Predictable processes: properties

- A predictable process $H(t)$ is adapted to the filtration $\mathcal{F}_t$

- If $H(t)$ is left continuous adapted process then $H(t)$ is a predictable process

- A measurable deterministic process is predictable

- $H(t)$ is a predictable process if and only if $H(t)$ is $\mathcal{F}_{t-}$ measurable

Integral process: properties

- $\int_0^t H(s,\omega)\,dB(s,\omega) = \lambda \int_0^t H(s,\omega)\,dA(s,\omega)$ if $B(s) = \lambda A(s)$ and $\lambda \geq 0$

- $\int_0^t H(s,\omega)\,dB(s,\omega) = \int_0^t H(s,\omega)\,dA_1(s,\omega) + \int_0^t H(s,\omega)\,dA_2(s,\omega)$
if $B(s) = A_1(s) + A_2(s)$

- $t \mapsto \int_0^t H(s,\omega)\,dA(s,\omega)$ is CADLAG

- $t \mapsto \int_0^t H(s,\omega)\,dA(s,\omega)$ is continuous if $t \mapsto A(t,\omega)$ is continuous

- $\int_0^t H(s)\,dA(s)$ is adapted to the filtration $\mathcal{F}_t$

- $\int_0^t H(s)\,dA(s)$ is predictable if $A(t)$ is predictable

Martingale theory

Doob-Meyer: $X(t)$ is a submartingale

- There exists a unique nondecreasing predictable process $A(t)$
- Under an integrability condition: $M(t) = X(t) - A(t)$ is a martingale and heuristic $\mathbf{E}(dM(t)|\mathcal{F}_{t-}) = 0$ and $dA(t) = \mathbf{E}(dX(t)|\mathcal{F}_{t-})$

$H(t)$ is a predictable process and $X(t)$ is a nondecreasing process

- Doob-Meyer: $M(t) = X(t) - A(t)$ is a martingale (under an integrability condition)
- $\mathbf{E}\Big(\int_0^t |H(s)|\, dA(s)\Big) < \infty$: $\int_0^t H(s)\, dM(s)$ is a martingale
- $\mathbf{E}\Big(\int_s^t H(u)\, dX(u)\Big|\mathcal{F}_s\Big) = \mathbf{E}\Big(\int_s^t H(u)\, dA(u)\Big|\mathcal{F}_s\Big)$
  (A general formula of Thm 6.1 and 8.1 in CMM)

Square integrable martingale

$M(t)$ square integrable martingale and a FV-process:

- Doob-Meyer: there exists a predictable nondecreaing process $\langle M\rangle(t)$
- Under some integrability condition: $M^2(t) - \langle M\rangle(t)$ is a martingale and $d\langle M\rangle(t) = \mathrm{var}(dM(t)|\mathcal{F}_{t-})$

$H(t)$ is a predictable process such that $\mathbf{E}\Big(\int_0^t H^2(s)\,d\langle M\rangle(s)\Big) < \infty$:

- $\langle\int_0^{\cdot} H(s)\,dM(s)\rangle(t) = \int_0^t H^2(s)\,d\langle M\rangle(s)$
- $\big(\int_0^t H(s)\,dM(s)\big)^2 - \int_0^t H^2(s)\,d\langle M\rangle(s)$ is a martingale

Counting process

$(N(t))_{t\geq 0}$ is a (one-dimensional) counting process if

- $N(t)$ is $\mathcal{F}_t$-adapted
- $N(0) = 0$ and $t \mapsto N(t,\omega)$ is nondecreasing, right continuous and takes integer values
- $\Delta N(t) = N(t) - N(t-)$ is either 0 or 1 for all $t \geq 0$

$N(t) = (N_1(t), \ldots, N_k(t))$ is a k-dimensional counting process if

- $N_h(t)$ is a counting process for $h = 1, \ldots, k$
- $\sum_{i\neq j} \Delta N_i(t)\, \Delta N_j(t) = 0$ for all t

Construction of predictable compensator

According to a result for the Doob-Meyer composition if

$$\sum_{i=1}^{l2^{k-n}} \mathbf{E}\Big(N\Big(i2^{-k}\Big) - N\Big((i-1)2^{-k}\Big)\Big|\mathcal{F}_{(i-1)2^{-k}}\Big)$$

has a limit in L^1 for $l, n \geq 1$ for $k \to \infty$

and the constructed process satisfies

- nondecreasing
- predictable

then it is a predictable compensator for the counting process