# Homology Representations

# of

# Braid Groups

*by*

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# HOMOLOGY REPRESENTATIONS OF BRAID GROUPS

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# Abstract

In this thesis, a topological construction of Hecke algebra representations associated with two-row Young diagrams is presented. These are the representations which appear in the one-variable Jones polynomial, looked at from the braid point of view. The construction used obtains these representations from monodromy representations on a vector bundle whose fibre is the homology of a complex manifold with a suitable, non-trivial, abelian local coefficient system. Alternatively, they are expressed as the monodromy representations obtained from the solutions of suitable systems of differential equations.

In the work of Tsuchiya & Kanie and Kohno, another construction of these representations can be found, in terms of the monodromy of $n$-point functions in conformal field theory. A comparison between the two constructions is made, which leads to a detailed correspondence, and the implications of this, in the context of conformal field theory, are very briefly discussed.

# Contents

# Acknowledgements

# 1: Introduction and overview

In this Thesis, it will be shown how Hecke algebra representations associated with two-row Young diagrams can be constructed topologically, using monodromy representations on homology with a non-trivial local coefficient system. Tsuchiya & Kanie have given (in [TK]) another construction for these same representations, but from the point of view of conformal field theory. These two constructions will be compared, and, using a Theorem of Kohno's, will be identified. The correspondence thus obtained leads to some speculation on the wider relations between conformal field theory and topology.

The main themes which motivated the work were the twin links with knot theory and physics. In §1.1, the scene will be set from the knot theory point of view. There are many different influences from physics which bear on the work, but the one which initially motivated it was the paper of Tsuchiya & Kanie [TK]. This conformal field theory aspect will be discussed in more detail in Chapter 8.

In this Chapter, some of the background material leading up to the results put forward in this Thesis, will be discussed. The interest in Hecke algebra representations comes from their presence in the Jones invariant of knots and links. In §1.2, the basic structure of the homology representations to be constructed, will be described. The main Theorems of the Thesis are announced here, albeit without the details of the constructions involved—see Chapter 3 for such details. Finally the basic structure of the work is outlined in §1.3.

## 1.1 Background knot theory

Over its history, knot theory has benefited from contacts with many different fields—some in mathematics and some in physics—at different times. Indeed, the theory of knots was begun in an attempt to solve a

problem in physics around 1870. That problem was the modelling of atoms and their spectra, and the model proposed by Lord Kelvin and J.C. Maxwell (see [Max], [Th], [Kn]) was known as the *vortex theory*. In this model , the presence of matter was considered as equivalent to the existence of a singularity in the motion of the æther. Such singularities, if point-like, can be identified and classified by the knots corresponding to the flow-lines of the æther in the vicinity of the singularity. Thus it was thought that one could classify atoms by classifying the corresponding knots, namely, all those knots satisfying some constraint associated with the physical viability ('dynamic stability' as it was then called) of the æther flow represented by the knot. The spectrum of the atom was thought to correspond to different modes of vibration of the knot, very much as present day string theory seeks to explain the energy level structure of elementary particles.

The task of classifying knots, up to isotopy, was started by P.G. Tait (see [Ta]). By largely combinatorial methods, Tait succeeded in classifying all knots with ten or fewer crossings. Until the last few years, purely topological methods were used to make progress on the classification of knots (see [R]). For example, the first knot polynomial discovered, the *Alexander polynomial* [Al 1] can be expressed in terms of the first homology of the infinite cyclic cover of the knot complement (we discuss this from our point of view in §5.1). Various techniques have been used, including considering manifolds which are covering spaces branched over the knot. Seifert surfaces have also played an important role (see [R]).

Another angle from which the study of knots and links has been approached is via braids. In §2.1, more details will be given of this approach, but for now we will only mention that for each $n$, there is an associated braid group $B_n$; and from any braid, a knot (or more correctly, a link) can be obtained by the operation of closure. Thus, the specification of an invariant, defined on the collection of links, up to isotopy, is equivalent to a specification of a representation of $B_n$, for each $n$, with certain constraints to ensure 'compatibility' of these representations. The classification of links can thus be reduced to the word problem on the braid groups. An algorithm for solving this word problem was produced by W.Haken. However, although in theory this solves the classification problem for links, in practice only knots up to 13 crossings can be classified using present computers. This is due to the complexity of the algorithm.

However, the braid approach to knots goes far beyond producing an algorithm for their classification. In 1984, while working in the theory of von Neumann algebras, V. Jones discovered a new invariant of links (see

[Jo 1]) and [Jo 2]). This invariant is a one-variable polynomial just like the Alexander polynomial. However, unlike the Alexander polynomial, the Jones polynomial is *chiral*; that is, it is capable of distinguishing some knots from their mirror images. This one-variable polynomial was soon extended to a two-variable polynomial, which also contained the Alexander polynomial within itself, as a one-dimensional specialisation. See [FYHLMO] for more details.

As a result of contributions from many branches of mathematics and physics, it was seen that the original one-variable Jones polynomial could be viewed as the first in a series of one-variable polynomial link invariants. For each Lie group $G$, representation $\rho$, and integer $k$ (known as the *level*), such a link invariant can be defined. The original Jones polynomial is then associated with $G = SU(2)$ and the standard two-dimensional (vector) representation of $G$. At level $k$, the invariant obtained is simply:

$$ V_L \left( \exp \left( 2\pi i / (k + 2) \right) \right) $$

where $V_L$ is the (one-variable) Jones polynomial of link $L$. Areas that contributed to this result include statistical mechanics (see [Jo 3], [Ba]) and quantum groups (see [D], [Ji], [L], [Tu], just to name a few of the many relevant references in this area), and their relations to the Yang-Baxter equation.

Many viewpoints on these invariants exist, and it is the process of attempting to understand the relations between them which has recently led to many new insights in mathematics. Until recently, however, all such approaches *either* used a two-dimensional projection of a knot, with over and under crossings, or used the braid approach (see §2.1), or a combination of these twin viewpoints. Both types of method involve viewing the knot in an essentially two-dimensional way. Since a knot (or link) intrinsically lives in three dimensions, it seems that one ought to be able to express invariants, such as the Jones polynomial, naturally in terms of the link as embedded in $S^3$.

Recently, E.Witten [W] interpreted the Jones polynomial and its generalisations, in terms of an expectation value in a *topological quantum field theory*. For any Lie group $G$, and level $k$, he defines a topological quantum field theory in which, for a three-dimensional manifold $M$, the fields are $G$-connections on $M$. That is, he constructs the space:

$$ \mathcal{A} = \left\{ G\text{-connections on } M \right\} . $$

An element $A \in \mathcal{A}$ is a Lie algebra valued 1-form on a trivial $G$-bundle over $M$. For any $A \in \mathcal{A}$, the action $S(A)$ is defined to be given by the Chern-Simons functional:

$$S(A) = \int_M \text{tr}(A \wedge dA + {}^2\!/_3 A \wedge A \wedge A)$$

where 'tr' denotes a suitably normalised inner product on $\mathfrak{g}$, the Lie algebra associated with the Lie group $G$. The partition function of this model is now constructed as:

$$Z(M) = \int_{\mathcal{A}/\mathcal{G}} e^{ikS(A)} \mathcal{D}A$$

where $\mathcal{A}/\mathcal{G}$ is the moduli space obtained by dividing out by the action of the gauge group $\mathcal{G}$. For a three-manifold $M$, $Z(M)$ is now meant to be an invariant of $M$ up to homotopy.

To obtain invariants of links, and not just invariants of three-manifolds, one starts with the general situation of a link $L$ embedded in a three-manifold $M$. Together with the data above, namely $G$ and $k$, it is also necessary to specify a representation of $G$ associated with each component of the link, together with framings of $M$ and $L$. For a given connection $A \in \mathcal{A}$ and representation $\rho$ of $G$, a *Wilson loop* can be defined for a component of $L$, by taking the trace of the holonomy of $A$ around the loop, in the given representation $\rho$. The expectation value of the product of the Wilson loops associated with the different components of $L$:

$$Z(M, L) = \int_{\mathcal{A}/\mathcal{G}} \prod \text{(Wilson loops)} \ e^{ikS(A)} \mathcal{D}A$$

is now meant to give an invariant of the pair $(M, L)$. Leaving aside any question as to the validity of such a functional integral, for the moment, this invariant gives a complex number, for each choice of $G$, $k$ and an appropriate collection of representations of $G$. This is Witten's formulation of (generalised) link invariants.

The Jones polynomial is obtained from the above formulation in the special case $G = SU(2)$, associating the vector representation to each component of the link. The value of $Z(S^3, L)$ obtained with this data turns out to be $V_L(q)$ where $q = \exp\left({}^{2\pi i}\!/\!(k+2)\right)$. Since $V_L$ is a polynomial, it is uniquely determined by the collection of its values at such roots of unity, $q$. Note that this is the very same evaluation of $V_L$ mentioned above in connection with quantum groups, etc.

However, there are many problems with this functional integral formulation of $V_L$. Firstly, it is not clear what $\mathcal{D}A$ means, as $\mathcal{A}/\mathcal{G}$ is an infinite-dimensional space. There are several approaches which attempt to

resolve these problems—see for example, [S 1], [H 1] and [H 2]. In the approach following along the lines of [S 1], the invariant is extended to a functor $Z$, which in the simplest case associates:

(a) to a Riemann surface, $\Sigma$, with marked points $\{p_i\}$, and representations $\{\lambda_i\}$ of $G$, one associated with each marked point, a vector space $Z\big(\Sigma, \{p_i\}, \{\lambda_i\}\big)$;

(b) to a three-dimensional manifold $M$, containing a curve $L$, together with representations of $G$, one assigned to each component of $L$, a vector $Z(M, L)$ in the vector space associated with the boundary data $(\partial M, \partial L)$.

It is assumed in (b) that $L$ is such that $\partial L \subseteq \partial M$, and the representations of $G$ associated with the marked points (in $\partial L$) on $\partial M$, are induced from those associated with components of $L$ by assigning $\rho$ or $\bar{\rho}$ to a boundary point of a component of $L$ to which the representation $\rho$ is attached. Whether it is $\rho$ or $\bar{\rho}$ which is picked, depends on the relative orientations, as indicated in Fig. 1.1.

component of $L$

$\rho$

•     marked point on $\partial M$

$\bullet \rho$       (in $\partial L$)

•

$\bullet \bar{\rho}$

$M$

$\partial M$
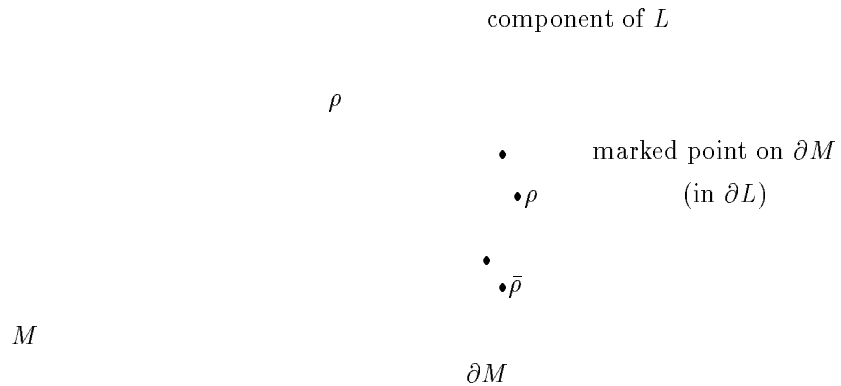
Figure 1.1

Many details have been omitted from the above structure; for example, the precise manner in which the framings on $M$ and $L$ enter the theory. The functor $Z$ must satisfy certain properties, which relate $Z$'s on manifolds whose orientations are opposite, or which have been obtained by surgery on some other manifold. For more details, see [S 2] and [S 3]. To define a functor $Z$ satisfying all these axioms, geometric quantisation

and loop groups have been employed, amongst other techniques. When $M$ is a closed manifold containing the link $L$, $Z(M, L)$ is a vector in the vector space associated with $\Sigma = \emptyset$. By one of the axioms, this vector space is $\mathbf{C}$, so that $Z(M, L)$ is a complex number, namely the invariant required.

There is another relationship with physics of interest to the present work—namely that with conformal field theory. In [TK], Tsuchiya & Kanie obtain representations of the braid group by considering conformal field theory on $\mathbf{P}^1$, the complex projective line. This work will be discussed in more detail in Chapter 8, but for now it suffices to say, that their construction also makes essential use of a punctured complex plane, since they construct $n$-point functions on such a plane. The representations which they obtain are the building blocks out of which the one-variable Jones polynomial can be constructed. These representations all factor through a quotient of the braid group known as the Hecke algebra. The Hecke algebra $H_n(q)$ can be thought of as a deformation of the symmetric group, and, as such, the representations involved are deformations of the representations of the symmetric group associated with two-row Young diagrams. See §2.1 for more details. Kohno [Ko] produces a flat connection on a vector bundle whose monodromy representation gives rise to similar Hecke algebra representations.

The two connections between knot theory and physics outlined above, namely the Witten theory, and the theory of Tsuchiya & Kanie/Kohno, are themselves related. Thus, the Witten theory gives rise to Hilbert spaces which are the conformal blocks of a suitable two-dimensional conformal field theory.

The present work was motivated by the constructions of Tsuchiya & Kanie and Kohno, and contains the essentials of their methods, but put into a topological context. The differential equations found for $n$-point functions in [TK], and used extensively in that work, will be identified with those satisfied by the homology used in the present work—see Chapter 8. Hence the theory described, not only gives rise to a Hecke algebra representation by purely topological techniques, but also may be identified with the construction of [TK]. Although such an identification has not been explicitly constructed, the fact that it exists, has many interesting implications—see Chapter 9.

## 1.2 Basic structure of homology representations

The braid group $B_n$ can be constructed as the fundamental group of the configuration space, $\widetilde{X}_n$, of $n$ unordered distinct points in a plane. Thus representations of $B_n$ are defined whenever a flat connection is given, on a vector bundle over $\widetilde{X}_n$. The vector bundle which we shall use is one whose fibre, over a point $\mathbf{w} \in \widetilde{X}_n$, is the middle homology of the configuration space of $m$ points in the punctured complex plane $\mathbf{C}\backslash\{w_1, \ldots, w_n\}$, with a suitable local coefficient system, depending on a complex parameter $q$ (see Chapter 3). The representation of the braid group so obtained, does not factor through the Hecke algebra $H_n(q^{-1})$, but generically has, as its largest quotient, an irreducible Hecke algebra representation. The representation so obtained, depends on the parameter $q$, as well as the integer $m \leq [n/2]$. It turns out to give, for $m = 0, 1, \ldots, [n/2]$, precisely the characters obtained in a decomposition of the braid group representation associated with the Jones polynomial—see Chapter 2 for more details, and in particular Theorem 2.3.

The homology construction of representations of the braid groups can be viewed dually as a cohomology construction. The basic situation is essentially the same as before, except that the fibres of the vector bundle used are cohomology spaces rather than homology spaces. The braid group representations obtained from the two procedures are dual to each other, and thus instead of obtaining a representation of $B_n$ which factors through $H_n(q^{-1})$ as a quotient, one gets a sub-representation factoring through $H_n(q)$ in the cohomology picture.

The monodromy representation of the braid group which gives rise to the Hecke algebra representation required, is one special member of a larger family of braid representations. For, a monodromy representation of the braid group can be constructed using a vector bundle over $X_n$ whose fibre is homology with a local coefficient system determined by two parameters $q$ and $\alpha$. The 'special' representations are given by $\alpha = q^{-2}$ and they are reducible, containing a sub-representation, being the required Hecke algebra representation. Generically, the braid group representation for $\alpha \neq q^{-2}$ is irreducible. It turns out that, for fixed $q$, the representation can be viewed as lying on a quotient $V/V_\alpha$ of some fixed vector space $V$, as $\alpha$ varies. Subject to various degeneracy conditions, it is shown in Chapter 7 (and in particular Theorem 7.2) how a *derived representation* can be obtained from this family, near to $\alpha = q^{-2}$. This representation exists on a subspace

of $V/V_\alpha$ transverse to the variation $dV_\alpha$ of $V_\alpha$ with respect to $\alpha$, at $\alpha = q^{-2}$. Hence a braid group action exists on a reduced space to $V/V_{q^{-2}}$. In the special case $m = 2$, it is verified in §7.2 that the representation of the braid group so obtained is the required Hecke algebra representation. For larger $m$, a slightly more refined construction is needed, and this is given in §7.3. This essentially consists of taking the totally symmetric part of the derived representation of the family of braid group representations obtained from a suitable parameterisation of the space of local coefficient systems. It will be shown (Theorem 7.5) that the representation obtained contains a Hecke algebra representation corresponding to a two-row Young diagram.

In Chapter 6, the required Hecke algebra representation will be constructed, from the special member of the family used, as an explicit sub-representation. This will be done in terms of the standard basis for homology, employing repeated loops as a basis at the level of chains.

Note that whenever we refer simply to the Jones polynomial, we mean the one-variable Jones polynomial of [Jo 2], which in the context of the more general link polynomials discussed in §1.1, is associated with $G = SU(2)$ and its vector representation.

Tsuchiya & Kanie also construct these same Hecke algebra repesentations, as monodromy representations, but they use a vector bundle whose fibre is spanned by $n$-point functions. They show that these $n$-point functions satisfy a system of differential equations, from which it is possible to compute the braid group action. In Chapter 8, Theorem 8.5, it will be shown that a similar differential equation can be obtained naturally in the homology picture. Since the two differential equations give rise to the same monodromy representation, and are of the same form, with identical behaviour near $q = 1$, it is possible to conclude that they are isomorphic systems of differential equations, using a Theorem of Kohno. For more details, see Chapter 8, and in particular Theorem 8.10. As will be seen later in Chapter 8, this leads to a precise correspondence between our approach and that of [TK].

### 1.3 Overview

In this section we will give an overview of the approach given in this Thesis. Chapter 2 describes the motivations for constructing Hecke algebra representations from the context of knot theory, starting in

§2.1 with a review of the basic concepts in the Jones theory. In §2.2, the motivations for the topological constructions of this Thesis are described in more detail than it was possible to do in this Chapter. Chapter 3 gives the basic definitions and notation used throughout the work. In §3.2, all the main Theorems are presented, in a more precise way than it was possible to give in the last section.

To study the monodromy action in more detail, the geometrical constructions of Chapter 3 are translated into algebra in Chapter 4. In that Chapter, recursion relations are obtained (Theorems 4.4, 4.5) from which matrices for all the relevant actions can be evaluated, using the basis for the space of chains in terms of repeated loops. The application of these formulae is illustrated in Chapter 5, where three special cases are considered in detail.

The monodromy representations obtained depend on two parameters; a non-zero complex number $q$, and an integer $m$. Here $q$ is the same as the variable appearing in the Jones polynomial. For a given $m$, the representation of the braid group $B_n$ obtained, is generically the irreducible Hecke algebra representation associated with the Young diagram of two rows whose lengths are $n - m$ and $m$, respectively. The Jones polynomial is a combination of characters of such representations, for $m = 0, 1, \ldots, [n/2]$. The special cases $m = 1, 2$ contain some of the essential points necessary for the general case, and therefore serve as useful examples for the general theory. These cases are dealt with in §§5.1, 5.2 respectively. When $q = 1$, the local coefficient system is trivial, and the construction produces a representation of the symmetric group $S_n$. This situation is investigated in detail in §5.3.

Chapter 6 deals with the case of general $m$. A quotient space of homology, or dually a subspace of cohomology, is explicitly constructed at the level of chains. It is then shown, using the recursion formulae of Chapter 4, and induction, that the monodromy action of $B_n$ on this space gives the Hecke algebra representaions required. See Theorems 6.1 and 6.6 for explicit statements of the action of $B_n$, using the concrete basis of homology specified in terms of iterated loops.

Chapter 7 discusses the 'limiting lemma' by which a *derived* representation is obtained from a suitable family of representations of $B_n$ deforming a particular 'special' representation. The derived representation obtained is then a sub-representation of the special representation. In the case $m = 2$, this lemma (The-

orem 7.2) is used to derive the subspace on which the Hecke algebra action appears. The point of this construction is that it shows how one may obtain the Hecke algebra naturally without having to pick it out explicitly, in terms of a basis of homology. In §7.3, various problems that arise when one attempts to generalise this construction to arbitrary $m$ are discussed, and a conjectured solution is given.

Chapter 8 aims to describe how our homology approach to obtaining braid group representations is related to the work of Tsuchiya & Kanie and Kohno. In §8.1, a review of their work, or at least of that part of interest in the present context, is given. It is found that the natural correspondence is with the dual picture on cohomology, rather than homology. In §8.3, Theorem 8.5 gives a system of differential equations satisfied by flat sections of the bundle from which we construct our monodromy representations. In the rest of Chapter 8, it is seen how a comparison of this system with that used in [TK] and [Ko] gives rise to an isomorphism between the two constructions. This leads to a detailed correspondence between the two approaches (see the table at the end of §8.4), and some implications of this are discussed in §8.6.

Finally, Chapter 9 contains some, as yet, open problems which arise out of the theories discussed. Some remarks on ways in which the constructions described in earlier Chapters can be extended and connected to other approaches to Hecke algebra representations (and knot polynomials in general) are made.

# 2: Motivation

## 2.1 Review of the Jones theory

In this section we will review some of the basic theory of knots and their invariants. Although there are many other approaches to knot theory, as mentioned in §1.1, we shall be concerned, in this work, with the *braid approach*. Those results of Wenzl, which will be used in later Chapters, have also been included.

We start by defining the full braid group. Let $X_n$ be the configuration space of $n$ (ordered points) in the complex plane $\mathbf{C}$. The fundamental group, $\pi_1(X_n)$, of $X_n$ is called the *pure braid group*, $P_n$. The symmetric group $S_n$ acts naturally on $X_n$ by:

$$\sigma(z_1, \ldots, z_n) = \left(z_{\sigma(1)}, \ldots, z_{\sigma(n)}\right).$$

The orbits of this action are all of order $n!$, and:

$$X_n/S_n \cong \widetilde{X}_n$$

where $\widetilde{X}_n$ is the configuration space of sets of n, unordered, distinct points in $\mathbf{C}$. The fundamental group of this reduced space is the *full braid group*, denoted by $B_n$.

Any element of $B_n$ can thus be represented by a curve:

$$\gamma \colon [0, 1] \longrightarrow X_n$$

with $\gamma(0) = (z_1, \ldots, z_n)$ being the chosen base-point in $X_n$ and $\left[\gamma(1)\right] = \left[\gamma(0)\right]$ as elements of $\widetilde{X}_n$. There is thus some $\sigma \in S_n$ such that:

$$\gamma(1) = \sigma\left(\gamma(0)\right)$$

as points in $X_n$. There is a natural map:

$$B_n \longrightarrow S_n$$

$$[\gamma] \longmapsto \sigma$$

and the kernel of this map is precisely $P_n$, the subset of $B_n$ consisting of elements $[\gamma]$ for which $\gamma(0) = \gamma(1)$

in $X_n$. Such a curve $\gamma$ in $X_n$ can be represented by $n$ curves in $\mathbf{R}^2 \times [0, 1]$ joining two sets of $n$ points, as

indicated by an example with $n = 3$ in Fig. 2.1 below.
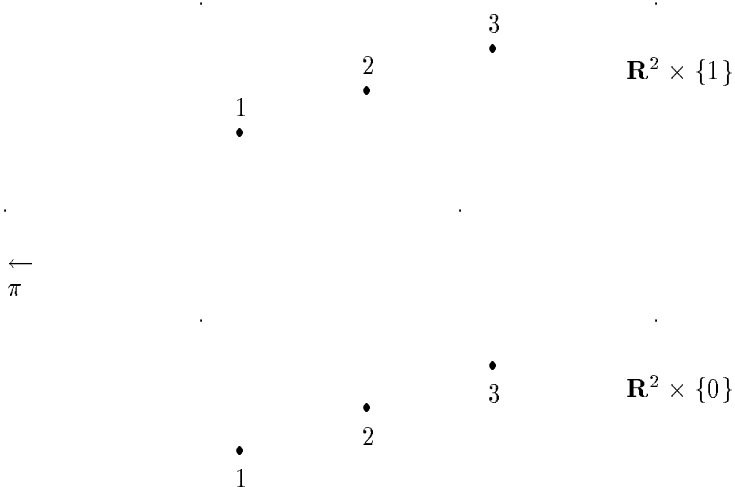


*Figure 2.1*

A horizontal slice through the $n$ curves in such a diagram, that is a slice $\mathbf{R}^2 \times \{t\}$ for some fixed $t$, will reveal

the $n$ points in $\mathbf{R}^2 \cong \mathbf{C}$ specified by $\gamma(t) \in X_n$. The usual pictorial representation of the braid $[\gamma] \in B_n$ is

now obtained by projecting onto a two-dimensional space, using the map:

$$\pi \colon \mathbf{R}^2 \times [0, 1] \longrightarrow \mathbf{R} \times [0, 1]$$

$$(z, t) \longmapsto \big(\Im(z), t\big) \, .$$

At positions where the projections of two of the curves in $\mathbf{R}^2 \times [0, 1]$ cross, it is recorded which curve was

associated with the smaller $\Re(z)$ by drawing over and under crossings. It may be assumed, without loss of

generality that at most two curves pass through any one point in $\mathbf{R} \times [0, 1]$; if this is not true in any particular

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case, then the curves in $\mathbf{R}^2 \times [0, 1]$ are deformed slightly so as to remove such multiple crossings. It is also assumed that the base-point $\left(z_1^0, z_2^0, \ldots, z_n^0\right) \in X_n$ (that is, $\gamma(0)$), is chosen such that $\Im(z_i^0)$ $(1 \leq i \leq n)$ is ordered in a monotonic increasing sequence.

The resulting description of a braid $[\gamma]$ is as a diagram in two-dimensions, joining two sets of $n$ points, with over and under crossings marked. The braid drawn above would then have a description as indicated in Fig. 2.2.

$$
\begin{array}{ccc}
1 & 2 & 3 \\
\bullet & \bullet & \bullet \\
\\
\\
\bullet & \bullet & \bullet \\
1 & 2 & 3
\end{array}
$$
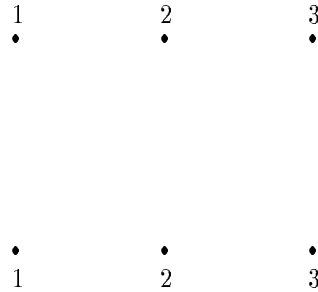
Figure 2.2

The composition of braids is effected by placing such diagrams end to end. Since $t$ increases upwards in such diagrams, the diagram for $[\gamma_1] \circ [\gamma_2]$ where $[\gamma_1], [\gamma_2] \in B_n$ is obtained by placing the diagram for $\gamma_1$ above that for $\gamma_2$.

$$
\begin{array}{ccccccc}
1 & 2 & & i & i+1 & n-1 & n \\
\bullet & \bullet & & \bullet & \bullet & \bullet & \bullet \\
& & \ldots & & & \ldots & \\
\bullet & \bullet & & \bullet & \bullet & \bullet & \bullet \\
1 & 2 & & i & i+1 & n-1 & n
\end{array}
$$
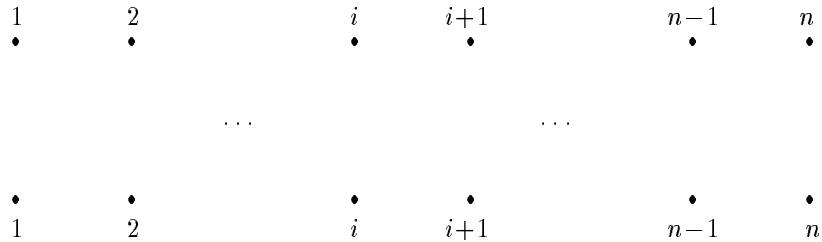
Figure 2.3

From this description it is clear that $B_n$ is generated by elements $\sigma_i$ for $1 \leq i \leq n-1$, which transpose the two points $z_i$ and $z_{i+1}$, as in Fig. 2.3. It is seen that the curve in $X_n$ corresponding to $\sigma_i$ is given by

fixing all $z_j$ for $j \neq i$, $i{+}1$, and making $z_i$ swap with $z_{i+1}$ by going around each other in a clockwise direction (see §§3.1, 4.1). The relations satisfied by these generators are found to be:

$$\sigma_i \sigma_{i+1} \sigma_i = \sigma_{i+1} \sigma_i \sigma_{i+1} \ ; \qquad \text{for } i = 1, 2, \ldots, n-2$$

$$\sigma_i \sigma_j = \sigma_j \sigma_i \ . \qquad \text{for } \left| i - j \right| > 1$$

These relations are both obvious geometrically; Fig. 2.4 illustrates the first of them.
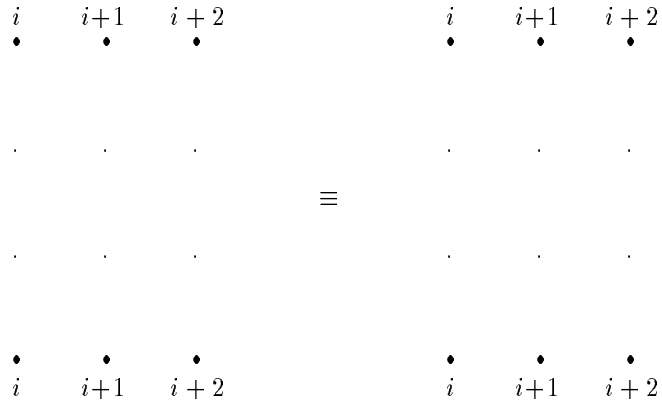


Figure 2.4

As mentioned above, the symmetric group $S_n$ may be viewed as a quotient of the braid group, namely that given by imposing the extra relations:

$$\sigma_i^2 = 1$$

for $i = 1, 2, \ldots, n-1$. Another quotient of $B_n$ which plays an important role in knot theory is the *Hecke algebra*, $H_n(q)$. It is defined for arbitrary complex $q$ by imposing the relations:

$$(\sigma_i - 1)(\sigma_i + q) = 0$$

for each $i \in \{1, 2, \ldots, n-1\}$. Some authors use relations requiring each $\sigma_i$ to have eigenvalues of $-1$, $q$ only, rather than $1$, $-q$ as we have used here. The map $\sigma_i \mapsto -\sigma_i$, which is well-defined as a homomorphism on $B_n$, converts between the two conventions. Note that when $q = 1$, $H_n(q)$ reduces to $S_n$. In general, $H_n(q)$ should be thought of as a deformation of the algebra of the symmetric group $S_n$. The work of Wenzl [We] shows that the representation theory of $H_n(q)$ is also very similar to that of $S_n$.

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As is well known, any Young diagram $\Lambda$, with $n$ squares, gives rise to an irreducible representation of $S_n$. In [We], Wenzl shows how to explicitly construct deformations of these representations as $q$ moves away from unity, which are representations of $H_n(q)$. When $q$ is not a root of unity, such representations are irreducible; but when $q$ is a root of unity they may be reducible, and Wenzl has shown how to explicitly produce an irreducible sub-representation. However, in the present work, we will not be concerned with this smaller representation. We shall denote by $\pi_\Lambda$ the representation of $H_n(q)$ associated with the Young diagram $\Lambda$, irrespective of whether $q$ is a root of unity or not. Hence it is only certain that $\pi_\Lambda$ is irreducible away from roots of unity.

There is a natural embedding of Hecke algebras:

$$H_n(q) \hookrightarrow H_{n+1}(q)$$

given by transforming the generators $\sigma_1, \ldots, \sigma_{n-1}$ of $H_n(q)$ to the corresponding generators of $H_{n+1}(q)$. The direct limit:

$$H_\infty(q) = \coprod H_n(q)$$

has generators $\{\sigma_i \mid i \in \mathbf{N}\}$ with relations:

$$
\left.
\begin{array}{ll}
\sigma_i \sigma_{i+1} \sigma_i = \sigma_{i+1} \sigma_i \sigma_{i+1} & \\[2mm]
\sigma_i \sigma_j = \sigma_j \sigma_i & \text{for } |i - j| > 1 \\[2mm]
(\sigma_i - 1)(\sigma_i + q) = 0 &
\end{array}
\right\}
$$

for any integers $i$ and $j$. Ocneanu showed (see [FYHLMO], [Jo 4]) that there is a unique trace, tr, defined on $H_\infty(q)$ such that:

$$
\left.
\begin{array}{l}
\operatorname{tr}(1) = 1 \\[2mm]
z \operatorname{tr}(g) = \operatorname{tr}(g\sigma_n)
\end{array}
\right\}
\tag{2.1.1}
$$

for all $g$ which are expressible as words in $\sigma_1, \ldots, \sigma_{n-1}$ only (that is, they can be thought of as elements of $H_n(q)$). Here $z$ is an arbitrary complex parameter.

The importance of all this to knot theory is due to the relation between braids and links, obtained from the operation of closure. Thus from a braid $\beta \in B_n$, a link $\widehat{\beta}$ can be constructed by canonically joining the endpoints of the braid. This operation is illustrated in Fig. 2.5.
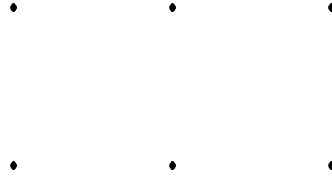
Figure 2.5

The following Theorems describe the nature of this closure map from braids to links.

**Theorem 2.1 (Alexander [Al 2])**  *For any link $L$, there exists $N \in \mathbf{N}$ and $\beta \in B_n$ for each $n \geq N$, such that $\widehat{\beta} \sim L$, where $\sim$ denotes isotopy equivalence.*

**Theorem 2.2 (Markov [Mar])**  *Suppose $\alpha \in B_m$ and $\beta \in B_n$, are two braids in possibly differently sized braid groups. Then their closures $\widehat{\alpha}$ and $\widehat{\beta}$ are isotopic if, and only if, there exists a sequence of moves from $\alpha$ to $\beta$, each one of which is of one of the following two types:*

$$\mathbf{I} \quad \gamma \longrightarrow \theta\gamma\theta^{-1} \qquad \text{with } \gamma,\ \theta \in B_r;$$

$$\mathbf{II} \ \gamma \longleftrightarrow \gamma\sigma_r^{\pm 1} \qquad \text{with } \gamma \in B_r \hookrightarrow B_{r+1}\ .$$

In this second Theorem, move **II** changes the order of the braid group either up of down by 1. Thus $\gamma$ is considered as an element of $B_r$, whereas $\gamma\sigma_r^{\pm 1}$ is considered as an element of $B_{r+1}$. The embedding of $B_r$ in $B_{r+1}$ is defined by adding an extra point $z_{r+1}$, which is fixed throughout a motion associated with a braid in $B_r$. This move is illustrated by Fig. 2.6.

Theorem 2.1 expresses the surjectivity of the map from braids to links, whereas Theorem 2.2 expresses the extent to which the operation of closure departs from an injective map. Thus, starting from a link $L$,

$\longleftrightarrow$
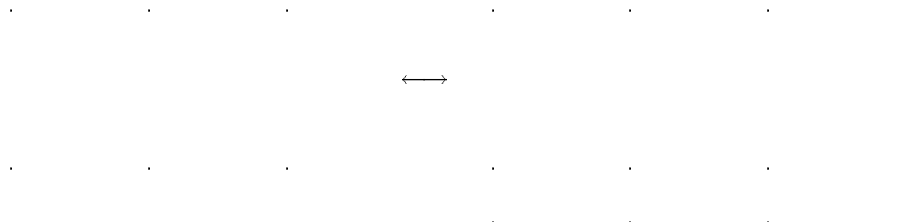
Figure 2.6

we can express it as the closure of some braid $\beta \in B_n$. Let:

$$\pi \colon B_n \longrightarrow H_n(q)$$

denote the map obtained by restricting to the group $B_n$, the natural quotient map from the group algebra of $B_n$ to $H_n(q)$. It maps generators $\sigma_i$ of $B_n$ (as a group) to the corresponding generators of $H_n(q)$ (as an algebra). Then Theorem 2.2 implies that $\mathrm{tr}\big(\pi(\beta)\big)$ is invariant under move **I** and multiplies by a factor of $z$ or $\big(q^{-1}(z-1)+1\big)$ under move **II** (according to whether the degree of the braid group increases or decreases by one). A suitable renormalisation now gives rise to an invariant polynomial—the *Jones polynomial*.

**Theorem 2.3**      *The expression:*

$$X_L(q,\lambda) = \left(\frac{\lambda q - 1}{\sqrt{\lambda}(1-q)}\right)^{n-1} \left(\sqrt{\lambda}\right)^e \, \mathrm{tr}\big(\pi(\beta)\big) \tag{2.1.2}$$

*defines an invariant of the link* $L = \widehat{\beta}$ *where* $\beta \in B_n$ *and* $e$ *is the exponent sum of the word for* $\beta$ *in* $\{\sigma_i \mid 1 \le i \le n-1\}$. *Here* $\lambda$, $q$, $z$ *are complex parameters related by the equation:*

$$z(1-\lambda q) = (q-1) \,. \tag{2.1.3}$$

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Here $X_L$ is known as the two-variable Jones polynomial. The original one-variable Jones polynomial, $V_L$, is the specialisation of this given by $q = \lambda$; this polynomial historically predates the two-variable polynomial. In the present work, we shall only be concerned with the one-variable polynomial. Before we discuss the significance of this specialisation, we shall consider the nature of the Ocneanu trace occuring in (2.1.2), in more detail.

The Ocneanu trace, defined by (2.1.1), can be thought of as a collection of traces, one on $H_n(q)$ for each $n \in \mathbf{N}$, and as such can be decomposed as a linear combination of characters $\chi_\Lambda$ corresponding to the (generically irreducible) representations $\pi_\Lambda$ of $H_n(q)$. When this is carried out, it can be shown that:

$$\mathrm{tr}(g) = \sum_\Lambda W_\Lambda(q, z)\chi_\Lambda(g) \tag{2.1.4}$$

where the sum is over all Young diagrams $\Lambda$, with $n$ squares, and $g \in B_n$. The coefficients $W_\Lambda(q, z)$ are given by:

$$W_\Lambda(q, z) = \prod_{(i,j)} \frac{(wq^i - zq^j)}{(1 - q^{l(i,j)})} \tag{2.1.5}$$

where $w = 1 - q + z$, and the product is over all squares $(i, j)$ in the Young diagram $\Lambda$; here $i, j$ label rows and columns starting with $i = j = 0$ at the top left hand corner of $\Lambda$. The *hook length* $l(i, j)$ is defined to be the number of squares of $\Lambda$ in the L-shaped hook with corner at $(i, j)$ and extending, in one direction to the right, and in the other direction vertically downwards. See [Jo 4] for more details. Fig. 2.7 illustrates this notation.

| | $i = 0$ | 1 | 2 | 3 | 4 | 5 |
|---|---|---|---|---|---|---|
| $j = 0$ | 8 | 6 | 5 | 3 | 2 | 1 |
| 1 | 4 | 2 | 1 | | | |
| 2 | 1 | | | | | |

Figure 2.7

The numbers inside the squares give the values of $l(i, j)$ in each square for a sample Young diagram with ten squares.

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When $q = \lambda$, the relations for $z$, $w$ above give $w = q^2 z$ (see (2.1.3)). From (2.1.5), we conclude that $W_\Lambda$ vanishes for any Young diagram $\Lambda$ with more than two rows. The two-row Young diagram, $\Lambda_m$, with rows of lengths $n - m$ and $m$, has:

$$W_{\Lambda_m}\left(q, -\frac{1}{1+q}\right) = \frac{1 - q^{n-2m+1}}{(1-q)(1+q)^n}$$

as can be seen from the hook lengths for $\Lambda_m$ illustrated in Fig. 2.8.

| $n - m + 1$ | $\cdots$ | $n - 2m + 2$ | $n - 2m$ | $\cdots$ | $1$ |
|---|---|---|---|---|---|
| $m$ | $\cdots$ | $1$ | | | |

Figure 2.8

However, in the case $q = \lambda$, (2.1.3) gives $z = -1/(1+q)$, and thus from Theorem 2.3, we conclude the following Theorem.

**Theorem 2.4**    *The one-variable Jones polynomial $V_L(q)$ of a link $L$ expressed as the closure of a braid $\beta \in B_n$ is given by:*

$$V_L(q) = \frac{(-1)^n (\sqrt{q})^{e-n+1}}{1 - q^2} \sum_{m=0}^{[n/2]} \left(1 - q^{n-2m+1}\right) \chi_{\Lambda_m}\left(\pi(\beta)\right)$$

*where $\pi(\beta) \in H_n(q)$ is the image of $\beta \in B_n$ under the natural map; and $e$ is the exponent sum of $\beta$ as a word in the generators $\sigma_1, \ldots, \sigma_{n-1}$.*

The importance of this result for the present work, is that it expresses $V_L$ as a combination of characters associated with two-row Young diagrams. No other specialisation of the two-variable polynomial $X_L$, can be expressed in this way, and thus for our purposes $V_L$ is the simplest specialisation of $X_L$ with which one can deal. The main constructions in the present work give rise to precisely the representation $\pi_{\Lambda_m}$ (see Chapters 3–6). Hence $V_L$ can be expressed as a combination of the characters $\chi_{\Lambda_m}$ which are naturally constructed in the following Chapters.

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The main motivation for wishing to obtain a topological interpretation of $\pi_{\Lambda_m}$ was the desire to better understand $X_L$. It seems that, as was remarked on p.364 of Jones [Jo 4], a better understanding of $X_L$ will only come once $V_L$ has been understood. Since $V_L$ is expressible as in Theorem 2.4, the first step is to investigate $\pi_{\Lambda_m}$. In Chapter 9 we will make some comments on possible methods for continuing the programme.

## 2.2 Translation into topology

As was discussed extensively in the last section, the problem of obtaining a better understanding of the one-variable Jones polynomial $V_L(q)$ leads naturally to the Hecke algebra representation $\pi_{\Lambda_m}$ associated with two-row Young diagrams $\Lambda_m$. In the work of Tsuchiya & Kanie [TK], Kohno [Ko] and Witten [W], these representations, $\pi_{\Lambda_m}$, appear in various differing contexts. Tsuchiya & Kanie obtained them by doing conformal field theory on the complex projective line $\mathbf{P}^1$. They obtained a basis that arose naturally from their constructions, and on which the action of the braid group, $B_n$, could be identified with that of $\pi_{\Lambda_m}$, as found in the work of Wenzl [We].

Kohno produced a connection on $X_n$, which gives rise to a monodromy representation of:

$$\pi_1(X_n) = P_n.$$

It is shown in [Ko] that this representation factors through the Hecke algebra $H_n(q)$, for a suitable choice of the parameters occuring in the connection. However, the representations obtained are not irreducible, although, in the simplest cases, they do give rise to combinations of $\pi_\Lambda$'s for Young Young diagrams $\Lambda$, with two rows only. It turns out that in this latter case, the multiplicities involved are such that the representation of $B_n$ is precisely the Pimsner-Popa-Temperley-Lieb representation required to obtain the Jones polynomial (see [Jo 4]).

Kohno's connection is naturally associated with the differential equations arising in Tsuchiya & Kanie's work. For more details on these constructions see Chapter 8. For now, it is sufficient to remark that the constructions are essentially based on picking out conformal blocks of a conformal field theory on the punctured sphere.

As was briefly outlined in §1.1, Witten's approach to the Jones polynomial also puts Riemann surfaces with punctured (or marked points) into a central role. At least, that part of Witten's construction which has been made rigorous in Segal's work (see [S 2] and [S 3]) is restricted to the situation in which such punctured Riemann surfaces bound manifolds containing curves.

All these results lead one to suspect that a simple topological approach should exist, producing the Hecke algebra representations $\pi_{\Lambda_m}$ naturally. In the next few Chapters, we will present a method by which this can be achieved. As in the other methods for producing such representations, it is obtained as the monodromy representation associated with some flat connection on a vector bundle over $X_n$.

$$E$$

$$\gamma \qquad\qquad X_n$$

Figure 2.9

The vector bundle $E_m(\mathbf{q})$ will be constructed in terms of branched coverings and using a twisted local coefficient system. The fibres will be homology spaces. Then $\pi_{\Lambda_m}$ will appear as the dual representation; equivalently, it is obtained as the monodromy action on a vector bundle whose fibres are cohomology spaces. Such cohomology spaces can be identified with spaces of functions, and this point of view will be found to

be useful later on.

In Chapter 8, we will return to the comparison between the constructions of Chapters 3–7 and the others mentioned above, in particular, those of Tsuchiya & Kanie. It will be shown that the connection defined in Chapter 3 gives rise to a differential equation of the same nature as that to be found in [TK] or [Ko]. There are further similarities, and they lead to a more intimate correspondence between the two constructions—see Chapters 8 and 9 for further details.

In Witten's approach, the representation $\pi_{\Lambda_m}$ appears, once one shifts from the '$S^3$' picture to the '$S^1 \times S^2$' picture. That is, in the former picture one considers a link $L$ embedded in $S^3$ (in some senses this is the most natural approach). In the latter picture, $L$ is considered as embedded in $S^1 \times S^2$; this approach is naturally related to the braid approach to links. The fact that a topological description of the '$S^3$' picture exists, and that this is related to the braid approach, was another motivation for the work.

# 3: Basic structure of theory

## 3.1 Definitions of basic spaces

We shall here define all the basic spaces necessary to enable us to state the main theorems in the next section.

Recall that in §2.1, $X_n$, $\widetilde{X}_n$ were defined to be:

$$X_n = \left\{ (w_1, \ldots, w_n) \mid \{w_i\} \text{distinct in } \mathbf{C} \right\} \qquad \text{(ordered points)}$$

$$\widetilde{X}_n = \left\{ \{w_1, \ldots, w_n\} \mid \{w_i\} \text{distinct in } \mathbf{C} \right\} \qquad \text{(unordered points)}$$

$$= X_n/S_n$$

We shall now construct a fibre bundle over $X_n$, whose fibre has dimension $m$ (complex dimension), where $m \in \mathbf{N}$ is arbitrary, but fixed. For any $\mathbf{w} \in X_n$, define:

$$Y_{\mathbf{w},m} = \left\{ (z_1, \ldots, z_m) \in X_m \;\middle|\; z_i \neq w_j \text{ for all } i \in \{1, 2, \ldots, m\} \text{ and } j \in \{1, 2, \ldots, n\} \right\}.$$

Then $Y_{\mathbf{w},m}$ defines a subset of $X_m$. In fact, the projection map:

$$X_{m+n}$$
$$\downarrow$$
$$X_n$$

given by taking the first $n$ points only, of a set of $m+n$ points in $\mathbf{C}$, representing a point in $X_{m+n}$, has fibre $Y_{\mathbf{w},m}$ over the point $\mathbf{w} \in X_n$. There is an obvious action of $S_m$ on $Y_{\mathbf{w},m}$ given by permuting $z_1, \ldots, z_m$; this action will be important later in this section.

Over each $\mathbf{w} \in X_n$, we now wish to define a branched covering space $\widetilde{Y_{\mathbf{w},m}}$ of $Y_{\mathbf{w},m}$. Such a covering space, or equivalently, a local coefficient system $\chi_{\mathbf{w},m}$, will be defined as a function of a finite number of complex parameters. Now, a local coefficient system on $Y_{\mathbf{w},m}$, modelled on $\mathbf{C}$, is specified by a map:

$$\pi_1(Y_{\mathbf{w},m}) \longrightarrow \mathbf{C}^* .$$

However, $\pi_1(X_m)$ is the pure braid group on $m$ strings; and $\pi_1(Y_{\mathbf{w},m})$ is the generalisation of this to the complex plane with $n$ points removed (namely the points $w_1, \ldots, w_n$). An element of $\pi_1(Y_{\mathbf{w},m})$ is given by $[\gamma]$ where:

$$\gamma \colon [0,1] \longrightarrow Y_{\mathbf{w},m}$$

is a collection of $m$ curves in $\mathbf{C} \backslash \{w_1, \ldots, w_n\}$, describing the motions of $z_1, \ldots, z_m$. Using the usual braid notation, we obtain a picture of $\gamma$ like that in Fig. 3.1, as an element of $B_{n+m}$ with the last $n$ strands straight, corresponding to $w_1, \ldots, w_n$ being fixed.



$$
\begin{array}{cccccccc}
z_1 & z_2 & \cdots & z_m & w_1 & w_2 & \cdots & w_n
\end{array}
$$

$$
\begin{array}{cccccccc}
z_1 & z_2 & \cdots & z_m & w_1 & w_2 & \cdots & w_n
\end{array}
$$
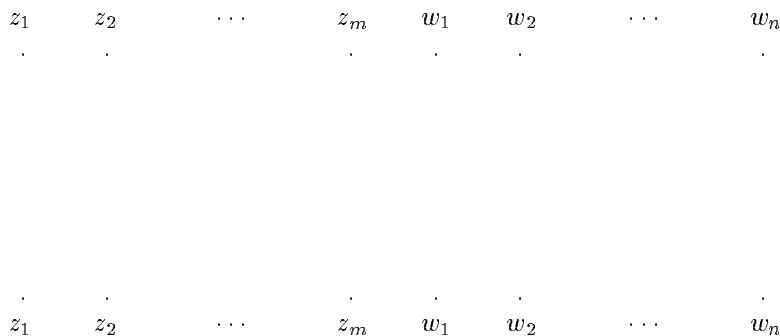
Figure 3.1

To make matters more precise at this stage, we will introduce some notation for particular elements of $\pi_1(Y_{\mathbf{w},m})$. Since $Y_{\mathbf{w},m}$ is unchanged when $\mathbf{w} \in X_n$ is changed to $\sigma(\mathbf{w})$ for any $\sigma \in S_n$, we may assume, without loss of generality, that $w_1, \ldots, w_n$ are ordered so that their imaginary parts are increasing. We will also choose a base point in $Y_{\mathbf{w},m}$, say $\mathbf{z}^0$, such that:

$$\Im(z_1^0) < \cdots < \Im(z_m^0) < \Im(w_1) < \cdots < \Im(w_n).$$

Let $\beta_{\lambda\mu}$ denote the element of $\pi_1(Y_{\mathbf{w},m})$ given by the curve fixing all $z_i \neq \lambda$, with $\lambda$ going round a curve in $\mathbf{C}$ which has winding number 1 about $\mu$, in a clockwise direction. Here,

$$\lambda \in \{z_1, \ldots, z_m\}$$

$$\text{and} \qquad \mu \in \{z_{j+1}, \ldots, z_m, w_1, \ldots, w_n\}$$

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where $z_j = \lambda$. The curve which $\lambda$ must follow, which corresponds to $\beta_{\lambda\mu}$, is defined by the statement that no point on the curve lies below any point in:

$$\{z_1, \ldots, \widehat{z_j}, \ldots, z_n, w_1, \ldots, w_n\} \backslash \{\mu\}$$

in the sense that $x$ is said to lie below $y$ (for $x, y \in \mathbf{C}$) if and only if $\Re(x) = \Re(y)$ and $\Im(x) < \Im(y)$. Thus we obtain a diagram for $\beta_{\lambda\mu}$ like that in Fig. 3.2.

curve followed
by $\lambda$

$\bullet$
$w_n$

$\bullet$
$\mu$

$\bullet$
$z_{j+1}^0$

$\bullet$
$z_j^0$

$\bullet$
$z_1^0$
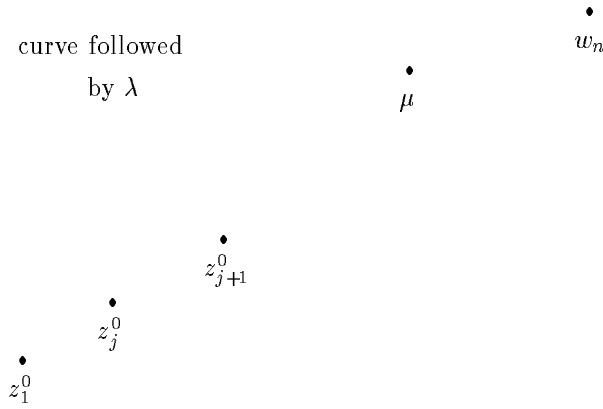
*Figure 3.2*

Any such element $[\gamma]$ of $\pi_1(Y_{\mathbf{w},m})$ defines an element of $B_{m+n}$ as mentioned above. It corresponds to a set of $m$ curves in $[0,1] \times \left(\mathbf{C} \backslash \{w_1, \ldots, w_n\}\right)$ given by $\left\{\left(t, \gamma_i(t)\right) \mid t \in [0,1]\right\}$ for $i = 1, 2, \ldots, m$. If we draw time, $t$, in a vertical direction, we obtain $m$ oriented curves in $\mathbf{R}^3$, connecting $m$ points on the plane $t = 0$ to the corresponding set of $m$ points on the plane $t = 1$. This picture may be viewed by projecting onto the vertical plane $[0,1] \times i\mathbf{R}$, as indicated in Fig. 3.3. This diagram gives the three-dimensional picture of $\beta_{\lambda\mu}$. When this is projected onto $[0,1] \times i\mathbf{R}$, we obtain, as illustrated in Fig. 3.4, the usual braid picture for $\beta_{\lambda\mu}$ (see §2.1). In this picture time moves upwards, giving a natural orientation to the curves.

The generalised pure braid group $\pi_1(Y_{\mathbf{w},m})$ is generated by $\left\{\beta_{\lambda\mu} \mid \lambda = z_j, \ j \in \{1, 2, \ldots, m\}\right\}$ and $\mu \in \{z_{j+1}, \ldots, z_m, w_1, \ldots, w_n\}\}$. These generators satisfy generalised braid group relations, the details of which we shall not go into here. A one-dimensional representation of $\pi_1(Y_{\mathbf{w},m})$ is thus given by specifying

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·$t = 1$

$\mu$

$\{1\} \times \mathbf{C}$

$\lambda$

$\longrightarrow$

graph of $\beta_{\lambda_m}$

·$t = 0$

$\{0\} \times \mathbf{C}$

$\mu$

$\lambda$

$[0, 1] \times i\mathbf{R}$
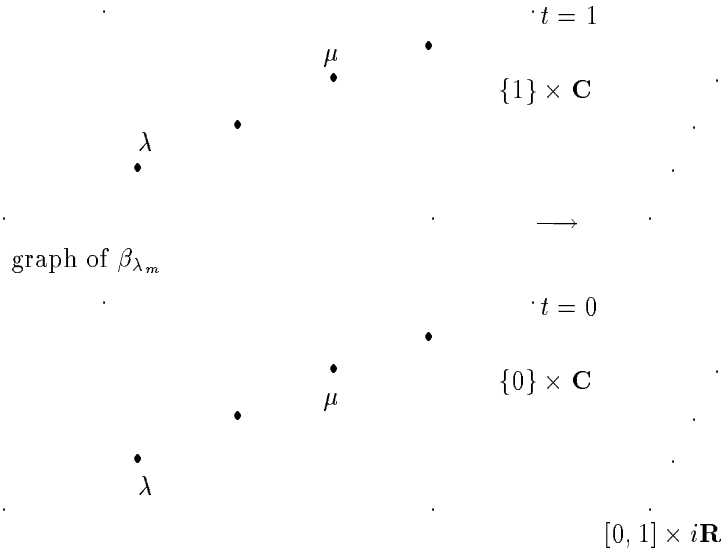
*Figure 3.3*



$\lambda$ $\mu$

$\lambda$ $\mu$
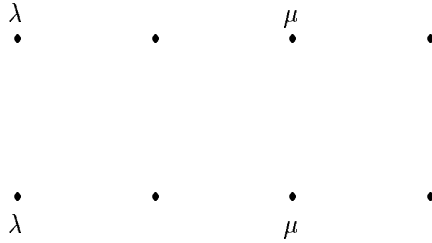
*Figure 3.4*

the images of these generators:

$$\pi_1(Y_{\mathbf{w},m}) \longrightarrow \mathbf{C}^*$$

$$\beta_{\lambda\mu} \longmapsto q_{\lambda\mu}^{-1}$$

where $q_{\lambda\mu}$ are non-zero complex numbers. There are no relations imposed on $q_{\lambda\mu}$, as can be seen by noting that:

$$\prod_{i=1}^{m} \left( \prod_{\substack{j=1 \\ j > i}}^{m} (z_i - z_j)^{k_{z_i z_j}^{-1}} \cdot \prod_{l=1}^{n} (z_i - w_l)^{k_{z_i w_l}^{-1}} \right)$$

defines an analytic function of $z_1, \ldots, z_m$ with branch points where $z_i = z_j$ or $z_i = w_l$. This function

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multiplies by $q_{\lambda\mu}^{-1}$ as $\lambda$ goes around $\mu$ along the curve $\beta_{\lambda\mu}$. Here $\{k_{\lambda\mu}\}$ is defined so that:

$$q_{\lambda\mu} = \exp\left(2\pi i/k_{\lambda\mu}\right).$$

For any given set of non-zero complex numbers $\{q_{\lambda\mu}\}$ we can therefore define a local coefficient system $\chi_{\mathbf{w},m}(\mathbf{q})$ on $Y_{\mathbf{w},m}$, or equivalently, a branched covering $\widetilde{Y_{\mathbf{w},m}}$. When $\{k_{\lambda\mu}\}$ are all integers, $\widetilde{Y_{\mathbf{w},m}}$ is a finite branched covering of the space $Y_{\mathbf{w},m}$. For most of the work we will deal with arbitrary $q_{\lambda\mu}$, and thus prefer to work with the twisted local coefficient system rather than the branched covering.

We have thus shown the following result:

**Lemma 3.1**     *Given any* $\mathbf{q} = \{q_{\lambda\mu}\}$ *with* $q_{\lambda\mu} \in \mathbf{C}\backslash\{0\}$ *for any* $\lambda$, $\mu$ *of the form:*

$$\lambda = z_j, \quad \text{with } j = \{1, 2, \ldots, m\}$$

$$\mu \in \{z_{j+1}, \ldots, z_m, w_1, \ldots, w_n\},$$

*there exists a well defined local coefficient system* $\chi_{\mathbf{w},m}(\mathbf{q})$ *which twists by* $q_{\lambda\mu}^{-1}$ *around the curve represented by* $[\beta_{\lambda\mu}] \in \pi_1(Y_{\mathbf{w},m})$.

Hence there is defined a homology space $H_m(Y_{\mathbf{w},m}, \chi_{\mathbf{w},m}(\mathbf{q}))$. We now define a vector bundle $E_m(\mathbf{q})$ over the space $X_n$ by defining the fibre over a point $\mathbf{w} \in X_n$ to be the vector space:

$$H_m(Y_{\mathbf{w},m}, \chi_{\mathbf{w},m}(\mathbf{q})) = E_{\mathbf{w},m}(\mathbf{q}).$$

Since homology is homotopy invariant, there is a natural flat connection on this vector bundle. The monodromy of this connection gives rise to a representation of $\pi_1(X_n) = P_n$ on the homology. However, to obtain a representation of $B_n = \pi_1(\widetilde{X}_n)$, it is only necessary to note that the natural action of $S_n$ on $X_n$ acts trivially on $E_{\mathbf{w},m}(\mathbf{q})$. That is, if $[\gamma] \in \pi_1(\widetilde{X}_n)$, then $\gamma$ is a curve:

$$[0, 1] \longrightarrow X_n$$

such that $\gamma(0)$, $\gamma(1) \in X_n$ differ from each other by a permutation. The natural connection on $E_m(\mathbf{q})$ then gives a parallel transport along $\gamma$ from the fibre over $\gamma(0)$ to that over $\gamma(1)$. However $\gamma(0)$, $\gamma(1)$ differ by a

permutation, and thus for suitable $\mathbf{q}$ (see below), we have:

$$E_{\gamma(0),m}(\mathbf{q}) \cong E_{\gamma(1),m}(\mathbf{q}) .$$

Together with the parallel transport defined above, this gives rise to an action of $\gamma$ on $E_{\gamma(0),m}(\mathbf{q})$, and hence a representation of $B_n$ on the twisted homology space.

As mentioned earlier in this section, there is a natural action of $S_m$ on $Y_{\mathbf{w},m}$ given by permuting $z_1, \ldots, z_m$. This carries over to an action on $\widetilde{Y_{\mathbf{w},m}}$ so long as the local coefficient system $\chi_{\mathbf{w},m}(\mathbf{q})$ is preserved by the action of $S_m$. In particular, this requires that $\mathbf{q}$ must be such that,

$$\left.\begin{array}{l} q_{z_i z_j} \text{ is independent of } i, j \in \{1, 2, \ldots, m\}, \text{ for } i < j; \\[2mm] q_{z_i w_j} \text{ is independent of } i \in \{1, 2, \ldots, m\}, \text{ for } j = 1, 2, \ldots, n . \end{array}\right\} \tag{3.1.1}$$

Going back to the action of $B_n$, if $[\gamma] \in \pi_1(\widetilde{X}_n)$ with $\gamma(1) = \sigma(\mathbf{w}_0)$, $\gamma(0) = \mathbf{w}_0$, then:

$$Y_{\mathbf{w},m} = Y_{\sigma(\mathbf{w}_0),m} .$$

The local coefficient system $\chi_{\sigma(\mathbf{w}_0),m}(\mathbf{q})$ on $Y_{\sigma(\mathbf{w}_0),m}$ is equivalent to a local coefficient system $\chi_{\mathbf{w}_0,m}(\sigma(\mathbf{q}))$ on $Y_{\mathbf{w}_0,m}$, where $\sigma(\mathbf{q})$ is defined by:

$$\left[\sigma(\mathbf{q})\right]_{\lambda\mu} = \begin{cases} q_{\lambda\sigma^{-1}(\mu)} & \text{for } \mu \in \{w_1, \ldots, w_n\} \\ q_{\lambda\mu} & \text{for } \mu \in \{z_1, \ldots, z_m\}\backslash\lambda \end{cases}$$

Hence if $\mathbf{q}$ is such that $\sigma(\mathbf{q}) = \mathbf{q}$, then there is a natural isomorphism between:

$$H_m\big(Y_{\mathbf{w}_0,m}, \ \chi_{\mathbf{w}_0,m}(\mathbf{q})\big) = E_{\gamma(0),m}(\mathbf{q})$$

$$\text{and} \qquad H_m\big(Y_{\sigma(\mathbf{w}_0),m}, \ \chi_{\sigma(\mathbf{w}_0),m}(\mathbf{q})\big) = E_{\gamma(1),m}(\mathbf{q}) .$$

The parallel transport induces a map:

$$E_{\gamma(0),m}(\mathbf{q}) \longrightarrow E_{\gamma(1),m}(\mathbf{q}) \cong E_{\gamma(0),m}(\mathbf{q})$$

which thus gives rise to an action on $E_{\gamma(0),m}(\mathbf{q})$. Hence we obtain an action of $B_n$ on the fibre $E_{\gamma(0),m}(\mathbf{q})$ so long as:

$$\sigma(\mathbf{q}) = \mathbf{q} \qquad \forall \sigma \in S_n . \tag{3.1.2}$$

This requires that $q_{z_i w_j}$ is independent of $j \in \{1, 2, \ldots, n\}$ for all $i \in \{1, 2, \ldots, m\}$. Hence we have:

**Lemma 3.2**      *The natural connection on $E_m(\mathbf{q})$ induces natural actions of $B_n$ and $S_m$ on the fibres of $E_m(\mathbf{q})$, whenever $\mathbf{q}$ satisfies conditions (3.1.1) and (3.1.2), respectively. Hence there is an action of $B_n \times S_m$ on the fibres of $E_m(\mathbf{q})$, whenever $\mathbf{q}$ is of the form:*

$$\left. \begin{array}{l} q_{z_i z_j} = \alpha \\[2ex] q_{z_i w_k} = q \end{array} \right\} \tag{3.1.3}$$

*where $i, j \in \{1, 2, \ldots, m\}$, $k \in \{1, 2, \ldots, n\}$ and $q, \alpha \in \mathbf{C}^*$.*

When $\mathbf{q}$ satisfies (3.1.2), the action of $B_n$ on the fibres of $E_m(\mathbf{q})$ may be expressed more simply as follows. Let $\widetilde{E}_m(\mathbf{q})$ be the vector bundle over $\widetilde{X}_n$ whose fibre over a point $[\mathbf{w}] \in \widetilde{X}_n$ is the vector space $E_{\mathbf{w}, m}(\mathbf{q})$. This is well-defined, so long as we identify the vector spaces corresponding to $\sigma(\mathbf{w})$ and $\mathbf{w}$ as outlined above. Then the natural connection on $E_m(\mathbf{q})$ induces a natural connection on $\widetilde{E}_m(\mathbf{q})$. The two vector bundles $E_m(\mathbf{q})$, $\widetilde{E}_m(\mathbf{q})$ have identical fibres, but their base spaces differ, being $X_n$ and $\widetilde{X}_n$, respectively. The action of $B_n$ on the fibres of $E_m(\mathbf{q})$ is now more simply expressed as the monodromy action of $\pi_1(\widetilde{X}_n) = B_n$ on the fibres of $\widetilde{E}_m(\mathbf{q})$.

Since the fibres of $E_m(\mathbf{q})$ and $\widetilde{E}_m(\mathbf{q})$ are identical, the action of $S_m$ on $E_m(\mathbf{q})$ naturally identifies with an action on $\widetilde{E}_m(\mathbf{q})$, so long as $\mathbf{q}$ satisfies (3.1.2), that is, for $\mathbf{q}$'s of the form (3.1.3).

## 3.2 Statement of main Theorems

Using the definitions of the preceding section, we are now in a position to give precise versions of the main Theorems.

The local coefficient system $\chi_{\mathbf{w}, m}(\mathbf{q})$ (defined in Lemma 3.1) in which $q$ takes the special values given by:

$$\left. \begin{array}{l} q_{z_i z_j} = q^{-2} \\[2ex] q_{z_i w_k} = q \end{array} \right\} \tag{3.2.1}$$

will be denoted by $\chi_{\mathbf{w}, m}(q)$. Here $\mathbf{q}$ refers to all the coefficients $q_{\lambda \mu}$, whereas $q$ indicates the special value of $\mathbf{q}$ given by (3.2.1). This special local coefficient system satisfies both the conditions in Lemma 3.2 (that is (3.1.1) and (3.1.2)). Thus, by Lemma 3.2, natural actions of $B_n$ and $S_m$ exist on the fibres of $E_m(\mathbf{q})$, and, equivalently on the fibres of $\widetilde{E}_m(\mathbf{q})$.

**Theorem 3.3** *The monodromy action of $B_n = \pi_1(\widetilde{X}_n)$ on the $S_m$-invariant part of the vector bundle $\widetilde{E}_m(q)$ contains, as a quotient, the representation of $B_n$ obtained from $\pi_{\Lambda_m}$ of $H_n(q^{-1})$. The remaining component of the monodromy representation has dimension of order $1/n$ times that of $\pi_{\Lambda_m}$.*

The proof of this Theorem occupies Chapters 4 and 6. In Chapter 6, the local coefficient system is restricted to that of the form (3.2.1). However, since it is no more complicated to do so, the results of Chapter 4 will be proved for arbitrary local coefficient systems $\chi_{\mathbf{w},m}(\mathbf{q})$ satisfying (3.1.1) and (3.1.2) as appropriate. As it stands, the extent to which $\pi_{\Lambda_m}$ can be *naturally* picked out of the larger monodromy representation on $E_{\mathbf{w},m}(q)^{S_m}$, is not clear. However the monodromy representation consists almost entirely of $\pi_{\Lambda_m}$, and in Chapter 7, a construction will be given which enables the sub-representation to be isolated, at least in certain cases.

For any local coefficient system $\chi_{\mathbf{w},m}(\mathbf{q})$ for which $\mathbf{q}$ satisfies (3.1.1), there is an action of $B_n$ on $E_{\mathbf{w},m}(q)$. Thus we have a family of representations of $B_n$, which contains the special case in which $\mathbf{q}$ is given by (3.2.1). There is an action of $S_m$ only on the two-dimensional sub-family obtained from those $\mathbf{q}$ of the form (3.2.1). In Chapter 7, a quotient representation of the special braid group representation is constructed from the family of braid group representations with neighbouring $\mathbf{q}$'s. This quotient representation will be referred to as the *derived representation* of the family. As was mentioned above, there is no action of $S_m$ defined on a general member of the family; however, the derived representation exists at the special value of $\mathbf{q}$ given by (3.2.1), and at this value of $\mathbf{q}$, an action of $S_m$ exists.

**Theorem 3.4** *The symmetric part of the derived representation of the family of monodromy representations of $B_n$ on the vector bundles $\widetilde{E}_m(\mathbf{q})$, for $\mathbf{q}$ satisfying (3.1.1), at the value of $\mathbf{q}$ given by (3.2.1), is $\pi_{\Lambda_m}$.*

This Theorem is proved in §7.2 in the special case of $m = 2$ (see Theorem 7.3). For general $m$, it is shown in §7.3 (see Theorem 7.5) that the derived representation referred to above contains the representation $\pi_{\Lambda_m}$ of $H_n(q^{-1})$. Theorem 3.4 has not been proved in general, but from Theorem 7.5, the proof would be complete if it was verified that the symmetric part of the derived representation contains nothing other than

$\pi_{\Lambda_m}$, by, for example, a dimension count.

Another point of view on the construction of $\pi_{\Lambda_m}$ is given in Chapter 8, where it is shown that the following result holds (see Corollary 8.7).

**Theorem 3.5** *The monodromy action defined in §3.1 is equivalent to that obtained from a system of differential equations of the form:*

$$\frac{\partial \mathbf{f}}{\partial w_i} = \sum_{j \neq i} \frac{\mathbf{c}_{ij} \mathbf{f}}{(w_i - w_j)}$$

*for a vector valued function $\mathbf{f}$ on $X_n$, where $\mathbf{c}_{ij}$ are constant matrices.*

In Chapter 8, the constructions of Tsuchiya & Kanie [TK] and Kohno [Ko], which also give rise to Hecke algebra representations, are discussed. One of the main themes of their methods is the reduction to the study of a system of differential equations of the form specified in Theorem 3.5; and in §8.4, a detailed dictionary correspondence between the two approaches will be given.

However, in Chapters 4,5 and 6 we will confine our attention to the homology construction of §3.1. In Chapter 4, the action of $B_n \times S_m$ on homology is determined by obtaining the full action on a suitable chain complex. The obvious basis for chains, in terms of repeated loops, is used, and recurrence relations are obtained from which matrices for all the actions can, in principle, be computed. The homology space is also constructed as the kernel of a certain map on the space of chains.

In Chapter 5, these formulae are applied in some special cases. To prove Theorem 3.3, a concrete basis for a subspace of cohomology is constructed, on which the actions of $B_n$ and $S_m$ are determined. This is carried out in Chapter 6, where it is found that working on cohomology is more convenient than homology. The action of $B_n$ on cohomology is dual to that on homology, and Theorem 3.3 is equivalent on cohomology to stating that a sub-representation of the monodromy representation of $B_n$ on cohomology, factors through $H_n(q)$ (rather than $H_n(q^{-1})$ as in Theorem 3.3). It turns out that, in terms of a concrete basis for the sub-representation, the action of $B_n$ is given in a particularly simple form (see Theorem 6.6), and it is then easy to deduce Theorem 3.3, or Theorem 6.1, which is a more precise form of the earlier Theorem.

In Chapter 9, some remarks are made on some possible extensions of the constructions employed in Chapters 3–7, as well as on the implications of the correspondence found, in Chapter 8, between our methods and those of Tsuchiya & Kanie.

# 4: Translation into algebra

In this Chapter we translate all the geometry of Chapter 3 into algebra. We will produce a concrete chain complex which gives rise to the homology:

$$H_m(Y_{\mathbf{w},m}, \chi_{\mathbf{w},m}(\mathbf{q}))$$

and on this complex find relations which define the actions of the braid group, $B_n$, and the symmetric group $S_m$. In §4.1 we construct the concrete chain complex, and in §4.3 give the boundaries necessary to construct the homology. The last two sections §§4.5, 4.6 contain the formulae for the actions of $B_n$, $S_m$. In §4.4 we use some geometry to obtain some relations which will be useful in later sections.

## 4.1 Concrete basis for chains

In this section we will construct a concrete chain complex on $Y_{\mathbf{w},m}$ with local coefficient system $\chi_{\mathbf{w},m}(\mathbf{q})$. On $\{z_1, \ldots, z_m, w_1, \ldots, w_n\}$ define an ordering so that:

$$z_i < z_j \quad \text{iff } i < j$$

$$w_k < w_l \quad \text{iff } k < l$$

$$z_i < w_k$$

for $i, j \in \{1, 2, \ldots, m\}$, $k, l \in \{1, 2, \ldots, n\}$. We start with a base-point $\mathbf{z}^0$, as in §3.1, at which:

$$\Im(\lambda) < \Im(\mu)$$

whenever $\lambda, \mu \in \{z_1, \ldots, z_m, w_1, \ldots, w_n\}$ and $\lambda < \mu$.

For $\lambda < \mu$, let $\alpha_{\lambda\mu}$ denote the motion in $Y_{\mathbf{w},m}$ in which $\nu \neq \lambda$, $\mu$ is fixed $\forall \nu$, and $\lambda$, $\mu$ move so that they transpose while following curves which are such that no point on them is ever below any point in:

$$\{z_1, \ldots, z_m, w_1, \ldots, w_n\} \backslash \{\lambda, \mu\}.$$

We suppose that $\lambda$, $\mu$ swap round by going around each other in a clockwise direction. We thus have the diagrams found in Fig. 4.1 for $\alpha_{\lambda\mu}$ as a motion in the complex plane, and in terms of the braid picture.
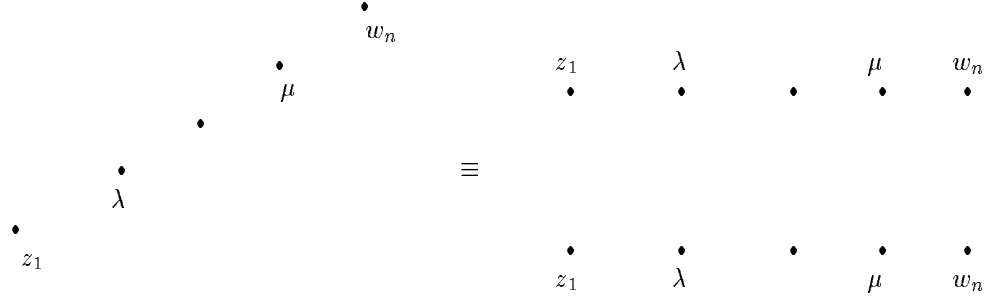


Figure 4.1

In §3.1, we similarly defined the curves $\beta_{\lambda\mu}$ to correspond to motions in which $\lambda$ went once around $\mu$ in a clockwise direction. It is now obvious that:

$$\beta_{\lambda\mu} = \alpha_{\lambda\mu}^2 .$$

**Definition** *For any* $\mathbf{w} \in X_n$, *and* $r \in \{1, 2, \ldots, m\}$, *let:*

$$\mathcal{S}_{\mathbf{w}}^r = \{\underline{\alpha} = (\alpha_1, \ldots, \alpha_r) | \alpha_i \in \{z_{i+1}, \ldots, z_m, w_1, \ldots, w_n\}\} \ ;$$

$$\mathcal{T}_{\mathbf{w}}^r = \{\underline{\alpha} \in \mathcal{S}_{\mathbf{w}}^r | \alpha_i \in \{w_1, \ldots, w_n\} \ and \ \alpha_i \ are \ all \ distinct\} \ ;$$

$$\mathcal{U}_{\mathbf{w}}^r = \{\underline{\alpha} \in \mathcal{T}_{\mathbf{w}}^r | \alpha_1 > \alpha_2 > \cdots > \alpha_r\} \ .$$

This definition gives rise to sets $\mathcal{S}_{\mathbf{w}}^r$, $\mathcal{T}_{\mathbf{w}}^r$, $\mathcal{U}_{\mathbf{w}}^r$ of orders:

$$(n + m - 1) \ldots (n + m - r)$$

$$n(n - 1) \ldots (n - r + 1)$$

$$1/r! \, n(n - 1) \ldots (n - r + 1)$$

respectively.

For each $\underline{\alpha} \in \mathcal{S}_{\mathbf{w}}^r$ we will now proceed to define an embedding of the $r$-dimensional torus $T^r$ in $Y_{\mathbf{w},m}$. This torus will have:

$$z_s = z_s^0$$

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whenever $s > r$. The map:

$$\gamma_{\underline{\alpha}} \colon T^r \longrightarrow Y_{\mathbf{w},m}$$

$$(t_1, \ldots, t_r) \longmapsto \gamma_{\underline{\alpha}}(t_1, \ldots, t_r)$$

is then defined by giving the $i^{\text{th}}$ component of $\gamma_{\underline{\alpha}}(t_1, \ldots, t_r)$ starting at $i = r$, and working back to $i = 1$. This definition will be such that, for all $i$, the $i^{\text{th}}$ component is independent of $t_1, \ldots, t_{i-1}$. So we start by setting:

$$(\gamma_{\underline{\alpha}}(t_1, \ldots, t_r))_r = \beta_{z_r \alpha_r}(t_r).$$

For a particular value of $t_r$, we have defined the value of the position of $z_r$. The loop defined by $z_i$ as $t_i$ increases from 0 to 1, with $t_{i+1}, \ldots, t_r$ fixed is defined so as to be a deformation of $\beta_{z_i \alpha_i}$. Suppose that $z_{i+1}, \ldots, z_r$ have already been defined as functions of $t_{i+1}, \ldots, t_r$. Then we deform $\beta_{z_i \alpha_i}$ continuously as $z_{i+1}, \ldots, z_r$ move from $z_{i+1}^0, \ldots, z_r^0$ due to the variation of $(t_{i+1}, \ldots, t_r)$ from $(0, \ldots, 0)$. The deformed curve is the curve we use to define the motion of $z_i$.

Thus for $t_1, \ldots, t_r$ small, the values of $(\gamma_{\underline{\alpha}}(t_1, \ldots, t_r))_i$ are given by:

$$\beta_{z_i \alpha_i}(t_i) \qquad \text{for } 1 \leq i \leq r \,.$$

When $t_1, \ldots, t_r$ are increased, we define $\gamma_{\underline{\alpha}}$ so as to give a continuous embedding in $Y_{\mathbf{w},m}$.

We can now think of $\gamma_{\underline{\alpha}}$ as a cycle on $Y_{\mathbf{w},m}$, whenever $\underline{\alpha} \in \mathcal{S}_{\mathbf{w}}^r$. When $\mathbf{w}$ moves along a curve in $X_n$, the torus $\gamma_{\underline{\alpha}}$ can be continuously deformed in a unique way (up to homotopy). This deformation corresponds at the level of homology to the natural connection discussed in §3.1. For each $\underline{\alpha} \in \mathcal{S}_{\mathbf{w}}^r$, it is now possible to lift the torus $\gamma_{\underline{\alpha}}$, which is embedded in $Y_{\mathbf{w},m}$, to $\widetilde{Y_{\mathbf{w},m}}$. When this is done, one obtains an embedding of $[0,1]^r$ in $\widetilde{Y_{\mathbf{w},m}}$ with base-point $\mathbf{z}^0$. Thus for any such $\underline{\alpha}$, $\gamma_{\underline{\alpha}}$ defines a chain on $\widetilde{Y}_{\mathbf{w},m}$.

The homology $H_m(Y_{\mathbf{w},m}, \chi_{\mathbf{w},m}(\mathbf{q}))$ may be computed in terms of the homology groups evaluated with a trivial local coefficient system, $\chi_0$, by using the following Lemma.

**Lemma 4.1** *The homology $H_m(Y_{\mathbf{w},m}, \chi_{\mathbf{w},m}(\mathbf{q}))$ can be computed from a finite dimensional chain complex $\mathcal{D}_r$ given by:*

$$\mathcal{D}_r = H_r(Y_{\mathbf{w},m}, \chi_0)$$

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*as the kernel of a suitably defined chain map* $\delta \colon \mathcal{D}_m \longrightarrow \mathcal{D}_{m-1}$ .

PROOF: Throughout this proof, $\mathbf{w} \in X_n$ will be fixed, and we will abbreviate $Y_{\mathbf{w},m}$ to $Y$ and $\chi_{\mathbf{w},m}(\mathbf{q})$ to $\chi$. It is also assumed that $\{\Im(w_i)\}$ are ordered as in §3.1. Consider the space:

$$Y_r = \left\{ (z_{r+1}, \ldots, z_m) \in X_{m-r} \;\middle|\; z_i \neq w_j \text{ for } r < i \leq m \text{ and } 1 \leq j \leq n \right\} .$$

Then $Y_0 = Y$ and for all $r = 1, 2, \ldots, m-1$, there is a fibration of $Y_{r-1}$ over $Y_r$, the fibres being one-dimensional. The fibre over $(z_{r-1}, \ldots, z_m) \in Y_r$ is the punctured plane $\mathbf{C} \backslash \{z_{r+1}, \ldots, z_m, w_1, \ldots, w_n\}$ . A filtration of this fibre is defined by:

$$K_0^{(r-1)} = \emptyset \,;$$
$$K_1^{(r-1)} = \bigcup_{\alpha > z_r} (\mathbf{R}^+ + \alpha) \,;$$
$$K_2^{(r-1)} = \mathbf{C} \backslash \{z_{r+1}, \ldots, z_m, w_1, \ldots, w_n\} \,.$$

where $K_1^{(r-1)}$ is a union of $(n + m - r)$ cuts emanating from $z_{r+1}, \ldots, z_m, w_1, \ldots, w_n$ and $\mathbf{R}^+$ denotes the positive real numbers. When $r = m$, the above defines a filtration of $Y_{m-1}$. This filtration defines a cell decomposition of the fibre, in which the $d$-dimensional cells are the components of $K_d^{(r-1)} - K_{d-1}^{(r-1)}$, so long as no two $z_i$'s $(i < r)$ have identical imaginary parts. Whenever two or more $z_i$'s have identical imaginary parts, we obtain non-distinct cuts, but it is still possible to define a filtration of the fibre by suitably deforming these cuts, in such a way that they no longer intersect.

Since there is a tower:

$$Y = Y_0 \longrightarrow Y_1 \longrightarrow \cdots \longrightarrow Y_{m-1} \longrightarrow 0 \,,$$

we can define a filtration, $\emptyset = K_0 \subseteq K_1 \subseteq \cdots \subseteq K_{2m} = Y$ of $Y$, in which $K_r$ is of dimension $r$, and is obtained as the union of:

$$K_{r_0}^{(0)} \times \cdots \times K_{r_{m-1}}^{(m-1)} \tag{4.1.1}$$

over all $r_0, \ldots, r_{m-1} \in \{0, 1, 2\}$ with sum $r$. Here the product is defined in the natural way, so as to give a subspace of $Y$, and the $s^{\text{th}}$ term gives the possible values of $z_s$ in this subspace, once the values of $z_{s+1}, \cdots, z_m$ are fixed. Now $K_i^{(r)} - K_{i-1}^{(r)}$ is a disjoint union of Euclidean spaces for $i = 1, 2$ and any $r$, and thus so also is $K_r - K_{r-1}$. Hence $\{K_r\}$ may be viewed as providing a cellular decomposition of $Y$, in which the $r$-dimensional cells are given by the components of $K_r - K_{r-1}$. This is a slightly unconventional use of the term 'cellular decomposition', since $Y$ is a non-compact space; however, $K_r - K_{r-1}$ is still a disjoint union of Euclidean spaces.

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Since $K_0^{(i)} = \emptyset$ for all $i$, so $K_r = \emptyset$ whenever $r < m$. Also, $K_m$ consists of a disjoint union of products of one-dimensional rays. The components of $K_m$ are indexed by $(\alpha_1, \ldots, \alpha_m)$ where $\alpha_i$ indexes the components of $K_1^{(i-1)}$. That is:

$$\alpha_i \in \{z_{i+1}, \ldots, z_m, w_1, \ldots, w_n\} \tag{4.1.2}$$

and hence the components of $K_m$ are indexed by $\mathcal{S}_\mathbf{w}^m$. It is also seen that $K_{m+1}$ is a disjoint union of products of the form (4.1.1), with all but one $r_i$ being 1, the remaining one being 2. Hence $K_{m+1}$ naturally splits into $m$ parts, the $k^{\text{th}}$ part having components indexed by $(\alpha_1, \ldots, \widehat{\alpha}_k, \ldots, a_m)$ where $\{\alpha_i\}$ satisfy (4.1.2). This part is given by $r_i = 1 + \delta_{ik}$.

The natural fibration of $Y_{r-1}$ over $Y_r$ has fibres homotopic to a wedge of $n + m - r$ circles. Hence if a tower:

$$Z = Z_0 \longrightarrow Z_1 \longrightarrow \cdots \longrightarrow Z_{m-1} \longrightarrow 0 \,,$$

is defined so that the fibres in the fibration of $Z_{r-1}$ over $Z_r$ are wedges of $n + m - r$ circles, for all $r$, then the homology of $Y$ can be computed from that of $Z$. It is also easily seen that $K_{s+m} - K_{s+m-1}$ consists of a union of products of $K_2^{(r)} - K_1^{(r)}$ and $K_1^{(r)}$ spaces, with $s$ of the former type in each product. (Once again the product refers to the subspace of $Y$ with the $z$'s as specified by the factors in each term.) Hence $K_{s+m} - K_{s+m-1}$ is homotopic to a wedge of $s$-torii. In particular, the components of $K_{s+m} - K_{s+m-1}$ are in 1–1 correspondence with those $s$-torii embedded in $Z$, obtained from one of the components of a slice of $Z$ by fixing one or more of the coordinates. The cohomology of $Y$ can be computed from the chain complex:

$$\mathcal{D}^r = H_c^r(K_r - K_{r-1})$$

where $H_c^*$ denotes cohomology with compact support. In the case of a compact manifold this would follow from the standard theorem giving cohomology in terms of a cell decomposition. Although $Y$ is not compact, its cohomology can still be computed in this way, with the compactly supported cohomology $H_c^*$ replacing ordinary cohomology, since the interesting structure of $Y$ comes from those points $\mathbf{z}$ for which $z_i$ is near to $z_{i+1}, \ldots, z_m$ or $w_1, \ldots, w_n$, for each $i$. As noted above, $\mathcal{D}^r = 0$ whenever $r < m$, while $\mathcal{D}^m \cong \langle \mathcal{S}_\mathbf{w}^m \rangle$ and $\mathcal{D}^{m+1}$ splits naturally as a product of $m$ spaces. In the dual picture, it is seen that the required homology is given by the homology of the chain complex $\mathcal{D}_r \cong \mathcal{D}^{2m-r}$. Hence the $m$-dimensional homology is given by:

$$\ker(\partial \colon \mathcal{D}_m \to \mathcal{D}_{m-1}) / \mathrm{Im}\,(\partial \colon \mathcal{D}_{m+1} \to \mathcal{D}_m) \cong \ker(\delta \colon \mathcal{D}^m \to \mathcal{D}^{m+1}) \,.$$

Since $\mathcal{D}_{m-1}$ naturally splits into a product of $m$ vector spaces, this homology space is the intersection of the kernels of $m$ maps on $\mathcal{D}^m \cong \langle \mathcal{S}_\mathbf{w}^m \rangle$.

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Finally, when the local coefficient system is trivial, all the boundary maps $\partial$ become trivial, and thus:

$$\mathcal{D}_r \cong H_r(Y, \chi_0)$$

as required. ∎

In §4.3, the precise form of $\delta$ will be investigated. In particular, the components of $\delta$ corresponding to the decomposition of $\mathcal{D}^{m+1}$ noted above, will be evaluated. The above analysis in terms of cellular decompositions has a more geometrical formulation in which each cell in $K_{2m-s}$ is represented by an embedding of a torus in $Y$, with base-point $\mathbf{z}^0$. Thus a cell in $K_{2m-s}$ is given by a choice of $\alpha_i$'s satisfying (4.1.2) for $i \in I$ where $I$ is some subset of $\{1, 2, \ldots, m\}$ of order $s$. The subset $I$ labels those $i$ for which $r_i = 1$ in (4.1.1), the rest of the $r_i$'s being 2. Such a choice of $\alpha_i$'s defines an embedding of $T^s$ in $Y$, as given by $\gamma_{\underline{\alpha}}$. When this is lifted to $\widetilde{Y}$, it defines an embedding of $[0, 1]^s$, and the different components of $\delta$, mentioned above, are obtained from the $s$ pairs of opposite faces in the boundary of such a hypercube.

The tower given by the spaces $Y_i$ will play a central role in all the calculations of the rest of this Chapter. Let $\mathcal{C}_r$ denote the vector space generated by formal $\mathbf{C}$-combinations of $\gamma_{\underline{\alpha}}$ for $\underline{\alpha} \in \mathcal{S}_{\mathbf{w}}^r$. Then $\mathcal{C}_r$ can be identified with a subspace of the space of $r$-chains on $Y$. As noted above $\mathcal{D}_m \cong \mathcal{C}_m$, and so $H_m(Y_{\mathbf{w},m}, \chi_{\mathbf{w},m}(\mathbf{q}))$ may be identified with a subspace of $\mathcal{C}_m$.

## 4.2 General definitions and notation

In the next section we shall embark on the process of determining first the boundary map, and then the actions of the braid group $B_n$ and symmetric group $S_m$ on the space of chains defined in the last section. In this section we shall set up some notation.

Whenever $\underline{\alpha} \in \mathcal{S}_{\mathbf{w}}^r$ and $s < r$, we denote by $\underline{\alpha}^s$, that element of $\mathcal{S}_{\mathbf{w}}^s$ given by truncating $\underline{\alpha}$:

$$\underline{\alpha}^s = (\alpha_1, \ldots, \alpha_s).$$

If $\underline{\alpha} \in \mathcal{S}_{\mathbf{w}}^r$ and $s > r$, then we shall denote the element $(\alpha_1, \ldots, \alpha_s) \in \mathcal{S}_{\mathbf{w}}^s$ by:

$$\underline{\alpha}.\alpha_{r+1}.\cdots.\alpha_s$$

For any $\underline{\alpha} \in \mathcal{S}_{\mathbf{w}}^r$, the $r$-torus $\gamma_{\underline{\alpha}}$ has $z_{r+1}, \ldots, z_m$ fixed, and a section on which $z_{r+1}$ is constant looks like $\gamma_{\underline{\alpha}}$, or a deformed version of it. In future sections we shall often write $\gamma_{\underline{\alpha}}.\alpha_{r+1}$ for $\gamma_{\underline{\alpha}.\alpha_{r+1}}$, where it is understood to mean that sections in which $z_{r+1}$ is constant are deformed versions of $\gamma_{\underline{\alpha}}$ and also that $z_{r+1}$ varies around a loop $\beta_{z_{r+1}\alpha_{r+1}}$.

Under the motions of $z_{r+1}, \ldots, z_m, w_1, \ldots, w_n$ specified by $\alpha_{\lambda\mu}, \beta_{\lambda\mu}$  $(z_{r+1} \leq \lambda \leq \mu \leq w_n)$, the elements of $\mathcal{S}_{\mathbf{w}}^r$ will transform to other chains, which are thus expressible as linear combinations of elements of $\mathcal{S}_{\mathbf{w}}^r$. We denote by $A_{\lambda\mu}^{(r)}$, $b_{\lambda\mu}^{(r)}$ these transformations; they will be square matrices of order $|\mathcal{S}_{\mathbf{w}}^r|$.

To obtain the homology, it is necessary to compute the boundary map:

$$\delta \colon \mathcal{D}_m \longrightarrow \mathcal{D}_{m-1} \ .$$

This map is specified by $\{\pi_i \circ \delta \colon i = 1, 2, \ldots, m\}$ where $\pi_i$ is the projection of $\mathcal{D}_{m-1}$ onto that part in which $z_i$ is fixed. Then $\pi_i \circ \delta$ gives rise to a matrix with $|\mathcal{S}_{\mathbf{w}}^m|$ rows; these matrices are denoted by $\mathbf{D}_i^{(m)}$. The corresponding matrices, for $r = i, i+1, \ldots, m$, representing the boundary map on $\mathcal{C}_r$, with $z_{r+1}, \ldots, z_m$ fixed will be denoted by $\mathbf{D}_i^{(r)}$.

The action of $S_m$ on the chains is specified in terms of the action of the generator which interchanges $z_i$ and $z_{i+1}$. This action is denoted by $j_{i\,i+1}^{(r)}$ on the chains $\gamma_{\underline{\alpha}}$ for $\underline{\alpha} \in \mathcal{S}_{\mathbf{w}}^r$ with $r \geq i+1$.

The above definition of the $A$'s means that the action of $B_n$ on the chain space $\mathcal{C}_m$ is given by:

$$\sigma_i \longmapsto b_{w_i w_{i+1}}^{(m)} \ .$$

Thus the aim of this Chapter is to produce formulae from which $A_{w_i w_{i+1}}^{(m)}$, $b_{w_i w_{i+1}}^{(m)}$, $j_{i\,i+1}^{(m)}$ and $\mathbf{D}_i^{(m)}$ can be computed. We note that at zeroth order,

$$b_{\lambda\mu}^{(0)} = q_{\lambda\mu}^{-1}$$

$$A_{\lambda\mu}^{(0)} = 1$$

for any $\lambda$, $\mu$. Note also that $A_{\lambda\mu}^{(r)}$ is only well defined if the chains at $\mathbf{w}$, and at the vector obtained from $\mathbf{w}$ by transposing $\lambda$ and $\mu$, can be identified. That is, only if the local twists $\mathbf{q}$ are invariant under a transposition of $\lambda$ and $\mu$. In the future, $\{\gamma_{\underline{\alpha}} \mid \underline{\alpha} \in \mathcal{S}_{\mathbf{w}}^m\}$ will often be referred to simply as $\mathcal{S}_{\mathbf{w}}^m$.

## 4.3 Form of the boundaries

The homology is the kernel of the boundary map on $\mathcal{C}_m$. When $\underline{\alpha} \in \mathcal{S}_{\mathbf{w}}^m$, corresponding to a basis element of $\mathcal{C}_m$, the $i^{\text{th}}$ component of $\delta(\gamma_{\underline{\alpha}})$ is given by:

$$\delta_i(\gamma_{\underline{\alpha}^i}.\alpha_i\alpha_{i+1}.\cdots.\alpha_m) = \delta_i(\gamma_{\underline{\alpha}^{i-1}}.\alpha_i).\alpha_{i+1}.\cdots.\alpha_m$$

Here we know that $\delta_i(\gamma_{\underline{\alpha}^i}.\alpha_i)$ is the boundary of the torus $T^i$ in which a section with $z_1, \ldots, z_{i-1}$ fixed is given by the loop $\beta_{z_i\alpha_i}$ for $\alpha_i$. Its boundary is thus the difference of two $(i-1)$-torii corresponding to $\underline{\alpha}^{i-1}$ and its deformation when $z_i$ has gone around $\beta_{z_i\alpha_i}$. Thus we obtain:

$$\delta_i(\gamma_{\underline{\alpha}}) = (b_{z_i\alpha_i}^{(i-1)} - 1)\gamma_{\underline{\alpha}^{i-1}}.\alpha_{i+1}.\cdots.\alpha_m$$

where this denotes an $(i-1)$-cycle in which $z_i$ is constant at $z_i^0$.

Hence a suitable matrix for $\delta$ is given by:

$$\mathbf{D}_i^{(i)} = \left( b_{z_iz_{i+1}}^{(i-1)} - 1 \middle| \cdots \middle| b_{z_iz_m}^{(i-1)} - 1 \middle| b_{z_iw_1}^{(i-1)} - 1 \middle| \cdots \middle| b_{z_iw_n}^{(i-1)} - 1 \right). \tag{4.3.1}$$

This is a partitioned matrix acting on $\langle \mathcal{S}_{\mathbf{w}}^i \rangle$, and mapping it to $\langle \mathcal{S}_{\mathbf{w}}^{i-1} \rangle$. The corresponding matrices $\mathbf{D}_i^{(r)}$ for $i \leq r \leq m$ acting on $\langle \mathcal{S}_{\mathbf{w}}^r \rangle$ are given by:

$$\mathbf{D}_i^{(r)} = \begin{pmatrix} \mathbf{D}_i^{(r-1)} & & \\ & \ddots & \\ & & \mathbf{D}_i^{(r-1)} \end{pmatrix} \text{ for } i < r \leq m. \tag{4.3.2}$$

In all cases, the matrices are partitioned according to the values of $\alpha_r \in \{z_{r+1}, \ldots, z_m, w_1, \ldots, w_n\}$.

We have now shown how the action of the boundary map on the space of chains $\mathcal{C}_m$ can be computed; it is given by $\mathbf{D}_i^{(m)}$. In the rest of this work, $\mathbf{D}_i^{(m)}$ will be referred to simply as $\mathbf{D}_i$. It is given by a diagonal matrix with blocks $\mathbf{D}_i^{(i)}$ down the diagonal, where the separation into blocks is specified by the values of $\alpha_{i+1}, \ldots, \alpha_m$. Thus we obtain the Lemma:

**Lemma 4.2**     The space $\bigcap_{i=1}^m \ker(\mathbf{D}_i) \subseteq \mathcal{C}_m$, where $\mathbf{D}_i$ are the matrices $\mathbf{D}_i^{(m)}$ specified by (4.3.1) and (4.3.2), is in 1-1 correspondence with the homology of $Y_{\mathbf{w},m}$ with the twisted local coefficient system $\chi_{\mathbf{w},m}(\mathbf{q})$.

The reason for the non-trivial boundary map is that the local coefficient system is non-trivial. Thus when $\gamma_{\underline{\alpha}}$ is lifted to $\widetilde{Y_{\mathbf{w},m}}$ it gives rise to an embedded m-cube:

$$[0,1]^m \longrightarrow \widetilde{Y_{\mathbf{w},m}}$$

and there are components in the boundary arising from each pair of opposite faces; that is from each $i \in \{1, 2, \ldots, m\}$. See Fig. 4.2.

Faces $z_i = $ constant

associated with the $i^{\text{th}}$ component of the boundary.

Figure 4.2

## 4.4 Some useful relations obtained using geometry

In this section we shall derive some relations which are satisfied by the $A^{(r)}$, $b^{(r)}$ matrices, by going back to the definitions of these matrices in §4.2, and using some geometry. The relations we consider are given by the following Lemma. So as to avoid unnecessary indices, $A_{\lambda\mu}^{(r)}$ and $b_{\lambda\mu}^{(r)}$ have been abbreviated to $A_{\lambda\mu}$ and $b_{\lambda\mu}$, where it is always understood that they are transformations on $\langle \mathcal{S}_{\mathbf{w}}^r \rangle$ for a common value of $r$.

**Lemma 4.3**

(i) $A_{jk}b_{ij} = b_{ik}A_{jk}$;

(ii) $b_{ij}b_{ik}A_{jk} = A_{jk}b_{ij}b_{ik}$;

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*(iii)* $A_{ij}A_{jk}A_{ij} = A_{jk}A_{ij}A_{jk}$;

*(iv)* $A_{ij}^2 = q_{ij}b_{ij}$;

*(v)* $A_{jk}$ commutes with $b_{i\lambda}$ whenever $\lambda < j$ or $\lambda > k$;

*(vi)* $b_{ij}^{-1}A_{jk}b_{ij}$ commutes with $b_{i\lambda}$ whenever $j < \lambda < k$;

*(vii)* $b_{jk}$ commutes with $b_{i\lambda}$ whenever $\lambda < j$ or $\lambda > k$;

*(viii)* $b_{ij}^{-1}b_{jk}b_{ij}$ commutes with $b_{i\lambda}$ whenever $j < \lambda < k$;

*(ix)* $b_{jk}b_{ij}b_{ik} = b_{ij}b_{ik}b_{jk} = b_{ik}b_{jk}b_{ij}$;

*and these relations hold between $A^{(r)}$ and $b^{(r)}$ matrices, for all $i, j, k, \lambda \in \{z_{r+1}, \ldots, z_m, w_1, \ldots, w_n\}$ and any $r \in \{0, 1, \ldots, m\}$.*

PROOF: We shall start with (iv) and then show that all the relations follow from (i), (ii), (iii), (v), (vi) about which we will then go into more detail.

The matrix $A_{ij}$ represents the action on $\langle \mathcal{S}_{\mathbf{w}}^r \rangle$ given by $\alpha_{ij} \in \pi_1(Y_{\mathbf{w},m})$, and similarly $b_{ij}$ corresponds to the action of $\beta_{ij}$. However,

$$\beta_{ij} = \alpha_{ij}^2$$

and so, at first sight, it would seem that $b_{ij}$ should be given by $A_{ij}^2$. However it must be recalled that the natural connection of §3.1 gives rise to a map from chains at $\{z_{r+1}^0, \ldots, z_m^0, w_1, \ldots, w_n\} = \mathbf{a}_r^0$ to those where $i$ and $j$ have been transposed, induced from $\alpha_{ij}$. There is an identification between the chain spaces obtained by using the natural identification which exists between the spaces $Y_{\mathbf{w},m}$ and $Y_{\sigma(\mathbf{w}),m}$, for any $\sigma \in S_n$. Hence $A_{ij}$ is defined to be the composition of the map induced by the connection with the identification map. The square of the map induced by the connection here is precisely $b_{ij}$. However the composition of the identification maps:

$$\{\text{chains at } \mathbf{a}_r^0\} \longleftrightarrow \{\text{chains at } (ij)(\mathbf{a}_r^0)\}$$

$$\text{and} \quad \{\text{chains at } (ij)(\mathbf{a}_r^0)\} \longleftrightarrow \{\text{chains at } \mathbf{a}_r^0\}$$

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which occur in the two applications of $A_{ij}$, leaves a residual factor of $q_{ij}$, coming from the fact that overall, $i$ and $j$ have gone round each other once in a clockwise direction. Here $(ij) \in S_{m+n}$ acts on $\mathbf{a}_r^0$ in the usual way. See Fig. 4.3.

multiplication by $q_{ij}$

$\{$space of chains $\langle \mathcal{S}_{\mathbf{a}}^r \rangle\}$

$\bullet (i\, j)\mathbf{a}_r^0$

$\{\mathbf{a}\}$

$\mathbf{a}_r^0$
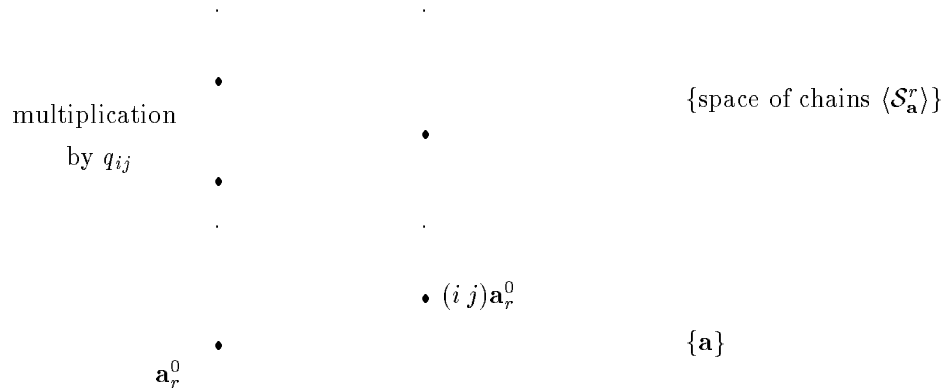
Figure 4.3

Hence we conclude that:

$$A_{ij} = q_{ij} b_{ij}$$

as required.

From this result, it is clear that (vii), (viii) follow directly from (v), (vi). Also, (ix) can be deduced from (i), (ii), since:

(a) from (ii), $b_{ij} b_{ik}$ commutes with $A_{jk}$, and thus by (iv), with $b_{jk}$, giving the first half of (ix);

(b) from (i), $b_{ik} A_{jk}^2 b_{ij} = A_{jk} b_{ij} b_{ik} A_{jk} = A_{jk}^2 b_{ij} b_{ik}$ using (ii), which by (iv), leads to the other half of (ix).

We are now left with (i), (ii), (iii), (v), (vi). However $A_{ij}$ and $b_{ij}$ are defined in terms of the actions of the braid group, as is illustrated in Figs. 4.4 and 4.5.

$z_{r+1}$     $i$        $\bullet$       $\bullet$     $j$    $w_n$

$A_{ij} \quad \longleftrightarrow$

$z_{r+1}$     $i$                  $j$    $w_n$

*Figure 4.4*

$z_{r+1}$     $i$             $j$    $w_n$

$b_{ij} \quad \longleftrightarrow$

$z_{r+1}$     $i$               $j$    $w_n$

*Figure 4.5*
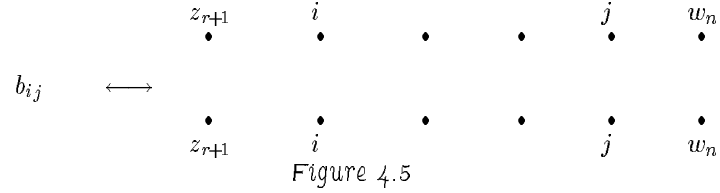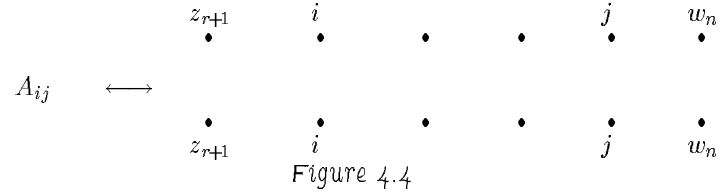
Thus we can verify all the relations by checking that the corresponding equations hold on the braid group. (The extra factors of $q_{ij}$ which may appear, due to the reason outlined above in the proof of (iv), are identical on either side for all of the relations considered, so that we need only consider the maps induced by the connection.)

We obtain Figs 4.6–4.9 for (i), (ii), (iii) and (iv). Since the relations can be checked at the braid group level, they can all be derived algebraically from the braid group relations. However it is nicer to derive them geometrically!
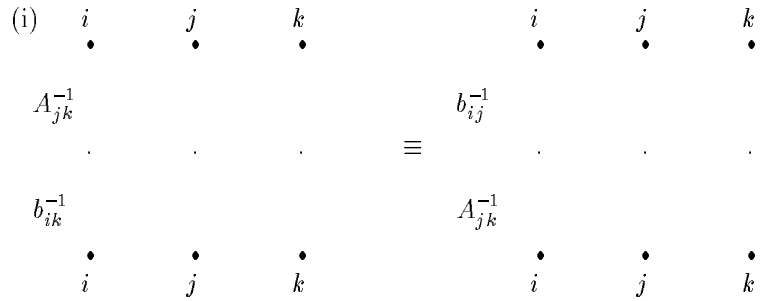
(i)    $i$      $j$      $k$            $i$      $j$      $k$

$A_{jk}^{-1}$                  $b_{ij}^{-1}$

                     $\equiv$

$b_{ik}^{-1}$                  $A_{jk}^{-1}$

     $i$      $j$      $k$           $i$      $j$      $k$

*Figure 4.6*

Finally (vi) states that for $j < \lambda < k$, $b_{i\lambda}$ commutes with $b_{ij}^{-1} A_{jk} b_{ij}$. This latter transformation is given by Fig. 4.10, and the commutativity of this with $b_{i\lambda}$ is given by Fig. 4.11. This equivalence follows by sliding the twist of $i$ around $\lambda$ down the diagram until it comes out at the base.

(ii)

$$
\begin{array}{ccc}
\overset{i}{\bullet} & \overset{j}{\bullet} & \overset{k}{\bullet} \\
b_{ik}^{-1} & & \\
\cdot & \cdot & \cdot \\
b_{ij}^{-1} & & \\
\cdot & \cdot & \cdot \\
A_{jk} & & \\
\underset{i}{\bullet} & \underset{j}{\bullet} & \underset{k}{\bullet}
\end{array}
\quad \equiv \quad
\begin{array}{ccc}
\overset{i}{\bullet} & \overset{j}{\bullet} & \overset{k}{\bullet} \\
A_{jk} & & \\
\cdot & \cdot & \cdot \\
b_{ik}^{-1} & & \\
\cdot & \cdot & \cdot \\
b_{ij}^{-1} & & \\
\underset{i}{\bullet} & \underset{j}{\bullet} & \underset{k}{\bullet}
\end{array}
$$

Figure 4.7

(iii)

$$
\begin{array}{ccc}
\overset{i}{\bullet} & \overset{j}{\bullet} & \overset{k}{\bullet} \\
A_{ij} & & \\
\cdot & \cdot & \cdot \\
A_{jk} & & \\
\cdot & \cdot & \cdot \\
A_{ij} & & \\
\underset{i}{\bullet} & \underset{j}{\bullet} & \underset{k}{\bullet}
\end{array}
\quad \equiv \quad
\begin{array}{ccc}
\overset{i}{\bullet} & \overset{j}{\bullet} & \overset{k}{\bullet} \\
A_{jk} & & \\
\cdot & \cdot & \cdot \\
A_{ij} & & \\
\cdot & \cdot & \cdot \\
A_{jk} & & \\
\underset{i}{\bullet} & \underset{j}{\bullet} & \underset{k}{\bullet}
\end{array}
$$

Figure 4.8

Hence the proofs of (i), (ii), (iii), (v) and (vi) are complete, and from the discussion at the start of the proof, it follows that the proof of the Lemma is complete. ∎

(v)

$i \quad j \quad k \quad \lambda \qquad\qquad i \quad j \quad k \quad \lambda$

$A_{jk}$ $\qquad\qquad\qquad\qquad b_{i\lambda}$

$\equiv$

$b_{i\lambda}$ $\qquad\qquad\qquad\qquad A_{jk}$

$i \quad j \quad k \quad \lambda \qquad\qquad i \quad j \quad k \quad \lambda$

$Figure\ 4.9$

$i \quad j \quad \lambda \quad k \qquad\qquad i \quad j \quad \lambda \quad k$

$b_{ij}^{-1}$

$A_{jk}$ $\qquad\qquad\qquad\qquad\qquad \equiv$

$b_{ij}$

$i \quad j \quad \lambda \quad k \qquad\qquad i \quad j \quad \lambda \quad k$

$Figure\ 4.10$

$i \quad j \quad \lambda \quad k \qquad\qquad i \quad j \quad \lambda \quad k$

$b_{i\lambda}$

$\qquad\qquad\qquad\qquad\qquad b_{ij}^{-1} A_{jk} b_{ij}$

$\equiv$

$b_{ij}^{-1} A_{jk} b_{ij}$

$\qquad\qquad\qquad\qquad\qquad b_{i\lambda}$

$i \quad j \quad \lambda \quad k \qquad\qquad i \quad j \quad \lambda \quad k$

$Figure\ 4.11$

## 4.5 Action of the braid group

In this section we shall derive recursion formulae for $A^{(r)}_{\lambda\mu}$ and $b^{(r)}_{\lambda\mu}$ in terms of the matrices $\{A^{(r-1)}_{\lambda\mu}\}$ and $\{b^{(r-1)}_{\lambda\mu}\}$. One can think of such relations as connecting transformation properties of embedded $r$-torii with $z_{r+1}, \ldots, z_m, w_1, \ldots, w_n$ fixed with those of embedded $(r-1)$-torii with $z_r, \ldots, z_m, w_1, \ldots, w_n$ fixed. One can think of the space $\mathcal{S}^r_{\mathbf{w}}$ as equivalent to $\mathcal{S}^m_{\mathbf{w}}$ where:

$$\{z_1, \ldots, z_m\} \text{ is replaced by } \{z_1, \ldots, z_r\};$$

$$\{w_1, \ldots, w_n\} \text{ is replaced by } \{z_{r+1}, \ldots, z_m, w_1, \ldots, w_n\}.$$

Passing from $\mathcal{S}^r_{\mathbf{w}}$ to $\mathcal{S}^{r-1}_{\mathbf{w}}$ is thus given by thinking of one of the $z$'s (namely $z_r$) as part of the set of parameters on the base (the $w$'s) rather than as a parameter on the fibre.

**Theorem 4.4**     *The actions of $A^{(r)}_{\lambda\mu}$ and $b^{(r)}_{\lambda\mu}$ on $\langle \mathcal{S}^r_{\mathbf{w}} \rangle$ are given in block form by the following matrices, where the blocks are separated by the value of $\alpha_r \in \{z_{r+1}, \ldots, z_m, w_1, \ldots, w_n\}$:*

$$
A^{(r)}_{\lambda\mu} =
\begin{pmatrix}
A & & & & & & & & \\
 & \ddots & & & & & & & \\
 & & A & & & & & & \\
 & & & 0 & \cdots & (b_j - 1)b_\lambda^{-1}Ab_\lambda & \cdots & b_\mu A & \\
 & & & & \ddots & & & & \\
 & & & & & b_\lambda^{-1}Ab_\lambda & & & \\
 & & & & & & \ddots & & \\
 & & & A & \cdots & A(1 - b_j) & \cdots & A(1 - b_\mu) & \\
 & & & & & & & A & \\
 & & & & & & & & \ddots \\
 & & & & & & & & & A
\end{pmatrix}
$$

with column $\lambda$, column $j$, and column $\mu$ indicated.

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*and:*



$$
b^{(r)}_{\lambda\mu} =
$$

where all entries vanish except those given above; $b_\lambda = b^{(r-1)}_{z_r\lambda}$, $b = b^{(r-1)}_{\lambda\mu}$, $A = A^{(r-1)}_{\lambda\mu}$.

PROOF: To determine the action of $A^{(r)}_{\lambda\mu}$ and $b^{(r)}_{\lambda\mu}$ on an $r$-torus $\gamma_{\underline{\alpha}}$, for $\underline{\alpha} \in \mathcal{S}^r_{\mathbf{w}}$, we consider the following four cases separately:

(i) $\alpha_r$ either greater than both $\lambda$, $\mu$ or less than both $\lambda$, $\mu$;

(ii) $\alpha_r = \lambda$;

(iii) $\alpha_r = \mu$;

(iv) $\lambda < \alpha_r < \mu$.

In each case we evaluate the actions of $A^{(r)}_{\lambda\mu}$ and $b^{(r)}_{\lambda\mu}$ on the $r$-torus $\gamma_{\underline{\beta}.\alpha_r}$ where $\underline{\beta} = \underline{\alpha}^{r-1} \in \mathcal{S}^{r-1}_{\mathbf{w}}$, in terms of the action of the braid group on $(r-1)$-torii.

Case (i): Either $\alpha_r > \lambda$, $\mu$ or $\alpha_r < \lambda$, $\mu$.

In this case, under transposition of $\lambda$ and $\mu$, the loop $\beta_{z_r \alpha_r}$ is unchanged, and thus $\gamma_{\underline{\alpha}}$ transforms according to $\gamma_{\underline{\beta}}$, that is:

$$A^{(r)}_{\lambda\mu}(\underline{\alpha}) = A^{(r-1)}_{\lambda\mu}(\underline{\beta}).\alpha_r.$$

When $\alpha_r < \lambda$, $\mu$ we have the situation illustrated in Fig. 4.12.

$\mu$

action of $\alpha_{\lambda\mu}$

motion of

$\lambda$

$z_r$

$\alpha_r$

$\beta_{z_r \alpha_r}$

$z_r^0$
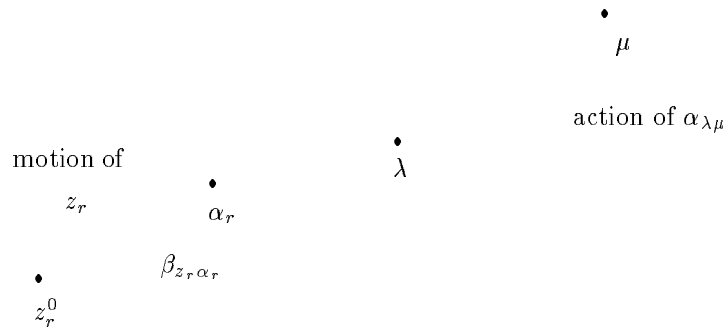
Figure 4.12

When $\alpha_r > \lambda$, $\mu$ we obtain Fig. 4.13.

motion
of $z_r$

$\alpha_r$

$\mu$

$\lambda$

action $\alpha_{\lambda\mu}$

$z_r^0$
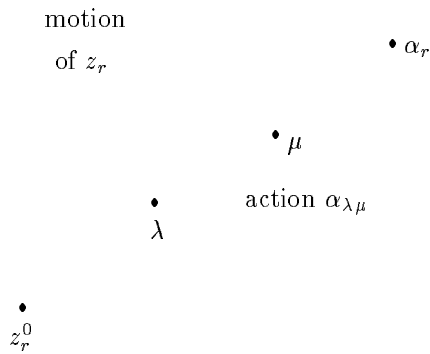
Figure 4.13

Similarly one obtains $b^{(r)}_{\lambda\mu}(\underline{\alpha}) = b^{(r-1)}_{\lambda\mu}(\underline{\alpha}_{r-1}.\alpha_r)$.

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*Case (ii):* $\alpha_r = \lambda$.

In this case the transformation given by $\alpha_{\lambda\mu}, \beta_{\lambda\mu}$ will deform the curve $\beta_{z_r\lambda}$ as indicated below. To obtain the deformed curve in terms of the basis loops, we cut the image loop up into three parts (under $\beta_{\lambda\mu}$). This gives rise to an image of the $r$-torus $\gamma_{\underline{\alpha}}$ which is:

$$b_{\lambda\mu}^{(r-1)}\underline{\beta}.\mu + b_{z_r\mu}^{(r-1)}b_{\lambda\mu}^{(r-1)}\underline{\beta}.\lambda - b_{z_r\mu}^{(r-1)}b_{z_r\lambda}^{(r-1)}b_{z_r\mu}^{(r-1)}b_{\lambda\mu}^{(r-1)}\underline{\beta}.\mu$$

under $b_{\lambda\mu}^{(r)}$; and under $A_{\lambda\mu}^{(r)}$ is:

$$A_{\lambda\mu}^{(r-1)}\underline{\beta}.\mu$$

For, when $z_r$ goes around $\lambda$, the cycle $\gamma_{\underline{\beta}}$ is transformed according to $b_{z_r\lambda}^{(r-1)}$; see Fig. 4.14.
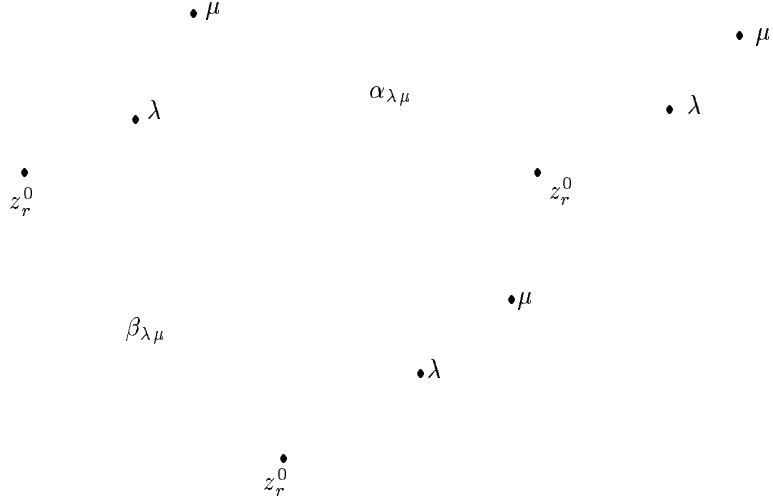


*Figure 4.14*

The dissection of the image of $\beta_{z_r\lambda}$ under $\beta_{\lambda\mu}$ is given by Fig. 4.15.

Hence if we write $b$, $b_\lambda$ and $A$ for $b_{\lambda\mu}^{(r-1)}$, $b_{z_r\lambda}^{(r-1)}$ and $A_{\lambda\mu}^{(r-1)}$, respectively, then we have:

$$\left.\begin{array}{l} A_{\lambda\mu}^{(r)}(\underline{\alpha}) = A\underline{\beta}.\mu \\[2mm] b_{\lambda\mu}^{(r)}(\underline{\alpha}) = (1 - b_\mu^{-1}b_\lambda b_\mu)b\underline{\beta}.\mu + b_\mu b\underline{\beta}.\lambda \end{array}\right\}$$

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$$\mu \bullet \qquad\qquad b\underline{\beta}.\mu$$

$$-b_{z_r\mu}^{(r-1)\,-1}\, b_{z_r\lambda}^{(r-1)}\, b_{z_r\mu}^{(r-1)}\, b\underline{\beta}.\mu$$

$$\bullet\,\lambda$$

$$b_{z_r\mu}^{(r-1)}\, b\underline{\beta}.\lambda$$

$$\therefore\, z_r^0$$

*Figure 4.15*

However, by Lemma 4.3(ix),

$$b_\lambda b_\mu b = b_\mu b b_\lambda$$

and so $(1 - b_\mu^{-1} b_\lambda b_\mu) b = b(1 - b_\lambda)$. This gives the $\alpha_r = \lambda$ columns of $A_{\lambda\mu}^{(r)}$, $b_{\lambda\mu}^{(r)}$ as required in the Theorem.

*Case (iii):* $\alpha_r = \mu$

The deformed versions of the loop $\beta_{z_r\mu}$ under the motions $\alpha_{\lambda\mu}$, $\beta_{\lambda\mu}$ are shown in Fig. 4.16.

We thus obtain:

$$\left.\begin{aligned}
A_{\lambda\mu}^{(r)}(\underline{\alpha}) &= (1 - b_\mu^{-1} b_\lambda b_\mu) A\underline{\beta}.\mu + b_\mu A\underline{\beta}.\lambda \\[2mm]
b_{\lambda\mu}^{(r)}(\underline{\alpha}) &= (1 - b_\lambda^{-1} b_\mu b_\lambda) b_\mu b\underline{\beta}.\lambda + (1 + b_\lambda b_\mu - b_\mu^{-1} b_\lambda^{-1} b_\mu b_\lambda b_\mu) b\underline{\beta}.\mu
\end{aligned}\right\}$$

Using the relations:

$$b_\lambda b_\mu A = b_\mu A b_\mu \qquad\qquad \text{(Lemma 4.3(i),(ii))}$$

$$b_\lambda b_\mu b = b b_\lambda b_\mu = b_\mu b b_\lambda \qquad \text{(Lemma 4.3(ix))}$$

we can reduce the above transformation laws to:

$$\left.\begin{aligned}
A_{\lambda\mu}^{(r)}(\underline{\alpha}) &= A(1 - b_\mu)\underline{\beta}.\mu + b_\mu A\underline{\beta}.\lambda \\[2mm]
b_{\lambda\mu}^{(r)}(\underline{\alpha}) &= b_\mu b(1 - b_\mu)\underline{\beta}.\lambda + b(1 + b_\lambda b_\mu - b_\mu)\underline{\beta}.\mu
\end{aligned}\right\}$$

This gives the same expressions for the $\mu^{\text{th}}$ columns of the matrices for $A_{\lambda\mu}^{(r)}$ and $b_{\lambda\mu}^{(r)}$ as are given in the Theorem.

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$\beta_{z_r\mu}$     $\bullet\,\mu$          $\bullet\,\mu$

$\bullet\,\lambda$     $\beta_{\lambda\mu}$     $\bullet\,\lambda$

$\bullet$
$z_r^0$     $\bullet$
$z_r^0$

$\bullet\mu$

$\alpha_{\lambda\mu}$     $\bullet\lambda$

$\bullet$
$z_r^0$
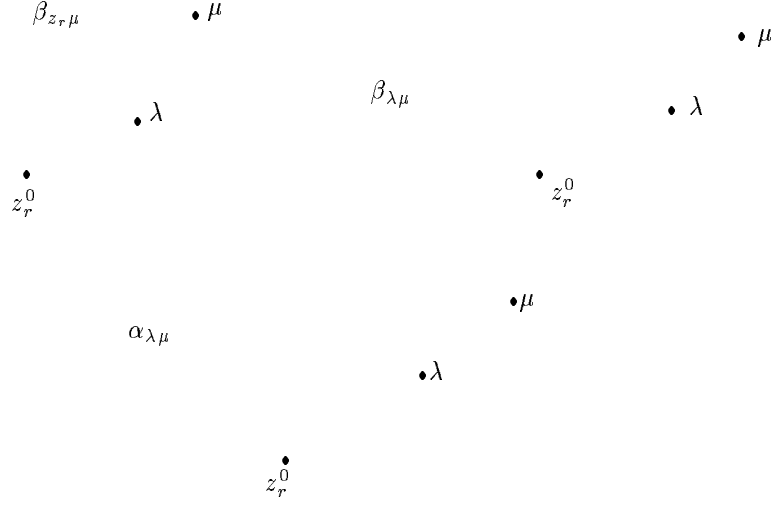
*Figure 4.16*

*Case (iv):* $\lambda < \alpha_r < \mu$

This is the last, and most complicated case. The actions of $\beta_{\lambda\mu}$ and $\alpha_{\lambda\mu}$ on $\beta_{z_r\mu}$ are illustrated in Fig. 4.17.

This gives rise to:

$$\left.\begin{array}{l} A_{\lambda\mu}^{(r)}(\underline{\alpha}) = (b_{\alpha_r}-1)b_\lambda{}^{-1}b_\mu A\underline{\beta}.\lambda + (1-b_\mu{}^{-1}b_\lambda b_{\alpha_r}b_\lambda{}^{-1}b_\mu)A\underline{\beta}.\mu + b_\lambda{}^{-1}b_\mu A\underline{\beta}.\alpha_r \\[2mm] b_{\lambda\mu}^{(r)}(\underline{\alpha}) = -(1-b_\lambda{}^{-1}b_\mu b_\lambda)(1-b_{\alpha_r})b_\lambda{}^{-1}b_\mu{}^{-1}b_\lambda b_\mu b\underline{\beta}.\lambda + b_\lambda{}^{-1}b_\mu{}^{-1}b_\lambda b_\mu b\underline{\beta}.\alpha_r \\[2mm] \qquad\qquad + (1-b_\mu{}^{-1}b_\lambda b_\mu)(1-b_\mu{}^{-1}b_\lambda{}^{-1}b_\mu b_\lambda b_{\alpha_r}b_\lambda{}^{-1}b_\mu{}^{-1}b_\lambda b_\mu)b\underline{\beta}.\mu \end{array}\right\}$$

However, by Lemma 4.3(ix),

$$b_\lambda{}^{-1}b_\mu{}^{-1}b_\lambda b_\mu b = b_\lambda{}^{-1}bb_\lambda = b_\mu bb_\mu{}^{-1}$$

which by 4.3(viii), commutes with $b_{\alpha_r}$. Thus the 'coefficients' of $\underline{\beta}.\mu$ and $\underline{\beta}.\lambda$ in $b_{\lambda\mu}^{(r)}(\underline{\alpha})$ reduce to:

$$(1-b_\mu{}^{-1}b_\lambda b_\mu)(b-bb_{\alpha_r}) = b(1-b_\lambda)(1-b_{\alpha_r})$$

and     $-(1-b_\lambda{}^{-1}b_\mu b_\lambda)b_\lambda{}^{-1}bb_\lambda(1-b_{\alpha_r}) = b_\mu b(1-b_\mu{}^{-1})(1-b_{\alpha_r})$

respectively. Also, by Lemma 4.3(i),

$$b_\lambda{}^{-1}b_\mu A = b_\lambda{}^{-1}Ab_\lambda$$

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$\bullet\mu$

$\beta_{z_r\alpha_r}$ $\bullet\alpha_r$ $\beta_{\lambda\mu}$ $\bullet\ \mu$

$\bullet\ \alpha_r$

$\bullet$
$\lambda$

$\bullet$
$\lambda$

$\bullet$
$z_r^0$

$\bullet$
$z_r^0$

$\mu\ \bullet$

$\alpha_{\lambda\mu}$

$\bullet$
$\alpha_r$

$\bullet$
$\lambda$

$z_r^0\ \bullet$
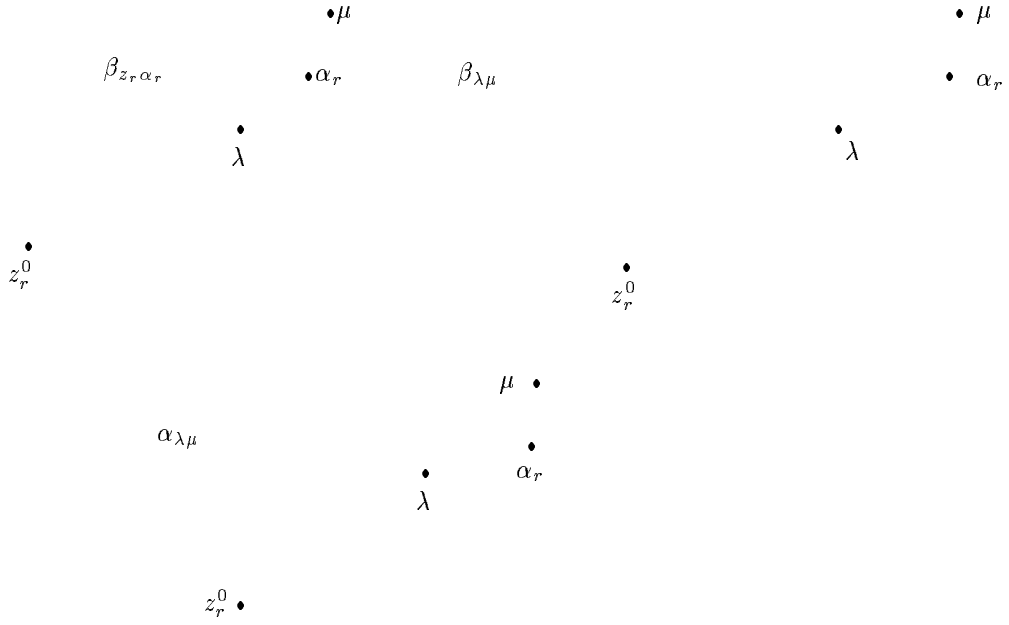
Figure 4.17

which, by Lemma 4.3(vi), commutes with $b_{\alpha_r}$. Thus:

$$(1 - b_\mu^{-1} b_\lambda b_{\alpha_r} b_\lambda^{-1} b_\mu)A = A - Ab_{\alpha_r}\ .$$

Hence the actions of $A_{\lambda\mu}^{(r)}$ and $b_{\lambda\mu}^{(r)}$ on $\underline{\alpha}$ are given by:

$$\left.\begin{aligned}
A_{\lambda\mu}^{(r)}(\underline{\alpha}) &= (b_{\alpha_r} - 1)b_\lambda^{-1} Ab_\lambda \underline{\beta}.\lambda + A(1 - b_{\alpha_r})\underline{\beta}.\mu + b_\lambda^{-1} Ab_\lambda \underline{\beta}.\alpha_r \\
b_{\lambda\mu}^{(r)}(\underline{\alpha}) &= b_\mu b(1 - b_\mu^{-1})(1 - b_{\alpha_r})\underline{\beta}.\lambda + b(1 - b_\lambda)(1 - b_{\alpha_r})\underline{\beta}.\mu + b_\lambda^{-1} b b_\lambda \underline{\beta}.\alpha_r
\end{aligned}\right\}$$

This gives rise to the columns of $A_{\lambda\mu}^{(r)}$ and $b_{\lambda\mu}^{(r)}$ between the $\lambda^{\text{th}}$ and $\mu^{\text{th}}$ columns, as given in the Theorem.

This completes the four cases required to prove the Theorem. ∎

Theorem 4.4 can be used to recursively compute the matrices $A_{\lambda\mu}^{(r)}$, $b_{\lambda\mu}^{(r)}$ for all $\lambda < \mu$ with:

$$\lambda,\ \mu \in \{z_{r+1}, \ldots, z_m, w_1, \ldots, w_n\}$$

starting from the zeroth order matrices:

$$\left.\begin{aligned}
A_{\lambda\mu}^{(0)} &= 1 \\
b_{\lambda\mu}^{(0)} &= q_{\lambda\mu}^{-1}.
\end{aligned}\right\}$$

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Using Theorem 4.4, it is now easy to check that $\ker(\mathbf{D}_r)$ is fixed under $A_{\lambda\mu}^{(m)}$ and $b_{\lambda\mu}^{(m)}$. For, using (4.3.1), it can be seen that:

$$\left.\begin{array}{c} \mathbf{D}_r^{(r)} A_{\lambda\mu}^{(r)} = A_{\lambda\mu}^{(r-1)} \mathbf{D}_r^{(r)} \\[2mm] \mathbf{D}_r^{(r)} b_{\lambda\mu}^{(r)} = b_{\lambda\mu}^{(r-1)} \mathbf{D}_r^{(r)} \end{array}\right\}$$

By applying (4.3.2), it may be observed that:

$$\left.\begin{array}{c} \mathbf{D}_r^{(m)} A_{\lambda\mu}^{(m)} = \overline{A_{\lambda\mu}^{(m-1)}} \mathbf{D}_r^{(m)} \\[2mm] \mathbf{D}_r^{(m)} b_{\lambda\mu}^{(m)} = \overline{b_{\lambda\mu}^{(m-1)}} \mathbf{D}_r^{(m)} \end{array}\right\} \tag{4.5.1}$$

where $\overline{A_{\lambda\mu}^{(m-1)}}$ and $\overline{b_{\lambda\mu}^{(m-1)}}$ denote the matrices obtained when $\alpha_{\lambda\mu}$, $\beta_{\lambda\mu}$ are applied to the space of chains $\langle \mathcal{S}_{\mathbf{w}}^m \rangle$, where we replace $\{z_1, \ldots, z_m\}$ by $\{z_1, \ldots, \widehat{z_r}, \ldots, z_m\}$. These latter matrices thus act on a space spanned by:

$$\underline{\alpha} = (\alpha_1, \ldots, \widehat{\alpha_r}, \ldots, \alpha_m)$$

with $\underline{\alpha} \in \{z_{i+1}, \ldots, z_m, w_1, \ldots, w_n\}$. It is clear from (4.5.1) that:

$$\left.\begin{array}{c} A_{\lambda\mu}^{(m)} \ker(\mathbf{D}_r^{(m)}) \subseteq \ker(\mathbf{D}_r^{(m)}) \\[2mm] b_{\lambda\mu}^{(m)} \ker(\mathbf{D}_r^{(m)}) \subseteq \ker(\mathbf{D}_r^{(m)}) \end{array}\right\}$$

Since by Lemma 4.2, the homology is given by the intersection of $\ker(\mathbf{D}_r)$ over $r = 1, 2, \ldots, m$, thus one may obtain the actions of $A_{\lambda\mu}^{(m)}$, $b_{\lambda\mu}^{(m)}$ on homology by restricting to the subspace $\ker(\mathbf{D}_r)$ at the $r^{\text{th}}$ stage of calculation. Under such a procedure, starting with $A^{(0)}$, $b^{(0)}$ matrices, we use Theorem 4.4 to obtain the $A^{(1)}, b^{(1)}$ matrices and then restrict to $\ker(\mathbf{D}_1)$. These reduced matrices are then used at the next stage, being substituted into Theorem 4.4 again. At the end of the procedure, the matrices obtained give only the actions of $A^{(m)}$ and $b^{(m)}$ matrices on homology (and not on all chains). For examples of these procedures, see Chapter 5.

## 4.6 Action of the symmetric group

As seen in §3.1, the symmetric group $S_m$ has a natural action on the homology space:

$$H_m(Y_{\mathbf{w},m}, \chi_{\mathbf{w},m}(\mathbf{q}))$$

so long as $\mathbf{q}$ is suitably chosen. This action is specified by the action of the generators $(i \, i{+}1)$ of $S_m$. Let $j_{i\,i+1}^{(r)}$ denote the action on $\langle \mathcal{S}_{\mathbf{w}}^r \rangle$ of the transposition $z_i \leftrightarrow z_{i+1}$ where $i{+}1 \leq r \leq m$. By this action we mean

the following: suppose that $\underline{\alpha} \in \mathcal{S}_{\mathbf{w}}^r$. Then $\underline{\alpha}$ defines an $r$-torus in $Y_{\mathbf{w},m}$; and lifts to an embedding of an $r$-dimensional cube in $Y_{\mathbf{w},m}$. Under the transposition $z_i \leftrightarrow z_{i+1}$, it maps to another $r$-dimensional cube in $Y_{\mathbf{w},m}$. However the transformed cube has a base-point given by:

$$(z_1^0, \ldots, z_{i-1}^0, z_{i+1}^0, z_i^0, \ldots, z_m^0)$$

and this base-point does not have its imaginary parts ordered in the natural way. To correspond this with a standard $r$-chain, it is necessary to move the base-point to $(z_1^0, \ldots, z_{i-1}^0, z_i^0, z_{i+1}^0, \ldots, z_m^0)$. When this is done, the $(i+1)$-torus given by $\underline{\alpha}^{(i+1)}$ transforms to another $(i+1)$-torus, given by:

$$j_{i\,i+1}^{(i+1)} \underline{\alpha}^{(i+1)}.$$

The whole $m$-torus $\gamma_{\underline{\alpha}}$ transforms according to $j_{i\,i+1}^{(m)}$ where:

$$j_{i\,i+1}^{(r)} = \begin{pmatrix} j_{i\,i+1}^{(r-1)} & & \\ & \ddots & \\ & & j_{i\,i+1}^{(r-1)} \end{pmatrix}$$

for $r = i+2, \ldots, m$; where the blocks are separated by the value of $\alpha_r$.

So, the important part of the matrix $j_{i\,i+1}^{(m)}$ is given by $j_{i\,i+1}^{(i+1)}$. The action on $\langle \mathcal{S}_{\mathbf{w}}^{i+1} \rangle$ is given by a partition matrix in terms of actions on $(i-1)$-torii of the braid group, by the following Theorem.

**Theorem 4.5** *The matrix for $j_{i\,i+1}^{(i+1)}$ as a partitioned matrix with blocks defined by the values of $\alpha_i$ and $\alpha_{i+1}$ is given by:*

$$\begin{pmatrix} (b_\lambda - 1)b'^{-1} - b_\lambda & (1-b_\lambda)b'^{-1}(b_\lambda'-1) + b_\lambda b_\lambda' & b_\lambda(b_\mu'-1) + (1-b_\lambda)b'^{-1}(b_\mu'^{-1}-1) & (b_\mu-1)b'^{-1} \\ \\ b'^{-1} - 1 & b'^{-1}(1-b_\lambda') + b_\lambda' & (b'^{-1}-1)(1-b_\mu') & 0 \\ \\ 0 & 0 & 0 & b'^{-1} \\ \\ 0 & 0 & 1 & 0 \end{pmatrix} A'$$

*where $A'$, $b'$, $b_\lambda$, $b_\lambda'$ denote $A_{z_i z_{i+1}}^{(i-1)}$, $b_{z_i z_{i+1}}^{(i-1)}$, $b_{z_i \lambda}^{(i-1)}$ and $b_{z_{i+1}\lambda}^{(i-1)}$, respectively. Here the entries correspond to $z_{i+1}\lambda$, $\lambda\lambda$, $\mu\lambda$ and $\lambda\mu$ where $\lambda > \mu$.*

PROOF: Using the results of Lemma 4.3, it can be shown that the above matrix satisfies the following properties:

(a) $j_{i\,i+1}$ preserves $\ker(\mathbf{D}_i) \cap \ker(\mathbf{D}_{i+1})$;

(b) $j_{i\,i+1}^2$ is a scalar, namely $q_{z_i z_{i+1}}$;

(c) $j_{i\,i+1}$ commutes with $A_{\lambda\mu}^{(i+1)}$ for all $\lambda$, $\mu$;

(d) $j_{i\,i+1}$ satisfy the braid relation.

As these are the only properties that we require of the matrices $j_{i\,i+1}$, at this stage it is not necessary to actually show that they correspond to the natural action of $S_m$. Of course, the natural action of $S_m$ will satisfy properties (a)–(d), and this follows immediately from the definitions of the $S_m$ and $B_n$ actions in §3.1. To actually derive the above matrix for the action of $S_m$, it is necesary to compute the images of the four types of elements of $\mathcal{S}_{\mathbf{w}}^{i+1}$ of the form: $\underline{\beta}.z_{i+1}\lambda$, $\underline{\beta}.\lambda\lambda$, $\underline{\beta}.\lambda\mu$, $\underline{\beta}.\mu\lambda$ where $\lambda > \mu$. When we swap $z_i$ and $z_{i+1}$ and then shift the transformed base-point back to a position in which the imaginary parts are in order, we obtain Figs. 4.18 and 4.19 for the first two of these cases.
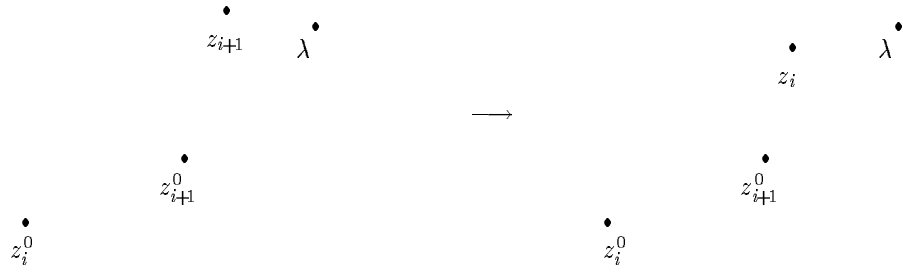
Figure 4.18

When the torii corresponding to the curves on the right hand side are expressed in terms of the basis

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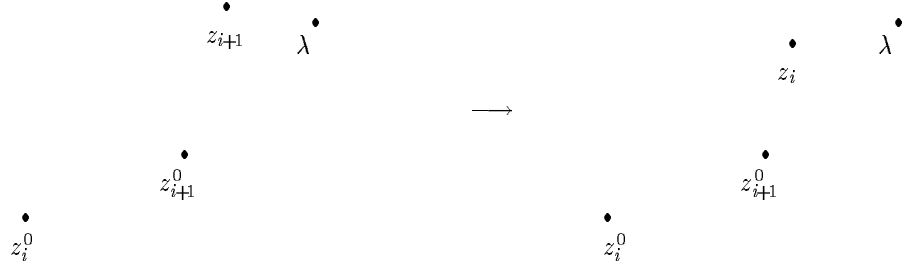
*Figure 4.19*

elements in $\mathcal{S}_{\mathbf{w}}^{m}$, we obtain:

$$\left.\begin{array}{l} \left((b_{\lambda}-1){b'}^{-1}-b_{\lambda}\right)A'\underline{\beta}.z_{i+1}\lambda + \left({b'}^{-1}-1\right)A'\underline{\beta}.\lambda\lambda \\[2mm] \left((1-b_{\lambda}){b'}^{-1}(b_{\lambda}'-1)+b_{\lambda}b_{\lambda}'\right)A'\underline{\beta}.z_{i+1}\lambda + \left({b'}^{-1}(1-b_{\lambda}')+b_{\lambda}'\right)A'\underline{\beta}.\lambda\lambda \end{array}\right\}$$

although the details required to obtain this result are not very simple.  ∎

The only part of the action of the braid group on homology with which we will be concerned is that which is on the part invariant under the action of the symmetric group $S_m$. By (c) above this is well defined. We now have the equipment necessary to explicitly compute this action; all the actions involved can be computed using Theorems 4.4, 4.5 together with Lemma 4.2.

Finally, we make the following remark about the matrices $j_{i\,i+1}$ obtained above. Since we are only interested in the action on homology, it is clear, using Lemma 4.2, that all the relevant parts of the action of $j_{i\,i+1}^{(i+1)}$ are contained in its action restricted to $\ker(\mathbf{D}_i^{(i+1)}) \cap \ker(\mathbf{D}_{i+1}^{(i+1)})$. However,

$$\mathbf{D}_i^{(i)} = \left( b'-1 \,\middle|\, \cdots \,\middle|\, b_{z_m}-1 \,\middle|\, b_{w_1} \,\middle|\, \cdots \,\middle|\, b_{w_n} \right)$$

by (4.3.1), and so:

$$A'{b'}^{-1}\mathbf{D}_i^{(i)} = \left( (1-{b'}^{-1})A' \,\middle|\, \cdots \,\middle|\, {b'}^{-1}(b_{\lambda}'-1)A' \,\middle|\, \cdots \right)$$

with the marked entry giving the $\lambda^{\text{th}}$ block. This means that $j_{i\,i+1}^{(i+1)}$'s action on $\ker(\mathbf{D}_i^{(i+1)}) \cap \ker(\mathbf{D}_{i+1}^{(i+1)})$ is

equivalently specified by the matrix:

$$\begin{pmatrix} (b_\lambda -1)b'^{-1} - b_\lambda & (1-b_\lambda)b'^{-1}(b_\lambda' -1) + b_\lambda b_\lambda' & b_\lambda(b_\mu' -1) + (1-b_\lambda)b'^{-1}(b_\mu'^{-1} -1) & (b_\mu -1)b'^{-1} \\ \\ 0 & b_\lambda' & (b_\mu' -1) & b'^{-1}(b_\lambda' -1) \\ \\ 0 & 0 & 0 & b'^{-1} \\ \\ 0 & 0 & 1 & 0 \end{pmatrix} A'$$

In later Chapters, this form of $j_{i\,i+1}$ will come in useful.

# 5: Examples

In this Chapter we will discuss the two special cases $m = 1$ and $m = 2$ in detail. These correspond to homology in one and two dimensions, respectively. The case $m = 1$ gives rise to the simplest non-trivial braid group representation, namely the Burau representation. The case $m = 2$ is the first case in which the action of the symmetric group is present. At the end of the Chapter, the case in which the local coefficient system, $\chi_{\mathbf{w},m}(\mathbf{q})$ is trivial, is also discussed. The examples of this Chapter should be borne in mind throughout the next Chapter, where we deal in detail with the case of general $m$.

## 5.1 $m = 1$ and the Alexander polynomial

When $m = 1$, we have precisely one $z_i$, so that $\mathcal{S}^1_{\mathbf{w}}$ is given by:

$$\left\{ (\lambda) \mid \lambda = w_1, \dots, w_n \right\}.$$

The chains thus form an n-dimensional space on which the braid group acts. We can now apply Theorem 4.4 with $r = 1$, starting with:

$$\left. \begin{aligned} A^{(0)}_{\lambda\mu} &= 1 \\ b^{(0)}_{\lambda\mu} &= 1 \\ b^{(0)}_{z_1\lambda} &= q^{-1} \end{aligned} \right\}$$

for all $\lambda$, $\mu \in \{w_1, \dots, w_n\}$. This gives rise to the following matrix for $A^{(1)}_{w_i w_{i+1}}$:

$$\begin{pmatrix} 1 & & & & & & & \\ & \ddots & & & & & & \\ & & 1 & & & & & \\ & & & 0 & q^{-1} & & & \\ & & & 1 & 1 - q^{-1} & & & \\ & & & & & 1 & & \\ & & & & & & \ddots & \\ & & & & & & & 1 \end{pmatrix} \qquad (5.1.1)$$

where the non-trivial $2 \times 2$ block occurs in the $i^{\text{th}}$ and $i+1^{\text{th}}$ rows and columns.

By (4.3.1), the matrix for $\mathbf{D}_1^{(1)}$ is the $1 \times n$ matrix:

$$\left( q^{-1} - 1, \ldots, q^{-1} - 1 \right).$$

Lemma 4.2 now gives the homology $H_1\left(Y_{\mathbf{w},1}, \chi_{\mathbf{w},1}(\mathbf{q})\right)$ as the subset of $\mathcal{C}_1$ given by $\ker \mathbf{D}_1$. Thus the homology space can be identified with the subset:

$$\left\{ (x_1, \ldots, x_n) \mid x_1 + \cdots + x_n = 0 \right\} \tag{5.1.2}$$

of $\mathcal{C}_1 = \langle \mathcal{S}_{\mathbf{w}}^1 \rangle \cong \mathbf{C}^n$. This subset is clearly preserved by the matrix $A_{w_i w_{i+1}}^{(1)}$ above, and thus the action of the $A^{(1)}$-matrices on homology gives rise to a representation of $B_n$ on an $(n-1)$-dimensional space.

The eigenvalues of the action of $A_{w_i w_{i+1}}^{(1)}$ on $\mathcal{C}_1$ are $q^{-1}$ and $1$ (with multiplicity $n-1$). The action of $B_n$ on homology thus factors through the Hecke algebra $H_n(q^{-1})$ (since all the $\sigma_i$ have eigenvalues $-q^{-1}$ and $1$ only). Its action is known as the *Burau representation*, $\pi_{\Lambda_1}$. This representation is important in knot theory in the context of the *Alexander polynomial*.

Suppose that $L$ is a link. By Theorem 2.1 (i), $L$ can be expressed as the closure $\widehat{\gamma}$ of some braid $\gamma \in B_n$, for suitably large $n$. The Alexander polynomial, $\Delta_L$, of the link $L$, can now be defined by:

$$\Delta_L = \frac{\det \left( 1 - \pi_{\Lambda_1}(\gamma) \right)}{1 + q^{-1} + \cdots + q^{1-n}} \tag{5.1.3}$$

in terms of the representation $\pi_{\Lambda_1}$. This defines a polynomial in the one variable $q$. See for example [Jo 4].

However, the original definition of $\Delta_L$ (see [Al 1]) was given in terms of covering spaces. Consider the complement of the link $S^3 \backslash L$. There is a natural map:

$$\pi_1(S^3 \backslash L) \longrightarrow \mathbf{Z}$$

$$[\Gamma] \longmapsto (\text{the linking number of } \Gamma \text{ and } L)$$

where $\Gamma$ is any closed curve in $S^3$, not intersecting the link $L$. This map defines an infinite cyclic covering $\widetilde{S^3 \backslash L}$ of $S^3 \backslash L$. There is a natural action, $T$, on $\widetilde{S^3 \backslash L}$ given by a translation in which each branch of the cover is translated into the next. Then $T$ induces an action on the first homology:

$$H_1(\widetilde{S^3 \backslash L}).$$

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This homology is finite dimensional, and the characteristic polynomial of the induced action, $T_*$, is the Alexander polynomial. This definition of $\Delta_L$ makes it obvious that $\Delta_L$ is invariant under continuous deformations of $L$, through non-self-intersecting curves.

The braid approach to $\Delta_L$, as given by (5.1.3), can be considered as corresponding to an embedding of $L$ in $S^1 \times S^2$, as opposed to an embedding in $S^3$. The latter approach is that used in the above topological interpretation of $\Delta_L$. For, a braid $\gamma \in B_n$ defines a map:

$$\gamma \colon S^1 \longrightarrow \widetilde{X}_n$$

where $\gamma(t) \in \widetilde{X}_n$ is given by $n$ points $\{w_1(t), \ldots, w_n(t)\}$, say. The subset of $S^1 \times S^2$ specified by:

$$\Big\{ \big(t, w_i(t)\big) \mid t \in S^1, \quad i = 1, 2, \ldots, n \Big\}$$

now gives the link $L = \widehat{\gamma}$, as embedded in $S^1 \times S^2$. Suppose next that $\Gamma$ is a closed curve in the complement of $L$, with base-point $(0, z^0) \in S^1 \times S^2$. We use the correspondence between $\mathbf{C} \cup \{\infty\}$ and the Riemann sphere $S^2$. Then $\Gamma$ is defined by a map:

$$\Gamma \colon [0, 1] \longrightarrow (S^1 \times S^2) \backslash L$$

with $\Gamma(0) = \Gamma(1) = (0, z^0)$. Such a curve $\Gamma$ is homotopic, in $(S^1 \times S^2) \backslash L$, to a combination of the curves $\Gamma_i$ $(0 \leq i \leq n)$ given by:

(i) for each $i = 1, 2, \ldots, n$, $\Gamma_i(t) \subseteq \{0\} \times S^2$ for all $t$, with the winding number of $\Gamma_i$, considered as embedded in $S^2 \cong \mathbf{C} \cup \{\infty\}$, around $w_j(0)$ being $\delta_{ij}$;

(ii) $\Gamma_0(t) = (t, z^0)$.

Thus $\pi_1\big((S^1 \times S^2)\backslash L\big)$ is generated by the $(n+1)$ elements associated with $\Gamma_0, \Gamma_1, \ldots, \Gamma_n$. Under the surgery $S^1 \times S^2 \rightsquigarrow S^3$, the situation transforms so that $\Gamma_0$ disappears as a non-trivial generator, and $\pi_1(S^3 \backslash L)$ is generated by $n$ elements, of similar form to $\Gamma_1, \ldots, \Gamma_n$. Hence, it is not surprising to find that the characteristic polynomial $\det(T^* - q^{-1})$ (where $T^*$ is the translation map on $\widetilde{S^3 \backslash L}$ defined above) is related to the local coefficient system on:

$$\mathbf{C} \backslash \big\{w_1(0), \ldots, w_n(0)\big\}$$

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with twistings of $q^{-1}$ around each $w_i$. This local coefficient system is precisely $\chi_{\mathbf{w},1}(q)$ as defined in §3.1. See Fig. 5.1.

$$w_3(t) \qquad w_1(t) \qquad w_2(t) \qquad \bullet \qquad \longleftrightarrow$$

$$w_3(0)$$
$$w_2(0)$$
$$w_1(0)$$

$$L$$

$$t = 0$$
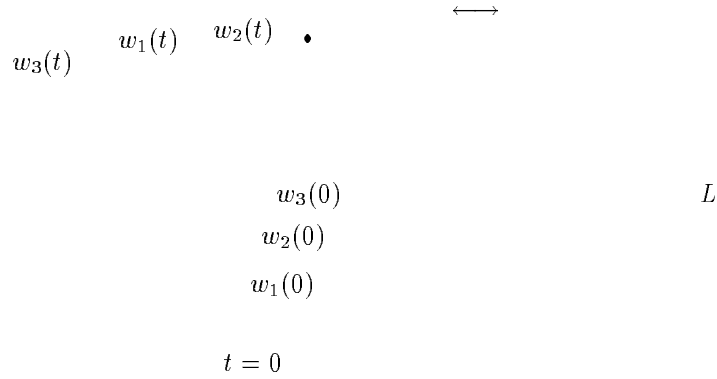
Figure 5.1

The action of $B_n$ on $H_1(Y_{\mathbf{w},1}, \chi_{\mathbf{w},1}(\mathbf{q}))$ is $\pi_{\Lambda_1}$, as noted above, and the precise relation between the $S^1 \times S^2$ and $S^3$ pictures is given by:

$$\frac{\det\left(1 - \pi_{\Lambda_1}(\gamma)\right)}{1 + q^{-1} + \cdots + q^{1-n}} = \det(T^* - q^{-1})$$

$$S^1 \times S^2 \text{ picture} \longleftrightarrow S^3 \text{ picture}$$

$$H_1(Y_{\mathbf{w},1}, \chi_{\mathbf{w},1}(\mathbf{q})) \text{ involved} \longleftrightarrow H_1(\widetilde{S^3 \backslash L}) \text{ involved}$$

$$\text{twist } \chi_{\mathbf{w}_1}(q) \longleftrightarrow \text{parameter } q^{-1}$$

The interplay between $S^1 \times S^2$ and $S^3$ also plays a major role in the discussion of the Jones polynomials $V_L$ and $X_L$, in Witten's theory (see [W]). Just as $V_L$ is a specialisation of the two-variable Jones polynomial $X_L$ (see §2.1), the Alexander polynomial $V_L(q)$ is another such one-variable specialisation.

It was the fact that $\Delta_L$ could be expressed purely topologically (that is, in terms of the link $L \subseteq S^3$, rather than using a plane projection of the link, or the braid approach) that was the initial motivation for

attempting to express the Jones polynomial in a similar way. To some extent [W] accomplishes this aim, although the remarks of §2.2 on the lack of a truly mathematically rigorous formulation, apply here.

## 5.2 The case $m = 2$ and symmetrisation

When $m = 2$, the space of chains $\mathcal{C}_2$ has a basis given by $\{ \gamma_{\underline{\alpha}} \mid \underline{\alpha} \in \mathcal{S}_\mathbf{w}^2 \}$ where:

$$\mathcal{S}_\mathbf{w}^2 = \left\{ (\alpha_1, \alpha_2) \mid \alpha_1 \in \{ z_2, w_1, \ldots, w_n \}, \ \alpha_2 \in \{ w_1, \ldots, w_n \} \right\}.$$

We shall use the local coefficient system $\chi_{\mathbf{w},2}(\mathbf{q})$ specified by the following parameters:

$$q_{z_i w_j} = q_i$$

$$q_{z_1 z_2} = \alpha$$

for $i \in \{1, 2\}$, $j \in \{1, 2, \ldots, n\}$. It is necessary that $q_{z_i w_j}$ is independent of $j$, for there to be an action of the braid group $B_n$ on $\mathcal{C}_2$. The local coefficient system defined above has three parameters, $q_1$, $q_2$, $\alpha$. For arbitrary non-zero values of these parameters, the definitions of §3.1 give rise to a representation of $B_n$. This is computed by using Theorem 4.4 and Lemma 4.2. An action of $S_m$ is only present if $q_1 = q_2$, but we shall avoid making this specialisation until later, in order to illustrate some points that will become relevant in Chapter 7.

We start from the following matrices for $i = 1, 2$; $\lambda$, $\mu \in \{1, 2, \ldots, n\}$:

$$\left.\begin{aligned}
A_{w_\lambda w_{\lambda+1}}^{(0)} &= 1 \\
b_{\lambda\mu}^{(0)} &= 1 \\
b_{z_i\lambda}^{(0)} &= q_i^{-1} \\
b_{z_1 z_2}^{(0)} &= \alpha^{-1}
\end{aligned}\right\}$$

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and we use Theorem 4.4 to obtain the following matrices for $A^{(1)}_{w_\lambda w_{\lambda+1}}$ and $b^{(1)}_{z_2 w_\lambda}$:

$$
A^{(1)}_{w_\lambda w_{\lambda+1}} = 
\begin{pmatrix}
1 & & & & & & \\
& \ddots & & & & & \\
& & 1 & & & & \\
& & & 0 & q_1^{-1} & & \\
& & & 1 & 1-q_1^{-1} & & \\
& & & & & 1 & \\
& & & & & & \ddots \\
& & & & & & & 1
\end{pmatrix}
\begin{array}{l}\text{column } z_2 \quad \text{column } w_\lambda \quad \text{column } w_{\lambda+1}\end{array}
$$

$$
b^{(1)}_{z_2 w_\lambda} =
\begin{pmatrix}
q_1^{-1}q_2^{-1} & \cdots & -q_2^{-1}(1-q_1^{-1})^2 & \cdots & q_2^{-1}q_1^{-1}(1-q_1^{-1}) & & \\
& \ddots & & & & & \\
& & q_2^{-1} & & & & \\
& & & \ddots & & & \\
q_2^{-1}(1-\alpha^{-1}) & \cdots & q_2^{-1}(1-\alpha^{-1})(1-q_1^{-1}) & \cdots & q_2^{-1}(1-q_1^{-1}+\alpha^{-1}q_1^{-1}) & & \\
& & & & & q_2^{-1} & \\
& & & & & & \ddots \\
& & & & & & & q_2^{-1}
\end{pmatrix}
$$

with columns labeled: column $z_2$; column $w_\mu$, $\mu < \lambda$; column $w_\lambda$.

By the Lemma 4.2, we can construct the homology $H_2\big(Y_{\mathbf{w},2},\ \chi_{\mathbf{w},2}(\mathbf{q})\big)$ as the subspace $\ker(\mathbf{D}_1) \cap \ker(\mathbf{D}_2)$ of the space $\langle \mathcal{S}^2_{\mathbf{w}} \rangle$ of chains. The action of $B_n$ on this subspace can be obtained by considering the matrix for $A^{(2)}_{w_\lambda w_{\lambda+1}}$ obtained from Theorem 4.4 using matrices $A^{(1)}$, $b^{(1)}$ restricted to $\ker(\mathbf{D}^{(1)}_1)$. By (4.3.1), $\mathbf{D}^{(1)}_1$ is given by the $1 \times (n+1)$ matrix:

$$
(\alpha^{-1} - 1, q_1^{-1} - 1, \ldots, q_1^{-1} - 1) \tag{5.2.1}
$$

where the first element corresponds to $z_2$. The actions of $A^{(1)}_{w_\lambda w_{\lambda+1}}$, $b^{(1)}_{z_2 w_\lambda}$ on the subspace $\ker(\mathbf{D}^{(1)}_1)$ can now be specified by $n \times n$ matrices defining the induced action on the space,

$$
\{(0, x_1, \ldots, x_n)\} \subseteq \langle \mathcal{S}^1_{\mathbf{w}} \rangle
$$

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under the projection:

$$\pi_1^2\colon (x_0, x_1, \ldots, x_n) \longrightarrow (0, x_1, \ldots, x_n).$$

This projection defines an isomorphism on $\ker(\mathbf{D}_1^{(1)})$. The matrices obtained for $A_{w_\lambda w_{\lambda+1}}^{(1)}$ and $b_{z_2 w_\lambda}^{(1)}$ are:

$$A_{w_\lambda w_{\lambda+1}}^{(1)} = \begin{pmatrix} 1 & & & & & & & \\ & \ddots & & & & & & \\ & & 1 & & & & & \\ & & & 0 & q_1^{-1} & & & \\ & & & 1 & 1-q_1^{-1} & & & \\ & & & & & 1 & & \\ & & & & & & \ddots & \\ & & & & & & & 1 \end{pmatrix}$$

(columns labeled: column $w_1$, column $w_\lambda$, column $w_{\lambda+1}$)

$$b_{z_2 w_\lambda}^{(1)} = \begin{pmatrix} \ddots & & & & & & \\ & q_2^{-1} & & & & & \\ & & \ddots & & & & \\ \cdots & q_2^{-1}\alpha^{-1}(q_1^{-1}-1) & \cdots & q_2^{-1}q_1^{-1}\alpha^{-1} & \cdots & q_2^{-1}(q_1^{-1}-1) & \cdots \\ & & & & \ddots & & \\ & & & & & q_2^{-1} & \\ & & & & & & \ddots \end{pmatrix}$$

(column $w_\lambda$ indicated)

All the non-zero elements in $b_{z_2 w_\lambda}^{(1)}$ occur in either the main diagonal or the $\lambda^{\text{th}}$ row.

These matrices may now be substituted into Theorem 4.4, to obtain the matrix for $A_{w_\lambda w_{\lambda+1}}^{(2)}$. The result obtained is shown in Fig. 5.2 below, where $h = 1 - q_1^{-1}q_2^{-1}\alpha^{-1}$. We can now restrict the action to the subspace $\ker(\mathbf{D}_2)$ in order to obtain the action on homology. The matrix elements given are only non-zero elements except for entries of 1 on the main diagonal, corresponding to basis elements $(w_\lambda w_\mu) \in \mathcal{S}_{\mathbf{w}}^2$ where $\lambda, \mu \in \{1, 2, \ldots, n\} \setminus \{i, i+1\}$. Here $j$, $k$ denote elements of $\{1, \ldots, i-1\}$, $\{i+2, \ldots, n\}$, respectively. The projection $\pi_1^2$ naturally gives a projection on $\langle \mathcal{S}_{\mathbf{w}}^2 \rangle$, and, when this causes no confusion, the same notation, $\pi_1^2$, will be used to refer to both.

|  | $w_iw_i$ | $w_{i+1}w_{i+1}$ | $w_iw_{i+1}$ | $w_{i+1}w_i$ | $w_iw_j$ | $w_jw_i$ | $w_{i+1}w_j$ | $w_jw_{i+1}$ | $w_iw_k$ | $w_kw_i$ | $w_{i+1}w_k$ | $w_kw_{i+1}$ |
|---|---|---|---|---|---|---|---|---|---|---|---|---|
| $w_iw_i$ |  | $q_1^{-1}q_2^{-1}$ |  |  |  |  |  |  |  |  |  |  |
| $w_{i+1}w_{i+1}$ | $1$ | $(1-q_1^{-1})h$ | $1-q_2^{-1}+q_2^{-1}\alpha^{-1}(1-q_1^{-1})^2$ | $1-q_1^{-1}$ |  |  |  | $q_2^{-1}\alpha^{-1}(1-q_1^{-1})^2$ |  |  |  | $q_2^{-1}(1-q_1^{-1})^2$ |
| $w_iw_{i+1}$ |  | $q_1^{-1}h$ | $q_1^{-1}q_2^{-1}\alpha^{-1}(1-q_1^{-1})$ | $q_1^{-1}$ |  |  |  | $q_1^{-1}q_2^{-1}\alpha^{-1}(1-q_1^{-1})$ |  |  |  | $q_1^{-1}q_2^{-1}(1-q_1^{-1})$ |
| $w_{i+1}w_i$ |  |  | $q_1^{-1}q_2^{-1}\alpha^{-1}$ |  |  |  |  | $q_2^{-1}\alpha^{-1}(q_1^{-1}-1)$ |  |  |  | $q_2^{-1}(q_1^{-1}-1)$ |
| $w_iw_j$ |  |  |  |  | $0$ |  | $q_1^{-1}$ |  |  |  |  |  |
| $w_jw_i$ |  |  |  |  |  | $0$ |  | $q_2^{-1}$ |  |  |  |  |
| $w_{i+1}w_j$ |  |  |  |  | $1$ |  | $1-q_1^{-1}$ |  |  |  |  |  |
| $w_jw_{i+1}$ |  |  |  |  |  | $1$ |  | $1-q_2^{-1}$ |  |  |  |  |
| $w_iw_k$ |  |  |  |  |  |  |  |  | $0$ |  | $q_1^{-1}$ |  |
| $w_kw_i$ |  |  |  |  |  |  |  |  |  | $0$ |  | $q_2^{-1}$ |
| $w_{i+1}w_k$ |  |  |  |  |  |  |  |  | $1$ |  | $1-q_1^{-1}$ |  |
| $w_kw_{i+1}$ |  |  |  |  |  |  |  |  |  | $1$ |  | $1-q_2^{-1}$ |

Figure 5.2

The matrix for $\mathbf{D}_2^{(2)}$ is specified by (4.3.1) as:

$$\mathbf{D}_2^{(2)} = \left(b_{z_2w_1}^{(1)}-1\Big|\cdots\Big|b_{z_2w_n}^{(1)}-1\right)$$

The $\lambda^{\text{th}}$ row of this matrix is given by:

$$\sum_{\mu=1}^{n}\left(b_{z_2w_\mu}^{(1)}-1\right)_{\text{row }w_\lambda}.w_\mu = \sum_{\mu\neq\lambda}\left((q_2^{-1}-1)(w_\lambda w_\mu)\right) + \sum_{j>\lambda}\left(q_2^{-1}(q_1^{-1}-1)(w_jw_\lambda)\right)$$
$$+ (q_1^{-1}q_2^{-1}\alpha^{-1}-1)(w_\lambda w_\lambda) + \sum_{j<\lambda}\left(q_2^{-1}\alpha^{-1}(q_1^{-1}-1)(w_jw_\lambda)\right). \tag{5.2.2}$$

The subspace $\ker(\mathbf{D}_2^{(2)})$ of $\left\langle(w_\lambda w_\mu)\big|1\leq\lambda,\ \mu\leq n\right\rangle$ has codimension $n$, and is given by $n$ relations, one corresponding to each of the rows of $\mathbf{D}_2^{(2)}$ given above. The projection :

$$\pi_2^2\colon \left\langle(w_\lambda w_\mu)\ \big|\ 1\leq\lambda,\ \mu\leq n\right\rangle \longrightarrow \left\langle(w_\lambda w_\mu)\ \big|\ 1\leq\lambda,\ \mu\leq n,\ \lambda\neq\mu\right\rangle$$

given by mapping $(w_\lambda w_\lambda)$ to $\mathbf{0}$, will be an isomorphism on the restriction $\ker(\mathbf{D}_2^{(2)})$ whenever:

$$q_1^{-1}q_2^{-1}\alpha^{-1} \neq 1 \ .$$

In this case, we can obtain an $n(n-1) \times n(n-1)$ matrix for the action of $A_{w_i w_{i+1}}^{(2)}$, from the induced action on:

$$\big\langle (w_\lambda w_\mu) \mid 1 \leq \lambda, \ \mu \leq n, \ \lambda \neq \mu \big\rangle$$

under the above projection. The matrix obtained is that given in Fig. 5.3 below, where we have omitted diagonal entries corresponding to $(w_\lambda w_\mu)$ with $\lambda, \ \mu \in \{1, 2, \ldots, n\} \backslash \{i\, i{+}1\}$, $\lambda \neq \mu$, which are all 1's.

|  | $w_iw_{i+1}$ | $w_{i+1}w_i$ | $w_iw_j$ | $w_jw_i$ | $w_{i+1}w_j$ | $w_jw_{i+1}$ | $w_iw_k$ | $w_kw_i$ | $w_{i+1}w_k$ | $w_kw_{i+1}$ |
|---|---|---|---|---|---|---|---|---|---|---|
| $w_iw_{i+1}$ |  | $q_1^{-1}q_2^{-1}$ |  |  | $q_1^{-1}(q_2^{-1}-1)$ |  |  |  | $q_1^{-1}(q_2^{-1}-1)$ |  |
| $w_{i+1}w_i$ | $q_1^{-1}q_2^{-1}\alpha^{-1}$ |  |  |  |  | $q_2^{-1}\alpha^{-1}(q_1^{-1}-1)$ |  |  |  | $q_2^{-1}(q_1^{-1}-1)$ |
| $w_iw_j$ |  |  | $0$ |  | $q_1^{-1}$ |  |  |  |  |  |
| $w_jw_i$ |  |  |  | $0$ |  | $q_2^{-1}$ |  |  |  |  |
| $w_{i+1}w_j$ |  |  | $1$ |  | $1-q_1^{-1}$ |  |  |  |  |  |
| $w_jw_{i+1}$ |  |  |  | $1$ |  | $1-q_2^{-1}$ |  |  |  |  |
| $w_iw_k$ |  |  |  |  |  |  | $0$ |  | $q_1^{-1}$ |  |
| $w_kw_i$ |  |  |  |  |  |  |  | $0$ |  | $q_2^{-1}$ |
| $w_{i+1}w_k$ |  |  |  |  |  |  | $1$ |  | $1-q_1^{-1}$ |  |
| $w_kw_{i+1}$ |  |  |  |  |  |  |  | $1$ |  | $1-q_2^{-1}$ |

*Figure 5.3*

The eigenvalues of this matrix are $1$, $-q_1^{-1}$, $-q_2^{-1}$, $\pm q_1^{-1}q_2^{-1}\alpha^{-1/2}$ . This is the matrix specifying the action of the generator $\sigma_i$ of the braid group $B_n$, with the three parameters $q_1$, $q_2$, $\alpha$. We can only proceed

further by specialising the values of $q_1$, $q_2$, $\alpha$. There is an $S_2$ action on the homology (and also on the space $\langle \mathcal{S}_\mathbf{w}^2 \rangle$ of chains) so long as $q_1 = q_2$. In this case, the matrix for $j_{12}$ as given by §4.6 (see Theorem 4.5) is:

|  | $z_2\lambda$ | $\lambda\lambda$ | $\mu\lambda$ | $\lambda\mu$ |
|---|---|---|---|---|
|  |  |  | $\overbrace{\hspace{4cm}}^{\lambda > \mu}$ |  |
| $z_2\lambda$ | $\alpha(q^{-1}-1)-q^{-1}$ | $q^{-2}-\alpha(1-q^{-1})^2$ | $(q^{-1}+(1-q^{-1})\alpha)(q^{-1}-1)$ | $\alpha(q^{-1}-1)$ |
| $\lambda\lambda$ | $\alpha-1$ | $\alpha(1-q^{-1})+q^{-1}$ | $(\alpha-1)(1-q^{-1})$ |  |
| $\mu\lambda$ |  |  |  | $\alpha$ |
| $\lambda\mu$ |  |  | $1$ |  |

as an action on the chains. Here we have substituted:

$$b' = \alpha^{-1}, \qquad A' = 1 \qquad b_\lambda = b'_\lambda = q^{-1}$$

where $q = q_1 = q_2$.

The natural orthogonal projection:

$$\langle \mathcal{S}_\mathbf{w}^2 \rangle \longrightarrow \langle (w_\lambda w_\mu) \mid 1 \le \lambda,\ \mu \le n,\ \lambda \ne \mu \rangle$$

is an isomorphism on $\ker(\mathbf{D}_1) \cap \ker(\mathbf{D}_2)$, and so there is an $n(n-1) \times n(n-1)$ matrix giving the action of $j_{12}$ on homology. This holds for $\alpha \ne q^{-2}$. The matrix of this induced action is:

|  | $\mu\lambda$ | $\lambda\mu$ |
|---|---|---|
| $\mu\lambda$ |  | $\alpha$ |
| $\lambda\mu$ | $1$ |  |

where $\lambda > \mu$. This matrix has eigenvalues $\pm\alpha^{1/2}$, with corresponding eigenvectors:

$$\alpha^{1/2}(\mu\lambda) \pm (\lambda\mu)$$

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for $\lambda > \mu$. Let us denote by $f_{\lambda\mu}$ the element of homology given by $\alpha^{1/2}(\mu\lambda) + (\lambda\mu)$, for each $\lambda > \mu$. Then the symmetric part of homology, under the action of $S_2$ given by $j_{12}$, is spanned by $f_{\lambda\mu}$ for $\lambda > \mu$. The action of $B_n$ on this space is specified by the action of its generators $\sigma_i$. From the matrix for the action of $\sigma_i$ on homology we obtain:

**Theorem 5.1**    *The action of $B_n$ on the symmetric part of the homology $H_2\big(Y_{\mathbf{w},2},\ \chi_{\mathbf{w},2}(\mathbf{q})\big)$ is given by the matrix below for the action of $\sigma_i$. This holds for all values of $\alpha$ and $q$ with $\alpha \neq q^{-2}$.*

|  | $f_{i+1\,i}$ | $f_{ij}$ | $f_{i+1\,j}$ | $f_{ki}$ | $f_{k\,i+1}$ |
|---|---|---|---|---|---|
| $f_{i+1\,i}$ | $q^{-2}\alpha^{-1/2}$ | | $q^{-1}(q^{-1}-1)\alpha^{-1/2}$ | | $q^{-1}(q^{-1}-1)$ |
| $f_{ij}$ | | $0$ | $q^{-1}$ | | |
| $f_{i+1\,j}$ | | $1$ | $1-q^{-1}$ | | |
| $f_{ki}$ | | | | $0$ | $q^{-1}$ |
| $f_{k\,i+1}$ | | | | $1$ | $1-q^{-1}$ |

On the other hand, for any $\alpha$, the symmetric part of the space of *chains* $\mathcal{S}_{\mathbf{w}}^2$ is found to be spanned by:

$$\left. \begin{aligned} \mathbf{f}_{\lambda\mu} &= \alpha^{1/2}(\mu\lambda) + (\lambda\mu) + \alpha^{1/2}(q^{-1}-1)(z_2\lambda) \\ \mathbf{f}_\lambda &= \big(q^{-1} + (q^{-1}-1)\alpha^{1/2}\big)(z_2\lambda) + (1+\alpha^{1/2})(\lambda\lambda) \end{aligned} \right\} \tag{5.2.3}$$

The subspace $\big\langle \mathbf{f}_{\lambda\mu}, \mathbf{f}_\lambda \big\rangle$ of the span of chains intersects $\ker(\mathbf{D}_1) \cap \ker(\mathbf{D}_2)$ in a space which is isomorphic to the symmetric part of the homology. However, the map:

$$\pi_2^2 \circ \pi_1^2 \colon \big\langle \mathcal{S}_{\mathbf{w}}^2 \big\rangle \longrightarrow \big\langle (w_\lambda w_\mu) \mid 1 \leq \lambda,\ \mu \leq n,\ \lambda \neq \mu \big\rangle$$

is an isomorphism on $\ker(\mathbf{D}_1) \cap \ker(\mathbf{D}_2)$ when $q^{-2} \neq \alpha$. In this case, $(\pi_2^2 \circ \pi_1^2)^{-1}(w_\lambda w_\mu)$ defines the element of $\big\langle \mathcal{S}_{\mathbf{w}}^2 \big\rangle$ given by some complicated expression, namely:

$$\left. \begin{aligned} (\lambda\mu) + \tfrac{1-q^{-1}}{q^{-2}\alpha^{-1}-1}\Big((\lambda\lambda)+q^{-1}(\mu\mu)\Big) + \tfrac{1-q^{-1}}{(\alpha^{-1}-1)(q^{-2}\alpha^{-1}-1)}\Big((1-q^{-1})(z_2\lambda)+(q^{-2}\alpha^{-1}-1+q^{-1}-q^{-2})(z_2\mu)\Big) \text{ for } \lambda > \mu \\ (\lambda\mu) + \tfrac{1-q^{-1}}{q^{-2}\alpha^{-1}-1}\Big((\lambda\lambda)+q^{-1}\alpha^{-1}(\mu\mu)\Big) + \tfrac{1-q^{-1}}{(\alpha^{-1}-1)(q^{-2}\alpha^{-1}-1)}\Big((1-q^{-1})(z_2\lambda)+(q^{-1}\alpha^{-1}-1)(z_2\mu)\Big) \qquad\quad \text{ for } \lambda < \mu \end{aligned} \right\} \tag{5.2.4}$$

In fact it can be seen that the inverse image of $\alpha^{1/2}(\mu\lambda) + (\lambda\mu)$ under this map is precisely:

$$\mathbf{f}_{\lambda\mu} + \frac{1 - q^{-1}}{(1 + \alpha^{1/2})(q^{-1}\alpha^{-1/2} - 1)}\left[\mathbf{f}_{\lambda} + \alpha^{1/2}\mathbf{f}_{\mu}\right] \qquad (5.2.5)$$

So, in the case $\alpha \neq q^{-2}$, when we refer to $\alpha^{1/2}(\mu\lambda) + (\lambda\mu)$ in the homology $H^2\big(Y_{\mathbf{w},2}, \chi_{\mathbf{w},2}(\mathbf{q})\big)$, as identified with $\big\langle (\lambda\mu) \mid \lambda, \mu \in \{w_1, \ldots, w_n\}, \lambda \neq \mu \big\rangle$ we are really referring to the element given by (5.2.5).

In the case $\alpha = q^{-2}$, all of this breaks down, since $\pi_2^2$ is no longer an isomorphism on $\ker(\mathbf{D}_2^{(2)})$. However, inside $\ker(\mathbf{D}_1) \cap \ker(\mathbf{D}_2)$, there is a subspace of dimension $\frac{1}{2}n(n-1)$ given by $j_{12} = \alpha^{1/2}$. There is no natural basis on which this action can given, unlike the case of $\alpha \neq q^{-2}$. However, as we shall see in the next Chapter, there is a natural action of $B_n$ on the quotient of the symmetric part of homology by a suitable subspace of dimension $n$.

This action can be obtained in another way, starting from the action of $B_n$ on the symmetric part of homology for $\alpha \neq q^{-2}$, given above. The matrix for the action of $\sigma_i$ given in the Theorem depends smoothly on $\alpha$ near $q^{-2}$, and the limiting matrix for this action is:

|  | $f_{i+1\,i}$ | $f_{ij}$ | $f_{i+1\,j}$ | $f_{ki}$ | $f_{k\,i+1}$ |
|---|---|---|---|---|---|
| $f_{i+1\,i}$ | $q^{-1}$ |  | $q^{-1} - 1$ |  | $q^{-1}(q^{-1} - 1)$ |
| $f_{ij}$ |  | $0$ | $q^{-1}$ |  |  |
| $f_{i+1\,j}$ |  | $1$ | $1 - q^{-1}$ |  |  |
| $f_{ki}$ |  |  |  | $0$ | $q^{-1}$ |
| $f_{k\,i+1}$ |  |  |  | $1$ | $1 - q^{-1}$ |

It can be seen that the subspace given by the kernel of the map:

$$\langle f_{ij} \rangle \longrightarrow \mathbf{C}$$

$$f_{ij} \longmapsto \big(0, \ldots, q^{-1}, \ldots, 1, \ldots 0\big)$$
$$\phantom{f_{ij} \longmapsto \big(0, \ldots,} j \phantom{{}^{-1}, \ldots,} i \phantom{, \ldots 0\big)}$$

is preserved by the limiting action of $\sigma_i$. Hence there is induced an action of $B_n$ on this subspace, and this

action of $\sigma_i$ is equivalent on the subspace to that of the matrix:

| | $f_{i+1\,i}$ | $f_{ij}$ | $f_{i+1\,j}$ | $f_{ki}$ | $f_{k\,i+1}$ |
|---|---|---|---|---|---|
| $f_{i+1\,i}$ | 1 | | | | |
| $f_{ij}$ | | 0 | $q^{-1}$ | | |
| $f_{i+1\,j}$ | | 1 | $1-q^{-1}$ | | |
| $f_{ki}$ | | | | 0 | $q^{-1}$ |
| $f_{k\,i+1}$ | | | | 1 | $1-q^{-1}$ |

This shows that:

**Theorem 5.2** *There is a quotient space of the symmetric part of the homology $H_2\big(Y_{\mathbf{w},2},\ \chi_{\mathbf{w},2}(\mathbf{q})\big)$ with $\alpha = q^{-2}$, on which the action of $B_n$ factors through the Hecke algebra $H_n(q^{-1})$.*

We have indicated above how this action may be obtained on either a quotient space of $(H_2)^{S_2}$ at $\alpha = q^{-2}$, or as a subspace of the limiting space $(H_2)^{S_2}$ for $\alpha \rightarrow q^{-2}$. In Chapter 7 we will discuss this in a more general context. The space obtained by either procedure has dimension $\frac{1}{2}n(n-1) - n = \frac{1}{2}n(n-3)$, and as will be seen in Chapter 7, the actions of $B_n$ obtained are identical on these two spaces. The action on the limiting subspace was seen above to factor through the Hecke algebra $H_n(q^{-1})$.

The work of Wenzl [W] showed how, for any Young diagram $\Lambda$ with $n$ squares, one could construct an irreducible representation $\pi_\Lambda$ of $H_n(q)$. These representations are deformations of the standard irreducible representations of $S_n$, as $q \rightarrow 1$. See §2.1 for more details. Consider the two-row Young diagram, $\Lambda_2$, with $n-2$ and 2 squares in its rows, as shown in Fig. 5.4. In each square, the integer indicates the hook length $l(i,\ j)$ as defined in §2.1. By the hook length formula the dimension of the corresponding irreducible representation of $S_n$ (and thus also that for $H_n(q)$) is:

$$\frac{n!}{((n-1)(n-2)(n-4)\cdots 1)(2.1)} = \tfrac{1}{2}n(n-3)\,.$$

| $n-1$ | $n-2$ | $n-4$ | $\cdots$ | 1 |
|-------|-------|-------|----------|---|
| 2 | 1 | | | |

*Figure 5.4*

A more careful examination of the transformation properties of $\langle f_{ij}\rangle$ reveals that the representation of $H_n(q^{-1})$ referred to in the Theorem above, is precisely the irreducible representation of $H_n(q^{-1})$ corresponding to the above Young diagram; see §6.3, p91–93 for more details. We conclude that:

**Theorem 5.3**    *There is an action of $B_n$ on the symmetric part of the homology $H_2\big(Y_{\mathbf{w},2},\, \chi_{\mathbf{w},2}(\mathbf{q})\big)$ for any local coefficient system $\mathbf{q}$ specified by two non-zero complex parameters $q$ and $\alpha$. When $\alpha = q^{-2}$, this action preserves an $n$-dimensional subspace of this $^1/_2 n(n-1)$-dimensional space, and the quotient action induced is the irreducible representation, $\pi_{\Lambda_2}$, of the Hecke algebra $H_n(q^{-1})$ corresponding to the two-row Young diagram with $n-2$ and 2 squares in its rows.*

### 5.3 Symmetric group representations for $q = 1$

In this section we will discuss the case when $q = 1$. In this case, the local coefficient system is trivial. Thus the $A_{\lambda\mu}^{(0)}$ and $b_{\lambda\mu}^{(0)}$ matrices are all just 1. Theorem 4.4 allows one to compute the matrices $A_{\lambda\mu}^{(r)}$, $b_{\lambda\mu}^{(r)}$ for arbitrary $\lambda$, $\mu$ and $r$, and it is found that:

$$b_{\lambda\mu}^{(r)} = \mathbf{I}$$

while $A_{\lambda\mu}^{(r)}$ is given by the following action on $\langle S_{\mathbf{w}}^r \rangle$:

$$A_{\lambda\mu}^{(r)}(\underline{\alpha}) = \underline{\beta}$$

where $\underline{\alpha},\ \underline{\beta} \in S_{\mathbf{w}}^r$, and:

$$\beta_i = \begin{cases} \alpha_i & \text{if } \alpha_i \neq \lambda,\, \mu \\ \mu & \text{if } \alpha_i = \lambda \\ \lambda & \text{if } \alpha_i = \mu. \end{cases}$$

The formulae given in §4.3 for the matrices $\mathbf{D}_i^{(r)}$ also simplify greatly. Thus, from (4.3.1), it is seen that:

$$\mathbf{D}_i^{(i)} = \mathbf{0}$$

for all $i$. Hence $\mathbf{D}_i^{(r)} = \mathbf{0}$, by (4.3.2), and so Lemma 4.2 reduces to the trivial statement that the homology $H_m(Y_{\mathbf{w},m}, \chi_{\mathbf{w},m}(\mathbf{q}))$ can be identified with $\mathcal{C}_m$. The action of $S_m$ on the space of chains is specified in Theorem 4.5 in terms of the matrices:

$$j_{i\,i+1}^{(i+1)} = \begin{pmatrix} -A' & A' & & \\ & A' & & \\ & & A' & \\ & & & A' \end{pmatrix} \tag{5.3.1}$$

where $A' = A_{z_i z_{i+1}}^{(i-1)}$ and the blocks of the above matrix are associated with the values $z_{i+1}\lambda$, $\lambda\lambda$, $\mu\lambda$, $\lambda\mu$ ($\lambda > \mu$) of the pair $\alpha_i\alpha_{i+1}$. The matrix gives the action on $\langle \mathcal{S}_{\mathbf{w}}^{i+1}\rangle$, and it is extended to give the action on $\mathcal{C}_m$, by putting blocks of $j_{i\,i+1}^{(i+1)}$ down the diagonal.

Since the representation of $B_n$ on homology, in this special case, gives rise to an action of $\sigma_i \in B_n$ of order 2, it factors through $S_n$. So all the representations obtained in this case, are representations of the symmetric group, $S_n$. The action of $\sigma \in S_n$ on $\mathcal{S}_{\mathbf{w}}^m$ is given by:

$$\sigma(\underline{\alpha}) = \underline{\beta}$$

where $\beta_i$ is obtained from $\alpha_i$ by the induced action of $\sigma$ on $\{z_{i+1}, \ldots, z_m, w_1, \ldots, w_n\}$. The character of the representation is thus given by:

$$\chi(\sigma) = (\sigma^1 + m - 1) \cdots (\sigma^1 + 1)\sigma^1$$

where $\sigma^r$ is the number of cycles of order $r$ in the disjoint cycle decomposition of $\sigma$.

However, the representation with which we are concerned here, is that on

$$V = \left[ H_m(Y_{\mathbf{w},m}, \chi_{\mathbf{w},m}(\mathbf{q})) \right]^{S_m}.$$

This totally symmetric part is the subspace of $\langle \mathcal{S}_{\mathbf{w}}^m \rangle$ on which $j_{i\,i+1} = 1$ for $i = 1, 2, \ldots, m-1$. By (5.3.1), the subspace given by $j_{i\,i+1} = 1$ is spanned by:

$$\begin{pmatrix} 0 \\ 0 \\ A'a \\ a \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} a \\ (A'+1)a \\ 0 \\ 0 \end{pmatrix}$$

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for arbitrary $a \in \mathcal{S}_{\mathbf{w}}^{i-1}$. The action of $A_{z_i z_{i+1}}^{(i-1)} = A'$ on $\mathcal{S}_{\mathbf{w}}^{i-1}$ is the natural one, under which $z_i$ and $z_{i+1}$ are interchanged. Hence $V$ consists of elements

$$\sum_{\underline{\alpha} \in \mathcal{S}_{\mathbf{w}}^{m}} \left( A_{\underline{\alpha}} \cdot \underline{\alpha} \right)$$

of $\langle \mathcal{S}_{\mathbf{w}}^{m} \rangle$ such that:

(a) $A_{\underline{\alpha}} = A_{\underline{\beta}}$ whenever $\underline{\alpha}$, $\underline{\beta}$ are both vectors of distinct elements, which can be obtained from each other by exchanging $z_i$ and $z_{i+1}$ while, at the same time, interchanging the $i^{\text{th}}$ and $i+1^{\text{th}}$ elements, for some $i$ with $1 \leq i < m$;

(b) all the $A_{\underline{\alpha}}$ are given by well defined linear combinations (which we will not give here) of those $A_{\underline{\alpha}}$ for which $\alpha_1, \ldots, \alpha_m$ are all distinct.

This implies that all the $A_{\underline{\alpha}}$ are determined by $\left\{ A_{\underline{\alpha}} \mid \underline{\alpha} \in V \right\}$ where $V$ is a suitable subset of $\mathcal{S}_{\mathbf{w}}^{m}$ of order:

$$(n + m - 1) \cdots (n+1)n/m! \,.$$

*Example 1*

Consider the case of $m = 2$. Then it is clear that:

$$V = \left\{ (w_i w_j) \mid 1 \leq i < j \leq n \right\} \cup \left\{ (z_2 w_i) \mid 1 \leq i \leq n \right\}$$

will do. The action of $S_n$ here is the natural action, and splits into a direct sum of representations:

(i) the induced representation coming from the identity on $S_2 \times S_{n-2} \subseteq S_n$ (of dimension $\frac{1}{2}n(n-1)$);

(ii) the natural representation of $S_n$ of dimension $n$ (associated with the identity action of $S_1 \times S_{n-1} \subseteq S_n$).

Both of these parts split into irreducible components, namely as $\pi_{\Lambda_2} \oplus \pi_{\Lambda_1} \oplus \pi_{\Lambda_0}$ and $\pi_{\Lambda_1} \oplus \pi_{\Lambda_0}$. Hence the total representation is $\pi_{\Lambda_2} \oplus 2\pi_{\Lambda_1} \oplus 2\pi_{\Lambda_0}$. Although it still contains $\pi_{\Lambda_2}$, it is by no means irreducible! The character of the represention of $S_n$ (or $B_n$) on the symmetric part of the homology is given by:

$$\chi(\sigma) = \frac{1}{2}\sigma^1(\sigma^1 + 1) + \sigma^2$$

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for $\sigma \in S_n$. This should be compared with $\chi_{\Lambda_2}(\sigma) = \frac{1}{2}\sigma^1(\sigma^1 - 3) + \sigma^2$.

*Example 2*

When $m = 3$, a suitable set $\mathcal{V}$ consists of all $\underline{\alpha} = (\alpha_1, \alpha_2, \alpha_3)$ in $\mathcal{S}_{\mathbf{w}}^3$, of one of the following forms:

$$(w_i w_j w_k) \qquad (i < j < k);$$

$$(z_2 w_i w_j) \qquad (i < j);$$

$$(z_3 w_i w_j) \qquad (i < j);$$

$$(z_2 z_3 w_i) \ .$$

This set has order:

$$\tfrac{1}{6}n(n-1)(n-2) + 2.\tfrac{1}{2}n(n-1) + n = \tfrac{1}{6}n(n+1)(n+2)$$

and the representation of $S_n$ so obtained is:

$$\pi_{\Lambda_3} \oplus 3\pi_{\Lambda_2} \oplus 4\pi_{\Lambda_1} \oplus 4\pi_{\Lambda_0} \ .$$

It is possible to prove, in the general case, the following Theorem.

**Theorem 5.4** *The representation of $B_n$ obtained in Theorem 3.3 when $q = 1$, factors through $S_n$, and as such, has the direct sum decomposition:*

$$\bigoplus_{k=0}^{m} A_k . \pi_{\Lambda_k}$$

*where $A_k = \sum\limits_{r=k}^{m} \binom{m-1}{r-1}$ for $k > 0$ and $A_0 = A_1$.*

Note that only two-row Young diagrams enter here. This would not be true if we expressed the representation of $B_n$ on the homology:

$$H_m\big(Y_{\mathbf{w},m}, \ \chi_{\mathbf{w},m}(q)\big)$$

with $q = 1$ (without restricting to the $S_m$-invariant part), in the form of a direct sum decomposition.

When $q$ differs from 1, but is nearby, the dimension of the homology is less than that at $q = 1$, since the boundaries are non-trivial in such a case. There is thus a discontinuity in the dimension of the representation

obtained at $q = 1$. Similar discontinuities exist at other roots of unity, but only a finite number of roots of unity are affected for a given $m$. In all cases, however, $\pi_{\Lambda_m}$ occurs with multiplicity 1, and is the major part of the representation.

# 6: The General Case

In this Chapter we will discuss the theory for the case of general $m$, and prove Theorem 3.3. Throughout this Chapter the local coefficient system is assumed to be given by:

$$q_{z_i z_j} = q^{-2}$$

$$q_{z_i w_k} = q$$

for $i$, $j \in \{1, 2, \ldots, m\}$, $k \in \{1, 2, \ldots, n\}$. This local coefficient system is suitably symmetric so as to enable both the actions of $B_n$ and $S_m$, as defined in §3.1, to exist on the space of homology.

The duality between homology and cohomology which exists via the natural pairing:

$$H_m \times H^m \longrightarrow \mathbf{C}$$

$$(\alpha, \omega) \longmapsto \int_\alpha \omega$$

induces natural actions of $B_n$ and $S_m$ on cohomology. In §6.1 we shall discuss the relation between the actions on homology and cohomology, and define a natural quotient of the space of chains. For the rest of the Chapter it is most convenient to work with the dual picture of actions on cohomology. The quotient space of homology is dual to a subspace of cohomology, and in §6.1 we specify a spanning set for this subspace.

Using the results of Chapter 4, the actions of $S_m$ and $B_n$ on this subspace are evaluated in §§6.2, 6.3 (see Lemma 6.3, Corollary 6.5 and Theorem 6.6). The action of $B_n$ on the symmetric part of the subspace is shown to be the required irreducible representaton of the Hecke algebra corresponding to the two-row diagram $\Lambda_m$. When $m > {}^n\!/_2$, the subspace of cohomology vanishes. The result proved is thus stronger than Theorem 3.3, since it identifies the subspace on which the Hecke algebra representation appears, with a concrete spanning set — see §6.1 (Theorem 6.1).

### 6.1: Concrete construction of homology quotient

The picture we have obtained up until now is that $B_n \times S_m$ acts on $\mathcal{C}_m$, a space of chains. This action preserves the subspace $\ker(\mathbf{D}) \subseteq \mathcal{C}_m$ where $\mathbf{D}$ denotes the matrix:

$$\begin{pmatrix} \mathbf{D}_1^{(m)} \\ \vdots \\ \mathbf{D}_m^{(m)} \end{pmatrix}$$

For, $\ker(\mathbf{D}) = \bigcap_{i=1}^{m} \ker(\mathbf{D}_i^{(m)})$, and by Lemma 4.2, there is an isomorphism between the homology:

$$H_m(Y_{\mathbf{w},m}, \chi_{\mathbf{w},m}(\mathbf{q}))$$

(which we shall in future abbreviate to $H_m$ since the context is clear) and $\ker(\mathbf{D})$.

In the dual picture, we have $B_n \times S_m$ acting on the space $\mathcal{C}^m$ dual to the space of chains $\mathcal{C}_m$. The action preserves the subspace:

$$\operatorname{Im}(\mathbf{D}') \subseteq \mathcal{C}^m$$

where $\mathbf{D}'$ corresponds to the differential map $d$, just as $\mathbf{D}$ corresponds to the boundary map $\delta$. The cohomology space $H^m$ is now isomorphic to $\mathcal{C}^m/\operatorname{Im}(\mathbf{D}')$. An element of $\mathcal{C}^m$ is specified by a vector:

$$\left\{ \int_{\underline{\alpha}} f \;\middle|\; \underline{\alpha} \in \mathcal{C}_m \right\}$$

The boundary map $\delta : \mathcal{C}_m \longrightarrow \mathcal{C}_{m-1}$ gives rise to the differential map $d : \mathcal{C}^{m-1} \longrightarrow \mathcal{C}^m$. This map is specified by the transpose of the matrix for $\delta$; that is $\mathbf{D}' = \mathbf{D}^{\mathrm{T}}$.

We wish to define a quotient $H_m/W$ of homology; or equivalently a subspace of cohomology. We will do this by defining a subspace $W'$ of $\mathcal{C}^m$, the dual to the space of chains. The situation which we now obtain is embodied in Fig. 6.1. The subspace $W' \subseteq \mathcal{C}^m$ gives a natural embedding $s'$. The map $r'$ induced by $s'$ on $W'/\operatorname{Im}(\mathbf{D}')$ gives the required subspace of $\mathcal{C}^m/\operatorname{Im}(\mathbf{D}) \cong H^m$. We can reach this subspace, alternatively, as the image of $j$, giving $W'/\operatorname{Im}(\mathbf{D}^{*\prime})$ where:

$$\mathbf{D}'_* : \mathbf{D}'^{-1}(W') \longrightarrow W'$$

is the resriction of $\mathbf{D}' : \mathcal{C}^{m-1} \longrightarrow \mathcal{C}^m$ to $\mathbf{D}'^{-1}(W')$.

$$\mathcal{C}_m/W$$

$$\nearrow^{j} \qquad\qquad \nwarrow^{s}$$

$$\ker(\mathbf{D})/W \quad\overset{r}{\longleftarrow}\quad \ker(\mathbf{D}) \quad\overset{i}{\longrightarrow}\quad \mathcal{C}_m$$

$$W'/\mathrm{Im}(\mathbf{D}') \quad\overset{r'}{\longrightarrow}\quad \mathcal{C}^m/\mathrm{Im}(\mathbf{D}') \quad\overset{i'}{\longleftarrow}\quad \mathcal{C}^m$$

$$\swarrow^{j'} \qquad\qquad \nearrow^{s'}$$

$$W'$$

Figure 6.1

In the dual picture, $W'$ is dual to a quotient $\mathcal{C}_m/W$ of the space of chains $\mathcal{C}_m$, giving a quotient map $s$. Restricted to $\ker(\mathbf{D})$, this gives the map $r$, whose image is $\ker(\mathbf{D})/W$, the required quotient of homology. Alternatively, one can obtain this result as a subspace of $\mathcal{C}_m/W$, using $j$; namely $\ker(\mathbf{D}^*)$ where:

$$\mathbf{D}^* \colon \mathcal{C}_m/W \longrightarrow \mathcal{C}_{m-1}/\mathbf{D}(W)$$

is a quotient of $\mathbf{D} \colon \mathcal{C}_m \longrightarrow \mathcal{C}_{m-1}$.

Now, to define $W' \subseteq \mathcal{C}^m$, we use the natural pairing between $\mathcal{C}^m$ and $\mathcal{C}_m$, and put:

$$W' = \left\{ f \in \mathcal{C}^m \ \middle| \ \int_{\gamma_{\underline{\alpha}}} f = 0 \text{ for all } \underline{\alpha} \in \mathcal{S}_{\mathbf{w}}^m \backslash \mathcal{T}_{\mathbf{w}}^m \right\}$$

where $\mathcal{T}_{\mathbf{w}}^m = \left\{ (\alpha_1, \ldots, \alpha_m) \ \middle| \ \alpha_i \in \{w_1, \ldots, w_n\} \text{ for all } i, \text{ and } \alpha_i \text{ are all distinct elements} \right\}$. From this definition it is clear that $\mathcal{T}_{\mathbf{w}}^m$, and thus also $W'$, is trivial when $m > n$. We will see later in this Chapter that $W'/\mathrm{Im}(\mathbf{D}')$ is also trivial when $m > n/2$.

The action of $B_n$ on $H_m$, given by the matrices $A_{w_i w_{i+1}}^{(m)}$ for the action of the generator $\sigma_i$, corresponds in the dual picture to the action of $B_n$ on $H^m$, given by a matrix $A'^{(m)}_{w_i w_{i+1}}$, where:

$$\left\langle A'^{(m)}_{w_i w_{i+1}} \mathbf{w} \ \middle| \ A^{(m)}_{w_i w_{i+1}} \mathbf{v} \right\rangle = \left\langle \mathbf{w} \ \middle| \ \mathbf{v} \right\rangle$$

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for all $\mathbf{v} \in H_m$, $\mathbf{w} \in H^m$, where $\langle \ | \ \rangle$ denotes the natural pairing between $H^m$ and $H_m$. Thus the matrix $A'^{(m)}_{w_i w_{i+1}}$ is the transpose of the inverse of $A^{(m)}_{w_i w_{i+1}}$; it gives rise to a representation of $B_n$ which is the dual of the representation obtained on $H_m$. Thus the statement of Theorem 3.3, namely that the action of $B_n$ on $H_m/W$ is an irreducible representation of $H_n(q^{-1})$ is equivalent to the statement below in terms of the action on cohomology.

**Theorem 6.1**    *There is a natural action of $B_n \times S_m$ on the subspace $\left(W'/\mathrm{Im}\,(\mathbf{D}')\right)$ of the cohomology:*

$$H^m\left(Y_{\mathbf{w},m}, \chi_{\mathbf{w},m}(\mathbf{q})\right)$$

*and the action of $B_n$ on that part of the space that is totally symmetric under the action of $S_m$, factors through the Hecke algebra $H_n(q)$. Moreover this action is irreducible and corresponds to the Young diagram with two rows of lengths $n-m$ and $m$, for $m \leq {}^n/_2$. When $m > {}^n/_2$, the subspace defined by $W$ is trivial.*

This is the form in which we will prove Theorem 3.3. In §6.2, we establish the action of $S_m$ on the space $W'$; and in §6.3, we obtain the form of the action of $B_n$ on the space $(W')^{S_m}$. The form of the space $\left(W'/\mathrm{Im}(\mathbf{D})\right)^{S_m}$ is also discussed in §6.3, and this discussion enables a proof of the Theorem above. We actually obtain a stronger result, namely that the action of $B_n$ is given on the basis for $W'$ in the standard form; see Theorem 6.6 in §6.3.

Finally note that $f \in \mathcal{C}^m$ is specified by the values of

$$\int_{\gamma_{\underline{\alpha}}} f$$

for $\underline{\alpha} \in \mathcal{S}^m_{\mathbf{w}}$ (i.e. thinking of $\mathcal{C}^m$ as $(\mathcal{C}_m)^*$). Then $W'$ has a basis consisting of $(\underline{\alpha})$ for each $\underline{\alpha} \in \mathcal{T}^m_{\mathbf{w}}$, where $\int_{\gamma_{\underline{\beta}}}(\underline{\alpha}) = \delta_{\underline{\alpha}\underline{\beta}}$ for all $\underline{\beta} \in \mathcal{S}^m_{\mathbf{w}}$. That is, dual to the basis $\mathcal{S}^m_{\mathbf{w}}$ for $\mathcal{C}_m$, we have a basis $\{(\underline{\alpha}) \mid \underline{\alpha} \in \mathcal{S}^m_{\mathbf{w}}\}$ for $\mathcal{C}^m$; and $W'$ is spanned by those $(\underline{\alpha})$ associated with $\underline{\alpha} \in \mathcal{T}^m_{\mathbf{w}}$.

## 6.2 Symmetrization

In this section we will evaluate the action of $S_m$ on the subspace of $H^m$ defined in §6.1, and will deduce a basis for that part of the space invariant under $S_m$ (see Corollary 6.5). Let us recall some notation:

$$\textbf{Definition} \begin{cases} \mathcal{S}_{\mathbf{w}}^r = \{\underline{\alpha} = (\alpha_1, \ldots, \alpha_r) | \alpha_i \in \{z_{i+1}, \ldots, z_m, w_1, \ldots, w_n\}\} \; ; \\[2mm] \mathcal{T}_{\mathbf{w}}^r = \{\underline{\alpha} \in \mathcal{S}_{\mathbf{w}}^r | \alpha_i \in \{w_1, \ldots, w_n\} \text{ and } \alpha_i \text{ are all distinct}\} \; ; \\[2mm] \mathcal{U}_{\mathbf{w}}^r = \{\underline{\alpha} \in \mathcal{T}_{\mathbf{w}}^r | \alpha_1 > \alpha_2 > \cdots > \alpha_r\} \; . \end{cases}$$

There is an obvious action of the symmetric group $S_m$ on $\mathcal{T}_{\mathbf{w}}^m$, given by:

$$\sigma(\underline{\alpha}) = \left(\alpha_{\sigma(1)} \, \alpha_{\sigma(2)} \cdots \alpha_{\sigma(m)}\right) .$$

Under this action,

$$\mathcal{T}_{\mathbf{w}}^m / S_m \cong \mathcal{U}_{\mathbf{w}}^m \; .$$

In this notation, the subspace $W' \subseteq \mathcal{C}^m$ of §6.1 is given by:

$$\left. \begin{aligned} \mathcal{C}^m &= \left\langle (\underline{\alpha}) \mid \underline{\alpha} \in \mathcal{S}_{\mathbf{w}}^m \right\rangle \\[2mm] W' &= \left\langle (\underline{\alpha}) \mid \underline{\alpha} \in \mathcal{T}_{\mathbf{w}}^m \right\rangle \end{aligned} \right\}$$

The main result of this section is that a basis exists for $(W')^{S_m}$, the fixed part of $W'$ under the action of the symmetric group, whose elements correspond to those of $\mathcal{T}_{\mathbf{w}}^m$ .

**Definition:** *For any $\sigma \in S_m$, define $\varepsilon(\sigma) \in \mathbf{N} \cup \{0\}$ by:*

$$\varepsilon(\sigma) = \sum_{i < j} H\left(\sigma(i) - \sigma(j)\right)$$

*where the sum runs over all pairs $(i, j)$ in $\{1, 2, \ldots, m\}$ with $i < j$; and $H(x)$ is the Heaviside function:*

$$H(x) = \begin{cases} 1 & \text{for } x \geq 0; \\ 0 & \text{for } x < 0. \end{cases}$$

Then $\varepsilon(\sigma)$ denotes the number of pairs of elements of $\{1, 2, \ldots, m\}$ whose numerical order is reversed under the action of $\sigma$.

**Definition:** *For each $\underline{\alpha} \in \mathcal{U}_{\mathbf{w}}^m$, define $f_{\underline{\alpha}}^m \in W'$ by:*

$$f_{\underline{\alpha}}^m = \sum_{\sigma \in S_m} q^{\varepsilon(\sigma)} . \left(\sigma(\underline{\alpha})\right)$$

**Lemma 6.2**    $\varepsilon(\sigma)$ *satisfies the following two relations which uniquely determine* $\varepsilon$:

$$\varepsilon(1) = 0 \; ;$$

$$\varepsilon\big(\sigma \circ (i\,i{+}1)\big) = \varepsilon(\sigma) + \mathrm{sgn}\big(\sigma(i{+}1) - \sigma(i)\big); \quad \forall \sigma \in S_m, \; 1 \le i \le m \; .$$

PROOF:    The two relations uniquely determine $\varepsilon$ since $S_m$ is generated by $(i\,i{+}1)$ for $1 \le i \le m-1$. Also for any pair $\lambda, \mu \in \{1, 2, \ldots, m\}$ with $\lambda < \mu$,

$$H\left\{ \big[\sigma \circ (i\,i{+}1)\big](\lambda) - \big[\sigma \circ (i\,i{+}1)\big](\mu) \right\} = H\big(\sigma(\lambda') - \sigma(\mu')\big)$$

where $\lambda' = (i\,i{+}1)(\lambda)$, $\mu' = (i\,i{+}1)(\mu)$. Then $\lambda' < \mu'$ unless $\lambda = i$ and $\mu = i{+}1$. Thus $\varepsilon\big(\sigma \circ (i\,i{+}1)\big)$ differs from $\varepsilon(\sigma)$ due only to the contribution of the pair $i$, $i{+}1$ to the two sums. This completes the proof.    ■

We claim that $\{f_{\underline{\alpha}}^m\}$ spans the symmetric part of $W'$. The proof of this result is split into the following two Lemmas.

**Lemma 6.3**    *For all* $\underline{\alpha} \in \mathcal{U}_{\mathbf{w}}^m$, $f_{\underline{\alpha}}^m$ *is preserved by the action of* $S_m$, *defined in §3.1, on* $\mathcal{C}^m$.

PROOF:    The action of the generator $(i\,i{+}1)$ of $S_m$ on $\mathcal{C}^m$ is the dual of the corresponding action on $\mathcal{C}_m$. However, as remarked in Theorem 4.5,

$$j_{i\,i{+}1}^2 = q_{z_i z_{i{+}1}} = q^{-2}$$

and thus the dual action of $\sigma_i$ is given by:

$$\big(q\,j_{i\,i{+}1}\big)^{-1\,\mathrm{T}} = q\,j_{i\,i{+}1}{}^{\mathrm{T}} \; .$$

However by Theorem 4.5, it is seen that $j_{i\,i{+}1}{}^{\mathrm{T}}$ preserves the vector space spanned by $(\underline{\alpha})$ where $\underline{\alpha} \in \mathcal{S}_{\mathbf{w}}^m$ and $\alpha_i$, $\alpha_{i{+}1} \in \{w_1, \ldots, w_n\}$ are distinct. Thus, if $\underline{\alpha} \in \mathcal{T}_{\mathbf{w}}^m$, then the action of $q\,j_{i\,i{+}1}^{(i{+}1)\,\mathrm{T}}$ on $(\underline{\alpha}^{i{+}1}) \in \mathcal{C}^{i{+}1}$ is given by:

$$\begin{array}{cc} \mu\lambda & \lambda\mu \end{array}$$
$$\begin{pmatrix} 0 & q\,A' \\ q^{-1}\,A'^{-1} & 0 \end{pmatrix}$$

where $A' = A_{z_i z_{i{+}1}}^{(i-1)}$ and the blocks are specified by the values of $\alpha_i$ and $\alpha_{i{+}1}$, with $\lambda > \mu$. In the next section it will be shown that the action of $A'$ on $\underline{\alpha}^{(i-1)}$ is trivial whenever $\underline{\alpha} \in \mathcal{T}_{\mathbf{w}}^m$ (see Lemma 6.8).

Hence the action of $qj_{i\,i+1}{}^{\mathrm{T}}$ on $(\underline{\alpha}) \in \mathcal{C}^m$ for $\underline{\alpha} \in \mathcal{T}_{\mathbf{w}}^m$ is given by:

$$\left(qj_{i\,i+1}{}^{\mathrm{T}}\right)(\underline{\alpha}) = \begin{cases} q^{-1}(\alpha_1 \cdots \alpha_{i-1}\alpha_{i+1}\alpha_i \cdots \alpha_m) & \text{for } \alpha_i < \alpha_{i+1} \\ q(\alpha_1 \cdots \alpha_{i-1}\alpha_{i+1}\alpha_i \cdots \alpha_m) & \text{for } \alpha_i > \alpha_{i+1} \end{cases}$$

Applying this to $\sigma(\underline{\alpha})$ where $\underline{\alpha} \in \mathcal{U}_{\mathbf{w}}^m$, we see that:

$$\left(qj_{i\,i+1}{}^{\mathrm{T}}\right)\left(\sigma(\underline{\alpha})\right) = q^{\pm 1}\left(\left(\sigma \circ (i\,i+1)\right)(\underline{\alpha})\right)$$

where the power $\pm 1$ is determined by:

$$\left(\sigma(\underline{\alpha})\right)_i \gtrless \left(\sigma(\underline{\alpha})\right)_{i+1} .$$

Since $\{\alpha_j\}$ is in decreasing order, thus $\left(\sigma(\underline{\alpha})\right)_i > \left(\sigma(\underline{\alpha})\right)_{i+1}$ is equivalent to $\sigma(i+1) > \sigma(i)$, and hence the image of $f_{\underline{\alpha}}^m$ is given by:

$$\left(qj_{i\,i+1}{}^{\mathrm{T}}\right)f_{\underline{\alpha}}^m = \sum_{\sigma \in S_m} q^{\varepsilon(\sigma)} q^{\operatorname{sgn}\left(\sigma(i+1) - \sigma(i)\right)} \left(\sigma \circ (i\,i+1)\right)(\underline{\alpha})$$

$$= \sum_{\sigma \in S_m} q^{\varepsilon\left(\sigma \circ (i\,i+1)\right)} \left(\sigma \circ (i\,i+1)\right)(\underline{\alpha})$$

$$= f_{\underline{\alpha}}^m$$

by Lemma 6.2. This shows that $f_{\underline{\alpha}}^m$ is invariant under the action of the generators of $S_m$; and hence the Lemma is proved. ∎

**Lemma 6.4** *The dimension of the symmetric part of $W'$ is $1/m!\, n(n-1) \cdots (n-m+1)$.*

PROOF: Now $\mathcal{T}_{\mathbf{w}}^m$ is a set of order $n(n-1) \cdots (n-m+1)$. When $q = 1$, we are using a trivial local coefficient system, and so the action of $S_m$ given by $j_{i\,i+1}{}^{\mathrm{T}}$ on $\langle (\underline{\alpha}) \mid \underline{\alpha} \in \mathcal{T}_{\mathbf{w}}^m \rangle$ is identical to the action of $S_m$ given by permuting $\{\alpha_i\}$. Hence the part of $\langle (\underline{\alpha}) \mid \underline{\alpha} \in \mathcal{T}_{\mathbf{w}}^m \rangle$ invariant under $S_m$ is spanned by the symmetrised elements:

$$1/m! \sum_{\sigma \in S_m} \left(\sigma(\underline{\alpha})\right) .$$

These are precisely the elements $f_{\underline{\alpha}}^m$ for $\underline{\alpha} \in \mathcal{U}_{\mathbf{w}}^m$, in the case $q = 1$. Hence for $q = 1$, the symmetrised part has dimension $\binom{n}{m}$. When $q$ is moved away from 1, this dimension cannot increase locally, since the symmetrised part can be thought of as an intersection of subspaces of $\mathcal{C}^m$:

$$\bigcap_{i=1}^{m-1} \left\{ \mathbf{v} \in \mathcal{C}^m \mid qj_{i\,i+1}\mathbf{v} = \mathbf{v} \right\} .$$

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However by Lemma 6.3, $\left\{ f_{\underline{\alpha}}^m \mid \underline{\alpha} \in \mathcal{U}_{\mathbf{w}}^m \right\}$ defines a set of $\binom{n}{m}$ linearly independent elements of the symmetric part, so that this dimension must be at least $\binom{n}{m}$. We conclude that the dimension of the symmetric part is precisely $\binom{n}{m}$. ∎

Hence, putting the two Lemmas together, we obtain:

**Corollary 6.5**    *The symmetric part of $W' \subseteq \mathcal{C}^m$ under the natural action of $S_m$ given by §3.1 is precisely $\left\langle f_{\underline{\alpha}}^m \mid \underline{\alpha} \in \mathcal{U}_{\mathbf{w}}^m \right\rangle$ .*

In Theorem 6.1, the space on which $B_n$ acts is the symmetric part of the subspace $W'/\mathrm{Im}(\mathbf{D}')$ of the cohomology space $\mathcal{C}^m/\mathrm{Im}(\mathbf{D}') = H^m$. By the above Corollary, this space is given by:

$$\left\langle i'(f_{\underline{\alpha}}^m) \mid \underline{\alpha} \in \mathcal{U}_{\mathbf{w}}^m \right\rangle$$

where, for $\mathbf{v} \in \mathcal{C}^m$, $i'(\mathbf{v})$ denotes the corresponding element of $H^m$.

### 6.3 The monodromy action

To calculate the monodromy action of $B_n$ on the subspace of cohomology given by the last section, we start by evaluating the action of $B_n$ on the corresponding chains. That is, we calculate the action of generators $\sigma_i \in B_n$ (for $i = 1, 2, \ldots, n-1$) on:

$$\left\langle f_{\underline{\alpha}}^m \mid \underline{\alpha} \in \mathcal{U}_{\mathbf{w}}^m \right\rangle \subseteq \mathcal{C}^m .$$

This action preserves the space $\mathrm{Im}(\mathbf{D})$, and thus induces an action on the corresponding subspace of cohomology, namely that referred to in Theorem 6.1.

At the level of $\mathcal{C}^m$, the action of $B_n$ is explicitly given by:

**Theorem 6.6**    *The generator $\sigma_i$ of the braid group $B_n$ acts on $\left\langle f_{\underline{\alpha}}^m \mid \underline{\alpha} \in \mathcal{U}_{\mathbf{w}}^m \right\rangle \subseteq \mathcal{C}^m$ by the natural monodromy representation, according to:*

$$f_{\underline{\alpha}}^m \longrightarrow \begin{cases} f_{\underline{\alpha}_i}^m + (1-q)f_{\underline{\alpha}}^m & \text{if } \{w_i, w_{i+1}\} \cap \{\alpha_1, \ldots, \alpha_m\} = \{w_i\} \\ q f_{\underline{\alpha}_i}^m & \text{if } \{w_i, w_{i+1}\} \cap \{\alpha_1, \ldots, \alpha_m\} = \{w_{i+1}\} \\ f_{\underline{\alpha}_i}^m & \text{otherwise} \end{cases}$$

*where $\underline{\alpha}_i$ denotes $\underline{\alpha}$ with any entry $w_i$ changed to $w_{i+1}$, and any entry $w_{i+1}$ changed to $w_i$.*

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Recall that the action of $B_n$ on $\mathcal{C}^m$ is the dual action to that on $\mathcal{C}_m$. The action of the generator $\sigma_i \in B_n$ on $\mathcal{C}_m$ is given by $A^{(m)}_{w_i w_{i+1}}$, and the dual action is thus given by:

$$\left(A^{(m)}_{w_i w_{i+1}}\right)^{-1\,\mathrm{T}}.$$

Hence the Theorem is equivalent to showing that the action of $\left(A^{(m)}_{w_i w_{i+1}}\right)^{\mathrm{T}}$ on the subspace of $\mathcal{C}^m$ spanned by $\{f^m_{\underline{\alpha}} \mid \underline{\alpha} \in \mathcal{U}^m_{\mathbf{w}}\}$ is given by:

$$f^m_{\underline{\alpha}} \longrightarrow \begin{cases} q^{-1} f^m_{\underline{\alpha}_i} & \text{if } \{w_i, w_{i+1}\} \cap \{\alpha_1, \ldots, \alpha_m\} = \{w_i\} \\ f^m_{\underline{\alpha}_i} + (1 - q^{-1}) f^m_{\underline{\alpha}} & \text{if } \{w_i, w_{i+1}\} \cap \{\alpha_1, \ldots, \alpha_m\} = \{w_{i+1}\} \\ f^{\overline{m}}_{\underline{\alpha}} & \text{otherwise} \end{cases}$$

The matrices $(A^{(m)}_{w_i w_{i+1}})$ for $i = 1,, 2, \ldots, n-1$, are given by the recursion formulae of Theorem 4.4 (§4.5), and the proof of Theorem 6.6 proceeds by applying induction on $m$ to prove many intermediate results. To avoid the necessity for using extra brackets, we shall in future use $\underline{\alpha}$ to refer to the corresponding element $(\underline{\alpha})$ of $\mathcal{C}^m$, as well as an elements $\mathcal{S}^m_{\mathbf{w}}$, so long as the context is clear.

**Lemma 6.7** For $\underline{\alpha} \in \mathcal{U}^r_{\mathbf{w}}$, then if $f^r_{\underline{\alpha}}$ is defined as in §6.2 it satisfies the folowing relation:

$$f^r_{\underline{\alpha}} = \sum_{a=1}^{r} q^{r-a}(f^{r-1}_{\underline{\alpha}(a)}.\alpha_a)$$

where $\alpha(a) \in \mathcal{U}^{r-1}_{\mathbf{w}}$ is obtained by removing the $a^{\mathrm{th}}$ element from $\alpha$, to give $(\alpha_1, \ldots, \alpha_{a-1}, \alpha_{a+1}, \ldots, \alpha_r)$.

PROOF: From the definition of $f^r_{\underline{\alpha}}$ as:

$$\sum_{\sigma \in S_r} q^{\varepsilon(\sigma)}.\sigma(\underline{\alpha})$$

we may split this sum up, according to the value of $a = \sigma(r)$, into $r$ parts. For any particular value $a$, $\exists \sigma' \in S_{r-1}$ such that:

$$\sigma(\underline{\alpha}) = \sigma'\big(\underline{\alpha}(a)\big).\alpha_a\,.$$

The correct $\sigma'$ to use here is given by:

$$\sigma'(i) = \begin{cases} \sigma(i) & \text{if } \sigma(i) < a \\ \sigma(i) - 1 & \text{if } \sigma(i) > a \end{cases}$$

However, it is easily verified from the definition of $\varepsilon$ that:

$$\varepsilon(\sigma) = \varepsilon(\sigma') + r - a\,.$$

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Thus one obtains:

$$f_{\underline{\alpha}}^r = \sum_{\sigma \in S_r} q^{r-a} . q^{\varepsilon(\sigma')} \sigma'\big(\underline{\alpha}(a)\big) . \alpha_a$$

$$= \sum_{a=1}^r q^{r-a} \big(f_{\underline{\alpha}(a)}^{r-1} . \alpha_a\big)$$

which completes the proof of the Lemma. ■

This Lemma enables us to prove theorems on the behaviour of the action of $B_n$ on $\langle f_{\underline{\alpha}}^r \rangle \subseteq \mathcal{C}^r$ in terms of its action on $\langle f_{\underline{\alpha}}^{r-1} \rangle \subseteq \mathcal{C}^r$.

**Lemma 6.8** *The actions on $b_{\lambda\mu}^{(r)}{}^{\mathrm{T}}$ and $A_{\lambda\mu}^{(r)}{}^{\mathrm{T}}$ on $(\underline{\alpha})$ are given by multiplication by $q_{\lambda\mu}^{-1}$ and 1 respectively, whenever $\underline{\alpha} \in \mathcal{S}_{\mathbf{w}}^r$ with $\alpha_i \notin \{z_1, \ldots, z_r, \lambda, \mu\}$ for all $i$. Here $r$ is an integer, $1 \le r \le m$.*

PROOF: We prove this Lemma by induction on $r$. For $r = 0$, it holds trivially. Assume the statement of the Lemma holds for $r-1$. Suppose $\underline{\alpha} \in \mathcal{S}_{\mathbf{w}}^r$, with $\alpha_i \notin \{z_1, \ldots, z_r, \lambda, \mu\}$ for all $i$. Then by Theorem 4.4:

$$A_{\lambda\mu}^{(r)}{}^{\mathrm{T}}(\underline{\alpha}^{r-1} . \alpha_r) = \begin{cases} A_{\lambda\mu}^{(r-1)}{}^{\mathrm{T}}(\underline{\alpha}^{r-1}) . \alpha_r & \text{if } \alpha_r < \lambda \text{ or } \alpha_r > \mu \\ \left[ b_{z_r\lambda}^{(r-1)}{}^{\mathrm{T}} A_{\lambda\mu}^{(r-1)}{}^{\mathrm{T}} \big(b_{z_r\lambda}^{(r-1)}\big)^{\mathrm{T}^{-1}} \right] (\underline{\alpha}^{r-1}) . \alpha_r & \text{if } \lambda < \alpha_r < \mu \end{cases}$$

$$\text{and } b_{\lambda\mu}^{(r)}{}^{\mathrm{T}}(\underline{\alpha}^{r-1} . \alpha_r) = \begin{cases} b_{\lambda\mu}^{(r-1)}{}^{\mathrm{T}}(\underline{\alpha}^{r-1}) . \alpha_r & \text{if } \alpha_r < \lambda \text{ or } \alpha_r > \mu \\ \left[ b_{z_r\lambda}^{(r-1)}{}^{\mathrm{T}} b_{\lambda\mu}^{(r-1)}{}^{\mathrm{T}} \big(b_{z_r\lambda}^{(r-1)}\big)^{\mathrm{T}^{-1}} \right] (\underline{\alpha}^{r-1}) . \alpha_r & \text{if } \lambda < \alpha_r < \mu \end{cases}$$

However, none of the entries of $\underline{\alpha}^{r-1}$ lie in the set $\{z_1, \ldots, z_r, \lambda, \mu\}$, and thus by the inductive assumption, $A_{\lambda\mu}^{(r-1)}{}^{\mathrm{T}}$, $b_{\lambda\mu}^{(r-1)}{}^{\mathrm{T}}$, $b_{z_r\lambda}^{(r-1)}{}^{\mathrm{T}}$ all act on $\underline{\alpha}^{r-1}$ as multiplication by factors of $1$, $q_{\lambda\mu}^{-1}$, $q_{z_r\lambda}^{-1}$ respectively. Thus we obtain:

$$\left. \begin{aligned} A_{\lambda\mu}^{(r)}(\underline{\alpha}) &= \underline{\alpha} \\ b_{\lambda\mu}^{(r)}(\underline{\alpha}) &= q_{\lambda\mu}^{-1}\underline{\alpha} \end{aligned} \right\}$$

which gives the general statement of the Lemma at $r$. Hence the proof of Lemma 6.8 is complete. ■

**Lemma 6.9** *Suppose that $\underline{\alpha}_1, \underline{\alpha}_2 \in \mathcal{T}_{\mathbf{w}}^r$ are such that $\alpha_1$ and $\alpha_2$ differ only in the $k^{\mathrm{th}}$ component where they are $w_i$, $w_{i+1}$ respectively, some $i$, $k$ with $1 \le k \le r$, $1 \le i \le n-1$. Then $b_{z_{r+1}w_{i+1}}^{(r)} b_{z_{r+1}w_i}^{(r)}$ preserves the element:*

$$(\underline{\alpha}_2) - q(\underline{\alpha}_1) \in W' .$$

PROOF: As in the last Lemma, we use induction on $r$. For $r = 0$, the result is trivial. Assume the statement of the Lemma holds for $r-1$. When $k < r$, the result follows immediately from the inductive hypothesis using Theorem 4.4 and Lemma 6.8, since:

$$(\underline{\alpha}_1)_r, \ (\underline{\alpha}_2)_r \notin \{w_i, w_{i+1}, z_1, \ldots, z_r\} \ .$$

When $k = r$, we have for some $\underline{\alpha} \in \mathcal{T}_{\mathbf{w}}^{r-1}$, that:

$$\underline{\alpha}_1 = \underline{\alpha}.w_i \ ;$$

$$\underline{\alpha}_2 = \underline{\alpha}.w_{i+1} \ .$$

By Theorem 4.4, since $\underline{\alpha}$ contains neither $w_i$ nor $w_{i+1}$,

$$b_{z_{r+1}w_i}^{(r)}{}^{\mathrm{T}}(\underline{\alpha}_2 - q\underline{\alpha}_1) = b_{w_i}'{}^{\mathrm{T}}(\underline{\alpha}).w_{i+1} - q\left\{(1 - b_{z_{r+1}}{}^{\mathrm{T}})b_{w_i}'{}^{\mathrm{T}}(\underline{\alpha}).z_{r+1} + \sum_{z_{r+1} < \lambda < w_i} \left[(1 - b_\lambda{}^{\mathrm{T}})(1 - b_{z_{r+1}}{}^{\mathrm{T}})b_{w_i}'{}^{\mathrm{T}}\right](\underline{\alpha}).\lambda \right.$$

$$\left. + \left[b_{w_i}'{}^{\mathrm{T}} + b_{w_i}'{}^{\mathrm{T}}b_{w_i}{}^{\mathrm{T}}(b_{z_{r+1}}{}^{\mathrm{T}} - 1)\right](\underline{\alpha}).w_i \right\}$$

where $b_\lambda \equiv b_{z_r\lambda}^{(r-1)}$, $b' \equiv b_{z_{r+1}\lambda}^{(r-1)}$. By Lemma 6.8 this can be reduced to:

$$(q^2 - 1)\underline{\alpha}.z_{r+1} + (q^{-1} - 1 - q)\underline{\alpha}.w_i + q^{-1}\underline{\alpha}.w_{i+1} + (q^2 - 1) \sum_{z_{r+1} < \lambda < w_i} (1 - b_\lambda{}^{\mathrm{T}})\underline{\alpha}.\lambda \ .$$

When $b_{z_{r+1}w_{i+1}}^{(r)}$ is applied to this vector, using Theorem 4.4 once more, together with Lemma 6.8, one obtains:

$$(q^2 - 1)\left\{ q^{-2}\underline{\alpha}.z_{r+1} + (1 - q^{-1})q^{-2}\underline{\alpha}.w_{i+1} + q^{-2}(1 - q) \sum_{z_{r+1} < j < w_{i+1}} (1 - b_j{}^{\mathrm{T}})\underline{\alpha}.w_j \right\} + (q^{-1} - 1 - q)q^{-1}\underline{\alpha}.w_i$$

$$+ q^{-1}\left\{ (1 - q^2)q^{-1}\underline{\alpha}.z_{r+1} + (1 - q^{-1} + q)q^{-1}\underline{\alpha}.w_{i+1} + q^{-1}(1 - q^2) \sum_{z_{r+1} < j < w_{i+1}} (1 - b_j{}^{\mathrm{T}})\underline{\alpha}.w_j \right\}$$

$$+ (q^2 - 1) \sum_{z_{r+1} < \lambda < w_i} q^{-1}(1 - b_\lambda{}^{\mathrm{T}})\underline{\alpha}.\lambda$$

which reduces to $\underline{\alpha}.w_{i+1} - q\underline{\alpha}.w_i = \underline{\alpha}_2 - q\underline{\alpha}_1$. This completes the proof of the Lemma. ∎

PROOF: (Theorem 6.6) As remarked at the start of this section, the Theorem is proved once it is shown that, for all $\underline{\alpha} \in \mathcal{U}_{\mathbf{w}}^r$:

$$A_{w_i w_{i+1}}^{(r)}{}^{\mathrm{T}}(f_{\underline{\alpha}}) = \begin{cases} q^{-1}f_{\underline{\alpha}_i}^r & \text{if } \{w_i, w_{i+1}\} \cap \{\alpha_1, \ldots, \alpha_m\} = \{w_i\} \\ f_{\underline{\alpha}_i}^r + (1 - q^{-1})f_{\underline{\alpha}}^r & \text{if } \{w_i, w_{i+1}\} \cap \{\alpha_1, \ldots, \alpha_m\} = \{w_{i+1}\} \\ f_{\underline{\alpha}}^r & \text{otherwise} \end{cases}$$

For $r = 1$, this follows directly from Theorem 4.4 applied at first order. Assume that the above action of $A_{w_i w_{i+1}}^{(r)}{}^{\mathrm{T}}$ on $\langle f_{\underline{\alpha}}^r \mid \underline{\alpha} \in \mathcal{U}_{\mathbf{w}}^r \rangle$ holds for $r-1$. Suppose $\underline{\beta} \in \mathcal{U}_{\mathbf{w}}^r$. By Lemma 6.7,

$$f_{\underline{\beta}}^r = \sum_{a=1}^r q^{r-a} \left(f_{\underline{\beta}(a)}^{r-1}.\beta_a\right) \ . \tag{6.3.1}$$

We now consider the action of $A_{w_i w_{i+1}}^{(r)}{}^{\mathrm{T}}$ on the terms in (6.3.1) separately.

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*Case(i)* $\beta_a \neq w_i, w_{i+1}$ .

By Theorem 4.4,

$$A_{w_i w_{i+1}}^{(r)\,\mathrm{T}} \left( f_{\underline{\beta}(a)}^{r-1} . \beta_a \right) = A_{w_i w_{i+1}}^{(r-1)\,\mathrm{T}} \left( f_{\underline{\beta}(a)}^{r-1} \right) . \beta_a$$

and thus by the inductive assumption, $f_{\underline{\beta}(a)}^{r-1} . \beta_a$ transforms under $A_{w_i w_{i+1}}^{(r)\,\mathrm{T}}$ in the natural way, according to whether $\underline{\beta}(a)$ contains $w_i$ and/or $w_{i+1}$ .


*Case(ii)* $\beta_a = w_i$ and $\underline{\beta}$ *does not contain both* $w_i$ *and* $w_{i+1}$ .

Here, $\underline{\beta}(a)$ does not contain either $w_i$ or $w_{i+1}$, and by Theorem 4.4 and Lemma 6.8,

$$A_{w_i w_{i+1}}^{(r)\,\mathrm{T}} \left( f_{\underline{\beta}(a)}^{r-1} . \beta_a \right) = A_{w_i w_{i+1}}^{(r-1)\,\mathrm{T}} b_{z_r w_{i+1}}^{(r-1)\,\mathrm{T}} \left( f_{\underline{\beta}(a)}^{r-1} \right) . w_{i+1}$$

$$= q^{-1} f_{\underline{\beta}(a)}^{r-1} . w_{i+1}$$

using the inductive assumption. This gives the natural transformation for $f_{\underline{\beta}(a)}^{r-1} . \beta_a$ .


*Case(iii)* $\beta_a = w_{i+1}$ and $\underline{\beta}$ *does not contain both* $w_i$ *and* $w_{i+1}$ .

Here, once again $\underline{\beta}(a)$ does not contain either $w_i$ or $w_{i+1}$, and :

$$A_{w_i w_{i+1}}^{(r)\,\mathrm{T}} \left( f_{\underline{\beta}(a)}^{r-1} . \beta_a \right) = A_{w_i w_{i+1}}^{(r-1)\,\mathrm{T}} \left( f_{\underline{\beta}(a)}^{r-1} \right) . w_i + (1 - b_{z_r w_{i+1}}^{(r-1)\,\mathrm{T}}) A_{w_i w_{i+1}}^{(r-1)\,\mathrm{T}} \left( f_{\underline{\beta}(a)}^{r-1} \right) . w_{i+1}$$

$$= f_{\underline{\beta}(a)}^{r-1} . w_i + (1 - q^{-1}) f_{\underline{\beta}(a)}^{r-1} . w_{i+1}$$

by Lemma 6.8 and the inductive assumption.


When $\underline{\beta}$ does not contain both $w_i$ and $w_{i+1}$, these three cases put together, using (6.3.1) give the required transformation properties of $f_{\underline{\beta}}^r$ at level $r$. The only case we are left with is that for which $\underline{\beta}$ contains both $w_i$ and $w_{i+1}$. By case (i), those terms in (6.3.1) with $\beta_a \neq w_i, w_{i+1}$ are preserved. Thus, to show that $f_{\underline{\beta}}^r$ is preserved, it is only necessary to show that the sum of the two terms in (6.3.1) corresponding to $a$'s such that $\beta_a = w_i, w_{i+1}$, is preserved by $A_{w_i w_{i+1}}^{(r)\,\mathrm{T}}$ .

Since $\underline{\beta} \in \mathcal{U}_{\mathbf{w}}^r$ then $\beta_s = w_{i+1}$ and $\beta_{s+1} = w_i$ for some $s$. Thus, it is only necessary to show that $A_{w_i w_{i+1}}^{(r)\,\mathrm{T}}$ preserves:

$$q f_{\underline{\beta}(s)}^{r-1} . w_{i+1} + f_{\underline{\beta}(s+1)}^{r-1} . w_i = q f_{\underline{\alpha}_1}^{r-1} . w_{i+1} + f_{\underline{\alpha}_2}^{r-1} . w_i$$

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where $\underline{\alpha}_1$, $\underline{\alpha}_2$ satisfy the conditions of Lemma 6.9. Since $\{\beta_1, \ldots, \beta_{s-1}, \beta_{s+2}, \ldots, \beta_r\}$ does not contain either $w_i$ or $w_{i+1}$, we deduce from Theorem 4.4, that:

$$A^{(r)}_{w_i w_{i+1}}{}^{\mathrm{T}}\left(q f^{r-1}_{\underline{\alpha}_1}.w_{i+1} + f^{r-1}_{\underline{\alpha}_2}.w_i\right)$$

$$= q\left\{A^{\mathrm{T}} f^{r-1}_{\underline{\alpha}_1}.w_i + \left[A^{\mathrm{T}}(1 - b_{w_{i+1}}{}^{\mathrm{T}} b_{w_i}{}^{\mathrm{T}} b_{w_{i+1}}^{-1}{}^{\mathrm{T}})\right] f^{r-1}_{\underline{\alpha}_1}.w_{i+1}\right\} + (A^{\mathrm{T}} b_{w_{i+1}}{}^{\mathrm{T}}) f^{r-1}_{\underline{\alpha}_2}.w_{i+1}$$

$$= q\left\{q^{-1} f^{r-1}_{\underline{\alpha}_2}.w_i + (q^{-1} f^{r-1}_{\underline{\alpha}_2} - q A^{\mathrm{T}} b_{w_{i+1}}{}^{\mathrm{T}} b_{w_i}{}^{\mathrm{T}} f^{r-1}_{\underline{\alpha}_1}).w_{i+1}\right\} + (A^{\mathrm{T}} b_{w_{i+1}}{}^{\mathrm{T}}) f^{r-1}_{\underline{\alpha}_2}.w_{i+1}$$

by the inductive hypothesis. (In the above we have used the simplified notation in which $A = A^{(r-1)}_{w_i w_{i+1}}$, $b_\lambda = b^{(r-1)}_{z_r \lambda}$.) Applying Lemma 6.8, we can reduce this to the form:

$$f^{r-1}_{\underline{\alpha}_2}.w_i + \left[f^{r-1}_{\underline{\alpha}_2} + A^{\mathrm{T}} b_{w_{i+1}}{}^{\mathrm{T}} b_{w_i}{}^{\mathrm{T}} (q f^{r-1}_{\underline{\alpha}_2} - q^2 f^{r-1}_{\underline{\alpha}_1})\right].w_{i+1}$$

$$= f^{r-1}_{\underline{\alpha}_2}.w_i + \left[f^{r-1}_{\underline{\alpha}_2} + A^{\mathrm{T}}(q f^{r-1}_{\underline{\alpha}_2} - q^2 f^{r-1}_{\underline{\alpha}_1})\right].w_{i+1} \qquad \text{by Lemma 6.9}$$

$$= f^{r-1}_{\underline{\alpha}_2}.w_i + q f^{r-1}_{\underline{\alpha}_1}.w_{i+1}$$

the last step again being a consequence of the inductive hypothesis.

Hence the proof of Theorem 6.6 is complete. ■

Finally to derive Theorem 6.1 it is now necessary to discuss the part of $\mathrm{Im}\,(\mathbf{D}')$ contained in:

$$\left\langle f^m_{\underline{\alpha}} \mid \underline{\alpha} \in \mathcal{U}^m_{\mathbf{w}} \right\rangle .$$

That is, it is necessary to derive the relations between $\{j'(f^m_{\underline{\alpha}}) | \underline{\alpha} \in \mathcal{U}^m_{\mathbf{w}}\} \subseteq H^m$ which exist due to taking out the 'boundary space' $\mathrm{Im}\,(\mathbf{D}')$.

**Lemma 6.10**      Suppose $\underline{\alpha} \in \mathcal{U}^r_{\mathbf{w}}$. Then in $\mathcal{C}^r$ we have

$$(b^{(r)}_{z_m w_{\alpha_j}}{}^{\mathrm{T}} - 1) f^r_{\underline{\alpha}} = \sum_{i \neq \alpha_k{}'s} \left(f^r_{(\alpha_1 \ldots \widehat{\alpha}_j \ldots \alpha_r w_i)}.q^{s_i - j}(q^{-1} - 1)\right)$$

where $s_i$ is such that $\alpha_{s_i - 1} > i > \alpha_{s_i}$ and $s_i \in \{1, 2, \ldots, r\}$.

PROOF:      By Lemma 6.7, we can express $f^r_{\underline{\alpha}}$ as a combination of terms:

$$\sum_{a=1}^r q^{r-a} f^{r-1}_{\underline{\alpha}(a)}.\alpha_a . \tag{6.3.2}$$

Those terms corresponding to $a \neq j$ transform under $\left(b_{z_r w_{\alpha_j}}^{(r)}{}^{\mathrm{T}} - 1\right)$ according to:

$$q^{r-a}\left(b_{z_m w_{\alpha_j}}^{(r-1)}{}^{\mathrm{T}} - 1\right)\left(f_{\underline{\alpha}(a)}^{r-1}\right).\alpha_a \qquad \text{if } a < j$$

$$q^{r-a}\left(b_{z_r z_m}^{(r-1)}{}^{\mathrm{T}} b_{z_m w_{\alpha_j}}^{(r-1)}{}^{\mathrm{T}} b_{z_r z_m}^{(r-1)}{}^{\mathrm{T}^{-1}} - 1\right)\left(f_{\underline{\alpha}(a)}^{r-1}\right).\alpha_a \quad \text{if } a > j$$

by Theorem 4.4. However, by Lemma 6.8, $f_{\underline{\alpha}(a)}^{r-1}$ multiplies by $q_{z_r z_m}^{-1}$ under $b_{z_r z_m}^{(r-1)}$, and so assuming the Lemma holds for $r-1$, the terms in (6.3.2) with $a \neq j$ transform to:

$$\sum_{\substack{a=1 \\ a \neq j}}^{r} \sum_{i \neq \alpha_k's} q^{r-a}\left(f_{(\alpha_1 \ldots \widehat{\alpha_j} \ldots \widehat{\alpha_a} \ldots \alpha_r w_i)}^{r-1}.\alpha_a\right) q^{s'_{ia}-j^a}\left(q^{-1}-1\right) \tag{6.3.3}$$

where $s'_{ia}$ is the value of $s_i$ corresponding to $w_i$ relative to $\{\alpha_1, \ldots, \widehat{\alpha_a}, \ldots, \alpha_r\}$; and $j^a$ is the position at which $\alpha_j$ occurs in $\{\alpha_1, \ldots, \widehat{\alpha_a}, \ldots, \alpha_r\}$. However, it is easily verified that:

$$s'_{ia} - s_i = a - j + j^a - a^j$$

where $a^j$ is the position $\alpha_a$ occurs in, when $\alpha_j$ is removed from $\{\alpha_1, \ldots, \alpha_r\}$. Thus:

$$(r-a) + (s'_{ia} - j^a) = (r - a^j) + (s_i - j) .$$

The expression on the right hand side of the equation in the Lemma may be written as a combination of $f^{r-1}$'s, by expanding out each $f^r$ as in Lemma 6.7. Those terms obtained when $\alpha_s$ $(s \neq j)$ is removed, in going from an $f^r$ to an $f^{r-1}$, give rise to the expression (6.3.3). The remaining component consists of those terms obtained when $w_i$ is removed. Hence to prove the Lemma for $r$, it remains to show that the remaining terms on either side are equal, namely that:

$$q^{r-j}\left(b_{z_m w_{\alpha_j}}^{(r)}{}^{\mathrm{T}} - 1\right)\left(f_{\underline{\alpha}(j)}^{r-1}.\alpha_j\right) = \sum_{i \neq \alpha_k's}\left(q^{r-s'_{ij}}\left(f_{(\alpha_1 \ldots \widehat{\alpha_j} \ldots \alpha_r)}^{r-1}.w_i\right) q^{s_i-j}\left(q^{-1}-1\right)\right) \tag{6.3.4}$$

However, by Theorem 4.4, the left hand side here is:

$$q^{r-j}\left\{ \sum_{k<j} \left[b_{z_m}{}^{\mathrm{T}}\left(b_{w_k}{}^{\mathrm{T}}-1\right)b^{\mathrm{T}}\right]\left(f_{\underline{\alpha}(j)}^{r-1}\right).\alpha_k + \left(b_{w_{\alpha_j}}{}^{\mathrm{T}} b_{z_m}{}^{\mathrm{T}} b^{\mathrm{T}} - 1\right)\left(f_{\underline{\alpha}(j)}^{r-1}\right).\alpha_j \right.$$

$$\left. + \sum_{k>j} \left[\left(b_{w_k}{}^{\mathrm{T}}-1\right)b^{\mathrm{T}}\right]\left(f_{\underline{\alpha}(j)}^{r-1}\right).\alpha_k \right\}$$

where $b$, $b_\lambda$ denote $b_{z_m w_{\alpha_j}}^{(r-1)}$ and $b_{z_r \lambda}^{(r-1)}$. By Lemma 6.7, $b$, $b_{z_m}$ and $b_{w_k}$ act on $f_{\underline{\alpha}(j)}^{r-1}$ as multiplication by $q^{-1}$, $q^2$ and $q^{-1}$ respectively, when:

$$k \neq \alpha_j \quad \forall j .$$

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Hence, the left hand side of (6.3.4) reduces to:

$$\sum_{i \neq \alpha_k's} q^{r-j} q^{\pm 1} \left( f_{\underline{\alpha}(j)}^{r-1}.w_i \right) (q^{-1} - 1)$$

where the sign $\pm$ depends on whether $i$ is greater or less than $\alpha_j$. This reduces to the right hand side of (6.3.4), since it is easily seen that the sign is precisely $s_i - s'_{ij}$. The inductive step in the proof of this Lemma is thus complete. The case $r = 1$ follows directly from the first order formula for $b_{z_m w_i}^{(1)}$ given by Theorem 4.4. Hence the proof of Lemma 6.10 is complete. ∎

However, $\mathrm{Im}\left( \mathbf{D}_i^{\mathrm{T}} \right)$ can be factored out of the space of chains $\mathcal{C}^m$ by setting the component of $(\underline{\alpha})$ to zero for all $\underline{\alpha} \in \mathcal{S}_{\mathbf{w}}^m$ with $\alpha_i = z_{i+1}$. This is possible for $i = 1, 2, \ldots, m-1$, so that:

$$\mathcal{C}^m / \langle \mathrm{Im}\left( \mathbf{D}_i^{\mathrm{T}} \right) \mid i+1, 2, \ldots, m-1 \rangle \cong \langle (\underline{\alpha}) \in \mathcal{S}_{\mathbf{w}}^m \mid \alpha_i \neq z_{i+1} \quad \text{for } i = 1, 2, \ldots, m-1 \rangle .$$

Hence the only part of $\mathrm{Im}\left( \mathbf{D}' \right)$ which imposes relations on $\left\{ j'(f_{\underline{\alpha}}^m) \mid \underline{\alpha} \in \mathcal{U}_{\mathbf{w}}^m \right\}$ comes from $\mathrm{Im}\left( \mathbf{D}_m^{\mathrm{T}} \right)$. Now:

$$\mathbf{D}_m^{\mathrm{T}}\left( f_{\underline{\alpha}}^{m-1} \right) = \sum_{i=1}^{n} \left( b_{z_m w_i}^{(m-1)^{\mathrm{T}}} - 1 \right) \left( f_{\underline{\alpha}}^{m-1} \right).w_i$$

$$= \sum_{i \neq \alpha_j} (q^{-1} - 1) \left( f_{\underline{\alpha}}^{m-1}.w_i \right) + \sum_{j=1}^{m} \sum_{i \neq \alpha_k's} \left( f_{\underline{\alpha}(j).w_i}^{m-1}.w_{\alpha_j} \right) q^{s_i - j}$$

by Lemmas 6.8 and 6.10. Lemma 6.7 reduces the right hand side to:

$$(q^{-1} - 1) \sum_{i \neq \alpha_j} q^{s_i} f_{(\alpha_1 \ldots \alpha_{m-1} w_i)}^m .$$

Thus the relations on the subset $j'(W') = \left\{ j'(f_{\underline{\alpha}}^m) \mid \underline{\alpha} \in \mathcal{U}_{\mathbf{w}}^m \right\}$ of $H^m$ are given by:

$$\sum_{i \neq \alpha_j} q^{s_i} f_{\underline{\alpha}.w_i}^m = 0 \tag{6.3.5}$$

for all $\underline{\alpha} \in \mathcal{U}_{\mathbf{w}}^{m-1}$, using the usual notation for $s_i$.

As was shown in Theorem 6.6, if we define:

$$g_{\underline{\alpha}} = \sum_{i \neq \alpha_j} q^{s_i} f_{\underline{\alpha}.w_i}^{(m)} \in \mathcal{C}^m \tag{6.3.6}$$

for $\underline{\alpha} \in \mathcal{U}_{\mathbf{w}}^{m-1}$, then under $\sigma_i \in B_n$, $\{g_{\underline{\alpha}}\}$ transforms according to:

$$g_{\underline{\alpha}} \longrightarrow \begin{cases} g_{\underline{\alpha}_i} + (1-q)g_{\underline{\alpha}} & \text{if } \{w_i, w_{i+1}\} \cap \{\alpha_1, \ldots, \alpha_m\} = \{w_i\} \\ q g_{\underline{\alpha}_i} & \text{if } \{w_i, w_{i+1}\} \cap \{\alpha_1, \ldots, \alpha_m\} = \{w_{i+1}\} \\ g_{\underline{\alpha}} & \text{otherwise} \end{cases}$$

since each term in (6.3.6) transforms in this way.

**Definition:** $V^m \equiv \left\langle \left\{ j' \left( f_{\underline{\alpha}}^m \right) \mid \underline{\alpha} \in \mathcal{U}_{\mathbf{w}}^m \right\} \right\rangle \subseteq H^m$ .

The action of $\sigma_i$ on $V^m$, as defined in §3.1, is the quotient of an action which factors through the Hecke algebra $H_n(q)$, as is given in Theorem 6.6. Lemma 6.10 shows that this quotient is by another Hecke algebra representation, in which $m$ is replaced by $m-1$. The dimension of $V^m$ is:

$$\left| \mathcal{U}_{\mathbf{w}}^m \right| - \left| \mathcal{U}_{\mathbf{w}}^{m-1} \right| = \binom{n}{m} - \binom{n}{m-1}$$

since there are $\left| \mathcal{U}_{\mathbf{w}}^{m-1} \right|$ relations satisfied by the spanning set $\left\{ j' \left( f_{\underline{\alpha}}^m \right) \mid \underline{\alpha} \in \mathcal{U}_{\mathbf{w}}^m \right\}$ of $V_m$ (see Lemma 6.10). The hook length formula [Jo 4] gives the dimension of the Hecke algebra representation $\pi_{\Lambda_m}$ where $\Lambda_m$ is the two-row Young diagram with rows of length $n - m$ and $m$, as:

$$\frac{n!}{(n - m + 1) \cdots (n - 2m + 2)(n - 2m) \cdots 1.m!} = \frac{1}{m!} n(n-1) \cdots (n - m + 2)(n - 2m + 1)$$

$$= \binom{n}{m} - \binom{n}{m-1}$$

$$= \dim V^m .$$

| $n - m + 1$ | $\cdots$ | $n - 2m + 2$ | $n - 2m$ | $\cdots$ | $1$ |
|---|---|---|---|---|---|
| $m$ | $\cdots$ | $1$ | | | |

Figure 6.2

The Hecke algebra representaion given in Theorem 6.6 corresponds to that obtained by symmetrising along the rows in $\Lambda_m$, but not anti-symmetrising down the columns. As in the case of the symmetric group $S_n$, such a representation of the Hecke algebra has character:

$$\chi_{\Lambda_m} + \cdots + \chi_{\Lambda_1} + \chi_{\Lambda_0}$$

where $\chi_{\Lambda_i}$ is the character of the irreducible representation of $H_n(q)$ corresponding to the Young diagram $\Lambda_i$. However $V^m$ is the quotient of $\left\langle \left( f_{\underline{\alpha}}^m \right) \mid \underline{\alpha} \in \mathcal{U}_{\mathbf{w}}^m \right\rangle$ by $\left\langle g_{\underline{\alpha}} \mid \underline{\alpha} \in \mathcal{U}_{\mathbf{w}}^{m-1} \right\rangle$, and the action of $B_n$ on $\left\langle g_{\underline{\alpha}} \right\rangle$ also

factors through $H_n(q)$, with character:

$$(\chi_{\Lambda_m} + \cdots + \chi_{\Lambda_0}) - (\chi_{\Lambda_{m-1}} + \cdots + \chi_{\Lambda_0}) = \chi_{\Lambda_m}$$

Going back to Wenzl's definition of the representation $\pi_{\Lambda_m}$, it is easy to see that the action on $V^m$ of $B_n$ is precisely that of $\pi_{\Lambda_m}$, and the basis that $\mathcal{U}_{\mathbf{w}}^m$ supplies at the level of chains, is the natural basis for this action. Hence Theorem 6.1 is proved. ∎

In this Chapter we have completed the proof of Theorem 6.1 and thus also of Theorem 3.3. However, the embedding of $V^m$ in the totally symmetric part of the cohomology:

$$H^m(Y_{\mathbf{w},m}, \chi_{\mathbf{w},m}(\mathbf{q}))$$

remains to be understood in more detail. Some remarks on this question are made in the next Chapter.

# 7: The selection of a quotient of homology

In this Chapter, we will discuss the construction of the quotient of homology defined in §6.1, namely $\ker(\mathbf{D})/_W$. As was observed in §6.1, this is equivalent, by duality, to the construction of a subspace of cohomology, namely $W'/\mathrm{Im}\,(\mathbf{D}')$. It is the braid group action induced on the part of this subspace invariant under the action of $S_m$, which factors through a Hecke algebra, giving rise to the representation $\pi_{\Lambda_m}$.

The general situation considered in §7.1 starts from a smooth family of linear transformations, each of which fixes a subspace, and satisfying certain suitable conditions near to one member of the family. There is thus also a family of linear transformations defined by the quotient actions. The main Theorem of §7.1 (Theorem 7.2) shows how a *derived* action can be obtained naturally from this data, which is a subaction of the chosen action at the 'special' member of the family.

Going back to the braid group representation obtained on homology, the representation used in Theorem 6.1 can be considered as a special member of a family of such representations, obtained using more general local coefficient systems. The derived representation of this family, at the special local coefficient system used in Theorem 6.1, is shown in §7.2 (Theorem 7.3) to give rise to $\pi_{\Lambda_m}$, in the special case $m = 2$. In §7.3, it is shown that $\pi_{\Lambda_m}$ is contained in the derived representation obtained from a suitable family (Theorem 7.5), and it is conjectured that the symmetric part of this derived representation gives precisely $\pi_{\Lambda_m}$. This would be resolved by a dimension count — however this is not easy to do here. In the next Chapter we will go on to discuss the theory of Tsuchiya & Kanie, and see how this is related to the construction embodied in Theorem 6.1.

## 7.1 Limiting lemma

The representation of the braid group on homology as defined in §3.1, is a function of $m$ and the local coefficient system $\mathbf{q}$. It is thus possible to obtain a braid group representation from any $\mathbf{q}$ written in the form:

$$q_{z_i z_j} = \alpha_{ij}$$

$$q_{z_i w_k} = q_i$$

where $q_i$, $\alpha_{ij}$ are non-zero complex numbers. Here $i < j$ and $i, j \in \{1, 2, \ldots, m\}$, $k \in \{1, 2, \ldots, n\}$. Thus there are $1/2 m(m+1)$ parameters available in the representation.

Of course, the action of the symmetric group $S_m$ on the space of homology only exists if $\alpha_{ij} = \alpha$ and $q_i = q$ are independent of $i$ and $j$. This reduces the number of parameters to two, and in Theorems 3.3, 6.1 it is seen that the special case given by $\alpha = q^{-2}$ is the one used. The Lemma which we discuss in this section can be applied before the symmetrisation procedure, to produce a derived representation of the braid group at:

$$\alpha_{ij} = q^{-2}, \qquad q_i = q \, .$$

This derived representation can now be symmetrised, and it is claimed that the result obtained is the action on $\left( H_m / W \right)^{S_m}$ referred to in §6.3. We will discuss this last part in more detail in §7.3, and in the special case $m = 2$, in §7.2. However, in this section we concentrate on the Theorem enabling the derived representation to be obtained.

Let us start with the simplest situation in which the lemma we prove, Theorem 7.2, is applicable. Suppose $\{\mathbf{A}(h)\}$ is a one parameter family of linear transformations on a vector space $V$, such that for each $h$, there is a subspace $V_h \subseteq V$ fixed by $\mathbf{A}(h)$. Then $\mathbf{A}(h)$ defines an action on the quotient space $V/V_h$. We suppose also that $\mathbf{A}(h)$ and $V_h$ depend smoothly on $h$, using a suitable definition of the smooth variation of a subspace. The first order variation of $V_h$ near $h = 0$ can be represented by a map:

$$V_0 \longrightarrow V/V_0$$

and it will be assumed that this map has maximal rank. (This is a *non-degeneracy* condition on $\{V_h\}$ near $h = 0$.) Assume also that there exists a subspace $W$ of $V$ such that, for all sufficiently small $\delta$:

(i) $W$, $V_h$ are transverse in $V$ for $0 < |h| < \delta$, and $W$ is a maximal space satisfying this condition;

(ii) $V_0 \subseteq W$.

The first condition states that $W$ and $V_h$ span $V$, with trivial intersection, for all sufficiently small non-zero values of $h$; it is always possible to find a suitable subspace $W \subseteq V$ satisfying this condition. However, to be able to find $W$ satisfying condition (ii) as well, it is necessary to use the non-degeneracy condition mentioned above.

Assuming that all the above conditions are satisfied, $W$ can be identified with $V/V_h$, for all $h \neq 0$ sufficiently small. The action of $\mathbf{A}(h)$ on $V$ induces an action on $W$, say $\mathbf{B}(h)$, for all sufficiently small $h \neq 0$. If $\mathbf{B}(h)$ depends continuously on $h$, and has a limiting value:

$$\mathbf{B}_0 = \lim_{h \to 0} \mathbf{B}(h)$$

then $\mathbf{B}_0$ is another action on $W$. However, $\mathbf{B}_0$ is simply the limiting action of $\{\mathbf{A}(h)\}$ on a quotient $V/V_h \cong W$ of $V$. The limit of the family $\{\mathbf{A}(h)\}$, namely $\mathbf{A}(0)$, preserves $V_0 \subseteq W$, and thus it is natural to expect that $\mathbf{B}_0$ preserves $V_0$ and hence acts on $W/V_0$. The following Lemma sets out the exact relation between $\mathbf{B}_0$ and $\mathbf{A}(0)$, together with the assumptions required.

**Lemma 7.1**    *Assume that $\mathbf{A}(h)$, $\mathbf{B}(h)$, $V_h$, $W$ are defined as described above where $W$ satisfies conditions (i) and (ii) above; and $\{V_h\}$ is non-degenerate near $h = 0$. Further assume that:*

$$\mathbf{B}_0 = \lim_{h \to 0} \mathbf{B}(h)$$

*exists. Then:*

*(a) $\mathbf{B}_0$ preserves $V_0$;*

*(b) $\mathbf{A}(0)$ preserves $W$;*

*(c) the quotient action of $\mathbf{B}_0$ on $W/V_0$, and the subaction of $\mathbf{A}(0)$ on $W/V_0 \subseteq V/V_0$ are identical.*

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PROOF: Choose a basis for $V$, $\{\mathbf{e}_1, \ldots, \mathbf{e}_N\}$ such that $V_0 = \langle \mathbf{e}_1, \ldots, \mathbf{e}_n \rangle$ and $V_h = \langle \mathbf{e}_i + h\mathbf{f}_i \mid i = 1, 2, \ldots, n \rangle$ up to first order in $h$. Then, by condition (ii), we can assume without loss of generality that:

$$W = \langle \mathbf{e}_1, \ldots, \mathbf{e}_m \rangle$$

some $n$, $m$ with $n < m < N$. Then $\langle \mathbf{f}_1, \ldots, \mathbf{f}_n \rangle \cap W = \{\mathbf{0}\}$ by condition (i), and the non-degeneracy condition.

We can now write $\mathbf{A}(h)$ as a matrix $\bigl(A_{\lambda\mu}(h)\bigr)$, when it is partitioned into a $3 \times 3$ matrix, using the following subdivision of the basis:

$$\{\mathbf{e}_1, \ldots, \mathbf{e}_n\}, \{\mathbf{e}_{n+1}, \ldots, \mathbf{e}_m\}, \{\mathbf{e}_{m+1}, \ldots, \mathbf{e}_N\}.$$

Since $\mathbf{A}(h)$ preserves $V_h$, for some matrix $\mathbf{C}(h)$:

$$\bigl(\mathbf{A}(0) + h\mathbf{A}'(0)\bigr)(\mathbf{e}_j + h\mathbf{f}_j) = C_{ij}(h)(\mathbf{e}_i + h\mathbf{f}_i) + O(h^2)\,.$$

Hence we have, using the Einstein summation convention:

$$\left.\begin{aligned}
\mathbf{A}(0)\langle \mathbf{e}_i \rangle &\subseteq \langle \mathbf{e}_i \rangle \\
\mathbf{A}(0)\mathbf{f}_j + \mathbf{A}'(0)\mathbf{e}_j &= C_{ij}(0)\mathbf{f}_i + C'_{ij}(0)\mathbf{e}_i
\end{aligned}\right\}$$

The correspondence $V/V_h \cong W$ maps $[\mathbf{v}]$, for $\mathbf{v} \in V$, to $(\mathbf{v} + V_h) \cap W$. As remarked above, since $\langle W, V_h \rangle = V$, with $W \cap V_h = 0$ for all $h \neq 0$, sufficiently small,

$$\langle \mathbf{f}_1, \ldots, \mathbf{f}_n \rangle \cap W = \{\mathbf{0}\}$$

and $W$ is maximal satisfying this condition. It also follows that $\mathbf{f}_i$ are linearly independent (otherwise there would be a non-trivial element of $V_h$ in $V_0 \subseteq W$). Hence without loss of generality, we may use the basis $\{\mathbf{e}_1, \ldots, \mathbf{e}_n, \mathbf{e}_{n+1}, \ldots, \mathbf{e}_m, \mathbf{f}_1, \ldots, \mathbf{f}_n\}$ for $V$. With respect to this basis, the matrix for $\mathbf{B}(h)$ on $W = \langle \mathbf{e}_1, \ldots, \mathbf{e}_m \rangle$ is given by:

$$\begin{pmatrix} A_{11}(h) - \tfrac{1}{h}A_{31}(h) & A_{12}(h) - \tfrac{1}{h}A_{32}(h) \\[2em] A_{21}(h) & A_{22}(h) \end{pmatrix}.$$

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This depends on $h$ in a smooth way as $h \to 0$ so long as $A_{31}(0) = A_{32}(0)$. Thus $\mathbf{A}(0)$ preserves $W$; this proves (b). The matrix for $\mathbf{B}_0$ can now be written as:

$$\begin{pmatrix} A_{11}(0) - A'_{31}(0) & A_{12}(0) - A'_{32}(0) \\ \\ A_{21}(0) & A_{22}(0) \end{pmatrix}.$$

However $\mathbf{A}(0)$ preserves $V_0$, and so $A_{21}(0) = 0$. Thus $\mathbf{B}_0(V_0) \subseteq V_0$, proving (a). The induced action of $\mathbf{B}_0$ on $W/V_0$ is thus given by $A_{22}(0)$ using the correspondence,

$$W/V_0 \cong \langle \mathbf{e}_{n+1}, \dots, \mathbf{e}_m \rangle.$$

We can now write $\mathbf{A}(0)|_W$ as:

$$\begin{pmatrix} A_{11}(0) & A_{12}(0) \\ \\ 0 & A_{22}(0) \end{pmatrix}.$$

Its action on $W/V_0$ is thus also given by $A_{22}(0)$, and thus (c) is proved. The proof of the Lemma is thus complete. $\blacksquare$

We will refer to the action of $\mathbf{B}_0$ on $W/V_0$ as the *derived action* of the family $\{\mathbf{A}(h)\}$ at $h = 0$. By part (c) of the above Lemma, it is identical to the sub-action of $\mathbf{A}(0)$ on $W/V_0 \subseteq V/V_0$.

*Example* Suppose $V = \mathbf{C}^3$, and $\mathbf{A}(h)$ is given by:

$$\begin{pmatrix} 1 & 0 & 0 \\ 1/2 & 1/2 & 0 \\ h^2 + 2h & -h & -h \end{pmatrix}.$$

Then the one dimensional subspace $V_h \subseteq V$ spanned by,

$$\begin{pmatrix} 1 \\ 1 \\ h \end{pmatrix}$$

is preserved by $\mathbf{A}(h)$. A suitable space $W$, satisfying the conditions of Lemma 7.1 is given by:

$$\left\langle \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \right\rangle.$$

The action of $\mathbf{A}(h)$ on $V/V_h \cong W$, for $h \neq 0$, is then given by the matrix:

$$\mathbf{B}(h) = \begin{pmatrix} -h - 1 & 1 \\ -h - 3/2 & 3/2 \end{pmatrix}.$$

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This is a smooth function of $h$, and it is seen that:

$$\mathbf{B}_0 = \begin{pmatrix} -1 & 1 \\ -3/2 & 3/2 \end{pmatrix}$$

preserves $\left\langle \begin{pmatrix} 1 \\ 1 \end{pmatrix} \right\rangle$, inducing an action on $W/V_0$ given by the scalar $1/2$.

On the other hand, the action of:

$$\mathbf{A}(0) = \begin{pmatrix} 1 & 0 & 0 \\ 1/2 & 1/2 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

on $V/V_0$ restricts to an action on $W/V_0$, given by:

$$\begin{pmatrix} 1 & 0 \\ 1/2 & 1/2 \end{pmatrix}$$

with $\left\langle \begin{pmatrix} 1 \\ 1 \end{pmatrix} \right\rangle$ removed, and again this is given by the scalar $1/2$.

$V_h$

$h \to 0$

$V_0$ $\qquad V$

$W$
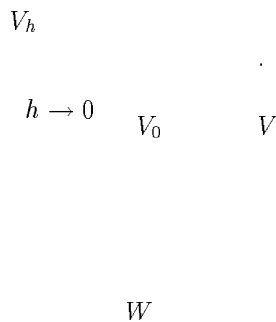
Figure 7.1

As we have observed during the proof of Lemma 7.1, the existence of the limit $\mathbf{B}_0$ is equivalent to (b) in that Lemma. Lemma 7.1 can be generalised to obtain derived actions from multi-dimensional families of linear transformations. In order to do so, it is necessary to set up some notation.

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**Definition:** *A map $f\colon M \longrightarrow \mathrm{Grass}(n, V)$ will be said to be non-degenerate at a point $x \in M$, if the derivative:*

$$df_x\colon T_x M \longrightarrow T_{f(x)}\mathrm{Grass}(n, V)$$

*when considered as a map:*

$$Df_x\colon T_x M \otimes f(x) \longrightarrow V/f(x)$$

*has maximal rank.*

In this definition, $M$ is a smooth manifold and $\mathrm{Grass}(n, V)$ denotes the set of all subspaces of $V$, of dimension $n$. Let $x_1, \ldots, x_k$ be local coordinates for $M$ near $x$, with $x_i = 0$ at $x \in M$. Then, locally, $f$ can be specified by $n$ maps:

$$\mathbf{v}_i\colon M \longrightarrow V$$

$$(x_1, \ldots, x_k) \longmapsto \mathbf{v}_i(x_1, \ldots, x_k)$$

This defines $n$ linearly independent sections of the trivial bundle over $M$ with fibre $V$ over each point. Then at $(0, \ldots, 0)$, the derivative $Df_x$ is a map:

$$\mathbf{R}^k \otimes \mathbf{R}^n \longrightarrow V$$

$$(\mathbf{h}, \mathbf{v}) \longmapsto (df_x.\mathbf{h})\mathbf{v}$$

given by:

$$(Df_x)(\mathbf{e}_j, \mathbf{e}_i) \equiv (df_x.\mathbf{e}_j)\mathbf{e}_i = \partial \mathbf{v}_i/\partial x_j$$

for $i \in \{1, 2, \ldots, n\}, j \in \{1, 2, \ldots, k\}$. The condition of non-degeneracy states that the $nk$ vectors $\partial \mathbf{v}_i/\partial x_j$ are linearly independent. It is much stronger than the statement that $df_x$, as a map $T_x M \longrightarrow T_{f(x)}\mathrm{Grass}(n, V)$ is non-degenerate. For, $df_x$ is given by:

$$(df_x)(\mathbf{e}_j) = \left( \partial \mathbf{v}_1/\partial x_j, \ldots, \partial \mathbf{v}_n/\partial x_j \right)$$

and non-degeneracy of this map means only these $k$, $n \times n$ matrices are linearly dependent.

The generalised version of Lemma 7.1 can now be stated.

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**Theorem 7.2**     *Suppose that* $\mathbf{A} \colon M \to \mathrm{End}\,(V)$ *is smooth family of linear transformations on* $V$, *and that:*

$$f \colon M \longrightarrow \mathrm{Grass}(n,\, V)$$

*defines a corresponding smooth family of subspaces of* $V$ *which are preserved by* $\mathbf{A}$. *That is,* $\mathbf{A}(x)f(x) \subseteq f(x)$. *Assume that* $f$ *is non-degenerate at* $x_0 \in M$, *and* $W \subseteq V$ *is a subspace such that:*

(i) $f(x_0) \subseteq W$, $\mathbf{A}(x_0)$ *preserves* $W$;

(ii) $W/f(x_0)$ *is transverse to* $\mathrm{Im}\,(Df_{x_0})$;

(iii) $W$ *is maximal satisfying (ii).*

*Then the family of actions* $\mathbf{A}(y)$ *(for* $y \in M \backslash \{x_0\}$) *on the quotients* $V/f(y)$ *defines a derived action* $\mathbf{B}_0$ *on* $W/f(x_0)$. *Furthermore, this action can be obtained in either of the following ways:*

(a) *the restriction of the quotient action of* $\mathbf{A}(x_0)$ *on* $V/f(x_0)$ *to* $W/f(x_0)$;

(b) *the quotient of a limiting action on* $W$.

Proof:     The main idea of the proof is to break the family up so as to consider only a one-dimensional variation at any one time; Lemma 7.1 may then be applied repeatedly. Suppose $x_1, \ldots, x_k$ are local coordinates on $M$ near $x_0$. Define a sequence of spaces $W_i \subseteq V$ for $i = 0, 1, \ldots, k$ as follows. For each $i$, $W_i$ depends on the parameters $x_{i+1}, \ldots, x_k$, and satisfies the conditions:

($\alpha$) $W_{i-1}(x_i, \ldots, x_k)/f(0, \ldots, 0, x_i, \ldots, x_k) \cong W_i(x_{i+1}, \ldots, x_k)$ for $x_j \neq 0$ sufficiently small;

($\beta$) $f(0, \ldots, 0, 0, x_{i+1}, \ldots, x_k) \subseteq W_i(x_{i+1}, \ldots, x_k)$.

We start with the initial space $W_0 = V$, and then $W_k = W$ without loss of generality.

At each stage, we suppose that $\{\mathbf{A}(x)\}$ has induced an action:

$$\mathbf{B}_{(i-1)}(x_i, \ldots, x_k) \in \mathrm{End}\,(W_{i-1}(x_i, \ldots, x_k))$$

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preserving the subspace $f(0, \ldots, 0, x_i, \ldots, x_k)$, where for $i = 1$, $\mathbf{B}_{(0)} = \mathbf{A}$. By Lemma 7.1, this induces an action $\mathbf{B}_{(i)}(x_{i+1}, \ldots, x_k)$ on $W_i(x_{i+1}, \ldots, x_k)$ where $x_{i+1}, \ldots, x_k$ are considered fixed and $x_i$ replaces the parameter $h$. This action corresponds to that of $\mathbf{B}_0$ on $W$ in Lemma 7.1. By Lemma 7.1(a), the limiting action $\mathbf{B}_{(i)}(x_{i+1}, \ldots, x_k)$ preserves the subspace:

$$f(0, \ldots, 0, x_i, \ldots, x_k) \subseteq W_i(x_{i+1}, \ldots, x_k) \,.$$

To sum up, we have here applied Lemma 7.1, with:

$$
\begin{array}{rcl}
\mathbf{A}(h) & \longrightarrow & \mathbf{B}_{(i-1)}(x_i, \ldots, x_k) \\
W & \longrightarrow & W_i(x_{i+1}, \ldots, x_k) \\
h & \longrightarrow & x_i \\
\mathbf{B}_0 & \longrightarrow & \mathbf{B}_{(i)}(x_{i+1}, \ldots, x_k) \\
V & \longrightarrow & W_{i-1}(x_i, \ldots, x_k) \\
V_h & \longrightarrow & f(0, \ldots, 0, x_i, \ldots, x_k)
\end{array}
$$

The spaces $W_i$ can be chosen to be independent of $x_{i+1}, \ldots, x_k$ for sufficiently small $x_j \neq 0$, and thus, at the $k^{\text{th}}$ stage, we observe that $\mathbf{B}_{(k)} \in \operatorname{End}(W_k)$ preserves $f(0, \ldots, 0) \subseteq W_k$. Thus in the notation of the Theorem, $W = W_k$ and $\mathbf{B}_0$ is the quotient action of $\mathbf{B}_{(k)}$ on $W/f(x_0)$. By applying part (c) of Lemma 7.1 at each stage, it may be seen that $\mathbf{B}_0$ can also be obtained as the reduced action of $\mathbf{A}(x_0)$ on the subspace $W/f(x_0) \subseteq V/f(x_0)$. This completes the proof of the Theorem. ∎

This Theorem should be thought of as involving the transposition of the operations of taking out fixed spaces, and taking a limit. Thus $\mathbf{B}_{(k)}$ should be considered as being obtained by *first* taking out the fixed space $f(x)$ from the action of $\mathbf{A}(x)$ on $V$, and *then* taking the limit $x \to x_0$. If instead we take the limit $x \to x_0$ first, we obtain an action of $\mathbf{A}(x_0)$ on $V$ fixing $f(x_0)$, and removing this fixed space gives an action on $V/f(x_0)$. The Theorem states that one obtains identical actions on the reduced space $W/f(x_0)$, namely the derived action, if one takes either

(i) a quotient of the action obtained by first taking out fixed spaces and then taking the limit; or

(ii) a restriction of the action obtained by taking the limit first.

The space $W/f(x_0)$ is isomorphic to $V/\langle\text{Im}\,(Df_{x_0}),\,f(x_0)\rangle$, and so should be thought of as being constructed from $V$ by removing not only the fixed space $f(x_0)$ but also the 'limiting space' given by $\text{Im}\,(Df_{x_0})$.

## 7.2 Special case of $m = 2$

In this section we will discuss the application of Lemma 7.1 to the action of $B_n$ on cohomology with $m = 2$. At the level of $\mathcal{C}^2$, $B_n$ acts on :

$$\left\langle (\underline{\alpha}) \mid \underline{\alpha} \in \mathcal{S}_{\mathbf{w}}^2 \right\rangle .$$

The cohomology is given by $\mathcal{C}^2/\langle\text{Im}(\mathbf{D}_1{}^{\mathrm{T}}),\,\text{Im}(\mathbf{D}_2{}^{\mathrm{T}})\rangle$. However as was noted in §5.2, $\mathcal{C}^2/\langle\text{Im}(\mathbf{D}_1{}^{\mathrm{T}})\rangle$ can be identified with:

$$\left\langle (\underline{\alpha}) \mid \alpha_1,\,\alpha_2 \in \{w_1,\ldots,w_n\} \right\rangle .$$

(In §5.2, we dealt with the dual situation of a basis for $\ker\mathbf{D}_1 \subseteq \mathcal{C}_2$ .) This gives a representation of $B_n$ on $\mathcal{C}^2/\langle\text{Im}(\mathbf{D}_1{}^{\mathrm{T}})\rangle$ which preserves $\text{Im}(\mathbf{D}_2{}^{\mathrm{T}})$ and depends on the parameters $q_1$, $q_2$ and $\alpha$.

Now one can apply Lemma 7.1 with $q_1$, $q_2$ fixed, and:

$$h = 1 - q_1^{-1}q_2^{-1}\alpha^{-1}$$

$$V = \left\langle (\underline{\alpha}) \mid \alpha_1,\,\alpha_2 \in \{w_1,\ldots,w_n\} \right\rangle$$

$$V_h = \text{Im}(\mathbf{D}_2{}^{\mathrm{T}}) = \langle \mathbf{e}_1,\ldots,\mathbf{e}_n\rangle$$

where $\mathbf{e}_i = (q_2^{-1}-1)\sum_{\lambda\neq i}(w_i w_\lambda) + q_2^{-1}(q_1^{-1}-1)\sum_{k>i}(w_k w_i) + q_2^{-1}\alpha^{-1}(q_1^{-1}-1)\sum_{j<i}(w_j w_i) + (q_1^{-1}q_2^{-1}\alpha^{-1}-1)(w_i w_i)$.

Then $W = \left\langle (\underline{\alpha}) \mid \alpha_1,\,\alpha_2 \in \{w_1,\ldots,w_n\};\alpha_1 \neq \alpha_2 \right\rangle$ is clearly transverse to $V_h$ whenever $q_1^{-1}q_2^{-1}\alpha^{-1}-1 \neq 0$, that is for $h \neq 0$. The action of $\mathbf{A}(h)$ is given by $A^{(2)}_{w_i w_{i+1}}{}^{\mathrm{T}\,-1}$, where $A^{(2)}_{w_i w_{i+1}}$ has the matrix form given in Fig. 5.2, §5.2. The matrix $\mathbf{B}(h)$ of Lemma 7.1 is then the action of $\sigma_i$ on the cohomology at $\alpha^{-1} = q_1 q_2(1 - h)$. It is given by the inverse of the transpose of the matrix of Fig. 5.3, §5.2 for $h \to 0$. Clearly, $\mathbf{B}(h)$ depends smoothly on $h$ near 0, and has a limit $\mathbf{B}_0$ as $h \to 0$. By Lemma 7.1, $\mathbf{B}_0$ preserves $V_0$, namely:

$$V_0 = \left\langle (q_2^{-1}-1)\sum_{\lambda\neq i}(w_i w_\lambda) + q_2^{-1}(q_1^{-1}-1)\sum_{k>i}(w_k w_i) + (1-q_1)\sum_{j<i}(w_j w_i) \,\middle|\, i = 1,2,\ldots,n \right\rangle \subseteq W$$

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and the action of $\mathbf{B}_0$ on $W/V_0$ is identical to the restriction of $\mathbf{A}(0)$ (i.e. the action of $\sigma_i$ on the cohomology for $\alpha = q_1^{-1}q_2^{-1}$) to $W/V_0 \subseteq V/V_0$. This gives an action of $\sigma_i$ on a space of dimension $n(n-1)$, since:

$$\dim W = n^2, \ \dim V_0 = n.$$

The action of $B_n$ now has two parameters, namely $q_1$ and $q_2$.

There is an action of $S_2$ on $\mathcal{C}^2$ or $H^2$, only when $q_1 = q_2$. In this case,

$$V_0 = \left\langle \sum_{\lambda \neq i}(w_i w_\lambda) + q^{-1}\sum_{k>i}(w_k w_i) + q\sum_{j<i}(w_j w_i) \ \middle| \ i = 1, 2, \ldots, n \right\rangle$$

lies within the symmetric part of $\mathcal{C}^2$, since the symmetric part of $\mathcal{C}^2$ is spanned by:

$$(w_i w_j) + q(w_j w_i) = \mathbf{f}_{ij} \qquad \text{for } i > j$$

$$2(w_i w_i) + (1-q)\sum_{j<i}(w_j w_i) + q^{-1}(q^{-1}-1)\sum_{k>i}(w_k w_i) = \mathbf{f}_i \qquad \text{for all } i.$$

Thus $V_0 = \left\langle \sum_{j<i}\mathbf{f}_{ij} + q^{-1}\sum_{k>i}\mathbf{f}_{ki} \ \middle| \ i = 1, 2, \ldots, n \right\rangle$.

Under the action of $qj_{12}$, $\mathcal{C}^2$ splits into two halves of dimension $\frac{1}{2}n(n+1)$; and similarly $H^2 \cong V/V_h \cong W$ splits into two equally sized spaces of dimension $\frac{1}{2}n(n-1)$. However, $V_0$ is contained in the half with $qj_{12} = 1$, and thus the dimension of the symmetric part of $W/V_0$ is:

$$\tfrac{1}{2}n(n-1) - n = \tfrac{1}{2}n(n-3).$$

It is spanned by $\{\mathbf{f}_{ij} \ | \ i > j\}$ considered as elements of $W/V_0$, with the $n$ relations:

$$q\sum_{j<i}\mathbf{f}_{ij} + \sum_{k>i}\mathbf{f}_{ki} = \mathbf{0}.$$

This space is precisely the subspace of cohomology specified in Theorem 6.6 for the case of $m = 2$; it is a subspace when considered after taking the limit $h \to 0$ (i.e. $\alpha \to q^{-2}$) *first*. When this limit is taken *after* the fixed space $\langle \mathrm{Im}(\mathbf{D}_2{}^{\mathrm{T}}) \rangle$ is removed, it is a quotient space.

Thus we have obtained the following Theorem:

**Theorem 7.3**    *Consider the family of representations of $B_n$ on the quotient of the chain space $\mathcal{C}^2$*

*obtained by dividing out by the boundaries $\mathrm{Im}(\mathbf{D}_1{}^\mathrm{T})$, and using the local coefficient system specified by $q$,*

*$\alpha$. Then the derived representation of this family at $\alpha = q^{-2}$, is a representation of $B_n$ on a subspace of the*

*cohomology $H^2$. Moreover, the part of the derived representation invariant under the action of $S_2$ factors*

*through $H_n(q)$, and is the irreducible representation $\pi_{\Lambda_2}$.*


## 7.3 Some general remarks

Theorem 7.2 (or at least its special case, Lemma 7.1) was successfully applied in the last section to

obtain the subspace of cohomology in Theorem 6.1 on which the braid group action factors through $H_n(q)$,

at least in the case $m = 2$.

For the case of general $m$ this procedure gets more complicated. Firstly, the initial representation of $B_n$

has $^1\!/_2 m(m+1)$ parameters; namely $\alpha_{ij}$, $q_i$. Theorem 7.2 can now be applied to the situation where:

$$V = \mathcal{C}^m\big/\!\big\langle \mathrm{Im}(\mathbf{D}_1{}^\mathrm{T}), \ldots, \mathrm{Im}(\mathbf{D}_{m-1}{}^\mathrm{T}) \big\rangle$$

$$\cong \big\langle (\underline{\alpha}) \ \big| \ \underline{\alpha} \in \mathcal{S}_{\mathbf{w}}^m, \ \alpha_i \neq z_{i+1} \quad \text{for } i = 1, 2, \ldots, m-1 \big\rangle$$

$$f(x) = \mathrm{Im}(\mathbf{D}_m{}^\mathrm{T}) \,.$$

In the above $x \in M$ refers to $(\alpha_{im}) \in \mathbf{C}^{m-1}$ with $q_i$ $(i = 1, 2, \ldots, m)$ and $\alpha_{ij}$ $(1 \leq i < j < m)$ fixed. Here,

the family $\mathbf{A}(x)$, which we use, is given by the induced action of $\sigma_i \in B_n$ on the quotient $V$ of $\mathcal{C}^m$ (that is,

$A_{w_i w_{i+1}}^{(m)}{}^{\mathrm{T}^{-1}}$, using the notation of §4.2). The point $x_0 \in M$ is given by $\alpha_{ij} = q_i^{-1} q_j^{-1}$.

From Theorem 7.2, we now obtain an action of $B_n$ on a space $W\!/f(x_0) \subseteq H^m$, depending on the $m$

parameters $q_1, \ldots, q_m$. There is an action of $S_m$ on $\mathcal{C}^m$ (and thus also on $H^m$) only when:

$$q_1 = \cdots = q_m = q \,.$$

In this case, the action:

$$S_m \longrightarrow \mathrm{End}\,(H^m)$$

$$(i\,i{+}1) \longmapsto q j_{i\,i+1}$$

defines a symmetriser:

$$R \colon H^m \longrightarrow H^m$$

$$\mathbf{v} \longmapsto \frac{1}{m!} \sum_{\sigma \in S_m} \sigma(\mathbf{v}) \,.$$

Thus $\mathrm{Im}(R)$ consists of that part of the cohomology $H^m$ invariant under the action of the symmetric group, $S_m$. The best way to construct the symmetric part of the reduced space $W/f(x_0)$, is as a subset of $\mathrm{Im}(R)$, transverse to:

$$\mathrm{Im}(R \circ Df_{x_0}) \ .$$

Here, we are using the following maps:

$$Df_{x_0} \colon T_{x_0}M \otimes f(x_0) \longrightarrow V/f(x_0) \cong H^m$$

$$R \circ Df_{x_0} \colon T_{x_0}M \otimes f(x_0) \longrightarrow H^m \ .$$

Of course in order to apply Theorem 7.2, it is necessary to check all the conditions of that Theorem. For generic $q$, $Df_{x_0}$ has maximal rank, as can be verified by using the recursion relations of Chapter 4, together with $q$ close to 1, but not equal to 1. To do this, it is necessary to identify $V$ with a suitable subspace of $\mathcal{C}^m$, and then to investigate the action of $\mathbf{D}_m{}^{\mathrm{T}}$ on this space.

Although $W$ needs to be introduced in Theorem 7.2, and this gives rise to an arbitrary element in the construction, which is not 'natural', of course the braid group action obtained, using different choices of $W$ satisfying the conditions of the Theorem, are isomorphic. Thus the derived representation finally obtained *is* natural. We now make the following conjecture:

**Conjecture 7.4**      *Consider the family of representations of $B_n$ on the cohomology with local coefficient system given by:*

$$q_{z_i z_j} = \alpha_{ij} \ , \qquad q_{w_k z_i} = q$$

*(for $1 \le i, j \le m$, $1 \le k \le n$), and parametrised by $\{\alpha_{ij}\}$. The symmetric part of the derived representation of this family at $\alpha_{ij} = q^{-2}$ factors through the Hecke algebra $H_n(q)$, giving the representation $\pi_{\Lambda_m}$.*

In the case $m = 2$, this conjecture has been proved in §7.2. In the general case, $V^m = \left\langle \{ j'(f_{\underline{\alpha}}^m) \mid \underline{\alpha} \in \mathcal{U}_{\mathbf{w}}^m \} \right\rangle$ is a subspace of the cohomology $H^m$ such that:

(a) $S_m$ preserves $V^m$; i.e. $V^m$ is contained in $\mathrm{Im}(R)$;

(b) $V^m$ is invariant under the action of $B_n$;

(c) for generic $q$, $\text{Im}\left(Df_{x_0}\right)$ is transverse to $V^m$.

The last result is obtained by considering $q$ close to 1, and using the basis for the space of chains, defined in §4.1. Since all the matrices involved depend on $q$ in an analytic way (indeed, they are polynomials in $q$ and $q^{-1}$), it is possible to infer results for generic $q$ from those which hold for all $q \neq 1$, sufficiently close to 1. A comparison of (a), (b) and (c) with the conditions of Theorem 7.2 shows that the derived action can be obtained on a space containing $V^m$; that is, $W$ can be chosen so that $W/f(x_0) \supseteq V^m$ while $W$ satisfies the conditions (i)–(iii) of the Theorem. By the remark above, the derived action is independent of $W$, and thus contains the action on $V^m$. Since $V^m$ is also preserved under the action of $S_m$, thus the action referred to in Conjecture 7.4 contains that of $B_n$ on $V^m$. By Theorem 6.1, we now obtain the following result.

**Theorem 7.5**    *The symmetric part of the derived action of Conjecture 7.4 contains $\pi_{\Lambda_m}$ for all $m$ and generic $q$.*

All that is necessary to obtain a proof of Conjecture 7.4 is to show that there is no other part to the totally symmetric piece of the derived representation. A dimension count would suffice for this; however, $\dim \text{Im}\left(R \circ Df_{x_0}\right)$ is not simple to compute!

# 8: Relations with conformal field theory

In Chapter 6, the main Theorem which was proved (Theorem 6.1) established the existence of an action of the Hecke algebra on a subspace of the cohomology of $Y_{\mathbf{w},m}$ with a suitably defined local coefficient system $\chi_{\mathbf{w},m}(\mathbf{q})$. In Chapter 7, a technique was described which, at least in certain special cases, gives this action naturally by considering the behaviour of a family of braid group representations in the vicinity of a 'special' representation.

In this Chapter, the theory of Tsuchiya & Kanie [TK] and Kohno [Ko] will be discussed. It will be seen that they also produce, initially, a representation on a large space, and then restrict it to a smaller space to obtain the (generically irreducible) Hecke algebra representation required. (However, in [TK], one of the approaches to this theory does give the resulting representation immediately, namely as the monodromy representation on a space of $n$-point functions. See §8.1 for more details.) Both approaches can be seen as being given by monodromy representations on vector bundles over $X_n$, with suitable flat connections. Equivalently, they can be viewed as monodromy representations of solutions to certain systems of differential equations (see Theorem 8.1 and Corollary 8.7). In §8.3, we will see that the differential equations giving rise to the representations of Chapters 3–7 are of the same form as those arising out of Tsuchiya & Kanie's work. Furthermore, it is deduced, using a Theorem of Kohno's, that these two systems of differential equations are isomorphic, since they give rise to identical braid group representations (see Theorem 8.10).

To establish the nature of the differential equations corresponding to the representations of Chapters 3–7, it is found to be more convenient to work with the action on cohomology rather than the dual action on homology. In §8.2, this duality is discussed, together with a concrete representation of the cohomology in terms of holomorphic functions.

Although, in §8.4, Theorem 8.10 proves the existence of a correspondence between the conformal field theory representation of [TK] and the homological construction of Chapters 3–7, it does not give the correspondence explicitly. However, in §8.5, the coefficients in the two systems of differential equations, corresponding to the two procedures, are evaluated in a special case.

In §8.6 some further remarks about the correspondence of Theorem 8.10, are made. In Chapter 9, some of the wider issues raised by this Theorem will be discussed.

## 8.1 Review of Tsuchiya-Kanie theory

In [TK], Tsuchiya & Kanie produced (generically irreducible) Hecke algebra representations from conformal field theory on the projective line $\mathbf{P}^1$. The representations obtained are, when $q$ is not a root of unity, those irreducible representations of $H_n(q)$ constructed by Wenzl [We], associated with Young diagrams with two rows. When $q$ is a root of unity, these representations constructed by Wenzl may be reducible, and Wenzl has shown how to pick out their largest irreducible pieces. These irreducible pieces are obtained by a natural construction in Tsuchiya & Kanie's theory, as we shall see later in this section. This section is devoted to reviewing the theory of [TK], picking out those points which will be of interest to us later in this Chapter.

*8.1.1 Vertex operators*

The Lie algebra $\mathfrak{sl}_2$ is generated by $H$, $E$, $F$ with the commutation relations:

$$\left.\begin{aligned} [H, E] &= 2E \\ [H, F] &= -2F \\ [E, F] &= H \end{aligned}\right\} \tag{8.1.1}$$

Let $V_j$ be the irreducible $\mathfrak{sl}_2$-module of highest weight $2j$. That is, the representation of $\mathfrak{sl}_2$ obtained on $V_j$ is the *spin j* representation, and $\dim V_j = 2j + 1$. The standard (vector) representation of $\mathfrak{sl}_2$, in which:

$$H = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \qquad E = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \qquad F = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \tag{8.1.2}$$

is then the (spin $^1\!/_2$) representation on $V_{1/2}$.

Let $\widehat{\mathfrak{g}}$ be the affine Lie algebra extending $\mathfrak{g} = \mathfrak{sl}_2$:

$$\widehat{\mathfrak{g}} = \mathfrak{g} \otimes \mathbf{C}[t, t^{-1}] \oplus \mathbf{C}c$$

where $c$ is the central element. For each $X \in \mathfrak{g}$, we can now define:

$$X(m) = X \otimes t^m$$

to be an element of $\widehat{\mathfrak{g}}$. With this is mind, the *energy-momentum tensor* can be defined by:

$$L(m) = \frac{1}{2(c+2)} \sum_{k \in \mathbf{Z}} \left\{ \; :E(-k)F(m+k): \; + \; :F(-k)E(m+k): \; +\frac{1}{2} \; :H(-k)H(m+k): \; \right\}$$

where the symbol $:\quad:$ denotes the *normal ordering*. That is,

$$:X(m)Y(n): := \begin{cases} X(m)Y(n) & \text{for } m < n \\ \frac{1}{2}\big(X(m)Y(n) + Y(n)X(m)\big) & \text{for } m = n \\ Y(n)X(m) & \text{for } m > n \end{cases}$$

More usually, the term 'energy-momentum tensor' is used to refer to the 'Fourier transform' of the above formula, that is, $\sum z^m L(m)$.

Denote by $\mathcal{H}_j$ the irreducible $\widehat{\mathfrak{g}}$-module of highest weight $2j$ with central charge $l$, where $l$ is a fixed real number and $2j \in \mathbf{N} \cup \{0\}$; $V_j$ can be embedded in $\mathcal{H}_j$. A *vertex operator* of spin $j$ is now defined to be an operator-valued function $\Phi(u; z) \colon \mathcal{H} \to \widehat{\mathcal{H}}$, such that:

$$\left. \begin{aligned} \big[X(m), \; \Phi(u; z)\big] &= z^m \Phi(Xu; z) \\ \big[L(m), \; \Phi(u; z)\big] &= z^m \left( z\frac{d}{dz} + (m+1)\Delta_j \right) \Phi(u; z) \end{aligned} \right\}$$

for all $m \in \mathbf{Z}$, $X \in \mathfrak{g}$ and $u \in V_j$. Here $\mathcal{H}$ and $\widehat{\mathcal{H}}$ denote the direct sums of $\{\mathcal{H}_j\}$ and the completions $\{\widehat{\mathcal{H}_j}\}$ for $2j \in \mathbf{N} \cup \{0\}$. The quantities $\Delta_j$ are given by:

$$\Delta_j = (j^2 + j)/(l + 2)$$

and are known as the conformal dimensions.

The space of vertex operators of spin $j$ forms an infinite dimensional vector space, and it is shown in [TK] that a spanning set exists (which is a basis when $l \notin \mathbf{Z}$) indexed by vertices:

$$\mathbf{v} = \begin{pmatrix} & j & \\ j_1 & & j_2 \end{pmatrix}$$

where $j_1, j_2$ are non-negative half-integers such that $|j_1 - j_2| \le j \le j_1 + j_2$. The vertex operator associated with $\mathbf{v}$ is non-trivial purely on $V_j \otimes \mathcal{H}_{j_1} \subseteq V_j \otimes \mathcal{H}$ and gives a map $V_j \otimes \mathcal{H}_{j_1} \mapsto \widehat{\mathcal{H}}_{j_2}$. Let (CG) denote the set of vertices $\mathbf{v}$ satisfying these conditions (this notation is used so as to conform with [TK]).

*8.1.2 Differential equations*

Suppose now that $j_1, \ldots, j_{n+1}$ are non-negative half-integers. Whenever $\Phi_i(z_i)$ are vertex operators of spin $j_i$, for $1 \leq i \leq n$, the $n$-point function:

$$\big\langle \, \mathrm{vac} \big| \Phi_1(z_1) \cdots \Phi_n(z_n) \big| u \big\rangle$$

can be defined, for fixed $z_1, \ldots, z_n$ with $|z_1| < \ldots < |z_n|$, giving a map:

$$V_{j_1} \otimes \cdots \otimes V_{j_n} \otimes V_{j_{n+1}} \longrightarrow \mathbf{C}$$

where $\big| \mathrm{vac} \big\rangle$ denotes a generator of the one-dimensional vector space $V_0$ in $\mathcal{H}$, and $\big| u \big\rangle \in V_{j_{n+1}}$. In [TK], it was shown that this function satisfies the following differential equations:

$$\left( \kappa \frac{\partial}{\partial z_i} - \sum_{\substack{k=1 \\ k \neq i}}^{n} \frac{\Omega_{ik}}{z_i - z_k} \right) \big\langle \Phi_1(z_1) \cdots \Phi_n(z_n) \big\rangle = 0 \tag{8.1.3}$$

where $\kappa = l + 2$. Here $\big\langle \Phi_1(z_1) \cdots \Phi_n(z_n) \big\rangle$ denotes the $n$-point function defined above, considered as a map:

$$X_n \longrightarrow \big( V_{j_1} \otimes \cdots \otimes V_{j_n} \otimes V_{j_{n+1}} \big)^{\check{}} = \check{V}$$

where $\check{}$ denotes the operation of dualising; and the function is defined on $X_n$ as a many-valued holomorphic function using analytic continuation. Also $\Omega_{ik}$ denotes the transformation on $\check{V}$ induced by the action of the polarisation of the Casimir operator $\Omega$ on the $i^{\mathrm{th}}$ and $k^{\mathrm{th}}$ factors. That is:

$$\Omega_{ik} = {}^1\!/\!_2 \pi_i(H) \pi_k(H) + \pi_i(E) \pi_k(F) + \pi_i(F) \pi_k(E)$$

where $\pi_i$ denotes the action of $\mathfrak{sl}_2$ on the $i^{\mathrm{th}}$ factor $\check{V}_{j_i}$ in $\check{V}$.

When $l$ is an integer, a basis for the space of vertex operators is indexed by a subset $(\mathrm{CG})_l \subseteq (\mathrm{CG})$ given by imposing the extra condition $j + j_1 + j_2 \leq l$. The $n$-point functions obtained from such vertex operators then satisfy, in addition to (8.1.3), the further relations:

$$\big\langle \Phi_1(u_1, z_1) \cdots \widehat{E}(-1)^{l-2j_i+1} \Phi_i\big(u_i(j_i), z_i\big) \cdots \Phi_n(u_n, z_n) \big\rangle = 0 \tag{8.1.4}$$

whenever $u_i \in V_{j_i}$, where $\widehat{E}(-1)$ denotes the operator with:

$$\widehat{E}(-1)A(z) = \frac{1}{2\pi} \oint_C \frac{d\zeta}{\zeta - z} E(\zeta) A(z) \, .$$

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Here $C$ is a closed curve enclosing $z$, with $0$ in its exterior.

Tsuchiya & Kanie show that the reduced $n$-point function, given by projecting $\check{V}$ onto its $\mathfrak{g}$-invariant part, $\check{V}_0$, satisfies (8.1.3), and, when $l$ is integral,

$$\sum_{|\mathbf{m}_i|=l_i} \binom{l_i}{\mathbf{m}_i} \prod_{k \neq i}(z_k - z_i)^{-m_k} \left\langle \Phi_1(E^{m_1}u_1, z_1) \cdots \Phi_i\big(u_j(j_i), z_i\big) \cdots \Phi_n(E^{m_n}u_n, z_n) \right\rangle = 0 \qquad (8.1.5)$$

(this is the analogue of (8.1.4)). Here the sum is over all $(n-1)$-tuples of non-negative integers $\mathbf{m}_i = (m_1, \ldots, \widehat{m}_i, \ldots, m_n)$ and $l_i = l - 2j_i + 1$, while $\binom{n}{\mathbf{m}}$ refers to the multinomial coefficient:

$$\frac{n!}{m_1! \cdots m_r!}$$

where it is assumed that $|\mathbf{m}| \equiv \sum_{i=1}^{r} m_i = n$. The combined system of equations (8.1.3) and (8.1.5) for functions $X_n \to \check{V}_0$ thus has solutions given by the reduced $n$-point functions for any set of $n$ vertices:

$$\mathbf{v}_i = \begin{pmatrix} & j_i & \\ p_{i-1} & & p_i \end{pmatrix} \in (\mathrm{CG})_l$$

where $p_0 = 0$, $p_n = j_{n+1}$.

*8.1.3 Braid group representations*

Since the solutions to (8.1.3), (8.1.5) are functions on $X_n$, there is a natural action $B_n$ (or, at least of $\pi_1(X_n) = P_n$) on the solution space. In [TK], it is shown that the set of all $n$-point functions spans the solution space. Firstly we fix:

$$j_1 = \cdots = j_n = {}^1\!/_2, \qquad j_{n+1} = t \ .$$

Then the $n$-point functions are indexed by $(n+1)$-tuples $(p_0, \ldots, p_n)$ with:

$$\left.\begin{array}{ll} 2p_i \in \mathbf{N} \cup \{0\} & \text{for } i = 0, 1, \ldots, n \\[2mm] |p_i - p_{i-1}| = {}^1\!/_2 & \text{for } i = 1, 2, \ldots, n \\[2mm] p_0 = 0, \qquad p_n = t \end{array}\right\}$$

For any such $n$-tuple, vertex operators are defined, from which first an $n$-point function, and then a reduced function $X_n \to \check{V}_0$ can be computed (see Fig. 8.1). It turns out that the number of possible $(n+1)$-tuples $(p_0, \ldots, p_n)$ is exactly $\dim \check{V}_0$; and that for different $p_i$'s, the corresponding reduced $n$-point functions are linearly independent. Thus there is an action of $B_n$ on the space of solutions of (8.1.3) and (8.1.5), or equivalently on $\check{V}_0$. The main result of the paper of Tsuchiya & Kanie can now be stated as follows.

$$u_1 \qquad \cdots \qquad u_i \qquad \cdots \qquad u_n$$

$$0 \quad {}^{1}\!/_{2} = j_1 \qquad\qquad {}^{1}\!/_{2} = j_i \qquad\qquad {}^{1}\!/_{2} = j_n \quad t = j_{n+1}$$
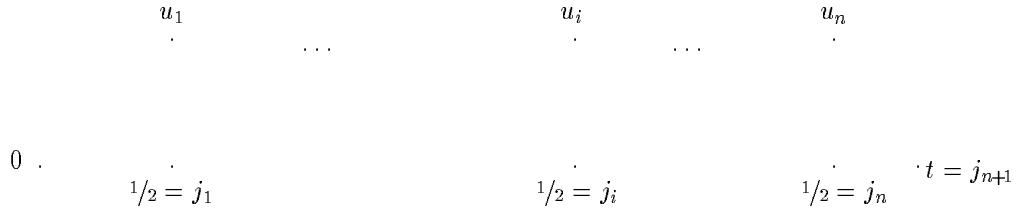
*Figure 8.1*

**Theorem 8.1**  *The monodromy representation of $B_n$ on the space of solutions to (8.1.3), for $l \notin \mathbf{Z}$, is $q^{-1/4}$ times the representation of the Hecke algebra $H_n(q)$ associated with the two-row Young diagram whose row lengths are ${}^{n}\!/_{2} + t$, ${}^{n}\!/_{2} - t$ respectively, where $q = e^{2\pi i/\kappa}$ and $\kappa = l + 2$.*

Note that here $\check{V}_0 = \left(V_{1/2}^{\otimes n} \otimes V_t\right)^{\check{}}$ and thus its dimension is given by the number of copies of $V_t$ in a direct sum decomposition of $V_{1/2}^{\otimes n}$. When $l \in \mathbf{Z}$, the $n$-point function obeys the further relations (8.1.5), and we obtain irreducible Hecke algebra representations once again; they are exactly those representations obtained by Wenzl.

**Theorem 8.1′**  *The monodromy representation of $B_n$ on the space of solutions to (8.1.3), (8.1.5) for integral $l$, is $q^{-1/4}$ times the representation of the Hecke algebra $H_n\!\left(e^{2\pi i/(l+2)}\right)$ associated with the two-row Young diagram $\Lambda_{n/2-t}$, and the integer $l$, in Wenzl's construction.*

The irreducible representation referred to in this Theorem is that constructed by Wenzl explicitly, on a space whose basis was indexed by Young tableaux. Equivalently, an element of the basis is specified by a sequence of Young diagrams $\Lambda_{(0)}, \Lambda_{(1)}, \ldots, \Lambda_{(n)}$ in which each $\Lambda_{(i)}$ is a Young diagram with $i$ squares and $\Lambda_{(i)} \subseteq \Lambda_{(i+1)}$. So $\Lambda_{(0)}$ is the empty Young diagram, and $\Lambda_{(n)}$ is the full two-row Young diagram $\Lambda_{n/2-t}$. On the vector space with a basis indexed on such sets, a representation of $H_n(q)$ is constructed by Wenzl. It is irreducible when $q = e^{2\pi i/(l+2)}$ and $l$ is not integral. When $l$ is an integer, an irreducible sub-representation is obtained on the subspace spanned by those Young tableaux for which the difference in the lengths of the two rows of $\Lambda_{(i)}$ is at most $l$, for all $i$.

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To summarise, the monodromy representation of $B_n$ on the space spanned by the reduced $n$-point functions always factors through a scaled version of the Hecke algebra $H_n\big(e^{2\pi i/(l+2)}\big)$, and is irreducible. It is equivalent to the monodromy representation obtained on the solution set of (8.1.3) when $l \notin \mathbf{Z}$, or of (8.1.3) and (8.1.5) when $l \in \mathbf{Z}$.

For the rest of this Chapter, it will be assumed except where explicitly stated otherwise, that $l \notin \mathbf{Z}$. Thus the monodromy representation is obtained on the space of solutions $f: X_n \to \check{V}_0$ to the equations of Knizhnik-Zamolodchikov:

$$\left(\kappa\frac{\partial}{\partial z_i} - \sum_{\substack{k=1 \\ k \neq i}}^{n} \frac{\Omega_{ik}}{z_i - z_k}\right)\mathbf{f}(z_1,\ldots,z_n) = 0 \tag{8.1.6}$$

where $\kappa = l+2$. The projections $\Omega_{ik}$ of the Casimir operator $\Omega$ act naturally on $V_{1/2}^{\otimes n}$ and thus on that part of $V_{1/2}^{\otimes n}$ which transforms as a spin-$t$ representation under $\mathfrak{sl}_2$; that is, on $\check{V}_0$. As such, the required scaled Hecke algebra representations are obtained as sub-representations of a much larger braid group representation, namely the monodromy representation on the solution space of (8.1.6) where $\mathbf{f}$ are functions with values in $V_{1/2}^{\otimes n}$. This latter representation has dimension $2^n$.

For the purposes of the section it will be found to be more convenient to replace (8.1.6) by a similar differential equation in such a way that the monodromy representation obtained will factor through the Hecke algebra $H_n(q)$, rather than a scaled version of it. Since the action of the generator $\sigma_i \in B_n$ associated with (8.1.6) is $q^{-1/4}$ times that associated with the representation $\pi_{\Lambda\, n/2-t}$ of $H_n(q)$, thus the monodromy action given by functions:

$$\mathbf{g}(z_1,\ldots,z_n) = \prod_{i<j}(z_i - z_j)^p.\mathbf{f}(z_1,\ldots,z_n),$$

where $\mathbf{f}$ satisfies (8.1.6), factors through $H_n(q)$ so long as:

$$e^{-\pi i p}.q^{-1/4} = 1.$$

It is also clear that $\mathbf{f}$ satisfies (8.1.6) if, and only if, $\mathbf{g}$ satisfies:

$$\left(\kappa\frac{\partial}{\partial z_i} - \sum_{\substack{k=1 \\ k \neq i}}^{n} \frac{\Omega_{ik} + p\kappa\mathbf{I}}{z_i - z_k}\right)\mathbf{g}(z_1,\ldots,z_n) = 0 \tag{8.1.6$'$}$$

From the above, $p = -1/2\kappa$ is suitable, and thus, in (8.1.6)$'$, $p\kappa = -1/2$ gives rise to monodromy representations of the Hecke algebra $H_n\big(e^{2\pi i/\kappa}\big)$ where $\kappa = (l+2)$. The functions $\mathbf{g}$ are defined on $X_n$ with values

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in $\check{V}_0$, just as for the functions $\mathbf{f}$ in (8.1.6). So (8.1.6)' is of the same form as (8.1.6), with $\mathbf{\Omega}_{ik}$ replaced by $\mathbf{\Omega}_{ik} + \frac{1}{2}\mathbf{I}$.

*8.1.4 Special case*

In the case n=2, it is particularly easy to compute the image of the generator $\sigma_1 \in B_2$ in the monodromy representation of $B_n$ on the space of solutions to (8.1.6) by functions with values in $V_{1/2}^{\otimes 2}$. The matrix for $\mathbf{\Omega} = \mathbf{\Omega}_{12}$ is given by:

$$\mathbf{\Omega} = \begin{pmatrix} 1/2 & & & \\ & -1/2 & 1 & \\ & 1 & -1/2 & \\ & & & 1/2 \end{pmatrix} \tag{8.1.7}$$

with respect to the basis 11, 12, 21, 22 of $V_{1/2} \otimes V_{1/2}$. In this case (8.1.6) reduces to:

$$\left. \begin{aligned} \kappa \frac{\partial \mathbf{f}}{\partial z_1} &= \frac{\mathbf{\Omega}_{12}}{z_1 - z_2}\mathbf{f} \\ \kappa \frac{\partial \mathbf{f}}{\partial z_2} &= \frac{\mathbf{\Omega}_{21}}{z_2 - z_1}\mathbf{f} \end{aligned} \right\} \tag{8.1.8}$$

where $\mathbf{\Omega}_{12} = \mathbf{\Omega}_{21} = \mathbf{\Omega}$ and $\mathbf{f}$ is a (4-dimensional) vector-valued function on $X_n$. The solutions to (8.1.8) are given by:

$$\mathbf{f} = \exp\left(\frac{1}{\kappa}\ln(z_1 - z_2).\mathbf{\Omega}\right).\mathbf{f}_0$$

where $\mathbf{f}_0 \in V_{1/2} \otimes V_{1/2}$ is a constant vector. The monodromy matrix associated with $\sigma_1 \in B_2$ is thus:

$$\exp\left(-\frac{\pi i}{\kappa}.\mathbf{\Omega}\right)$$

where the action of $\sigma_1$ on $\widetilde{X}_2$ is as defined in §2.1, namely, a clockwise motion of $z_1$ and $z_2$ around each other, under which they transpose.

However, it should be noted that $z_1$ and $z_2$ have now been interchanged, and thus the two factors of $V_{1/2}$ should properly be interchanged when the spaces $V_{1/2} \otimes V_{1/2}$ over the two points $(z_1, z_2)$ and $(z_2, z_1)$ in $X_2$, are identified. Thus the image of $\sigma_1 \in B_2$ under the monodromy representation associated with (8.1.8), is given by the matrix:

$$\mathbf{P} \exp\left(-\frac{\pi i}{\kappa}.\mathbf{\Omega}\right)$$

where:

$$\mathbf{P} = \begin{pmatrix} 1 & & & \\ & 0 & 1 & \\ & 1 & 0 & \\ & & & 1 \end{pmatrix}.$$

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It is easily verified, using (8.1.7), that this matrix is:

$$\begin{pmatrix} e^{-\pi i/2\kappa} & & & \\ & \tfrac{1}{2}(e^{-\pi i/2\kappa} - e^{3\pi i/2\kappa}) & \tfrac{1}{2}(e^{-\pi i/2\kappa} + e^{3\pi i/2\kappa}) & \\ & \tfrac{1}{2}(e^{-\pi i/2\kappa} - e^{3\pi i/2\kappa}) & \tfrac{1}{2}(e^{-\pi i/2\kappa} + e^{3\pi i/2\kappa}) & \\ & & & e^{-\pi i/2\kappa} \end{pmatrix} \qquad (8.1.9)$$

The eigenvalues of this matrix are $e^{-\pi i/2\kappa}$ and $-e^{3\pi i/2\kappa}$. However, $q = e^{2\pi i/\kappa}$, and thus these eigenvalues are $q^{-1/4}$ and $-q^{3/4}$. This verifies that the representation obtained in this case is $q^{-1/4}$ times a Hecke algebra representation.

In order to discuss the monodromy representations produced from solutions of (8.1.3), it is necessary to investigate the solutions in more detail: see [TK] for this. However, it is hoped that the above calculation gives a flavour of the way in which equation (8.1.3) can be used to obtain information about the structure of the monodromy representation.

Equation (8.1.6) may also be viewed as stating that $\mathbf{f}$ defines a flat section of a vector bundle over $X_n$ with fibre $\check{V}_0$, with respect to a suitable (flat) connection. The connection required is given in Kohno's work [Ko]. The associated 1-form is given by:

$$\nabla(e_\lambda) = -\sum_{\mu=1}^{N} \omega_{\mu\lambda} \otimes e_\mu \qquad (8.1.10)$$

where $N = \dim \check{V}_0$, $(e_\lambda)$ is a basis for $\check{V}_0$, and $\boldsymbol{\omega} = (\omega_{\lambda\mu})$ is the matrix valued 1-form defined by:

$$\boldsymbol{\omega} = \sum_{i<j} \boldsymbol{\Omega}_{ij} d\ln(z_i - z_j) \qquad (8.1.11)$$

the sum being over all $i, j \in \{1, 2, \ldots, n\}$ with $i < j$.

For convenience we collect together in the following Theorem various results obtained from this point of view.

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**Theorem 8.2** *The connection $\nabla$ defined by (8.1.10) is integrable, where $\omega$ is the matrix valued 1-form given in (8.1.11), if and only if:*

$$\big[\boldsymbol{\Omega}_{ij},\ \boldsymbol{\Omega}_{ik} + \boldsymbol{\Omega}_{jk}\big] = 0$$

$$\big[\boldsymbol{\Omega}_{ij},\ \boldsymbol{\Omega}_{kl}\big] = 0$$

*for all $i$, $j$, $k$, $l$ distinct. Here $\boldsymbol{\Omega}_{ij} \equiv \boldsymbol{\Omega}_{ji}$ when $i < j$, $i, j \in 1, 2, \ldots, n$. Furthermore, when these conditions are satisfied, the monodromy representation of $B_n$ obtained is given by:*

$$\gamma \longmapsto \mathbf{I} + \int_\gamma \boldsymbol{\omega} + \int_\gamma \boldsymbol{\omega}\boldsymbol{\omega} + \int_\gamma \boldsymbol{\omega}\boldsymbol{\omega}\boldsymbol{\omega} + \cdots$$

*where $\gamma \in \pi_1(X_n)$ and $\int_\gamma \boldsymbol{\omega}\boldsymbol{\omega}\cdots\boldsymbol{\omega}$ denotes Chen's iterated integral.*

For more details, see Kohno [Ko]. The last part of this Theorem is due to Chen. Note that Chen's iterated integral differs from the repeated loop integrals mentioned in Chapter 6, since the region over which Chen integrates is always a simplex, whereas in the case of repeated loop integrals, it is a cuboid.

In [Ko], the above Theorem is used to show that the monodromy representation associated with the equations (8.1.6), when solutions $\mathbf{f}\colon X_n \to V_{1/2}^{\otimes n}$ are considered, gives rise to a Hecke algebra representation when $\boldsymbol{\Omega}_{ik}$ is given by the Casimir operator. This provides an alternative proof of part of Theorem 8.1, to that presented in [TK].

## 8.2 Duality between homology and cohomology

In §3.1, natural actions of $B_n$ and $S_m$ on the homology space:

$$H_m(Y_{\mathbf{w},m},\ \chi_{\mathbf{w},m}(\mathbf{q}))$$

were defined. As mentioned in §6.1, there is naturally defined a dual action of $B_n \times S_m$ on the cohomology space:

$$H^m(Y_{\mathbf{w},m},\ \chi_{\mathbf{w},m}(\mathbf{q})).$$

To obtain useful information from this point of view, it is necessary to express this cohomology space in terms of functions. Since $Y_{\mathbf{w},m}$ is the complement of a complex algebraic hypersurface in $\mathbf{C}^m$, it is a Stein manifold. For any Stein manifold, the cohomology can be calculated as the cohomology of the the complex

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of holomorphic differentials. This result also holds when an abelian local coefficient system is introduced. Thus:

$$H^m(Y_{\mathbf{w},m},\ \chi_{\mathbf{w},m}(\mathbf{q})) \qquad \text{(abbreviated to } H^m\text{)}$$

can be computed in terms of the space:

$$\mathcal{O} = \left\{ f \colon Y_{\mathbf{w},m} \longrightarrow \mathbf{C} \mid f \text{ is holomorphic and twists according to } \chi_{\mathbf{w},m}(\mathbf{q}) \right\}.$$

That is, $\mathcal{O}$ consists of those holomorphic functions $f$ for which:

$$f\big(\gamma(1)\big) = \big(\chi_{\mathbf{w},m}(\mathbf{q})\big)(\gamma).f\big(\gamma(0)\big)$$

for all $\gamma \in \pi_1(Y_{\mathbf{w},m})$. Hence:

$$H^m \cong \mathcal{O} \Big/ \left\{ \sum_{i=1}^{m} \frac{\partial f_i}{\partial z_i} \ \middle|\ f_i \in \mathcal{O} \text{ for } i = 1, 2, \ldots, m \right\} \tag{8.2.1}$$

By a Theorem of Grothendieck (see [A]), This cohomology can be computed as the cohomology of algebraic differential forms. Any function $f \in \mathcal{O}$ can always be written as:

$$f = g.\left( \prod_{i=1}^{m} \prod_{j=1}^{n} (z_i - w_j)^b \right) \left( \prod_{\substack{i,k=1 \\ i<k}}^{m} (z_i - z_k)^a \right) \equiv g.g^0 \tag{8.2.2}$$

where $b$, $a$ are such that $e^{2\pi i b} = q$, $e^{2\pi i a} = \alpha$. Thus $a = -2b$ in the situation considered in Theorem 3.3. Here $g$ is a holomorphic function $Y_{\mathbf{w},m} \to \mathbf{C}$, since $g^0$ contains all the twisting required of $f$. The space $\mathcal{R} \subseteq \mathcal{O}$, of algebraic differential forms, in the case in which the local coefficient system is non-trivial, is given by:

$$\left\{ gg^0 \ \middle|\ g \text{ is a finite linear combination of terms of the form } \left( \prod_{i=1}^{m} \prod_{j=1}^{n} (z_i - w_j)^{-\mu_{ij}} \right) \left( \prod_{\substack{i,k=1 \\ i<k}}^{m} (z_i - z_k)^{-\lambda_{ik}} \right) \right\} \tag{8.2.3}$$

where $\{\lambda_{ik}\}$ and $\{\mu_{ij}\}$ are all integers.

For any map:

$$\underline{\alpha} \colon \{1, 2, \ldots, m\} \longmapsto \{z_1, \ldots, z_m, w_1, \ldots, w_n\}$$

which is such that, thought of as an $m$-tuple, $\underline{\alpha}$ is an element of $\mathcal{S}_{\mathbf{w}}^m$, define:

$$g_{\underline{\alpha}}^0 = \left( \prod_{i=1}^{m} (z_i - \alpha_i)^{-1} \right).g^0 \in \mathcal{R} \tag{8.2.4}$$

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Just as for each $\underline{\alpha} \in \mathcal{S}_{\mathbf{w}}^m$, the chain $\gamma_{\underline{\alpha}}$ was defined in §4.1, with $\{\gamma_{\underline{\alpha}} \mid \underline{\alpha} \in \mathcal{S}_{\mathbf{w}}^m\}$ defining a spanning set for $\mathcal{C}_m$, and ultimately for the homology $H_m$, similarly we have the following result for cohomology.

**Lemma 8.3** *The subset $\{g_{\underline{\alpha}}^0 \mid \underline{\alpha} \in \mathcal{S}_{\mathbf{w}}^m\}$ of $\mathcal{R}$, when projected onto the cohomology $H^m$ is a spanning set.*

PROOF: By Gothendieck's Theorem, $H^m$ is the cohomology of the complex of differential forms based on the subset $\mathcal{R}$ of $\mathcal{O}$ defined in (8.2.3). It is thus given by a similar expression to (8.2.1), in which $\mathcal{O}$ has been replaced by $\mathcal{R}$. To verify the Lemma it is thus sufficient to show that the function defined by (8.2.2), with:

$$g = \left( \prod_{i=1}^m \prod_{j=1}^n (z_i - w_j)^{-\mu_{ij}} \right) \left( \prod_{\substack{i,k=1 \\ i<k}}^m (z_i - z_k)^{-\lambda_{ik}} \right) \tag{8.2.5}$$

(where $\{\lambda_{ik} \mid 1 \leq i \leq k \leq n\}$, $\{\mu_{ij} \mid 1 \leq i \leq m,\ 1 \leq j \leq n\}$ are subsets of the integers) is equivalent to a combination of $\gamma_{\underline{\alpha}}$'s with $\underline{\alpha} \in \mathcal{S}_{\mathbf{w}}^m$, up to the equivalence '$\sim$' in which:

$$\frac{\partial f}{\partial z_i} \sim 0 \qquad \text{whenever } f \in \mathcal{R} \,. \tag{8.2.6}$$

This result is obtained by repeated application of Lemma 8.4. At each stage, $\{a_{ik}\}$ and $\{b_{ij}\}$ differ from $a$ and $b$, by integer values, and the Lemma is used to reduce $gg^0$ (where $g$ is defined in (8.2.2)) to a combination of similar functions,

$$\left\{ hg^0 . \left( \prod_{i=1}^m (z_i - \alpha_i)^{-1} \right) \ \middle|\ \underline{\alpha} \in \mathcal{S}_{\mathbf{w}}^m \right\}$$

where $h$ is given by (8.2.5) with $\lambda_{ik}$, $\mu_{ij}$ replaced by smaller integers (at least, integers no larger than $\lambda_{ik}$, $\mu_{ij}$).

In the rest of this Chapter, the element $g.g^0 \in \mathcal{R}$ defined by (8.2.5) will be denoted by $[\boldsymbol{\lambda}, \boldsymbol{\mu}]$ where $\boldsymbol{\lambda}$ is an upper triangular $m \times m$ matrix with zero diagonal entries, and $\boldsymbol{\mu}$ is an $m \times n$ matrix. If all the elements of the $m \times (m+n)$ integer matrix $(\boldsymbol{\lambda} \mid \boldsymbol{\mu})$ are 0 or 1, with precisely one '1' entry in each row, then $g.g_0$ is a $g_{\underline{\alpha}}^0$, for some $\underline{\alpha}$, and we are finished. If an element of the matrix is 2 or more, then one may apply Lemma 8.4

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with:

$$a_{ik} = 2\delta_{ix}\delta_{ly} - \lambda_{ik} + a \, ;$$

$$b_{ij} = -\mu_{ij} + b \, ;$$

if $\lambda_{xy} \geq 2$. Similarly, if the element $\geq 2$ is $\mu_{xy}$, we may apply Lemma 8.4 with:

$$a_{ik} = -\lambda_{ik} + a \, ;$$

$$b_{ij} = -\mu_{ij} + 2\delta_{ix}\delta_{jy} + b \, .$$

In either case, Lemma 8.4 reduces the function to a combination of functions $[\boldsymbol{\lambda}', \boldsymbol{\mu}']$ where $(\boldsymbol{\lambda}' \mid \boldsymbol{\mu}')$ is still a matrix of integers with at least one '1' in each row, but now the sum of all the integers in this matrix is one less than that in $(\boldsymbol{\lambda} \mid \boldsymbol{\mu})$. This process must therefore terminate, with a combination of $[\boldsymbol{\lambda}, \boldsymbol{\mu}]$ in which the matrices $(\boldsymbol{\lambda} \mid \boldsymbol{\mu})$ all contain exactly one '1' in each row, with all the remaining entries vanishing. Such matrices, as was remarked earlier, give rise to functions in $\mathcal{R}$ of the form $g_{\underline{\alpha}}^0$ some $\underline{\alpha} \in \mathcal{S}_{\mathbf{w}}^m$. Hence the proof is complete. ∎

In the proof of the last Lemma, repeated uses of the following Lemma were made, which will also play an important role in the next section.

**Lemma 8.4**     *Suppose that $f$ is a function of the following form:*

$$f = \left( \prod_{i=1}^{m} \prod_{j=1}^{n} (z_i - w_j)^{b_{ij}} \right) \left( \prod_{\substack{i,k=1 \\ i<k}}^{m} (z_i - z_k)^{a_{ik}} \right) \in \mathcal{R}$$

*where $\{a_{ik}\}$ and $\{b_{ij}\}$ differ from $a$ and $b$ by integers only. For $\underline{\alpha} \in \mathcal{S}_{\mathbf{w}}^m$, define $f_{\underline{\alpha}}$ by an equation similar to (8.2.4) in which $g^0$ is replaced by $f$. Then, up to the equivalence of (8.2.6), $f_{\underline{\alpha}}(z_x - \lambda)^{-1}$ can be expressed as a combination of $f_{\underline{\beta}}$'s, where $w_1, \ldots, w_n$ are thought of as fixed and $\lambda \in \{z_{x+1}, \ldots, z_m, w_1, \ldots, w_n\}$. Furthermore, the coefficients of $f_{\underline{\beta}}$ in $f_{\underline{\alpha}}(z_x - w_j)^{-1}$ can be expressed as constant linear combinations of $(w_j - w_l)^{-1}$ over $l$'s not equal to $j$.*

PROOF:     We shall prove the last part of the Lemma first. There are three different cases here:

(a) $\alpha_x \in \{w_1, \ldots, \widehat{w}_j, \ldots, w_n\}$;

(b) $\alpha_x = w_j$;

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(c) $\alpha = z_k$ some $k \in \{1, 2, \ldots, m\}$.

In case (a), the result follows immediately from the partial fraction decomposition:

$$f_{\underline{\alpha}}(z_x - w_j)^{-1} = f.\left(\prod_{\substack{i=1 \\ i \neq x}}(z_i - \alpha_i)^{-1}\right)\left(\frac{1}{z_x - w_j} - \frac{1}{z_x - \alpha_x}\right).\frac{1}{w_j - \alpha_x}$$

Thus $f_{\underline{\alpha}}(z_x - w_j)^{-1}$ can be expressed as $(w_i - \alpha_x)^{-1}$ times the difference of two $f_{\underline{\beta}}$'s, and we are complete. Note that, as it stands, this doesn't complete the proof in case (c), for in that case $\alpha_x$ is a $z_i$, so that $(w_i - \alpha_x)^{-1}$ is not a 'constant' for fixed $w$'s.

In case (b), note that in the expansion of:

$$\frac{\partial}{\partial z_x}\left(f.\prod_{i=1}^{m}(z_i - \alpha_i)^{-1}\right)$$

the following terms appear:

$$f.\prod_{i=1}^{n}(z_i - \alpha_i)^{-1}\left\{\sum_{y \neq x}\frac{a_{xy}}{z_x - z_y} + \sum_{l=1}^{n}\frac{b_{xl} - \delta_{l\alpha_x}}{z_x - w_l}\right\}$$

where, for convenience $a_{xy}$ is defined for $x > y$ so as to make it symmetric. The term corresponding to $l = j$ is precisely $(b_{xy} - 1)f_{\underline{\alpha}}(z_x - w_j)^{-1}$. Hence under the equivalence '$\sim$', $f_{\underline{\alpha}}(z_x - w_j)^{-1}$ is equivalent to a constant combination of terms of the form:

$$f_{\underline{\alpha}}(z_x - w_l)^{-1} \qquad \text{for } l \in \{1, 2, \ldots, n\};$$

$$f_{\underline{\alpha}}(z_x - z_y)^{-1}.$$

By case (a), we have already dealt with the first of these forms.

Before proceeding further, we shall introduce a graphical notation for the functions occuring. A function $f_{\underline{\alpha}}$ will be denoted by a set of lines joining $n + m$ points, representing $z_1, \ldots, z_m, w_1, \ldots, w_n$. Each $z_i$ is joined to the point $\alpha_i$ and an arrow is inserted. In general an edge joining $\lambda$ and $\mu$, with an arrow from $\lambda$ to $\mu$ is used to denote $(\lambda - \mu)^{-1}$. Thus for $\underline{\alpha} \in \mathcal{S}_{\mathbf{w}}^m$, the diagram for $f_{\underline{\alpha}}$ consists of $m$ oriented lines, and at each $z_i$ precisely one line emanates. Multiplying by $(z_x - \lambda)^{-1}$ is equivalent to placing an extra edge on the diagram, from $z_x$ to $\lambda$. By the reasoning above, it can be seen that any diagram in which there are two edges emerging from $z_x$ and going to the same vertex $\lambda$ ($> z_x$) can be replaced by a constant combination of diagrams in which one copy of the duplicate edge is replaced by an edge from $z_x$ to $\mu$ for some $\mu$ in:

$$\{z_1, \ldots, \widehat{z_x}, \ldots, z_m, w_1, \ldots, w_n\}\backslash\lambda.$$

See Fig. 8.2.

$$\bullet\lambda \qquad\qquad\qquad\qquad \mu \\ \qquad\qquad\qquad\qquad\qquad \bullet$$

$$\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \bullet\lambda$$

$$\longrightarrow \sum_{\mu} \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \text{(I)}$$
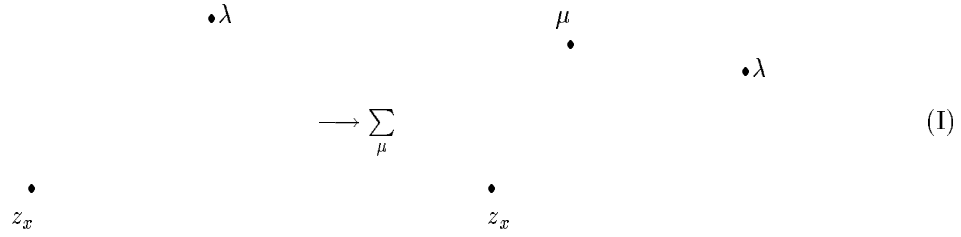
$$\bullet \qquad\qquad\qquad\qquad\qquad\qquad \bullet \\ z_x \qquad\qquad\qquad\qquad\qquad\qquad z_x$$

*Figure 8.2*

Next observe that:

$$\frac{1}{(z_x - \lambda)(z_x - \mu)} = \frac{1}{(z_x - \mu)(\mu - \lambda)} - \frac{1}{(z_x - \lambda)(\mu - \lambda)}$$

and diagramatically, this gives the relation depicted in Fig. 8.3.

$$\lambda \qquad\qquad\qquad\qquad \lambda \qquad\qquad\qquad\qquad \lambda \\ \bullet \qquad\qquad\qquad\qquad \bullet \qquad\qquad\qquad\qquad \bullet$$

$$\bullet \qquad \longrightarrow \qquad\qquad \bullet\mu \qquad - \qquad\qquad \bullet\mu \quad \text{(II)} \\ \mu$$

$$\bullet \qquad\qquad\qquad\qquad \bullet \qquad\qquad\qquad\qquad \bullet \\ z_x \qquad\qquad\qquad\qquad z_x \qquad\qquad\qquad\qquad z_x$$
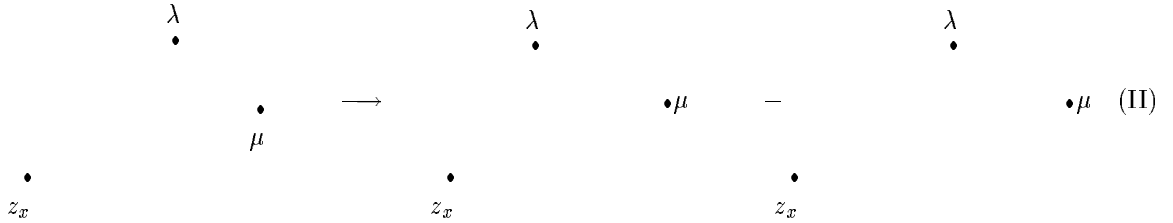
*Figure 8.3*

From a particular point $z_x$, the arrows on the edges of the diagram associated with $f_{\underline{\alpha}}$ may lead along a long chain of points, eventually landing up at $w_l$, for some $l$. Note that there is no limit on the number of arrows entering any particular vertex, although only one arrow may leave any vertex in a diagram of $f_{\underline{\alpha}}$ for $\underline{\alpha} \in \mathcal{S}_{\mathbf{w}}^m$. When the extra edge from $z_x$ to $\lambda$ is added, the diagram so obtained is not that of an $f_{\underline{\beta}}$ since now *two* edges issue from $z_x$; see Fig. 8.4.

Move (II) can be used repeatedly to obtain diagrams in which the node from which two edges issue is moved first from $z_x$ to $\alpha_x$, and then to the next node on the chain, etc. This process will get stuck at a node where the next node in the chain coincides with $\lambda$, unless such a node doesn't exist, in which case it ends with the last node but one, $z_k$, say. This is because move (II) only works when all three nodes involved

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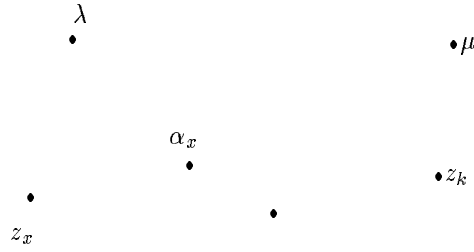
Figure 8.4

are distinct. However, if one gets stuck in the former way, the diagram involved will have a reduced chain with the first node being connected to $\lambda$ twice. Move (I) can then be used to 'unstick' the situation and then repeated applications of move (II) will move up the chain. In the end, we are left with a combination of diagrams, all of which are associated with functions of the form $f_{\underline{\beta}}.(z_k - \lambda)^{-1}$, in which the chain in the diagram associated with $\underline{\beta}$, starting at the point $z_k$ is of length one, as in Fig. 8.5.
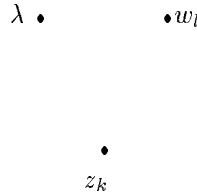


Figure 8.5

When $\lambda = w_j$, the above arguments have reduced case (c) to cases (a) and (b). As was discussed earlier, it is easy to deal with case (a). Case (b), that is $l = j$, can be reduced by move (I) to a combination of diagrams for which the only node at which two edges emerge is $z_k$, the edges being from $z_k$ to $\lambda$ and $\mu$ where $\mu \in \{z_1, \ldots, \widehat{z}_k, \ldots, z_m, w_1, \ldots, w_n\}$ (see Fig. 8.6).

Under all the moves used, the total number of edges joining $z$'s will either reduce, or, at most, stay the same. We can now solve for $f_{\underline{\alpha}}.(z_x - \lambda)^{-1}$ in terms of diagrams with one less edge joining two $z$'s. The diagrams are now those associated with $f_{\underline{\beta}}.(z_k - \mu)^{-1}$ where $\mu$ is a $w$, and distinct from $\lambda$. This falls into case (a) again, and thus the proof is complete; this is the second half of the Lemma.

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$\lambda$ •

        •$\mu$

      •

$z_k$
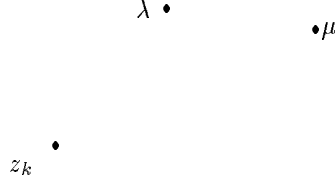
Figure 8.6

Finally we are left with the case in which $\lambda$ is a $z$, say $z_y$. In such a situation, the above arguments, used for $\lambda = w_j$, carry over to give a reduction to diagrams in which there is only one node, $z_k$, from which two edges issue, and these edges are to $\lambda = z_y$ and some other node $\mu \in \{w_1, \ldots, w_n\}$. However, this is precisely the case dealt with above, and has been expressed in the required form by the second half of the Lemma. The coefficients $f_{\underline{\beta}}$ entering, may be expressed as combinations of $(w_i - w_j)^{-1}$, but this time it is possible that all pairs $(i, j)$ become involved. This completes the proof of the Lemma. ∎

From the last two Lemmas, it is apparent that the action of the braid group $B_n$ on cohomology can be computed from its action on $\left\{ g_{\underline{\alpha}}^0 \mid \underline{\alpha} \in \mathcal{S}_{\mathbf{w}}^m \right\}$. Just as the homology can be embedded in $\left\langle \left\{ \gamma_{\underline{\alpha}} \mid \underline{\alpha} \in \mathcal{S}_{\mathbf{w}}^m \right\} \right\rangle$, similarly, in the cohomology $H^m$, $\left\{ g_{\underline{\alpha}}^0 \mid \underline{\alpha} \in \mathcal{S}_{\mathbf{w}}^m \right\}$ is not a linearly independent set, although it does span $H^m$. Since $H_m$ and $H^m$ are dual,

$$\dim H_m = \dim H^m$$

and thus the number of relations that exist between $\left\{ \left[ g_{\underline{\alpha}}^0 \right] \right\}$ is identical to the dimension of the image of the boundary map $\delta \colon \mathcal{C}_m \to \mathcal{C}_{m-1}$ (whose kernel is $H_m$).

## 8.3 Differential equations for cohomology

In this section we will obtain a system of differential equations whose monodromy action is identical to that defined in §3.1. As $\mathbf{w}$ follows a path in $X_n$, the flat connection defined in §3.1 enables elements of the fibres over different points $\mathbf{w}$ to be identified, using parallel transport. In Chapter 3, it was seen how such an identification could lead to a representation of $B_n$ (and not just $P_n = \pi_1(X_n)$). This was accomplished using the natural identification of $Y_{\mathbf{w},m}$ and $Y_{\mathbf{w}',m}$ which exists when $\mathbf{w}'$ lies in the orbit of $\mathbf{w}$ under the action of $S_n$.

The functions $g_{\underline{\alpha}}^0$ of §8.2 are defined for all $\underline{\alpha} \in \mathcal{S}_{\mathbf{w}}^m$, and give rise, over each $\mathbf{w} \in X_n$, to elements of the fibre $H^m(Y_{\mathbf{w},m}, \chi_{\mathbf{w},m}(\mathbf{q}))$ of $E_m(\mathbf{q})$. These functions vary holomorphically with $\mathbf{w}$. By Lemma 8.3 of the last section, any element of the cohomology can be represented as $[f]$, where:

$$f = \sum_{\underline{\alpha} \in \mathcal{S}_{\mathbf{w}}^m} A_{\underline{\alpha}} g_{\underline{\alpha}}^0 \tag{8.3.1}$$

for some coefficients $A_{\underline{\alpha}}$. In this relation , $\mathbf{w} \in X_n$ is fixed.

**Theorem 8.5**     *For suitable constant matrices $\mathbf{C}_{jk}$, defined for each distinct pair of elements $j$, $k$ of $1, 2, \ldots, n$, the system of differential equations:*

$$\frac{\partial \mathbf{A}}{\partial w_j} - \left( \sum_{k \neq j} \frac{\mathbf{C}_{jk}}{w_j - w_k} \right) \mathbf{A} = \mathbf{0} \tag{8.3.2}$$

*for vector valued functions $\mathbf{A}$ on $X_n$, with $\left| \mathcal{S}_{\mathbf{w}}^m \right|$ components has, as a solution, $\mathbf{A} = (A_{\underline{\alpha}})$ only if the function $f$ defined by (8.3.1), is such that the associated elements $[f]$ of $H^m(Y_{\mathbf{w},m}, \chi_{\mathbf{w},m}(\mathbf{q}))$ define a flat section of the cohomology, with respect to the flat connection induced by that of §3.1 on homology.*

PROOF:     Suppose $[f]$ defines a flat section of cohomology with respect to the natural flat connection, induced by duality, from that on $E_m(\mathbf{q})$ defined in §3.1. Then, by definition, as $\mathbf{w}$ moves in the base,

$$\frac{\partial}{\partial w_j} \left( \int_{\gamma_{\underline{\alpha}}} f \right) = 0$$

for all $j \in 1, 2, \ldots, n$ and $\underline{\alpha} \in \mathcal{S}_{\mathbf{w}}^m$. Thus:

$$\int_{\gamma_{\underline{\alpha}}} \left( \frac{\partial f}{\partial w_j} \right) = 0 \tag{8.3.3}$$

for all $\underline{\alpha}$, so that $\frac{\partial f}{\partial w_j} \sim 0$ with respect to the equivalence relation '$\sim$' of (8.2.6). By (8.3.1),

$$\frac{\partial f}{\partial w_j} = \sum_{\underline{\alpha} \in \mathcal{S}_{\mathbf{w}}^m} \left( \frac{\partial A_{\underline{\alpha}}}{\partial w_j} g_{\underline{\alpha}}^0 + A_{\underline{\alpha}} \frac{\partial g_{\underline{\alpha}}^0}{\partial w_j} \right) . \tag{8.3.4}$$

However the definition of $g_{\underline{\alpha}}^0$ in (8.2.4) gives rise to the following relation:

$$\frac{\partial g_{\underline{\alpha}}^0}{\partial w_j} = g_{\underline{\alpha}}^0 \cdot \left\{ \sum_{i=1}^m \frac{b - \delta_{\alpha_i w_j}}{w_j - z_i} \right\} .$$

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The second half of Lemma 8.4 states that there exists constants $C_{jk}^{(i)}(\underline{\beta}, \underline{\alpha})$ such that:

$$g_{\underline{\alpha}}^0 (z_i - w_j)^{-1} = \sum_{\underline{\beta}} \sum_{\substack{k=1 \\ k \neq j}}^n C_{jk}^{(i)}(\underline{\beta}, \underline{\alpha}) \frac{g_{\underline{\beta}}^0}{w_j - w_k}$$

where the first sum is over all $\underline{\beta} \in \mathcal{S}_{\mathbf{w}}^m$. Thus:

$$\frac{\partial g_{\underline{\alpha}}^0}{\partial w_j} = - \sum_{\substack{k=1 \\ k \neq j}}^n \sum_{\underline{\beta}} C_{jk}(\underline{\beta}, \underline{\alpha}) \frac{g_{\underline{\beta}}^0}{w_j - w_k}$$

where $C_{jk}(\underline{\alpha}, \underline{\beta}) = \sum_{i=1}^m (b - \delta_{\beta_i w_j}) C_{jk}^{(i)}(\underline{\alpha}, \underline{\beta})$. Equation (8.3.4) now reduces to:

$$\frac{\partial f}{\partial w_j} = \sum_{\underline{\alpha}} \left( \frac{\partial A_{\underline{\alpha}}}{\partial w_j} - \sum_{\substack{k=1 \\ k \neq j}}^n \sum_{\underline{\beta}} \frac{A_{\underline{\beta}} C_{jk}(\underline{\alpha}, \underline{\beta})}{w_j - w_k} \right) g_{\underline{\alpha}}^0$$

The condition for the flatness of the section $[f]$, namely (8.3.3), is now seen to follow from the differential equations:

$$\frac{\partial A_{\underline{\alpha}}}{\partial w_j} = \sum_{\substack{k=1 \\ k \neq j}}^n \frac{A_{\underline{\beta}} C_{jk}(\underline{\alpha}, \underline{\beta})}{w_j - w_k}$$

This completes the proof of the Theorem. The constant matrices $\mathbf{C}_{jk}$ referred to in the Theorem are square matrices of order $\left| \mathcal{S}_{\mathbf{w}}^m \right|$ whose entries are $C_{jk}(\underline{\alpha}, \underline{\beta})$. ∎

Theorem 8.5 shows that for any solution $\mathbf{A}$ of (8.3.2), the corresponding element of cohomology defines a flat section of the vector bundle $E^m(\mathbf{q})$. Here $E^m(\mathbf{q})$ denotes the vector bundle over $X_n$ whose are fibres dual to those of $E_m(\mathbf{q})$. The flat connection on $E_m(\mathbf{q})$ defined in §3.1, induces a flat connection on $E^m(\mathbf{q})$, by duality, and this is the one referred to in the Theorem. However, the dimension of $W'/\mathrm{Im}(\mathbf{D}')$ is less than the size of the vectors $\mathbf{A}$ in (8.3.2). Thus, the monodromy representation of $B_n$ given by (8.3.2) will be defined on a vector space, $X$, of dimension $\left| \mathcal{S}_{\mathbf{w}}^m \right|$, and will be much larger than the representation of Theorem 6.1.

Choose a fixed point $\mathbf{w}^0 \in X_n$. Then (8.3.2) has a unique solution $\mathbf{A}$ for which:

$$\mathbf{A}(\mathbf{w}^0) = \mathbf{A}^0$$

for arbitrary given $\mathbf{A}^0 \in X$. That is, (8.3.2) has $\left| \mathcal{S}_{\mathbf{w}}^m \right|$ linearly independent solutions. Let $Z$ denote the subspace of $X$ consisting of those $\mathbf{A}$ for which the corresponding element, $[f]$ of cohomology vanishes, where $f$ is given by (8.3.1). Then the solutions of (8.3.2) associated with $\mathbf{A}^0 \in Z$ are all such that:

$$\mathbf{A}(\mathbf{w}) \in Z$$

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for all $\mathbf{w} \in X_n$ in the orbit of $\mathbf{w}^0$, under the action of $S_n$ on $X_n$. The monodromy action obtained from (8.3.2) thus preserves $Z$, and the induced action of $B_n$ on $X/Z$ is identical to the monodromy representation obtained on cohomology.

Let $Y$ denote the subspace of $X$ consisting of those $\mathbf{A} \in X$ for which:

$$\left( \int_{\gamma_{\underline{\alpha}}} f \right) = \mathbf{y} + \mathbf{z} \tag{8.3.5}$$

where $f$ is the associated element of $\mathcal{C}^m$ defined by (8.3.1); $\mathbf{y}$ lies in the image of $\mathrm{Im}\,(\mathbf{D}') \subseteq \mathcal{C}^m$ under the map $g \to \left( \int_{\gamma_{\underline{\alpha}}} g \right)$; and $\mathbf{z}$ is a vector whose components associated with $\underline{\alpha} \in \mathcal{S}_{\mathbf{w}}^m \backslash \mathcal{T}_{\mathbf{w}}^m$ vanish. In this definition, $\mathbf{y}$ and $\mathbf{z}$ are vectors with $\left| \mathcal{S}_{\mathbf{w}}^m \right|$ components. This rather elaborate definition is analogous to that of $W' \subseteq \mathcal{C}^m$ in §6.1. Indeed, $[f] \in W'/\mathrm{Im}(\mathbf{D}')$ is equivalent to $\mathbf{A} \in Y$.

There is a natural action of $S_m$ on $X$ induced by the action on homology in §3.1. As was true of the actions in §3.1, the action of $S_m$ preserves the subspaces $Z$ and $Y$. The space $(Y)^{S_m}/(Z)^{S_m} \cong \left( Y/Z \right)^{S_m}$ has the same dimension as the space $W'/\mathrm{Im}(\mathbf{D}')$, on which the monodromy action of Theorem 6.1 acts.

From the above definitions of $Z$ and $Y$, it follows that the monodromy action of Theorem 6.1 is obtained from that given by (8.3.2), by inducing the action of $B_n$ on $X$, onto $\left( Y/Z \right)^{S_m}$ at $\mathbf{w} = \mathbf{w}_0$. Since (8.3.2) ensures that, for any solution, the associated function $f$ has:

$$\int_{\gamma_{\underline{\alpha}}} f$$

constant in $\mathbf{w}$, for all $\underline{\alpha} \in \mathcal{S}_{\mathbf{w}}^m$ (see the proof of Theorem 8.5), it is clear that the induced monodromy action preserves both $Z$ and $Y$.

**Lemma 8.6**     *The subspaces $Z$ and $Y$ of $X$ are independent of $\mathbf{w}$.*

PROOF:     The subspace $Z \subseteq X$ consists of the 'boundaries'. That is, its elements give those linear combinations of functions $g_{\underline{\alpha}}^0$, for $\underline{\alpha} \in \mathcal{S}_{\mathbf{w}}^m$, which can be written in the form:

$$\sum_{i=1}^{m} \frac{\partial f_i}{\partial z_i} .$$

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Such boundaries are removed by relations (I) and (II) of §8.2 (see Figs. 8..2 and 8.3). The relations so imposed on $\{g_{\underline{\alpha}}^0\}$ involve no dependence on $\mathbf{w}$, and thus $z$ must be independent of $\mathbf{w}$.

The definition of the subspace $Y \subseteq X$, given by (8.3.5), is clearly symmetric in the $w_i$ 's. It is also a natural definition, in that it is independent of the base-point, $\mathbf{z}^0$, chosen for the chains $\gamma_{\underline{\alpha}}$. The spaces of allowed $\mathbf{y}$ and $\mathbf{z}$ vectors in (8.3.5) are independent of $\mathbf{w}$, and hence, $Y$ is independent of $\mathbf{w}$. ∎

Note also that the action of $S_m$ on $X$ is independent of $\mathbf{w}$; in fact, the matrices giving the action of $\sigma \in S_m$ on $X$, in terms of the standard basis, have all their elements 1, -1 or 0. Thus $\left(Y/Z\right)^{S_m}$ is a constant quotient of a subspace of $X$. Since this space is invariant under the flow defined by (8.3.2), then the induced monodromy action of $B_n$ on $\left(Y/Z\right)^{S_m}$ is given by the total monodromy action of a similar system of differential equations to (8.3.2). In particular $\mathbf{C}_{jk}$ gives a well defined action on $\left(Y/Z\right)^{S_m}$. We thus deduce:

**Corollary 8.7** *The representation of Theorem 6.1 obtained by restricting the monodromy action on cohomology to $\left(W'/\mathrm{Im}(\mathbf{D}')\right)^{S_m} \subseteq H^m$, can also be obtained as the monodromy representation associated with the system of equations:*

$$\frac{\partial \mathbf{A}}{\partial w_j} - \left(\sum_{j \neq k} \frac{\mathbf{c}_{jk}}{w_j - w_k}\right)\mathbf{A} = 0 \qquad (8.3.6)$$

*where $\mathbf{A}$ is a vector-valued function on $X_n$, with constant matrices $\mathbf{c}_{jk}$.*

The matrices $\mathbf{c}_{jk}$ are obtained from $\mathbf{C}_{jk}$ by first taking $X$ to the quotient $X/Z$, and then restricting to $\left(Y/Z\right)^{S_m}$. That is, $\mathbf{C}_{jk}$ preserves $Z$, and so can be thought of as a map $X/Z \rightarrow X/Z$. This reduced linear transformation preserves the subspace $Y/Z$ of $X/Z$; it also preserves $\left(Y/Z\right)^{S_m}$, and its restriction to $\left(Y/Z\right)^{S_m}$ gives the matrix $\mathbf{c}_{jk}$.

We have thus shown that the monodromy representation of Theorem 6.1 can be associated with a system of differential equations (8.3.6), of the same form as those used by Tsuchiya & Kanie (see (8.1.6)). In the next section it will be seen that the two systems of differential equations are isomorphic. In the rest of this section we will illustrate Theorem 8.5, and the Lemma used in its proof (namely, Lemma 8.4), by considering the special case $m = 2$.

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When $m = 2$, we shall denote by $g(\lambda, \boldsymbol{\mu}^1, \boldsymbol{\mu}^2)$ the function given by:

$$g^0 (z_1 - z_2)^{-\lambda} \prod_{i=1}^{n} (z_1 - w_i)^{-\mu_i^1} (z_2 - w_i)^{-\mu_i^2} \,.$$

Then the equivalence (8.2.6) gives rise to the following relations:

$$\left.\begin{aligned}
\sum_{i=1}^{n} (b - \mu_i^1) g(\lambda, \boldsymbol{\mu}^1 + \mathbf{e}_i, \boldsymbol{\mu}^2) \sim (\lambda - a) g(\lambda + 1, \boldsymbol{\mu}^1, \boldsymbol{\mu}^2) \\
\sum_{i=1}^{n} (b - \mu_i^2) g(\lambda, \boldsymbol{\mu}^1, \boldsymbol{\mu}^2 + \mathbf{e}_i) \sim (a - \lambda) g(\lambda + 1, \boldsymbol{\mu}^1, \boldsymbol{\mu}^2)
\end{aligned}\right\} \tag{8.3.7}$$

It is also clear from the definition of $g(\lambda, \boldsymbol{\mu}^1, \boldsymbol{\mu}^2)$ that:

$$g(\lambda + 1, \boldsymbol{\mu}^1 - \mathbf{e}_i, \boldsymbol{\mu}^2) - g(\lambda + 1, \boldsymbol{\mu}^1, \boldsymbol{\mu}^2 - \mathbf{e}_i) = g(\lambda, \boldsymbol{\mu}^1, \boldsymbol{\mu}^2) \tag{8.3.8}$$

and when combined with (8.3.7), these relations become:

$$\left.\begin{aligned}
\sum_{i=1}^{n} (b - \mu_i^1) g(\lambda, \boldsymbol{\mu}^1 + \mathbf{e}_i, \boldsymbol{\mu}^2 - \mathbf{e}_i) \sim (bn + 1 + a - \lambda - |\boldsymbol{\mu}^1|) g(\lambda, \boldsymbol{\mu}^1, \boldsymbol{\mu}^2) \\
\sum_{i=1}^{n} (b - \mu_i^2) g(\lambda, \boldsymbol{\mu}^1 - \mathbf{e}_i, \boldsymbol{\mu}^2 + \mathbf{e}_i) \sim (bn + 1 + a - \lambda - |\boldsymbol{\mu}^2|) g(\lambda, \boldsymbol{\mu}^1, \boldsymbol{\mu}^2)
\end{aligned}\right\} \tag{8.3.9}$$

Here $\mathbf{e}_j$ denotes the $j^{\text{th}}$ standard basis vector, so that:

$$g_{\underline{\alpha}}^0 = g(0, \mathbf{e}_{\alpha_1}, \mathbf{e}_{\alpha_2})$$

while $|\ |$ denotes the sum of all the components of the vector concerned. Equation (8.3.7) is equivalent to move (I) of the proof of Lemma 8.4; and in the same way, (8.3.8) is associated with move (II).

The derivative $\frac{\partial}{\partial w_j}$ maps $g_{\underline{\alpha}}^0$ to:

$$\left( \frac{\delta_{\alpha_1 w_j} - b}{(z_1 - \alpha_1)(z_2 - \alpha_2)(z_1 - w_j)} + \frac{\delta_{\alpha_2 w_j} - b}{(z_1 - \alpha_1)(z_2 - \alpha_2)(z_2 - w_j)} \right) g^0 \tag{8.3.10}$$

Case (i): $w_j \neq \alpha_1, \alpha_2$

Using partial fractions (8.3.10) can be expressed in the required form as:

$$\frac{b}{\alpha_1 - w_j} \left( g_{(w_j \alpha_2)}^0 - g_{(\alpha_1 \alpha_2)}^0 \right) + \frac{b}{\alpha_2 - w_j} \left( g_{(\alpha_1 w_j)}^0 - g_{(\alpha_1 \alpha_2)}^0 \right) \,.$$

*Case (ii): $w_j = \alpha_1 \neq \alpha_2$*

The first equation in (8.3.9) can be used to express $g^0(z_1 - \alpha_1)^{-2}(z_2 - \alpha_2)^{-1}$ as a combination of $g_{\underline{\beta}}^0$'s. The expression obtained for (8.3.10) is then:

$$\frac{a + 2b}{\alpha_2 - \alpha_1} g_{(\alpha_1 \alpha_1)}^0 + \frac{b}{\alpha_2 - \alpha_1} g_{(\alpha_2 \alpha_1)}^0 + \left( \frac{a + 3b}{\alpha_1 - \alpha_2} + \sum_{i \neq \alpha_1, \alpha_2} \frac{b}{\alpha_1 - i} \right) g_{(\alpha_1 \alpha_2)}^0$$

$$+ \frac{b}{\alpha_2 - \alpha_1} \sum_{i \neq \alpha_1, \alpha_2} \left\{ g_{(i\alpha_1)}^0 - g_{(i\alpha_2)}^0 \right\} - \sum_{i \neq \alpha_1, \alpha_2} \frac{b}{\alpha_1 - i} g_{(i\alpha_2)}^0 .$$

*Case (iii): $w_j = \alpha_2 \neq \alpha_1$*

This case is similar to case (ii), and the answer is obtained by interchanging $\alpha_1$ and $\alpha_2$, while converting $g_{(xy)}^0$ to $g_{(yx)}^0$ for all $x$, $y$.

*Case (iv): $w_j = \alpha_1 = \alpha_2$ (= $\alpha$, say)*

Applying (8.3.9) we obtain:

$$(a + bn)(z_1 - \alpha)^{-1}(z_2 - \alpha)^{-2} g^0 + (1 - b)(z_1 - \alpha)^{-2}(z_2 - \alpha)^{-1} g^0 \sim b \sum_{j \neq \alpha} \frac{z_2 - j}{(z_1 - \alpha)(z_2 - \alpha)^2 (z_1 - j)} g^0$$

$$(a + bn)(z_1 - \alpha)^{-2}(z_2 - \alpha)^{-1} g^0 + (1 - b)(z_1 - \alpha)^{-1}(z_2 - \alpha)^{-2} g^0 \sim b \sum_{j \neq \alpha} \frac{z_1 - j}{(z_1 - \alpha)^2 (z_2 - \alpha)(z_2 - j)} g^0$$

Adding these two relations, it is apparent that (8.3.10) is equivalent to:

$$\frac{b(1 - b)}{a + 1 + b(n-1)} \sum_{j \neq \alpha} \left( \frac{z_2 - j}{(z_1 - \alpha)(z_1 - j)(z_2 - \alpha)^2} + \frac{z_1 - j}{(z_1 - \alpha)^2 (z_2 - \alpha)(z_2 - j)} \right) g^0$$

However, these two terms have already been dealt with in cases (ii) and (iii) above, and thus (8.3.10) can be reduced to the required form in this case also. In the terminology of Lemma 8.4, this case illustrates the situation in which move (II) has been applied repeatedly, and it is necessary to apply move (I) before move (II) can be applied again.

The matrices $\mathbf{C}_{jk}$, even in the simplest case, of $m = 2$, are non-trivial; however all the entries are simple rational combinations of $a$ and $b$. It is readily seen that when $a = b = 0$, $\mathbf{C}_{jk} = \mathbf{0}$ and thus $\mathbf{c}_{jk} = \mathbf{0}$, at least in this special case; this observation plays a central role in the next section.

## 8.4 Correspondence between CFT and topology

In this section a detailed correspondence between the methods of [TK], [Ko] and the homology representation techniques of Chapters 3–7 is proposed. This correspondence is based on the similarities exhibited in the preceding sections. Both techniques for producing braid group representations come down to constructing a flat connection on a vector bundle over $X_n$, and the representation required appears as the monodromy representation. The fibres of these vector bundles in the two cases are:

$$\check{V}_0 \quad \text{and} \quad \left( W'/\mathrm{Im}(\mathbf{D}') \right)^{S_m} \subseteq H^m(Y_{\mathbf{w},m}, \chi_{\mathbf{w},m}(\mathbf{q}))$$

as defined in §8.1 and §6.1, respectively.

**Lemma 8.8** $\quad \dim \check{V}_0 = \dim \left( W'/\mathrm{Im}(\mathbf{D}') \right)^{S_m}$ where $m = {}^n\!/_2 - t$.

PROOF: It was shown in Chapter 6 that $\left( W'/\mathrm{Im}(\mathbf{D}') \right)^{S_m}$ may be identified with the vector space spanned by the $\binom{n}{m}$ vectors $\{ \mathbf{f}_{\underline{\alpha}} \mid \underline{\alpha} \in \mathcal{U}_{\mathbf{w}}^m \}$, with $\binom{n}{m-1}$ relations existing between them. Thus:

$$\dim V^m = \binom{n}{m} - \binom{n}{m-1}.$$

However $\check{V} = \left( V_{1/2}^{\otimes n} \otimes V_t \right)^{\check{}}$ and thus $\dim \check{V}_0$ is the multiplicity of $V_t$ in the direct sum decomposition of $V_{1/2}^{\otimes n}$. Using the standard relation, obtained for adding angular momenta:

$$V_{j_1} \otimes V_{j_2} = V_{|j_1 - j_2|} \oplus V_{|j_1 - j_2|+1} \oplus \cdots \oplus V_{j_1 + j_2}$$

it can be shown that this gives:

$$\dim \check{V}_0 = \binom{n}{{}^n\!/_2 - t} - \binom{n}{{}^n\!/_2 - t - 1}$$

as required. ∎

In the next section an example will be given which illustrates this Lemma. The monodromy representation on the subspace of cohomology given by $\left( W'/\mathrm{Im}(\mathbf{D}') \right)^{S_m}$ is that associated with differential equations of the form specified in Corollary 8.7, but with $\mathbf{c}_{ij}$ being matrices of dimension $\dim \check{V}_0 = \dim V^m$.

Theorem 8.2 (see §8.1) gives the monodromy representation of $B_n$ in terms of the matrices $\Omega_{ij}$. In [Ko], Kohno shows the following Lemma:

**Lemma 8.9** *The monodromy representation $\theta_{\boldsymbol{\Omega}}$ of $B_n$ associated with the connection given by (8.1.10) and (8.1.11), with the $\Omega_{ij}$ matrices small, depends on $\{\Omega_{ij}\}$ injectively. That is, if $\theta_{\boldsymbol{\Omega}}$ and $\theta_{\boldsymbol{\Omega}'}$ are identical then $\boldsymbol{\Omega} = \boldsymbol{\Omega}'$, so long as $\Omega_{ij}$, $\Omega'_{ij}$ are small, in the sense that the maximum entries in $\Omega_{ij}$ and $\Omega'_{ij}$ are all sufficiently close to zero.*

From this Lemma we now deduce the following result.

**Theorem 8.10** *There exists an isomorphism $\alpha\colon \check{V}_0 \to (W')^{S_m}$ such that:*

$$\kappa^{-1}\alpha \circ (\boldsymbol{\Omega}_{ij} - {}^1\!/_2\mathbf{I}) = \mathbf{c}_{ij} \circ \alpha$$

*where $\mathbf{c}_{ij}$ are the matrices of Corollary 8.7, where $m = {}^n\!/_2 - t$, $q = e^{2\pi i/(l+2)}$.*

PROOF: The differential equations $(8.1.6)'$ and $(8.3.2)$ (or at least, the reduced system $(8.3.6)$, given by Corollary 8.7) both give rise to representations of $B_n$ which factor through $H_n(q)$. They both correspond to two-row Young diagrams, with $n$ squares in total, namely $\Lambda_{{}^n\!/_2 - t}$ and $\Lambda_m$, respectively. Thus when $m = {}^n\!/_2 - t$, the two braid group representations are isomorphic.

When $l \to \infty$, $\kappa^{-1}\Omega_{ij} \to 0$ in the Tsuchiya-Kanie side of the story. On the other hand, this is associated with $q \to 1$ and in this limit the local coefficient system is trivial. Thus in the proof of Theorem 8.5,

$$\frac{\partial g^0_{\underline{\alpha}}}{\partial w_j} \sim \sum_{i=1}^m \frac{\delta_{\alpha_i w_j} g^0_{\underline{\alpha}}}{z_i - w_j} \ .$$

The only terms that occur here are thus of the form:

$$(z_1 - \alpha_1)^{-1} \cdots (z_{i-1} - \alpha_{i-1})^{-1} (z_i - \alpha_i)^{-2} (z_{i+1} - \alpha_{i+1})^{-1} \cdots (z_m - \alpha_m)^{-1}$$

and such terms are equivalent to 0 under '$\sim$' of (8.2.6). Thus $\frac{\partial g^0_{\underline{\alpha}}}{\partial w_j} \sim 0$ for all $\underline{\alpha} \in \mathcal{S}^m_{\mathbf{w}}$ and $j \in \{1, 2, \ldots, n\}$. Hence $\mathbf{C}_{jk} = \mathbf{0}$ in (8.3.2), and so $\mathbf{c}_{jk} = \mathbf{0}$ in Corollary 8.7. Since $\mathbf{c}_{jk}$ are rational functions of $q$, thus, for sufficiently small $q$, $\kappa^{-1}\Omega_{jk}$ and $\mathbf{c}_{jk}$ will both be small and give rise to the same monodromy representations of $B_n$. By Lemma 8.9, the matrices $\kappa^{-1}\Omega_{jk}$ and $\mathbf{c}_{jk}$ must therefore be equivalent. ∎

We conclude this section with a dictionary of the correspondence between Tsuchiya-Kanie theory and the homology theory of Chapters 3–7, as derived in Theorem 8.10.

| Tsuchiya-Kanie theory | Homology theory |
|:---:|:---:|
| $t$ | $n/2 - m$ |
| $\check{V}_0$ | $\left(Y/Z\right)^{S_m} \cong \left(W'/\mathrm{Im}(\mathbf{D}')\right)^{S_m} \cong V^m$ |
| $\Omega_{ij}/\kappa$ | $\mathbf{c}_{ij}$ |
| $\kappa = l + 2$ | $2\pi i/(\ln q)$ |
| Theorem 8.1 | Theorem 6.1 |
| Equation (8.1.6) | Theorem 8.5 |
| Kohno connection | Natural connection on cohomology |

## 8.5 Example

In this section some aspects of the above correspondence will be discussed in the special case $m = 2$, $n = 5$. In this case, $t = n/2 - m = 1/2$, and so:

$$\check{V}_0 = \left(\left(V_{1/2}^{\otimes 5} \otimes V_{1/2}\right)\right)_0$$

where $(\ )_0$ denotes the $\mathfrak{sl}_2$-invariant part. The spin $1/2$ systems may be adjoined one at a time. This gives:

$$V_{1/2} \otimes V_{1/2} = V_0 \oplus V_1$$

$$V_{1/2} \otimes V_{1/2} \otimes V_{1/2} = V_{1/2} \oplus \left(V_{1/2} \oplus V_{3/2}\right)$$

$$V_{1/2} \otimes V_{1/2} \otimes V_{1/2} \otimes V_{1/2} = \left(V_0 \oplus V_1\right) \oplus \left(V_0 \oplus V_1\right) \oplus \left(V_1 \oplus V_2\right)$$

$$V_{1/2}^{\otimes 5} = V_{1/2} \oplus \left(V_{1/2} \oplus V_{3/2}\right) \oplus V_{1/2} \oplus \left(V_{1/2} \oplus V_{3/2}\right) \oplus \left(V_{1/2} \oplus V_{3/2}\right) \oplus \left(V_{3/2} \oplus V_{5/2}\right)$$

There are five copies of $V_{1/2}$ in $V_{1/2}^{\otimes 5}$ and so $\dim \check{V}_0 = 5$. A natural basis for $\check{V}_0$ may be constructed from the five occurences of $V_{1/2}$ in $V_{1/2}^{\otimes 5}$. Let $\mathbf{w}_1, \ldots \mathbf{w}_5$ be the basis so constructed, when spins are added in order, as was done above. Then it can be shown that the matrices $\Omega_{jk}$ are as shown below, with respect to the normalised basis of $\mathbf{w}$'s.

$$\Omega_{12} = \begin{pmatrix} -3/2 & & & & \\ & -3/2 & & & \\ & & 1/2 & & \\ & & & 1/2 & \\ & & & & 1/2 \end{pmatrix} ; \qquad \Omega_{13} = \begin{pmatrix} & & \sqrt{3}/2 & & \\ & & & \sqrt{3}/2 & \\ \sqrt{3}/2 & & -1 & & \\ & \sqrt{3}/2 & & -1 & \\ & & & & 1/2 \end{pmatrix}$$

$$\Omega_{14} = \begin{pmatrix} & & -\sqrt{3}/2 & & \\ & & & 1/\sqrt{12} & \sqrt{2/3} \\ -\sqrt{3}/2 & & -1 & & \\ & 1/\sqrt{12} & & 1/3 & -\sqrt{2}/3 \\ & \sqrt{2/3} & & -\sqrt{2}/3 & -5/6 \end{pmatrix} ; \qquad \Omega_{15} = \begin{pmatrix} & & & 1/2 & -1/\sqrt{2} \\ & & 1/2 & -1/\sqrt{3} & -1/\sqrt{6} \\ & 1/2 & & 1/\sqrt{3} & 1/\sqrt{6} \\ 1/2 & -1/\sqrt{3} & 1/\sqrt{3} & -2/3 & 1/3\sqrt{2} \\ -1/\sqrt{2} & -1/\sqrt{6} & 1/\sqrt{6} & 1/3\sqrt{2} & -5/6 \end{pmatrix}$$

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$$\Omega_{23} = \begin{pmatrix} & & -\sqrt{3}/2 & & \\ & & & -\sqrt{3}/2 & \\ -\sqrt{3}/2 & & -1 & & \\ & -\sqrt{3}/2 & & -1 & \\ & & & & 1/2 \end{pmatrix} \; ; \qquad \Omega_{24} = \begin{pmatrix} & & \sqrt{3}/2 & & \\ & & & -1/\sqrt{12} & -\sqrt{2/3} \\ \sqrt{3}/2 & & -1 & & \\ & -1/\sqrt{12} & & 1/3 & -\sqrt{2/3} \\ & -\sqrt{2/3} & & -\sqrt{2/3} & -5/6 \end{pmatrix}$$

$$\Omega_{25} = \begin{pmatrix} & & & -1/2 & 1/\sqrt{2} \\ & & -1/2 & 1/\sqrt{3} & 1/\sqrt{6} \\ & -1/2 & & 1/\sqrt{3} & 1/\sqrt{6} \\ -1/2 & 1/\sqrt{3} & 1/\sqrt{3} & -2/3 & 1/3\sqrt{2} \\ 1/\sqrt{2} & 1/\sqrt{6} & 1/\sqrt{6} & 1/3\sqrt{2} & -5/6 \end{pmatrix} \; ; \qquad \Omega_{34} = \begin{pmatrix} -3/2 & & & & \\ & 1/2 & & & \\ & & 1/2 & & \\ & & & -1/6 & 2\sqrt{2}/3 \\ & & & 2\sqrt{2}/3 & -5/6 \end{pmatrix}$$

$$\Omega_{35} = \begin{pmatrix} & \sqrt{3}/2 & & & \\ \sqrt{3}/2 & -1 & & & \\ & & & -1/\sqrt{12} & -\sqrt{2/3} \\ & & -1/\sqrt{12} & 1/3 & -\sqrt{2/3} \\ & & -\sqrt{2/3} & -\sqrt{2/3} & -5/6 \end{pmatrix} \; ; \qquad \Omega_{45} = \begin{pmatrix} & -\sqrt{3}/2 & & & \\ -\sqrt{3}/2 & -1 & & & \\ & & & -\sqrt{3}/2 & \\ & & -\sqrt{3}/2 & -1 & \\ & & & & 1/2 \end{pmatrix}$$

These ten matrices define the equations (8.1.6) from which the representation of $H_5(q)$ corresponding to the two-row Young diagram $\Lambda_2$ is obtained.

From the point of view of the homology representations, it is seen that:

$$\left| \mathcal{S}_\mathbf{w}^2 \right| = 30, \qquad \left| \mathcal{T}_\mathbf{w}^2 \right| = 20, \qquad \left| \mathcal{U}_\mathbf{w}^2 \right| = 10 \, .$$

The action of $S_2$ on the cohomology space gives rise to an action of $B_5$ on the 5-dimensional space $V^2 \cong \left( {}^{W'}/_{\mathrm{Im}(\mathbf{D}')} \right)^{S_2}$. The subspace ${}^{W'}/_{\mathrm{Im}(\mathbf{D}')} \subseteq H^2$ is of dimension 10, and $H^2$ itself has dimension 20. At the level of chains, there is an action on $B_5$ on the 15-dimensional space given by the symmetric half of the chain space. There is a five-dimensional boundary, totally contained within the symmetric half of the chain space, so that an action of $B_5$ on a 10-dimensional subspace of $H^2$ is obtained. Only the reduced action on the 5-dimensional space $\left( {}^{W'}/_{\mathrm{Im}(\mathbf{D}')} \right)^{S_2}$ factors through $H_5(q)$.

A natural spanning set for $V^2$ is indexed by $\mathcal{U}_\mathbf{w}^2$. However, even in this simple case, the correspondence, $\alpha$, between the two theories has no obvious description.

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## 8.6 Further remarks

In §8.4, it was shown that there exists an isomorphism between the vector spaces (and thus the vector bundles) used in the Tsuchiya-Kanie approach, and that of Chapters 3–7. However, this proof (see Theorem 8.10) was entirely non-constructive. It is hoped to give a more constructive form for this isomorphism in a future work.

Throughout the last few sections, it has been assumed that $q$ is not a root of unity. Equivalently, $l$ is not an integer. When $l$ is integral, Tsuchiya & Kanie showed how introducing the extra system of equations (8.1.5) gives rise to *irreducible* Hecke algebra representations, once again. This suggests that a similar such construction should exist in terms of the homology picture of Chapters 3–7. In §5.3, it was observed that in the special case $q = 1$, the representation of $S_n$ constructed on $(W')^{S_m}$ is larger than the irreducible representation $\pi_{\Lambda_m}$. At other roots of unity a similar degeneracy can occur, when the homology is computed; that is, $\mathbf{D}'$ may not have maximal rank. This is to be compared with the situation discussed above, at roots of unity, in the theory of Tsuchiya & Kanie.

In both approaches to the construction of $\pi_{\Lambda_m}$, the Hecke algebra representation required appears as a sub-representation of a much larger braid group representation. In Tsuchiya & Kanie's approach, the natural action of $B_n$ on $V_{1/2}^{\otimes n}$ gives the larger space. In the homology approach, the larger representation appears on the symmetric part of the cohomology (with the dual representation on the symmetric part of the homology). The dimension of the reduced representation is:

$$\binom{n}{m} - \binom{n}{m-1}$$

and those of the larger representations are $2^n$ and $1/m!(n + m - 2) \cdots n(n - 1)$, respectively.

In Chapter 7 it was shown how, in certain special cases, a reduced representation of $B_n$ could be constructed from the family of representations with parameter $\alpha$, by looking near to $\alpha = q^{-2}$. The representation of $B_n$ associated with generic values of $\alpha$ and $q$, is irreducible. When $\alpha = q^{-2}$, it is reducible, one part being the required Hecke algebra representation. There is a similarity here with the behaviour of the Hecke algebra representations as $q$ varies. Generically $\pi_{\Lambda_m}$ is irreducible. However, when $q$ is a root of unity, this representation may be reducible, and when it is, the representation constructed by Tsuchiya & Kanie is a

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large irreducible piece of it. This leads one to speculate on how such an irreducible piece may be isolated. Tsuchiya & Kanie do this by adding an extra system of equations. If the analogy is valid, one would expect that it may be possible to select the sub-representation out by using a limiting lemma, along the lines of Theorem 7.2.

As far as the Jones polynomial itself is concerned, the expression for $V_L$ given in Chapter 2 (in particular Theorem 2.3), is in terms of the characters $\chi_{\Lambda_m}$. When $q$ is not a root of unity, these characters correspond to irreducible representations; but when $q$ is a root of unity, it is still these characters, and not their decompositions into irreducible parts, which play the central role. Thus although it should be possible to construct, in a natural topological manner, the irreducible parts of $\chi_{\Lambda_m}$, for $q$ a root of unity, this would have no significance as far as a topological interpretation of $V_L$ is concerned. However, the generalisations of the Jones polynomial given in Witten's theory [W] are in terms of the (smaller) irreducible representations. In the case of the Jones polynomial, the extra parts of the representations cancel out, and so, we can equally well use the larger representation, $\pi_{\Lambda_m}$, as its irreducible sub-representation. This indicates that a better topological understanding of the generalisations of $V_L$ (and probably not even of $V_L$ itself) will not be obtained until the irreducible representations themselves have been interpreted topologically.

# 9: Some further remarks

In Witten's approach [W], the expansion of $V_L(q)$ as a linear combination of characters of $H_n(q)$ comes out naturally by considering the interplay between the link $L$ embedded in $S^3$, and in $S^1 \times S^2$. The latter theory is associated with the braid approach. Witten's theory has data consisting of a Lie algebra $\mathfrak{g}$, level $k$, and a representation of $\mathfrak{g}$ for each component of $L$. When $\mathfrak{g} = \mathfrak{sl}_2$, and the representations are all just the standard vector representation, the invariant has value:

$$V_L\left(e^{2\pi i/(k+2)}\right).$$

Thus $q$ and $k$ are related by $q = e^{2\pi i/(k+2)}$. In Witten's theory $V_L$ is only evaluated at roots of unity. However, when $k$ is sufficiently large, such a root of unity behaves in a similar way to generic $q$. That is, the characters of $H_n(q)$ which appear are $\chi_{\Lambda_m}$ which are irreducible. For small $k$, the representation $\pi_{\Lambda_m}$ is not irreducible, and in that case it is an irreducible part of $\pi_{\Lambda_m}$ which will appear in Witten's theory. An important problem is, therefore, to understand how this reduced representation can be naturally picked out of $\pi_{\Lambda_m}$, in terms of the topological construction of Chapters 3–7.

Just as Witten's theory can be extended to arbitrary Lie algebras $\mathfrak{g}$, it seems likely that our homology theory can also be so extended. In fact, the paper of Kohno [Ko] is framed in the context of an arbitrary Lie algebra. When $\mathfrak{sl}_2$ is generalised to $\mathfrak{sl}_N$, the shift $k \to k+2$ is generalised to a shift $k \to k+N$. When this generalisation is carried out in Tsuchiya & Kanie's theory, the Hecke algebra representations of $H_n\left(e^{2\pi i/(k+N)}\right)$ which are obtained are those associated with Young diagrams with $n$ squares, and $\leq N$ rows. Such Hecke algebra representations should be obtained using a suitable generalisation of the constructions presented in Chapters 3–6. One possible approach is to replace the set of points $z_1, \ldots, z_m$, by several such sets of points. The set $\{z_i\}$ was introduced, in Chapter 2, in order to give local coordinates in the fibre $Y_{\mathbf{w},m}$

of the vector bundle:

$$\begin{array}{ccc} X_{m+n} & & Y_{\mathbf{w},m} \\ \downarrow & & \downarrow \\ X_n & & \mathbf{w} \end{array}$$

and would be generalised, so as to give rise to local coordinates, in a suitable (more complicated) vector bundle. In the construction of the subspace of cohomology (namely $\left(W'/\mathrm{Im}(\mathbf{D}')\right)^{S_m}$) it is seen that an element of the spanning set $\left\{f_{\underline{\alpha}}^m\right\}$ (where $\underline{\alpha} \in \mathcal{U}_{\mathbf{w}}^m$) of this subspace is given by a map:

$$\underline{\alpha} \colon \left\{1, 2, \ldots, m\right\} \longrightarrow \left\{w_1, \ldots, w_n\right\}$$

or, at least, by a subset of $\left\{w_1, \ldots, w_n\right\}$ of order $m$. Such a map can be thought of as a pairing off of the points $z_1, \ldots, z_m$ with $m$ points in $\left\{w_1, \ldots, w_n\right\}$. This gives rise to the Young diagram $\Lambda_m$. In a similar way, one might imagine that $N$ sets of $z_i$'s would give rise to a space with a spanning set given by a pairing off of the $N$ sets of $z_i$'s with the $w_j$'s. Thus one may conjecture that a Hecke algebra representation with $(N+1)$ rows could be obtained in this way.

In Chapter 8, the factor $q^{-1/4}$ appears in Theorems 8.1 and 8.1'. This translates into a shift:

$$\Omega_{ik} \longrightarrow \Omega_{ik} - {}^1\!/_2\mathbf{I}$$

The '$-1/4$' and '$1/2$' appearing here all arise from the choice of $\mathfrak{g} = \mathfrak{sl}_2$ and its vector representation. It should be expected thus, that in the generalised theory, just as the shift $k \to k+2$ generalises to $k \to k+N$, so the shift '$1/2\mathbf{I}$' will generalise, except that in this case it will depend on the representation used, as well as the chosen Lie algebra.

The relations between the homology theory presented, and the structure of conformal field theory on $\mathbf{P}^1$, as presented in Chapter 8, lead naturally to a much more general conjecture. Thus we conjecture that the structure of vertex operators can be totally specified using elementary topology and branched covers, etc., without the need to bring in complex topological concepts such as geometric quantisation. A better understanding of the topological description of the Hecke algebra representations should then lead to an understanding of the Jones polynomial, since the Jones polynomial is simply a combination of the characters of these representations. To do this, it is necessary to obtain, in topological terms, an understanding of the significance of the combination in which these characters occur. It seems likely that a close analysis of

Witten's interpretation of $V_L$ in [W], and the relation between the braid and plait pictures of a link will lead to this understanding.

So, as discussed above, one should view this work as the simplest case of a general structure which we intend to investigate in future work. Finally, it is interesting to speculate on possible connections with the work of N. Hitchin, see [H 2]. The constructions alluded to above, in which $\mathfrak{sl}_{N+1}$-theory is obtained by using $N$ sets of points in the fibre, with *abelian* local coefficient systems on a suitable branched cover, have the same ingredients as in Hitchin's work. Namely, they both reduce a complex, non-commutative system, to a commutative one on a much larger space. In the homology picture described, it is the fibres of $E_m(\mathbf{q})$ which play the role of the larger space. Beyond this comment, however, it is not clear how, or if, a detailed correspondence between these two approaches exists.

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# Table of notation

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