An Introduction to the Theory of Hitches and Knots

Jeremy Stolarz
16-741 - Mechanics of Manipulation

Abstract

A review of the basic theory of hitches and knots used to determine whether a given hitch will hold or slip. The topology of the knot, along with the coefficients of friction, determine the constraints that must be satisfied in order for a hitch or knot to withstand an arbitrarily strong pull.

1 Introduction

For millennia humans have been using knots and hitches for all sorts of tasks, from sailing and building to hanging clothes and tying things onto tops of cars. However, only within the last few decades have knots and hitches been analyzed mathematically to see precisely under what conditions a knot will hold. Whereas before one just had an intuition that an extra loop around a pole could make things more secure, know one can understand the reasons why.

This paper will explain the basic method, first proposed by Bayman [1], by which hitches and knots can be modeled mathematically, and how these models are used to predict under what conditions a knot or hitch will hold. To understand knots we must first examine hitches, so that model will be shown first along with some example analyses of hitches. The extension of the theory to knots will then be presented.

2 Composition of a Hitch

A hitch is basically a rope wrapped around a pole such that one end of the rope is tucked under one or more turns of the rope around the pole. A successful hitch is one that can resist a strong force applied to the other end of the rope without slipping. This is determined in large part by the friction of the rope against the pole and the topology of the hitch.

When a rope lays across a pole, the frictional force between them is proportional to the surface area of the contact, thus the more wraps around a pole, the stronger the frictional force. Figure 1 shows a simple case where a rope has two tensions on it, $T_1$and T_2. Let μbe the coefficient of friction between the rope and the pole. The pole can apply a tangential force on the force such that the rope will not slip when one tension is greater than the other. For example, if $T_2 > T_1$, then the rope will not slip as long as the following inequality holds:

$$T_2 \leq T_1 e^{\mu\theta} \qquad (T_2 > T_1). \tag{1}$$

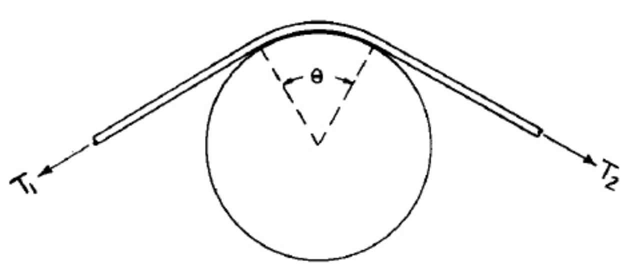

Figure 1: Friction between the pole and the rope allow for a difference between tensions $T_1$and T_2.

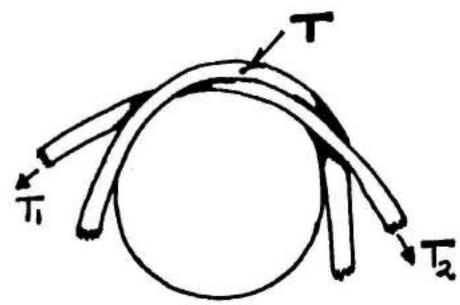

Figure 2: Upper segment with tension T pinches rope below it, splitting it into two segments with tensions $T_1$and T_2.

Here $\theta$is the angle (in radians) subtended at the axis of the pole by the arc along which the rope and pole are in contact. Since $\theta$is a multiple of 2π, (1) can be rewritten as

$$T_2 \leq T_1 \epsilon^n, \tag{2}$$

where n is the number of turns around the pole and $\epsilon = \mathrm{e}^{2\pi\mu}$.

Figure 2 shows the situation where a segment of rope T crosses over another one and squeezes the lower segment against the pole. The frictional forces involved here allow for a difference in tensions on either side of the crossing. The maximum value of this difference is proportional to the force exerted by T on the lower segment perpendicular to the surface of the pole, which in turn is proportional to the tension in T:

$$T_2 \leq T_1 + \eta F_T \qquad (T_2 > T_1). \tag{3}$$

The constant η depends on the coefficients of friction between the rope and pole and between the rope segments as well as upon the ratio of the diameters of the rope and pole. It is important to note the following assumptions have been made here: 1) the diameter of the rope is much smaller than the diameter of the pole, 2) the friction between the rope segments is much less than the friction between the rope and pole. These two assumptions lead to the third assumption that tension does not change in a segment when it crosses over another one.

We can now use this model to analyze a simple but commonly used hitch.

3 The Clove Hitch

Figure 3 shows a clove hitch. Let the tensions increase as we follow the rope around the hitch such that $t_0 \leq t_1 \leq t_2 \leq t_3 \leq t_4$. Using the above equations yield the following conditions that must be met for the hitch not to slip:

$$t_1 \leq t_0 + \eta t_2 \tag{4}$$
$$t_2 \leq \epsilon t_1 \tag{5}$$
$$t_3 \leq \epsilon t_2 \tag{6}$$
$$t_4 \leq t_3 + \eta t_2 \leq (\epsilon + \eta) t_2 \tag{7}$$

Combining these inequalities we obtain

$$t_2(1 - \eta\epsilon) \leq \epsilon t_0. \tag{8}$$

We now have two cases corresponding to low friction ($\eta\epsilon < 1$)and high friction ($\eta\epsilon > 1$). When there is low friction the value on the left-hand side of (8) is positive, so we can combine (8) and (7), resulting in

$$t_4 \leq \frac{\epsilon(\epsilon + \eta)}{1 - \eta\epsilon} t_0. \tag{9}$$

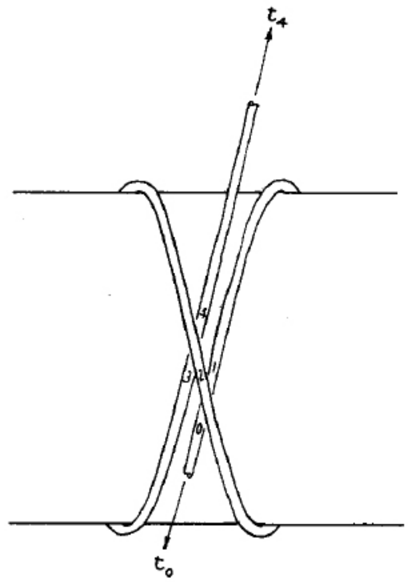

Figure 3: The clove hitch.

Thus the hitch will hold as long as $t_4$is not greater than $t_0$by the factor given in (9). When there is high friction in the hitch, the left-hand side of (8) is negative, making (8) valid for any non-negative tensions in $t_0$and t_2. Thus, in this situation, the clove hitch will hold against any arbitrarily strong pull on one end while the other end remains loose.

4 The General Case for a Hitch

4.1 Method

The general case for a hitch is best resolved using matrices based on the system of inequalities describing the various tensions in a hitch. The method is as follows:

(a) The hitch is split into segments starting at the free end (the 0th segment). The first segment begins where the free end passes under one of the turns in the hitch. It continues until it passes under another turn, marking the end of the first segment and the beginning of the second. The hitch continues to be split in this manner, where each segment begins where the rope passes under one of the turns of the hitch. Let T_i be the tension in the rope at the beginning of the ith segment. The final segment, which has the highest tension, shall be denoted segment q.

(b) Let n_i be the number of turns around the pole made by the ith segment. The tension at the end of the ith segment is therefore $\epsilon^{n_i}T_i$.

(c) Define $b_i$as the number of the segment under which the ith segment begins.

(d) Let m_i be the number of turns from the start of segment b_i to the place where it passes over the ith segment. The tension at the beginning of segment b_i is T_{b_i}, thus the tension where it passes over the ith segment is $\epsilon^{m_i}T_{b_i}$. Therefore, from (2), we get

$$T_i \leq \epsilon^{n_{i-1}}T_{i-1} + \eta\epsilon^{m_i}T_{b_i} \qquad (T_{i-1} \leq T_i) \tag{10}$$

for $i = 2, 3, ..., q$. See Figure 4. If $T_0$is the tension of the free end of the rope (just before the first segment), then we can rewrite the above inequality

$$\sum_{j=1}^{q} A_{ij}T_j \leq \delta_{i,1}T_0 \qquad (T_{i-1} \leq T_i) \tag{11}$$

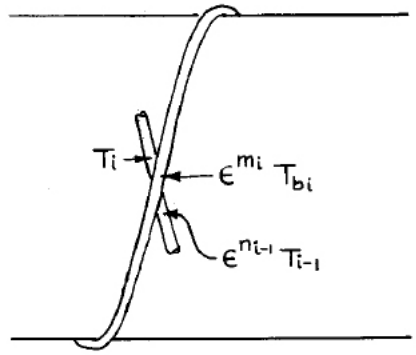

Figure 4: Conditions at the ith segment of rope.

where the matrix A is defined by

$$A_{ij} \equiv B_{ij} - \eta C_{ij}, \tag{12}$$
$$B_{ij} \equiv \delta_{ij} - \epsilon^{n_j} \delta_{i-1,j}, \tag{13}$$
$$C_{ij} \equiv \epsilon^{m_i} \delta_{b_i j}. \tag{14}$$

The Kronecker delta, δ_{ij}, is defined as $\delta_{ij} = 1$ when $i = j$ and $\delta_{ij} = 0$ if $i \neq j$.

We can now find the determinant of A, set it equal to zero and solve for what conditions must be met for the hitch to hold. Let η_c be the value of η which causes the determinant to be zero. In other words

$$\det A > 0 \qquad \text{if } 0 \le \eta < \eta_c \tag{15}$$
$$\det A = 0 \qquad \text{if } \eta = \eta_c \tag{16}$$
$$\det A < 0 \qquad \text{if } 0 \le \eta_c < \eta \tag{17}$$

It is shown in <citation> that if the determinant is greater than zero, the hitch will only hold for certain values of η and if $T_0 > 0$. If the determinant is less than zero than the hitch will hold against an arbitrary force.

4.2 Ground-line Hitch Example

The ground-line hitch (Fig. 5) is very similar to the clove hitch and will be analyzed using the above general method. The A matrix for this hitch is

$$A = \begin{bmatrix} 1 & -\eta\epsilon^1 \\ -\epsilon^1 - \eta\epsilon^0 & 1 \end{bmatrix}, \tag{18}$$

so

$$\det A = 1 - \eta\epsilon(\eta + \epsilon). \tag{19}$$

Therefore, the ground-line hitch holds when $\eta\epsilon(\eta + \epsilon) \ge 1$. Compared to the clove hitch, there is a range of η

$$\frac{1}{\epsilon(\eta + \epsilon)} < \eta < \frac{1}{\epsilon} \tag{20}$$

where the ground-hitch will hold fast whereas the clove hitch will slip. Thus this analysis shows that the ground-line hitch is superior to the clove hitch.

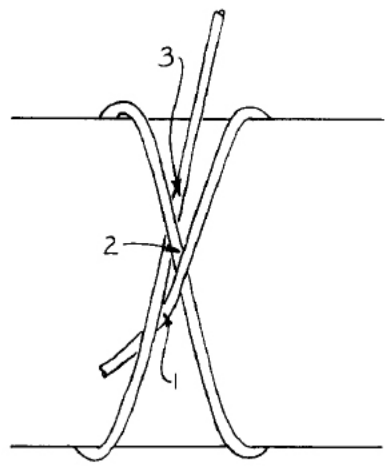

Figure 5: The gound-line hitch.

5 Application to Knots

Maddocks and Keller [2] show that many of the principles that apply to knots are the same as those that apply to hitches. As with hitches, the tensions of the segments of a knot must increase from the loose end to the loaded end in order for the hitch to hold. This variance in tension is caused by the friction between two segments of the same rope as well as the friction between two different pieces of rope (assuming the knot is used to tie two pieces of rope together). In essence, when applying the above model to knots, one treats the piece of rope being wrapped around as a pole. We know analyze the square knot (Fig. 6), most often used to tie two pieces of rope together.

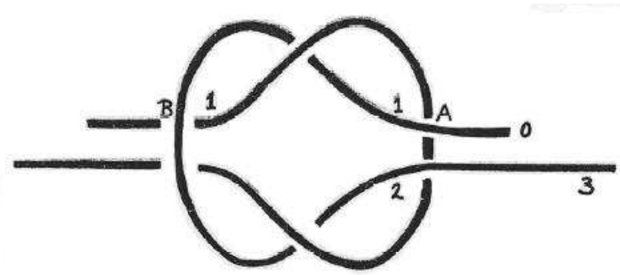

Figure 6: The Square Knot.

Assume the tension jumps from zero to T_1just left of the crossover at point A. Let N be the normal force that keeps the two ropes together at point A. Then Coulomb's law gives us

$$T_1 \leq 2\mu N. \tag{21}$$

The factor 2 comes from the fact that the segment of rope being analyzed is in contact with two other pieces of rope at point A. The normal force N is essentially the tension in the vertical rope at A. To find N we assume the tension in the segment increases from $T_1$by the factor $e^{\mu\pi}$ at point B, which is analogous to the rope being wrapped with a half turn around a cylinder (i.e. along an arc which turns through $\pi radians$). Therefore

$$N = T_1 e^{\mu\pi}, \tag{22}$$

which combined with (21) and solved for μgives

$$\mu \geq \frac{1}{2} e^{\pi\mu}.$$

This is the condition on $\mu$for the square knot to hold. Since a square knot is symmetrical, the same condition is found when analyzing the jump between $T_2$and T_3.

6 Conclusion

The models in this paper serve to give a first-order understanding of how knots and hitches work. They take into account the largest factors such as friction created by turns and crossings but gloss over various details. To make a more accurate model, one could take into account the ratio of the diameters between pole and rope, the weight of the rope, the change in tension in the top segment of a cross-over, non-cylindrical poles, and many other subtleties. In [2] after the basic analysis of knots the authors go on to make models of the first three assumptions mentioned. Work continues to be performed on perfecting our models of this very important tool.

References

[1] Bayman, B. F. *Theory of Hitches*, Amer. J. Phys., 45 (1977), pp. 185-190.

[2] Maddocks, J.H. and Keller, J. B. *Ropes in Equilibrium*, SIAM J Appl. Math., 47 (1987), pp. 1185-1200.

[3] Krauel, M. The Mathematical Theory of Hitches, Paper presented in Introduction to Knot Theory course, University of California, Los Angeles (2005).