spaces. It was written by active researchers coming to the topic from both perspectives and provides a concise overview of many of the classical results, while also beginning to explore the new directions of research that have emerged recently. As one highlight, it contains a research monograph by M. Abouzaid which proves a strengthened version of Viterbo's isomorphism between the homology of the free loop space of a manifold and the symplectic cohomology of its cotangent bundle, following a new strategy.
The book grew out of a learning seminar on free loop spaces held at Strasbourg University and should be accessible to a graduate student with a general interest in the topic. It focuses on introducing and explaining the most important aspects rather than offering encyclopedic coverage, while providing the interested reader with a broad basis for further studies and research.