CHAPTER 3

METRICAL GEOMETRY

> Space is another framework which we impose on the world. Whence are the first principles of geometry derived? Are they imposed on us by logic? Lobatschewsky, by inventing non-Euclidean geometries, has shown that this is not the case. Is space revealed to us by our senses? No; for the space revealed to us by our senses is absolutely different from the space of geometry. Is geometry derived from experience? Careful discussion will give the answer--no! We therefore conclude that the principles of geometry are only conventions; but these conventions are not arbitrary, and if transported into another world (which I shall call the non-Euclidean world, and which I shall endeavor to describe), we shall find ourselves compelled to adopt more of them.
>
> In mechanics we shall be led to analogous conclusions, and we shall see that the principles of this science, although more directly based on experience, still share the conventional character of the geometrical postulates. So far, nominalism triumphs; but we now come to the physical sciences, properly so called, and here the scene changes. We meet with hypotheses of another kind, and we full grasp how fruitful they are. No doubt at the outset theories seem unsound, and the history of science shows us how ephemeral they are; but they do not entirely perish, and of each of them some traces still remain. It is these traces which we must try to discover, because in them and in them alone is the true reality. H. Poincare [1905, pp. xxv-xxvi]

Cayley has shown how metrical concepts may be introduced into geometry on a purely projective basis. That is, a figure such as a quadric surface is designated as a fixed reference, the Absolute, and metrical properties, are those properties of figures which take on significance in relation to the Absolute. This is the starting point for the systematic development of metrical

geometries in Section 3.1. Metrical relations are developed using projective coordinates and hence the seemingly self-contradictory name of projective metrics. This section simultaneously treats the common properties of hyperbolic, Euclidean and elliptic geometries in a general manner using the notion of an Absolute polarity as an invariant connection of dual elements in space. Using a definition by Clifford, an analytical generalization for determining the pitches and axes of screws is given which apparently may be also found in Buchheim [1884b]. Metrical collineations are defined as those which leave the Absolute invariant and form a subgroup of projective collineations. Norms are then introduced as functions of the Absolute and enables the development of metrical coordinates where the components themselves are significant not just their ratios. Elements of projective space are then assigned a norm of unity although, in the general case, this leads to two sets of metrical coordinates for an element which differ in sign. Elements with a nonunity norm are defined as new types of space elements which have an associated weight or magnitude.

Since the properties of the various metrical geometries vary considerably, Section 3.2 deals exclusively with elliptic geometry. First, the elliptic polarity is introduced which has a close connection with interpreting the coordinates of a space element in terms of dual coordinates and forms a basis for some of the developments in Chapter 4. It is shown that properties which are often erroneously associated with n-dimensional "Euclidean" spaces, such as "orthogonality," are actually properties of elliptic space when homogeneous coordinates are utilized. This is particularly important with respect to some later developments dealing with the "orthogonality" of screws which actually signifies that two screws are elliptic conjugates. Specializing a previous formulation, it is shown that screws in elliptic space have two axes which are elliptic polars and two respective pitches that are reciprocal. These results agree with ones given by Clifford [1873] along with the notion that screws of pitch ±1 have properties of free vectors. Other investigations dealing with screws in elliptic space are given by Buchheim [1884a, 1884b], Cox [1882], Heath [1885] and Ball [1900].

In collineation form, the elliptic polarity of lines and screws is similar in appearance to the important identical relation between ray coordinates and axis coordinates which is a symmetrical correlation. Table 3.2.1 summarizes a number of relations in elliptic geometry which appear very similar to expressions in projective geometry. This exemplifies why it is necessary to have an unambiguous notation to delineate collineations from correlations and ray coordinates from axis coordinates. A systematic development commencing with projective geometry makes it possible to delineate the distinct but similar expressions.

Euclidean geometry is distinguished by the fact that the Absolute polarity is singular and consequently many relations must be approached as limiting cases. As shown in Section 3.3, the singularity introduces an asymmetric character to dual expressions which does not exist for projective or elliptic geometry. The general formulation in Section 3.1 is specialized for an interesting development of the pitch and axis of a screw which is expressible as a linear combination of a unique line and its polar. It is also shown that what is often referred to as the "dual" operator ω, where $\omega^2 = 0$, is merely the Euclidean polarity expressed in form amenable to biquaternions. Based on the Euclidean Absolute, norms are defined which are then used to introduce Euclidean metrical coordinates. Elements of projective space are assigned norms of unity and points are given a unique set of coordinates, unlike planes, lines and screws which have two sets of coordinates that differ in sign. New space elements are defined which have the property of weight or magnitude since their norms are nonunity.

It is typical to study a geometry in terms of examining its group of transformations. However, the group of transformations in Euclidean space always appears to be an entity given a priori from which subsequent geometric properties are then derived. Here, beginning with the general group of projective collineations and Cayley´s Absolute, it is shown how the corresponding group of

Euclidean transformations may be deduced as those which leave the Euclidean Absolute invariant. Although the procedure is not complex, it appears to have been previously overlooked.

In Section 3.4 polar and axial vectors in Euclidean space are introduced by way of Klein's second principle given at the beginning of Section 2.3. Then using Hamilton's vectors, a polar vector is defined as the difference between two points. By introducing vectors and making the point a more fundamental element in Euclidean space than the plane, the ambiguity of signs for the metrical coordinates of planes, lines and screws is examined. The ambiguity is only resolvable for new space elements that are then introduced namely, plane-sects, geometric couples, line vectors and screw vectors which are all distinguished by a magnitude and an unambiguous associated direction. The ray coordinates and axis coordinates of line vectors and screw vectors are then expressed in a formulation which is typical of modern presentations, especially the ones using dual vectors such as in Brand [1947]. Finally, in application to the area of mechanics, twists and wrenches are introduced along with the formulation of virtual work. When a body is in static equilibrium under impressed wrenches its virtual work vanishes, a property which is shown to be analogous to the projective property of incidence.

Section 3.1 Projective Metrics

> I remark in conclusion, that, in my own point of view, the more systematic course in the present introductory memoir on the geometrical part of the subject of quantics, would have been to ignore altogether the notions of distance and metrical geometry; for the theory in effect is, that the metrical properties of a figure are not the properties of the figure considered per se apart from everything else, but its properties when considered in connexion with another figure, viz the conic termed the Absolute. The original figure might comprise a conic; for instance, we might consider the properties of the figure formed by two or more conics, and we are then in the region of pure descriptive geometry: we pass out of it into metrical geometry by fixing upon a conic of the figure as a standard of reference and calling it the Absolute. Metrical geometry is thus a part of descriptive geometry, and descriptive geometry is all geometry, and reciprocally; and if this be admitted, there is no ground for the consideration, in an introductory memoir, of the special subject of metrical geometry; but as the notions of distance and of metrical geometry could not, without explanation, be thus ignored, it was necessary to refer to them in order to show that they are thus included in descriptive geometry.
> Arthur Cayley [1859, pp. 592]

In the preceding chapter, it has been demonstrated that homogeneous coordinates may be introduced into geometry without recourse to a form of measure, or in other words, a metric. Homogeneous coordinates are well-suited for examining incidence relations which comprises the domain of projective geometry, or "descriptive geometry" as Cayley referred to it. Metrical geometries such as elliptic, Euclidean and hyperbolic may be developed from projective geometry by establishing one or more figures as a fixed reference, which Cayley called the Absolute. In three-dimensional space, various metrical or so-called Cayley-Klein geometries may be developed by defining the Absolute to be a point locus together with a plane envelope of a quadric surface. Projective homogeneous coordinates may be adapted for metrical geometries and in doing so they may also be endowed with the additional property of magnitude, which is a function of the Absolute.

It is most interesting to note, that prior to the landmark paper "A Sixth Memoir on Quantics," Cayley [1859], projective geometry was considered merely to be a somewhat poorer subject in what was then the all-pervasive geometry of Euclid. Initially, not even Cayley recognized the scope of his dictum, "descriptive geometry is all geometry," since he had only considered the geometries known to him at that time, Euclidean and spherical, the former of which he presented for only one and two dimensions. It was left to F. Klein [1871, 1873], some twelve years hence, to demonstrate that the elliptic geometry of Riemann and the hyperbolic geometry of Lobatchewsky and Boylai, the so-called non-Euclidean geometries, may be developed by selecting the Absolute to be respectively an imaginary or real figure.

In this section, metrical geometries are developed in a general format which may then be specialized to yield hyperbolic, Euclidean and elliptic geometries, the latter two of which are investigated in the succeeding sections. Instead of commencing the development here with an Absolute quadric, it is preferred to first establish an Absolute polarity and its adjoint, Coxeter [1965].

A correlation is a linear transformation which maps each element of space into a dual element. A polarity is a symmetric correlation which can be represented by a symmetrical matrix and may be utilized to establish an invariant connection of space between dual elements. This is essentially equivalent to the Cayley-Klein development since there is generally a one-to-one relationship between polarities and quadric surfaces. It should be particularly noted that since metrical geometries are specializations within projective geometry, that metrical geometries must preserve projective properties, in particular, relations of incidence.

The diagonal polarity $\tilde{\pi}$ and its adjoint $\tilde{\Pi}$ where

$$\tilde{\pi} = \begin{bmatrix} \varepsilon & \cdot & \cdot & \cdot \\ \cdot & 1 & \cdot & \cdot \\ \cdot & \cdot & 1 & \cdot \\ \cdot & \cdot & \cdot & 1 \end{bmatrix} \qquad (1)$$

$$\tilde{\Pi} = \begin{bmatrix} 1 & \cdot & \cdot & \cdot \\ \cdot & \varepsilon & \cdot & \cdot \\ \cdot & \cdot & \varepsilon & \cdot \\ \cdot & \cdot & \cdot & \varepsilon \end{bmatrix} \qquad (2)$$

are used to establish elliptic, Euclidean and hyperbolic geometries for $\varepsilon = 1,0,-1$ respectively. Since the polarities become singular in the Euclidean case, the appropriate development considers the limiting case $\varepsilon \rightarrow 0$.

For the polar relations,

$$X' = \tilde{\Pi}x \qquad (3)$$

$$x' = \tilde{\pi} X \qquad (4)$$

the plane X' is said to be the polar of point x and the point x' is said to be the pole of plane X. Two points x,y (two planes X,Y) are said to be conjugate when each is incident with the other's polar (pole) and

$$x^T \tilde{\Pi} y = 0 \qquad (5)$$

$$X^T \tilde{\pi} Y = 0. \qquad (6)$$

Sometimes conjugate points and conjugate planes are referred to respectively as $\tilde{\Pi}$-orthogonal or $\tilde{\pi}$-orthogonal. The Absolute (quadric) is defined as the locus of self-conjugate points and the envelope of self-conjugate planes,

$$x^T \tilde{\Pi} x = 0 \qquad (7)$$

$$X^T \tilde{\pi} X = 0. \qquad (8)$$

The polarity of points and planes induces a corresponding polarity of lines. In Fig. 3.1.1, the join of points x,y is the line p and the meet of their polar planes X', Y' defines the polar line P'. Alternatively, the meet of planes T,U also defines the same line P and the join of their poles t',u' also defines the same polar line p'. Therefore P' and p', the axis and ray coordinates of the polar line, are given respectively by the nonsquare determinants (see Section 2.2),

$$P' = |\tilde{\Pi} x \ \tilde{\Pi} y| \qquad (9)$$

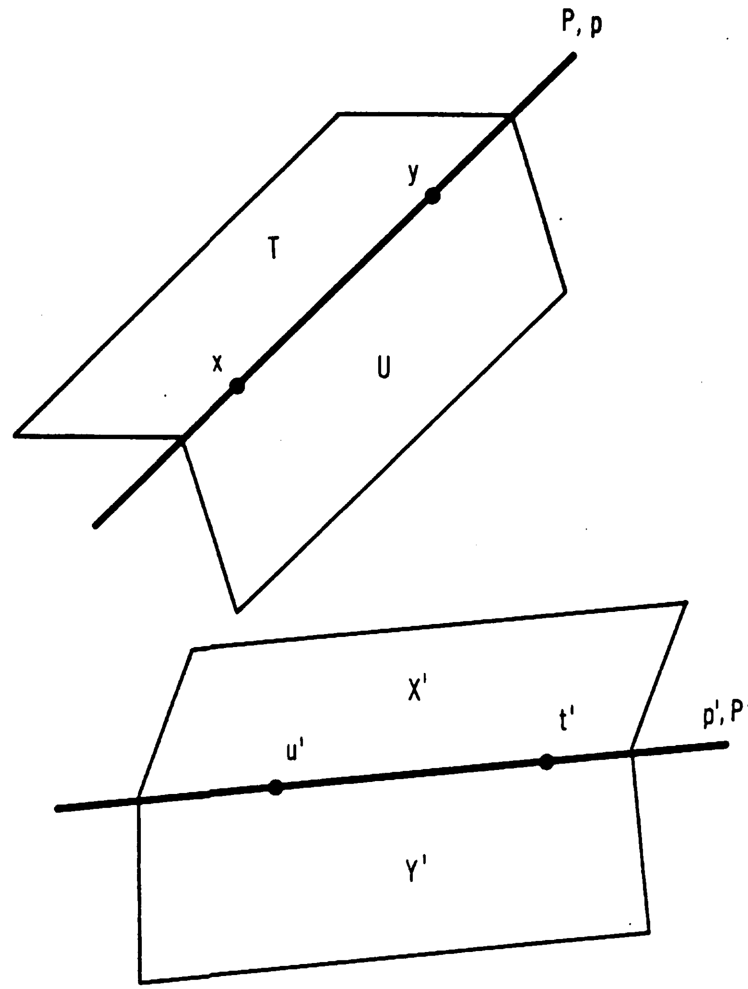

Figure 3.1.1 A pair of polar lines.

$$p' = |\tilde{\pi}X \; \tilde{\pi}Y|. \tag{10}$$

Substituting, (1), (2) in (9), (10) and expanding yields the induced polar relations

$$P' = \tilde{\Gamma}p \tag{11}$$

$$p' = \tilde{\gamma}P \tag{12}$$

where

$$\tilde{\Gamma} = \begin{bmatrix} I_3 & \cdot \\ \cdot & \varepsilon I_3 \end{bmatrix} \tag{13}$$

$$\tilde{\gamma} = \begin{bmatrix} \varepsilon I_3 & \cdot \\ \cdot & I_3 \end{bmatrix}. \tag{14}$$

In (13) the common factor ε has been removed which is necessary for the Euclidean case where $\varepsilon \to 0$.

Alternatively, (13) may be derived by substituting the relations between ray and axis coordinates (2.2.48), (2.2.49) in (14) to yield

$$\tilde{\Gamma} = \tilde{\Delta}^T \tilde{\gamma} \tilde{\Delta}. \tag{15}$$

Conversely, (15) can be rearranged as

$$\tilde{\gamma} = \tilde{\Delta}^T \tilde{\Gamma} \tilde{\Delta}. \tag{16}$$

Since $\tilde{\Gamma}$, $\tilde{\gamma}$ are symmetrical and are adjoints, then to a scalar multiple ε,

$$\tilde{\Gamma} \tilde{\gamma} = \hat{I}_6 \tag{17}$$

which may be used with (15),(16) to yield the tetrahedron relationships (see Section 2.3),

$$\tilde{\Gamma}^T \tilde{\Delta} \tilde{\Gamma} = \tilde{\Delta} \tag{18}$$

$$\tilde{\gamma}^T \tilde{\Delta} \tilde{\gamma} = \tilde{\Delta}. \tag{19}$$

Two lines p,q (P,Q) are said to be conjugate when each is incident with the other's polar line and

$$p^T \tilde{\Gamma} q = 0 \tag{20}$$

$$P^T \tilde{\gamma} Q = 0. \tag{21}$$

Alternatively, lines which are conjugate are sometimes referred to as $\tilde{\Gamma}$-orthogonal or $\tilde{\gamma}$-orthogonal. Lines which are self-conjugate are incident with their own polars and their assemblage forms the tangent lines tc the Absolute. In line coordinates, the Absolute is given by the quadratic complex, Jessop [1903],

$$p^T \tilde{\Gamma} p = 0 \tag{22}$$

$$P^T \tilde{\gamma} P = 0. \tag{23}$$

As previously defined, a linear combination of lines is in general a screw

$$p = \lambda_1 q + \ldots \lambda_n r \quad , \quad P = \lambda_1 Q + \ldots \lambda_n R. \tag{2.2.65}$$

Analogous to (11), (12), the polar of a screw is given by

$$P' = \lambda_1 \tilde{\Gamma} q + \ldots \lambda_n \tilde{\Gamma} r ,$$

$$p' = \lambda_1 \tilde{\gamma} Q + \ldots \lambda_n \tilde{\gamma} R \tag{24}$$

and two screws which satisfy a bilinear relation of the form (20) or (21) are also said to be conjugate.

In describing screws in elliptic space, Clifford [1873, pp. 193] asserted that a screw can be expressed uniquely as the sum of a line and its polar line and that this polar pair represents the axes of the screw. Here this result is generalized and, using ray coordinates, a screw p is expressed as a linear combination of a unique line q and its polar q'

$$p = \lambda q + \lambda' q' \tag{25}$$

where it is necessary to determine the scalars λ, λ' and the pair of polar lines q, q'. The line q' may be expressed in ray coordinates by

$$q' = \tilde{\Delta} Q' = \tilde{\Delta} \tilde{\Gamma} q \tag{26}$$

which is substituted in (25) to yield

$$p = (\lambda \hat{I}_6 + \lambda' \tilde{\Delta} \tilde{\Gamma}) q. \tag{27}$$

Provided that the matrix in (27) is nonsingular, then using (13) q is easily solved for and

$$q = \frac{1}{(\lambda^2 - \lambda'^2 \varepsilon)} (\lambda \hat{I}_6 - \lambda' \tilde{\Delta} \tilde{\Gamma}) p. \tag{28}$$

Since q is a line, it must satisfy the identical relation

$$q^T \tilde{\Delta} q = 0 \tag{29}$$

which is used to eliminate q. Substituting (28) in (29) yields, after some rearranging,

$$\frac{1}{(\lambda^2 - \lambda'^2 \varepsilon)^2} [(p^T \tilde{\Delta} p)\lambda^2 - 2(p^T \tilde{\Gamma} p)\lambda\lambda' + \varepsilon(p^T \tilde{\Delta} p)\lambda'^2] = 0 \tag{30}$$

For $\varepsilon = -1, 0, 1$, (30) may be solved for the ratio $\lambda:\lambda'$ which is then substituted back in (28) to determine q and subsequently q' from (26). By extension of Clifford's definition, the ratios $\lambda':\lambda$ resulting from (30) are called the pitches of the screw with respect to its axes. In the following sections the solution is detailed for both elliptic and Euclidean geometries.

Metrical collineations are defined as those which leave the form of the Absolute invariant. For points and planes, the projective collineations of space (see Section 2.3) are given by

$$y = Kx \quad , \quad Y = kX . \tag{31}$$

In the image space, the Absolute must be expressible in the same form as (7), (8) and

$$y^T \tilde{\Pi} y = 0 \tag{32}$$

$$Y^T \tilde{\pi} Y = 0. \tag{33}$$

Substituting (31) in (32) and (33) yields

$$x^T(K^T \tilde{\Pi} K)x = 0 \qquad (34)$$

$$X^T(k^T \tilde{\pi} k)X = 0. \qquad (35)$$

Comparing (34), (35) with (7), (8) for general x,X gives

$$K^T \tilde{\Pi} K = \mu \tilde{\Pi} \qquad (36)$$

$$k^T \tilde{\pi} k = \mu \tilde{\pi} \qquad (37)$$

where μ is a nonzero scalar multiple. Equations (36), (37) express the required conditions for a nonsingular collineation to leave the Absolute invariant. Collineations which satisfy (36) or (37) are referred to as metrical collineations and they form a subgroup within the general group of projective collineations. A projective collineation is expressed using 16 elements and since only the ratios are significant, it is thus determined by 15 independent parameters. However, a metrical collineation must also satisfy (36) or (37), and by symmetry either matrix equation represents a set of 10 nonhomogeneous scalar equations from which μ may be eliminated by considering the 9 ratios of equations. Thus the 15 parameters of a general projective collineation are related by 9 constraint equations to yield 15-9=6 independent parameters for the specification of a metrical collineation.

Briefly, for the induced collineations of lines or screws, let $\hat{K}$ and $\hat{k}$ be respectively the ray and axis transformations,

$$q = \hat{K}p \qquad (38)$$

$$Q = \hat{k}P. \qquad (39)$$

The equations of the Absolute are

$$q^T \tilde{\Gamma} q = p^T(\hat{K}^T \tilde{\Gamma} \hat{K})p = 0 \qquad (40)$$

$$Q^T \tilde{\gamma} Q = P^T(\hat{k}^T \tilde{\gamma} \hat{k})P = 0 \qquad (41)$$

and comparing with (22), (23) yields

$$\hat{K}^T \tilde{\Gamma} \hat{K} = \mu\tilde{\Gamma} \qquad (42)$$

$$\hat{k}^T \tilde{\gamma} \hat{k} = \mu\tilde{\gamma} \qquad (43)$$

which are the conditions for an induced collineation to be a metrical collineation.

The establishment of an Absolute enables the introduction of a projective norm which is useful in the development of metrical coordinates. For points and planes respectively, the norms are given by the scalar functions

$$\|x\| = (x^T \tilde{\Pi} x)^{\frac{1}{2}} \qquad (44)$$

$$\|X\| = (X^T \tilde{\pi} X)^{\frac{1}{2}} \qquad (45)$$

and for both lines and screws, the norms in terms of ray and axis coordinates are

$$\|p\| = (p^T \tilde{\Gamma} p)^{\frac{1}{2}} \qquad (46)$$

$$\|P\| = (P^T \tilde{\gamma} P)^{\frac{1}{2}}. \tag{47}$$

Normed elements are defined as elements whose norms are unity.

Once an Absolute is established, it is possible to remove the restriction from homogeneous coordinates that only the ratios are significant. This transition from projective homogeneous coordinates to metrical homogeneous coordinates is initiated by first normalizing all elements. For example, the projective homogeneous coordinates of a point x are normalized by

$$\frac{x}{\|x\|} = \frac{x}{(x^T \tilde{\Pi} x)^{\frac{1}{2}}} . \tag{48}$$

Replacing x with a scalar multiple λx, which also designates the same point in projective coordinates, yields

$$\frac{\lambda x}{\|\lambda x\|} = \frac{\lambda x}{\|\lambda\| \cdot \|x\|} = \pm \frac{x}{\|x\|} \tag{49}$$

where the norm of the scalar is

$$\|\lambda\| = (\lambda^2)^{\frac{1}{2}}. \tag{50}$$

Therefore in general, the ∞^1 sets of projective homogeneous coordinates which correspond to a single element and differ by a nonzero scalar factor, are transformed by the normalization into two sets of homogeneous metrical coordinates which designate the same element yet differ in sign. By including

further constraints, it is possible to resolve this ambiguity of signs completely for hyperbolic geometry, only partially for Euclidean geometry and not at all for elliptic geometry, Busemann and Kelly [1953]. For the case of elliptic geometry, this situation has interesting consequences which are detailed in Section 3.2 together with the Euclidean case in Section 3.3.

It is convenient to sometimes refer to elements of projective space, i.e. points, planes, lines and screws, as unit or unweighted elements since in terms of metrical coordinates they have unity norms. Since all the elements of projective space have a representation in terms of normalized metrical coordinates, it is reasonable to assign a new meaning to metrical coordinates with a nonunity norm. Such coordinates are said to represent weighted elements, Forder [1940], which are purely metrical in nature, i.e. they have no projective representation and are thus a new species of space element. Every weighted element may be represented as a scalar multiple of an unweighted element, e.g. for a weighted point

$$x = \|x\| \cdot \frac{x}{\|x\|} \tag{51}$$

where the weight is simply the norm. Oftentimes the terms weight and magnitude are used synonymously. A common physical example of a weighted element is a point mass.

Metrical collineations have already been defined as a subgroup of collineations which preserve the form of the Absolute and have been formulated using projective coordinates in (36), (37), (42), (43). However, these formulations only define collineations uniquely to a scalar multiple and are thus not suitable when it is desired to employ metrical coordinates with associated weights. For this purpose, it is required that a collineation also preserves the norms of weighted elements. Letting K and k be respectively point and plane collineations, then it is required that the relations

$$K^T \tilde{\Pi} \; K = \tilde{\Pi} \tag{52}$$

$$k^T \tilde{\pi} \; k = \tilde{\pi} \tag{53}$$

be satisfied identically, not just to a scalar factor.

For distinctiveness, a metrical collineation which preserves the norm may be referred to as a unit or unweighted collineation or if the context is clear, simply as a collineation. Weighted collineations do not preserve the norm and are not considered here further. Since a unit collineation may not be multiplied by an arbitrary nonzero scalar factor, it represents 16 parameters. Further, because (52) and (53) are symmetrical, either relation represents 10 independent scalar equations and thus, a metrical collineation is specified by 16-10=6 independent parameters, a result that agrees with the previous one employing projective coordinates.

For the induced unit collineations of lines and screws, let $\hat{K}$ and $\hat{k}$ be respectively the ray and axis transformations then it is required that the relations

$$\hat{K}^T \tilde{\Gamma} \hat{K} = \tilde{\Gamma} \tag{54}$$

$$\hat{k}^T \tilde{\gamma} \hat{k} = \tilde{\gamma} \tag{55}$$

be satisfied identically, not just to a scalar factor. It may be demonstrated that the collineations induced from unit collineations are also unit collineations. First, taking the determinants of (52) - (54) yields

$$|K|^2 = 1 \quad , \quad |k|^2 = 1 \tag{56}$$

$$|\hat{K}|^2 = 1 \quad , \quad |\hat{k}|^2 = 1 \tag{57}$$

which are the necessary and sufficient conditions for the collineations to be unweighted. Using (2.3.2) and (2.3.4), the identical relations (2.3.31) and (2.3.32) may be expressed as

$$\hat{K}^T \tilde{\Delta} \hat{K} = |K| \tilde{\Delta} \tag{58}$$

$$\hat{k}^T \tilde{\Delta} \hat{k} = |k| \tilde{\Delta} \tag{59}$$

and taking the determinants of these equations yields

$$|\hat{K}|^2 = |K|^6 \tag{60}$$

$$|\hat{k}|^2 = |k|^6 \; . \tag{61}$$

Substituting (56) in (60), (61) yields the desired relations (57).

The properties of hyperbolic, Euclidean and elliptic space vary considerably and the preceding general analysis only uncovers relations which are common to all. In the following two sections, specific properties are detailed which, in particular, distinguish elliptic and Euclidean geometries.

Section 3.2 Elliptic Geometry

> Consider any vertical line, and a series of horizontal planes cutting it at right angles. In ordinary or Euclidean geometry these planes intersect on the horizon, which is a straight line infinitely distant. In the geometry of a space of constant positive curvature, or elliptic geometry, the horizon is at a certain finite distance in all directions from the vertical line with which we started; it belongs to that particular line, which is called its polar, and is not the same for all vertical lines. Although it appears to be a great circle when viewed from the neighborhood of its polar, yet if we were to go to it and examine it we should find it straight. Points of it which are in opposite directions from a point on the polar are really identical; and every straight line in this space resembles a circle in being of finite length, so that if we travel far enough along it we shall arrive at our starting point. Every straight line has a polar line, which is the intersection of all planes at right angles to it.
>
> Let us take a very small circle on a sphere, and suppose it to expand, keeping always the same centre. At the beginning the circle will be concave inside and convex outside; but when the expansion has gone on far enough it will become a great circle of the sphere, which is of the same shape on both sides, or is straight so far as the surface of the sphere is concerned. So if in Euclidian space we take a sphere and suppose it to expand, keeping always the same centre, it will continue to be concave inside and convex outside so long as it is finite; but

> when the radius has become infinite, the inside in one direction is the same as the outside in the opposite direction, opposite points being identical; thus the sphere is of the same shape on both sides, or is a plane, viz., the plane at infinity. In elliptic space, just as in geometry on the surface of a sphere, this takes place for a finite length of the radius, not for an infinite length; for every point there is a sphere having its centre at that point, which is also a plane. Or, which is the same thing, every point has a polar plane which is the locus of all points situate at a certain distance from it; this distance is called a quadrant. So also every plane has a certain point, called its pole, which is distant a quadrant from every point in the plane. All lines and planes perpendicular to the plane pass through its pole, and conversely. The polar lines of all lines in the plane pass through its pole, and so do the polar planes of all points in the plane.
>
> When two lines are polars of one another, every point of one is distant a quadrant from every point of the other; hence the polar planes of all points on one pass through the other. Every line which is at right angles to one meets the other, and conversely. W.K. Clifford [1876, pp. 390-391]

In relation to hyperbolic and Euclidean geometries, elliptic geometry has the simplest and most symmetrical properties. The results of the previous section are recounted for elliptic geometry by setting $\varepsilon=1$.

The elliptic polarity for points and planes is

$$\tilde{\Pi}_1 = \tilde{I}_4 \quad (1)$$

$$\tilde{\pi}_1 = \tilde{I}_4 \quad (2)$$

and the polar relations are given by

$$x' = \tilde{\Pi}_1 x = \tilde{I}_4 x \quad (3)$$

$$x' = \tilde{\pi}_1 X = \tilde{I}_4 X. \tag{4}$$

Two points or planes which are elliptic conjugates satisfy respectively

$$x^T \tilde{\Pi}_1 y = x^T \tilde{I}_4 y = 0 \tag{5}$$

$$X^T \tilde{\pi}_1 Y = X^T \tilde{I}_4 Y = 0. \tag{6}$$

The locus of self-conjugate points and the envelope of self-conjugate planes define the elliptic Absolute

$$x^T \tilde{\Pi}_1 x = x^T \tilde{I}_4 x = 0 \tag{7}$$

$$X^T \tilde{\pi}_1 X = X^T \tilde{I}_4 X = 0. \tag{8}$$

For ray and axis coordinates, the induced elliptic polarity is given by

$$\tilde{\Gamma}_1 = \tilde{I}_6 \tag{9}$$

$$\tilde{\gamma}_1 = \tilde{I}_6 \tag{10}$$

and the polar relations are

$$P' = \tilde{\Gamma}_1 p = \tilde{I}_6 p \tag{11}$$

$$p' = \tilde{\gamma}_1 P = \tilde{I}_6 P. \tag{12}$$

Two lines which are elliptic conjugates satisfy

$$p^T \tilde{\Gamma}_1 q = p^T \tilde{I}_6 q = 0 \tag{13}$$

$$P^T \tilde{\gamma}_1 Q = P^T \tilde{I}_6 Q = 0 \tag{14}$$

and the line equations of the elliptic Absolute are

$$p^T \tilde{\Gamma}_1 p = p^T \tilde{I}_6 p = 0 \tag{15}$$

$$P^T \tilde{\gamma}_1 P = P^T \tilde{I}_6 P = 0. \tag{16}$$

By linearity, the elliptic polarity of screws is given by

$$P' = \lambda_1 \tilde{\Gamma}_1 q + \ldots \lambda_n \tilde{\Gamma}_1 r = \lambda_1 \tilde{I}_6 q + \ldots \lambda_n \tilde{I}_6 r \tag{17}$$

$$p' = \lambda_1 \tilde{\gamma}_1 Q + \ldots \lambda_n \tilde{\gamma}_1 R = \lambda_1 \tilde{I}_6 Q + \ldots \lambda_n \tilde{I}_6 R \tag{18}$$

and two screws which satisfy (13) or (14) are also said to be elliptic conjugates.

For points, planes, lines and screws the elliptic polarity is specified by a correlation which is the identity matrix, either $\tilde{I}_4$ in (1), (2) or $\tilde{I}_6$ in (9), (10). Thus, the elliptic polar of an element is determined by interpreting its coordinates in terms of dual coordinates, (3), (4), (11), (12), (17), (18). Two elements which are elliptic conjugates (5), (6), (13), (14), are also said to be either elliptic-orthogonal or more specifically prefixed by the polarity such as for points, $\tilde{\Pi}_1$-orthogonal or $\tilde{I}_4$-orthogonal. From the equations of the Absolute, (7), (8), (15), (16) it is clear that the points, planes and lines which form the

Absolute are not real and thus it is often referred to as an imaginary or virtual quadric surface. The elements forming the Absolute are said to be elliptic self-conjugate or alternatively, elliptic-isotropic.

For lines and screws, it is very important to carefully distinguish projective relations from the metrical elliptic relations which are similar in appearance due to the elementary form of the elliptic polarity. The projective correlation between ray coordinates and axis coordinates

$$P = \tilde{\Delta}p \qquad (2.2.60)$$

$$p = \tilde{\Delta}P \qquad (2.2.61)$$

may be substituted in (11), (12) to express the elliptic polarity as a collineation

$$P' = \tilde{I}_6\, p = \tilde{I}_6\, \tilde{\Delta}\, P = \hat{\Delta}P \qquad (19)$$

$$p' = \tilde{I}_6\, P = \tilde{I}_6\, \tilde{\Delta}\, p = \hat{\Delta}p \qquad (20)$$

where the product of correlations $\tilde{I}_6\tilde{\Delta}$ yields the collineation

$$\tilde{\Delta} = \begin{bmatrix} \cdot & I_3 \\ I_3 & \cdot \end{bmatrix}. \qquad (21)$$

Further, substituting (2.2.60) and (2.2.61) in (13), (14) yields alternative formulations for elliptic-orthogonality,

$$p^T\, \tilde{I}_6\, q = P^T\, \tilde{\Delta}\, \tilde{I}_6\, q = P^T\, \hat{\Delta}\, q = 0 \qquad (22)$$

$$P^T\, \tilde{I}_6\, Q = p^T\, \tilde{\Delta}\, \tilde{I}_6\, Q = p^T\, \hat{\Delta}\, Q = 0. \qquad (23)$$

Table 3.2.1 summarizes the projective relations along side the elliptic relations which are very similar in form. As throughout, a tilda is used to signify that a transformation is a correlation, e.g. $\tilde{I}_6$, whereas a caret denotes a collineation, e.g. $\hat{I}_6$. In the table, each expression contains either a correlation or a collineation and its correspondent contains the other. Further, for each pair of expressions, one of the two screws (or lines) which correspond are in dual coordinates while the other two are in the same coordinates. Without a clear notational distinction between correlations and collineations and between ray and axis coordinates these relations are easily confused and misinterpreted.

It was first discovered by Clifford [1873], that in elliptic space a screw p can in general be expressed as a linear combination of a unique line q and its polar q',

$$p = \lambda q + \lambda' q' \qquad (3.1.25)$$

both of which are called the axes of the screw. The ratios $\lambda':\lambda$ and $\lambda:\lambda'$ are respectively the pitches of the screw with respect to q and q'. Setting $\varepsilon=1$ in the general formulations (3.1.27), (3.1.28) and (3.1.30) yields

$$p = (\lambda\hat{I}_6 + \lambda'\hat{\Delta})q \qquad (24)$$

Table 3.2.1 Projective and metrical relations that are similar in appearance

Projective	Metrical
1. Transformations of ray and axis coordinates	1. Transformations of the elliptic polarity
$P = \tilde{\Delta} p$	$p' = \hat{\Delta} p$
$p = \tilde{\Delta} P$	$P' = \hat{\Delta} P$
$p = \hat{I}_6\ p$	$P' = \tilde{I}_6\ p$
$P = \hat{I}_6\ P$	$p' = \tilde{I}_6\ P$
2. Reciprocity	2. Elliptic-orthogonality
$p^T\ \tilde{\Delta}\ q = 0$	$p^T\ \hat{\Delta}\ q = 0$
$p^T\ \tilde{\Delta}\ Q = 0$	$p^T\ \hat{\Delta}\ Q = 0$
$p^T\ \hat{I}_6\ Q = 0$	$p^T\ \tilde{I}_6\ Q = 0$
$p^T\ \hat{I}_6\ q = 0$	$p^T\ \tilde{I}_6\ q = 0$

$$q = \frac{1}{(\lambda^2 - \lambda'^2)}\ [\lambda \hat{I}_6 - \lambda' \hat{\Delta}]p \tag{25}$$

$$\frac{1}{(\lambda^2 - \lambda'^2)}\ [(p^T \tilde{\Delta}\, p)\lambda^2 - 2(p^T\ \tilde{I}_6\ p)\lambda\lambda' + (p^T \tilde{\Delta}\, p)\lambda'^2] = 0. \tag{26}$$

Assuming first that $\lambda^2 \neq \lambda'^2$ and $p^T \tilde{\Delta}\, p \neq 0$ then (26) may be expressed by

$$\lambda^2 - 2b\lambda\lambda' + \lambda'^2 = 0 \tag{27}$$

where

$$b = \frac{p^T\ \tilde{I}_6\ p}{p^T\ \tilde{\Delta}\ p}\ . \tag{28}$$

The solutions to (27) are

$$\frac{\lambda'}{\lambda} = b + (b^2 - 1)^{\frac{1}{2}}\ ,\ b - (b^2 - 1)^{\frac{1}{2}} \tag{29}$$

or equivalently

$$\frac{\lambda'}{\lambda} = \mu,\ \frac{1}{\mu} \tag{30}$$

where

$$\mu = b + (b^2 - 1)^{\frac{1}{2}}\ . \tag{31}$$

Substituting in turn the roots (30) back in (25) yields the corresponding pair of axes

$$\lambda q_\mu = \frac{1}{(1-\mu^2)}\ [\hat{I}_6 - \mu\hat{\Delta}]p \tag{32}$$

$$\lambda q_{1/\mu} = \frac{\mu\hat{\Delta}}{(1-\mu^2)} [\hat{I}_6 - \mu\hat{\Delta}]p \qquad (33)$$

and since

$$(\mu\hat{\Delta})q_\mu = q_{1/\mu} \qquad (34)$$

the two axes are elliptic polars where the subscripts denote the associated pitches.

When $p^T \tilde{\Delta} p = 0$, p is a line and assuming that $p^T \tilde{I}_6 p \neq 0$ in (26), then either $\lambda=0$ or $\lambda'=0$ and thus from (3.1.25), respectively either $p=\lambda' q'$ or $p=\lambda q$.

The exceptional case occurs when $\lambda^2=\lambda'^2$ and thus the pitch of the screw is either +1 or -1. The form of this screw p may be determined by first taking the polar of (3.1.25),

$$p' = \lambda q' + \lambda' q \qquad (35)$$

and then subtracting and adding in turn (3.1.25) and (35) yields

$$(p'-p) = (\lambda-\lambda')(q'-q) \qquad (36)$$

$$(p'+p) = (\lambda+\lambda')(q'+q). \qquad (37)$$

When the pitch is +1 then (36) yields

$$p=p' \quad \text{or} \quad p=\tilde{\Delta}p \qquad (38)$$

and thus the first and last three components of p are the same. When the pitch is -1 then (37) yields

$$p = -p' \quad \text{or } p = -\hat{\Delta}p$$

and the first and last three components are the same except that they have opposite signs. As a consequence of these special forms, screws of pitch +1 may be expressed in an infinite number of ways as the sum of a line and its polar. The axes of such screws are thus indeterminant and for this reason Clifford referred to them as vectors, a right vector when the pitch is +1 and a left vector when the pitch is -1. Clifford also made the observation that a screw p, of any pitch, may be expressed as the sum of a right vector and left vector uniquely,

$$p = \frac{(p+p')}{2} + \frac{(p-p')}{2}. \qquad (39)$$

Vectors are closely related to right and left parallel lines in elliptic space which are also called paratactics, Sommerville [1929], or Clifford parallels, but are not developed here.

The transition from projective coordinates to elliptic metrical coordinates is made by first introducing elliptic norms. By setting $\varepsilon=1$ in (3.1.44) - (3.1.47), the elliptic norms for points, planes and lines or screws become

$$\|x\|_1 = (x^T \tilde{I}_4 x)^{\frac{1}{2}} \qquad (40)$$

$$\|X\|_1 = (X^T \tilde{I}_4 X)^{1/2} \tag{41}$$

$$\|p\|_1 = (p^T \tilde{I}_6 p)^{1/2} \tag{42}$$

$$\|P\|_1 = (P^T \tilde{I}_6 P)^{1/2}. \tag{43}$$

As discussed in Section 3.1, by normalizing projective coordinates, elements in projective space become unit or unweighted elements in terms of metrical coordinates. In general, an unweighted element has two representations in metrical coordinates which differ only in sign. For elliptic space, there is no additional criteria that can be imposed which enables a definitive selection of sign without ambiguity. The resolution to this predicament is merely to associate the pair of coordinates with each element, e.g. x and -x for a unit point. For the elliptic plane this has an interesting consequence since it enables modeling in Euclidean space by a unit sphere. That is, a point on the elliptic plane is modeled either by a pair antipodal points on a unit sphere or equivalently by a diametrical line through these points. A line on the elliptic plane is represented by a great circle. The analogy in space is somewhat more involved, however Clifford's description given at the beginning of this section is highly suggestive.

Elliptic collineations are defined as the subgroup of projective collineations which leave the elliptic Absolute invariant. In terms of metrical coordinates, the unweighted collineations for points, planes and lines or screws satisfy

the relations given by setting $\varepsilon=1$ in (3.1.52) - (3.1.55),

$$K^T \tilde{I}_4 K = \tilde{I}_4 \tag{44}$$

$$k^T \tilde{I}_4 k = \tilde{I}_4 \tag{45}$$

$$\hat{K}^T \tilde{I}_6 \hat{K} = \tilde{I}_6 \tag{46}$$

$$\hat{k}^T \tilde{I}_6 \hat{k} = \tilde{I}_6. \tag{47}$$

Analogous to the convention for elements, a collineation has two distinct representations which differ by sign, e.g. K and -K for point collineations, see Busemann and Kelly [1953]. From the above relations it is easy to deduce that the determinant of an elliptic collineation is +1 or -1. However, both representations of a collineation have the same determinant since the matrices are of an even order, e.g. $|-K| = (-1)^4|K| = |K|$. Further, it is readily deduced from (44) - (47) that the inverse of an elliptic collineation is equal to its transpose, e.g. $K^{-1} = K^T$. In the literature such matrices are usually referred to as orthogonal or orthonormal. However, for distinctiveness elliptic collineations are referred to here as elliptic-orthogonal or prefixed with the polarity such as $\tilde{I}_4$-orthogonal.

Section 3.3 <u>Euclidean Geometry</u>

> We shall find throughout this period, that almost every important proposition, though misleading in its obvious interpretation, has nevertheless, when rightly interpreted, a wide philosophical bearing. So it is with the work of Cayley, the pioneer of the projective method.

The projective formula for angles, in Euclidean Geometry, was first obtained by Laguerre, in 1853. This formula had, however, a perfectly Euclidean character, and it was left for Cayley to generalize it so as to include both angles and distances in Euclidean and non-Euclidean systems alike.

Cayley was, to the last, a staunch supporter of Euclidean space, though he believed that non-Euclidean Geometries could be applied, within Euclidean space, by a change in the definition of distance. He has thus, in spite of his Euclidean orthodoxy, provided the believers in the possibility of non-Euclidean spaces with one of their most powerful weapons. In his "Sixth Memoir upon Quantics" (1859), he set himself the task of "establishing the notion of distance upon purely descriptive principles." He showed that, with the ordinary notion of distance, it can be rendered projective by reference to the circular points and the line at infinity, and that the same is true of angles. Not content with this, he suggested a new definition of distance, as the inverse sine or cosine of a certain function of the coordinates; with this definition, the properties usually known as metrical become projective properties, having reference to a certain conic, called by Cayley the Absolute. (The circular points are, analytically, a degenerate conic, so that ordinary Geometry forms a particular case of the above.) He proves that, when the Absolute is an imaginary conic, the Geometry so obtained for two dimensions is spherical Geometry. The correspondence with Lobatchewsky, in the case where the Absolute is real, is not worked out; indeed there is, throughout, no evidence of acquaintance with non-Euclidean systems. The importance of the memoir, to Cayley, lies entirely in its proof that metrical is only a branch of descriptive Geometry.

The connection of Cayley's Theory of Distance with Metageometry was first pointed out by Klein. Klein showed in detail that, if the Absolute be real, we get Lobatchewsky's (hyperbolic) system; if it be imaginary, we get either spherical Geometry or a new system, analogous to that of Helmholtz, called by Klein elliptic; if the Absolute be an imaginary point-pair, we get parabolic Geometry, and if, in particular, the point-pair be the circular points, we get ordinary Euclid. . . .

Since these systems are all obtained from a Euclidean plane, by a mere alteration in the definition of distance, Cayley and Klein tend to regard the whole question as one, not of the nature of space, but of the definition of distance. Since this definition, on their view, is perfectly arbitrary, the philosophical problem vanishes--Euclidean space is left in undisputed possession, and the only problem remaining is one of convention and mathematical convenience. Bertrand Russell [1897, pp. 29-30]

In contrast to elliptic space, for which the elementary form of the polarity yields many symmetrical relations amongst dual elements, the singular nature of the Euclidean Absolute yields highly unsymmetrical dualistic properties. As a consequence, various general formulas which may be specialized for elliptic and hyperbolic geometry, do not appear applicable to Euclidean geometry unless they are treated as limiting cases where $\varepsilon \to 0$. One example is the formula for the distance between two points which yields an indeterminant result unless an infinitesmal device is utilized as in Klein [1908]. However, for most of the developments here it is possible to set $\varepsilon = 0$ in the general formulations of Section 3.1.

The Euclidean polarity for points and planes is given by

$$\tilde{\Pi}_0 = \begin{bmatrix} I_1 & \cdot \\ \cdot & \cdot \end{bmatrix} \quad (1)$$

$$\tilde{\pi}_0 = \begin{bmatrix} \cdot & \cdot \\ \cdot & I_3 \end{bmatrix} . \quad (2)$$

(As throughout, zero elements in an array are often denoted by periods and I_1 is used here in place of 1 for greater symmetry.) The polar relations are

$$X' = \tilde{\Pi}_0 x = \begin{bmatrix} I_1 & \cdot \\ \cdot & \cdot \end{bmatrix} x = \begin{bmatrix} x_0 \\ \cdot \end{bmatrix} \quad (3)$$

$$x' = \tilde{\pi}_0 X = \begin{bmatrix} \cdot & \cdot \\ \cdot & I_3 \end{bmatrix} X = \begin{bmatrix} \cdot \\ \underline{X} \end{bmatrix} \quad (4)$$

where the notation $\underline{X}$ and $\underline{x}$ is introduced,

$$\underline{X} = [X_1 \; X_2 \; X_3]^T \quad (5)$$

$$\underline{x} = [x_1 \; x_2 \; x_3]^T. \quad (6)$$

From (3), for a point x not on the plane at infinity, $x \neq [0 \; x_1 \; x_2 \; x_3]^T$, its polar plane X' is in fact the plane at infinity, $X' = [x_0 \; 0 \; 0 \; 0]^T$. For a point x on the plane at infinity, $x = [0 \; x_1 \; x_2 \; x_3]^T$, its polar plane X' is indeterminate since all of its components vanish, $X' = [0 \; 0 \; 0 \; 0]^T$.

From (4), for a plane X which itself is not the plane at infinity, $X \neq [X_0 \; 0 \; 0 \; 0]^T$, then its pole x' is a point on the plane at infinity, $x' = [0 \; X_1 \; X_2 \; X_3]^T$. For X itself being the plane at infinity, $X = [X_0 \; 0 \; 0 \; 0]^T$, its pole x' is indeterminate $x' = [0 \; 0 \; 0 \; 0]^T$.

Two points or planes which are Euclidean conjugates satisfy respectively,

$$x^T \tilde{\Pi}_0 \, y = x^T \begin{bmatrix} I_1 & \cdot \\ \cdot & \cdot \end{bmatrix} y = x_0 \, y_0 = 0 \quad (7)$$

$$X^T \tilde{\pi}_0 \, Y = X^T \begin{bmatrix} \cdot & \cdot \\ \cdot & I_3 \end{bmatrix} Y = \underline{X}^T \underline{Y} = 0. \quad (8)$$

In (7) two points are Euclidean conjugates if either lies on the plane at infinity. In (8), two planes which are Euclidean conjugates are said to be Euclidean-orthogonal and thus meet at right angles.

The angles of self-conjugate points and the envelope of self-conjugate planes define the Euclidean Absolute

$$x^T \tilde{\Pi}_0 \, x = x^T \begin{bmatrix} I_1 & \cdot \\ \cdot & \cdot \end{bmatrix} x = x_0^2 = 0 \quad (9)$$

$$X^T \tilde{\pi}_0 \, X = X^T \begin{bmatrix} \cdot & \cdot \\ \cdot & I_3 \end{bmatrix} X = \underline{X}^T \underline{X} = X_1^2 + X_2^2 + X_3^2. \quad (10)$$

Equation (9) represents the plane at infinity taken twice which is a rank one quadric locus. Equation (10) represents what is referred to as the imaginary spherical circle since it is the intersection of every sphere with the plane at infinity and further, because it is a rank three quadric envelope it represents a conic, see Klein [1908], Sommerville [1934]. Thus unlike elliptic geometry, the locus and envelope of the Euclidean Absolute are two distinct figures which is a consequence of the Euclidean polarity being singular.

In terms of ray and axis coordinates, the induced Euclidean polarity is respectively given by,

$$\tilde{\Gamma}_0 = \begin{bmatrix} I_3 & \cdot \\ \cdot & \cdot \end{bmatrix} \qquad (11)$$

$$\tilde{\gamma}_0 = \begin{bmatrix} \cdot & \cdot \\ \cdot & I_3 \end{bmatrix} \qquad (12)$$

and the corresponding polar relations are

$$p' = \tilde{\Gamma}_0 \; p = \begin{bmatrix} I_3 & \cdot \\ \cdot & \cdot \end{bmatrix} \begin{bmatrix} \underline{p} \\ \underline{p}_0 \end{bmatrix} = \begin{bmatrix} \underline{p} \\ \underline{0} \end{bmatrix} \qquad (13)$$

$$p' = \tilde{\gamma}_0 \; P = \begin{bmatrix} \cdot & \cdot \\ \cdot & I_3 \end{bmatrix} \begin{bmatrix} \underline{P}_0 \\ \underline{P} \end{bmatrix} = \begin{bmatrix} \underline{0} \\ \underline{P} \end{bmatrix} \qquad (14)$$

where the 3×1 arrays $\underline{p}$, $\underline{p}_0$, $\underline{P}$, $\underline{P}_0$, $\underline{0}$ are introduced,

$$\underline{p} = [p_{01}\; p_{02}\; p_{03}]^T \;,\; \underline{p}_0 = [p_{23}\; p_{31}\; p_{12}]^T \qquad (15)$$

$$\underline{P} = [P_{23}\; P_{31}\; P_{12}]^T \;,\; \underline{P}_0 = [P_{01}\; P_{02}\; P_{03}]^T \qquad (16)$$

$$\underline{0} = [0\;\; 0\;\; 0]^T. \qquad (17)$$

Considering firstly (14), it may be shown that the polar of any line P not on the plane at infinity is in fact a line on the plane at infinity, $p' = [0\; 0\; 0\; P_{23}\; P_{31}\; P_{12}]^T$. Let the line P be the meet of two finite planes, $P = |XY|$. The polar line p' is equivalent to the join of the two poles of X and Y which are points on the plane at infinity, $p' = |X'\; Y'|$ and thus represents a line on the plane at infinity.

This type of reasoning is not effective for (13). Let line p be the join of two points $p = |xy|$ and the polars of x and y are both the plane at infinity. Since the meet of any plane with itself vanishes identically, this does not yield the line given by (13). It is not clear why this synthetic argument fails. However, it should be noted that (13) was derived from (3.1.13) where it was necessary to delete a common factor ε before ε was set to zero to yield (13). Alternatively, (13) may be derived directly from (14) be multiplying throughout by $\tilde{\Delta}$,

$$\tilde{\Delta} p' = \tilde{\Delta} \begin{bmatrix} \underline{0} \\ \underline{P} \end{bmatrix} \qquad (18)$$

which yields the desired result

$$p' = \begin{bmatrix} \underline{p} \\ \underline{0} \end{bmatrix}. \qquad (19)$$

It also follows from both (13) and (14) that the polar of any line on the plane at infinity is indeterminate since its coordinates vanish. Thus the Euclidean polar of the Euclidean polar of a line or screw vanishes. This is exactly equivalent to a double application of the Euclidean polar operator ω, where $\omega^2=0$, to a rotor or motor (line or screw) and was invented by Clifford [1873] in the development of biquaternions. For elliptic space, Clifford used the polar operator ω, where $\omega^2=1$, which is equivalent to (3.2.19) and (3.2.20). Clifford did not discuss hyperbolic geometry in this context but the corresponding polar operator has the property $\omega^2 = -1$. An elegant development of biquaternions and the three related polar operators is given by Veblen and Young [1917].

Two lines or screws which are Euclidean conjugates satisfy

$$p^T \tilde{\Gamma}_0 q = p^T \begin{bmatrix} I_3 & \cdot \\ \cdot & \cdot \end{bmatrix} q = \underline{p}^T \underline{q} = 0 \qquad (20)$$

$$P^T \tilde{\gamma}_0 Q = P^T \begin{bmatrix} \cdot & \cdot \\ \cdot & I_3 \end{bmatrix} Q = \underline{P}^T \underline{Q} = 0 \qquad (21)$$

and are said to be Euclidean-orthogonal and are at right angles.

The line envelopes of the Euclidean Absolute are

$$p^T \tilde{\Gamma}_0 p = p^T \begin{bmatrix} I_3 & \cdot \\ \cdot & \cdot \end{bmatrix} p = \underline{p}^T \underline{p} = 0 \qquad (22)$$

$$P^T \tilde{\gamma}_0 P = P^T \begin{bmatrix} \cdot & \cdot \\ \cdot & I_3 \end{bmatrix} P = \underline{P}^T \underline{P} = 0 \qquad (23)$$

which both represent the plane at infinity taken twice and the imaginary spherical circle. Equations (22) and (23) may be shown equivalent by using the transformation between ray and axis coordinates and (3.1.16),

$$\begin{aligned} p^T \tilde{\Gamma}_0 p &= (\tilde{\Delta} P)^T \tilde{\Gamma}_0 (\tilde{\Delta} P) \\ &= P^T (\tilde{\Delta} \tilde{\Gamma}_0 \tilde{\Delta}) P \\ &= P^T \tilde{\gamma}_0 P. \end{aligned} \qquad (24)$$

In general, the points, planes and lines which form the Absolute are said to be Euclidean self-conjugate or alternately Euclidean isotropic.

By linear principles, the Euclidean polarity of screws may be expressed by

$$p' = \lambda_1 \tilde{\Gamma}_0 q + \ldots \lambda_n \tilde{\Gamma}_0 r \qquad (25)$$

$$p' = \lambda_1 \tilde{\gamma}_0 Q + \ldots \lambda_n \tilde{\gamma}_0 R. \qquad (26)$$

In Euclidean space, the pitch of a screw is determined by expressing it as a linear combination of a unique line and its polar.

$$p = \lambda q + \lambda' q' \qquad (3.1.25)$$

where the ratio $\lambda':\lambda$ is the pitch of the screw. Setting $\varepsilon=0$ in the general formulation (3.1.30) yields

$$\frac{1}{\lambda^3} [(p^T \tilde{\Delta} p)\lambda - 2(p^T \tilde{\Gamma}_0 p)\lambda'] = 0. \qquad (27)$$

For $\lambda \neq 0$, the pitch is unique and is given by

$$h = \frac{\lambda'}{\lambda} = \frac{1}{2} \frac{p^T \tilde{\Delta} p}{p^T \tilde{\Gamma}_0 p}. \qquad (28)$$

The corresponding axis of the screw is also unique and is given by substituting $\varepsilon=0$ in (3.1.28)

$$\lambda q = \begin{bmatrix} I_3 & \cdot \\ -hI_3 & I_3 \end{bmatrix} p \qquad (29)$$

which is equivalent to

$$\lambda q = \begin{bmatrix} \underline{p} \\ \underline{p}_0 - h\underline{p} \end{bmatrix}. \tag{30}$$

When $\lambda=0$, then it is necessary to use (3.1.27) with $\varepsilon=0$ to yield

$$p = \begin{bmatrix} \underline{p} \\ \underline{p}_0 \end{bmatrix} = \lambda' \begin{bmatrix} \underline{0} \\ \underline{q} \end{bmatrix}, \ \lambda' \neq 0 \tag{31}$$

and thus p is a line on the plane at infinity and has infinite pitch

$$h = \frac{\lambda'}{\lambda} = \frac{\lambda'}{0} = \infty. \tag{32}$$

In terms of axis coordinates, for $h\neq\infty$ the pitch is given by,

$$h = \tfrac{1}{2}\frac{P^T \tilde{\Delta} P}{P^T \tilde{\gamma}_0 P} \tag{33}$$

and the axis of the screw is

$$\lambda Q = \begin{bmatrix} \underline{Q}_0 - h\underline{Q} \\ \underline{Q} \end{bmatrix}. \tag{34}$$

When $h=\infty$ then the screw is a line on the plane at infinity and

$$P = \begin{bmatrix} \underline{P}_0 \\ \underline{P} \end{bmatrix} = \lambda' \begin{bmatrix} \underline{Q} \\ \underline{0} \end{bmatrix}, \ \lambda' \neq 0. \tag{35}$$

Euclidean metrical coordinates are developed from projective coordinates by first introducing Euclidean norms. Setting $\varepsilon=0$ in (3.1.44) - (3.1.47) yields the Euclidean norms for points, planes and lines or screws

$$\|x\|_0 = (x^T \tilde{\Pi}_0 x)^{\frac{1}{2}} = (x_0^2)^{\frac{1}{2}} \tag{36}$$

$$\|X\|_0 = (X^T \tilde{\pi}_0 X)^{\frac{1}{2}} = (\underline{X}^T \underline{X})^{\frac{1}{2}} \tag{37}$$

$$\|p\|_0 = (p^T \tilde{\Gamma}_0 p)^{\frac{1}{2}} = (\underline{p}^T \underline{p})^{\frac{1}{2}} \tag{38}$$

$$\|P\|_0 = (P^T \tilde{\gamma}_0 P)^{\frac{1}{2}} = (\underline{P}^T \underline{P})^{\frac{1}{2}}. \tag{39}$$

Projective coordinates are normalized by dividing through with the norm. Elements in projective space become unit or unweighted elements in terms of metrical coordinates. Generally, the normalization process leads to two sets of unweighted coordinates for an element which differ in sign. However for finite points, $x_0\neq 0$, the normalization may be made unique

$$\frac{x}{\sigma(x_0)\,\|x\|} = \frac{x}{x_0} \tag{40}$$

where $\sigma(x_0)$ is the sign of x_0. Thus the first coordinate of a normalized finite point is always positive unity, $x_0=1$.

When a point lies on the plane at infinity, $x_0=0$, it is not possible to normalize it in this manner. It is however possible to adopt a suitable normalization by first noting that a point on the plane at infinity may be regarded as the pole of a finite plane

$$x' = \tilde{\pi}_0 X. \tag{41}$$

Therefore, it is useful to define an unweighted point on the plane at infinity as the pole of an unweighted plane

$$\tilde{\pi}_0 \frac{X}{\|X\|} = \frac{\tilde{\pi}_0 X}{(X^T \tilde{\pi}_0 X)^{\frac{1}{2}}} = \frac{\tilde{\pi}_0 X}{((X^T \tilde{\pi}_0)\tilde{\pi}_0(\tilde{\pi}_0 X))^{\frac{1}{2}}} = \frac{x'}{(x'^T \tilde{\pi}_0 x')^{\frac{1}{2}}}$$

$$= \frac{x'}{(\underline{x}'^T \underline{x}')^{\frac{1}{2}}} \tag{42}$$

where

$$(\tilde{\pi}_0)^3 = \tilde{\pi}_0. \tag{43}$$

In distinction to finite points, points on the plane at infinity do not have a unique representation since like plane coordinates, the sign is not determinate.

The only instance for which the plane norm (37) vanishes is for the plane at infinity which is the polar of every finite point. Thus in a normalized form it may be considered to be the polar of any finite unweighted point such that its coordinates are $[1\ 0\ 0\ 0]^T$.

The norm of a line or screw (38), (39) vanishes only for a line on the plane at infinity which is the polar of a finite line or screw,

$$P' = \tilde{\Gamma}_0 P \tag{44}$$

$$p' = \tilde{\gamma}_0 P. \tag{45}$$

Thus an unweighted line on the plane at infinity may be defined as the polar of a finite unweighted line or screw, which in analogy with the result for points (42), is given for axis coordinates by

$$\frac{\tilde{\Gamma}_0 P}{\|P\|} = \frac{P'}{(P'^T \tilde{\Gamma}_0 P')^{\frac{1}{2}}} = \frac{1}{(\underline{P}_0'^T \underline{P}_0')^{\frac{1}{2}}} \begin{bmatrix} \underline{P}_0' \\ \underline{0} \end{bmatrix} \tag{46}$$

and is given for ray coordinates by

$$\frac{\tilde{\gamma}_0 P}{\|P\|} = \frac{p'}{(p'^T \tilde{\gamma}_0 p')^{\frac{1}{2}}} = \frac{1}{(\underline{p}_0'^T \underline{p}_0')^{\frac{1}{2}}} \begin{bmatrix} \underline{0} \\ \underline{p}_0 \end{bmatrix}. \tag{47}$$

Using incidence relations in Section 2.3, the relationship between projective point and plane collineation was deduced as

$$k = \mu K^{-T} \tag{48}$$

where for simplicity, the arbitrary nonzero scalar μ was set to unity, (2.3.9). Unweighted metrical collineation preserve the norms of elements and as such their determinants are ± 1. However, since points in Euclidean space may be assigned a unique representation, it becomes necessary for the relations between K and k to explicitly include the signs of their determinants.

For the unweighted point collineation

$$K = [A\ B\ C\ D]^T\ ,\ |K|^2 = 1 \qquad (49)$$

the corresponding unweighted plane collineation is defined by replacing each element of K by its cofactor

$$k = [|B\ C\ D|\ \ |C\ A\ D|\ \ |A\ B\ D|\ \ |B\ A\ C|]^T,\ |k|^2 = 1. \qquad (50)$$

Using the definition of a matrix inverse in terms of cofactors then

$$k = |K|K^{-T}\ ,\ |K|^2 = 1. \qquad (51)$$

For $|K| = -1$, this result differs from (48) with $\mu=1$. Taking the determinant of (51) yields

$$|k| = |K|^3 = |K| \qquad (52)$$

and then inverting (51) using (52) gives

$$K = |k|k^{-T}\ ,\ |k|^2 = 1. \qquad (53)$$

The unweighted line collineations corresponding to (51) and (53) are derived by forming the ray and axis coordinates for the edges of the corresponding tetrahedron as in Section 2.3 and

$$\hat{k} = |K|^{12}\ \hat{K}^{-T}\ ,\ |K|^2 = 1 \qquad (54)$$

$$\hat{K} = |k|^{12}\ \hat{k}^{-T}\ ,\ |k|^2 = 1 \qquad (55)$$

which are clearly equivalent to the previously used form

$$\hat{k} = \hat{K}^{-T}\ . \qquad (2.2.30)$$

Further, using (52) and (2.2.30) it is easily shown that the identities (2.3.31) - (2.3.35) remain applicable.

Euclidean collineations are defined as the subgroup of projective collineations which leaves the Euclidean Absolute invariant. Using metrical coordinates, the unweighted collineations for points, planes and lines or screws must satisfy the relations (3.1.52) - (3.1.55) with $\varepsilon=0$,

$$K^T\ \tilde{\Pi}_0\ K = \tilde{\Pi}_0\ ,\ K^T \begin{bmatrix} I_1 & \cdot \\ \cdot & \cdot \end{bmatrix} K = \begin{bmatrix} I_1 & \cdot \\ \cdot & \cdot \end{bmatrix} \qquad (56)$$

$$k^T\ \tilde{\pi}_0\ k = \pi_0\ ,\ k^T \begin{bmatrix} \cdot & \cdot \\ \cdot & I_3 \end{bmatrix} k = \begin{bmatrix} \cdot & \cdot \\ \cdot & I_3 \end{bmatrix} \qquad (57)$$

$$\hat{K}\ \tilde{\Gamma}_0\ \hat{K} = \tilde{\Gamma}_0\ ,\ \hat{K}^T \begin{bmatrix} I_3 & \cdot \\ \cdot & \cdot \end{bmatrix} \hat{K} = \begin{bmatrix} I_3 & \cdot \\ \cdot & \cdot \end{bmatrix} \qquad (58)$$

$$\hat{k}\ \tilde{\gamma}_0\ \hat{k} = \tilde{\gamma}_0\ ,\ \hat{k}^T \begin{bmatrix} \cdot & \cdot \\ \cdot & I_3 \end{bmatrix} \hat{k} = \begin{bmatrix} \cdot & \cdot \\ \cdot & I_3 \end{bmatrix}\ . \qquad (59)$$

In order to deduce the form of the point and plane collineations, it is useful to express K as

$$K = \begin{bmatrix} f & g \\ \underline{t} & h \end{bmatrix} \qquad (60)$$

where h is a 3×3 array and $\underline{t}$ is 3×1. Substitution of (60) in (56) yields

$$\begin{bmatrix} I_1 & \cdot \\ \cdot & \cdot \end{bmatrix} = \begin{bmatrix} f^T f & f^T g \\ g^T f & g^T g \end{bmatrix} \qquad (61)$$

for which it is easily determined that

$$I_1 = f^T f \text{ or } f = \pm I_1 \qquad (62)$$

$$0 = g^T g \text{ or } g = 0. \qquad (63)$$

The remaining elements of K are determined by substituting the identity $k = K^{-T}$ in (57) and rearranging to yield

$$\tilde{\pi}_0 = K\, \tilde{\pi}_0\, K^T. \qquad (64)$$

Then substitution of (60) in (64) yields

$$\begin{bmatrix} \cdot & \cdot \\ \cdot & I_3 \end{bmatrix} = \begin{bmatrix} gg^T & gh^T \\ hg^T & hh^T \end{bmatrix}. \qquad (65)$$

from which it is deduced that

$$hh^T = I_3 \text{ or } h^T = h^{-1}. \qquad (66)$$

The orthonormal matrices of (66) may be divided into two distinct sets which are characterized by the determinant being either +1 or -1. Since the matrix is of an odd

degree, a member of one set may be transformed into a member of the other by multiplying with -1 or equivalently $-I_3$. Thus, h may be expressed in the form

$$h = \sigma_3 e = e\sigma_3 \qquad (67)$$

where e is orthonormal with a positive determinant

$$ee^T = I_3\ ,\ |e| = 1 \qquad (68)$$

and where σ_n may assume either of the two values

$$\sigma_n = I_n \text{ or } \sigma_n = -I_n. \qquad (69)$$

Therefore using (62), (63), (66), (67) in (60) yields the form of the Euclidean point collineation

$$K = \begin{bmatrix} I_1 & \cdot \\ \underline{t} & e\sigma_3 \end{bmatrix}. \qquad (70)$$

In (70) the positive value for f in (62) has been selected since it is necessary for K to transform a normalized unweighted point into the same. Additionally, since the equations that determine K are independent of $\underline{t}$, then it is clear that $\underline{t}$ may assume arbitrary values.

It may be easily verified that Euclidean collineations (70), form a group. Further, three distinct subgroups of Euclidean collineations may be identified by expressing (70) in the form

$$K = K_t\, K_e\, K_\sigma$$

$$= \begin{bmatrix} I_1 & \cdot \\ \underline{t} & I_3 \end{bmatrix} \begin{bmatrix} I_1 & \cdot \\ \cdot & e \end{bmatrix} \begin{bmatrix} I_1 & \cdot \\ \cdot & \sigma_3 \end{bmatrix}. \qquad (71)$$

The first subgroup K_t represents the group of Euclidean translations which is in general characterized by the property that no finite points remain fixed (the exception is for $\underline{t} = \underline{0}$ and thus $K_t = I_4$). The second subgroup K_e represents the group of Euclidean rotations which are all characterized by the property that the origin always remains fixed. Additionally, for each rotation there is a unique line which remains pointwise invariant and is called the axis of rotation. Again, the exceptional case is for $K_e = I_4$ in which all lines are pointwise invariant. The third subgroup K_σ consists of only two collineations for which one is the identity $\sigma_3 = I_3$, and the other is a reflection through the origin $\sigma_3 = -I_3$. In such a reflection, spheres centered about the origin remain invariant since antipodal points are mapped into each other.

In Euclidean space, rigid body collineations or equivalently rigid body motions are a subgroup of the Euclidean collineations (63) and consist of only translations K_t and rotations K_e. Reflection through the origin K_σ must be excluded since it is a discontinuous transformation in relation to the identity collineation, see Klein [1908].

The corresponding Euclidean collineation of planes are determined from (71) using the relation between unweighted

point and plane collineations (51) and are tabulated in Table 3.3.1 along with the point collineations.

The induced line or screw collineations for ray coordinates may be determined directly from (70) by forming the six edges of the corresponding tetrahedron as in (2.3.21). Then using various determinant identities, the general induced collineation may be expressed as the product of a translation, a rotation and a reflection through the origin. However, it is preferred to use the factored form of (71) and apply an identity for the product of two general collineations K and J,

$$|KJx\ KJy| = \hat{K}|Jx\ Jy|$$

$$= \hat{K}\,\hat{J}\,|xy|. \qquad (72)$$

Thus applying (72) to (71) yields

$$\hat{K} = \hat{K}_t\,\hat{K}_e\,\hat{K}_\sigma$$

$$= \begin{bmatrix} I_3 & \cdot \\ t^* & I_3 \end{bmatrix} \begin{bmatrix} e & \cdot \\ \cdot & e \end{bmatrix} \begin{bmatrix} I_3 & \cdot \\ \cdot & \sigma_3 \end{bmatrix} \qquad (73)$$

where t* is the skew-symmetric array

$$t^* = \begin{bmatrix} \cdot & -t_3 & t_2 \\ t_3 & \cdot & -t_1 \\ -t_2 & t_1 & \cdot \end{bmatrix}. \qquad (74)$$

Table 3.3.1 Euclidean collineations for point and plane coordinates

$$K = \begin{bmatrix} I_1 & \cdot \\ \underline{t} & e\sigma_3 \end{bmatrix} = \begin{bmatrix} I_1 & \cdot \\ \underline{t} & I_3 \end{bmatrix} \begin{bmatrix} I_1 & \cdot \\ \cdot & e \end{bmatrix} \begin{bmatrix} I_1 & \cdot \\ \cdot & \sigma_3 \end{bmatrix} = K_t \; K_e \; K_\sigma$$

$$K^{-1} = \begin{bmatrix} I_1 & \cdot \\ -\sigma_3 e^T \underline{t} & \sigma_3 e^T \end{bmatrix} = \begin{bmatrix} I_1 & \cdot \\ \cdot & \sigma_3 \end{bmatrix} \begin{bmatrix} I_1 & \cdot \\ \cdot & e^T \end{bmatrix} \begin{bmatrix} I_1 & \cdot \\ -\underline{t} & I_3 \end{bmatrix} = K_\sigma^{-1} \; K_e^{-1} \; K_t^{-1}$$

$$k = \begin{bmatrix} \sigma_1 & -\underline{t}^T e \\ \cdot & e \end{bmatrix} = \begin{bmatrix} I_1 & -\underline{t}^T \\ \cdot & I_3 \end{bmatrix} \begin{bmatrix} I_1 & \cdot \\ \cdot & e \end{bmatrix} \begin{bmatrix} \sigma_1 & \cdot \\ \cdot & I_3 \end{bmatrix} = k_t \; k_e \; k_\sigma$$

$$k^{-1} = \begin{bmatrix} \sigma_1 & \sigma_1 \underline{t}^T \\ \cdot & e^T \end{bmatrix} = \begin{bmatrix} \sigma_1 & \cdot \\ \cdot & I_3 \end{bmatrix} \begin{bmatrix} I_1 & \cdot \\ \cdot & e^T \end{bmatrix} \begin{bmatrix} I_1 & \underline{t}^T \\ \cdot & I_3 \end{bmatrix} = k_\sigma^{-1} \; k_e^{-1} \; k_t^{-1}$$

The induced collineations for axis coordinates may be easily determined using $\hat{k} = \hat{K}^{-T}$, (2.2.30). The Euclidean collineations for the ray and axis coordinates of lines or screws are presented in Table 3.3.2.

For completeness, Euclidean measurement formulas are briefly summarized. It was Cayley who first formulated the measurement of distances and angles on a purely projective basis with his introduction of the Absolute. Later, Klein [1871, 1873] reformulated Cayley's work in terms of the logarithms of a cross ratio which is a function of the Absolute. He also extended the results to include elliptic and hyperbolic geometries.

Since the derivations of the Euclidean measurement formulas are quite lengthy, they are presented without detail. The distance between two unweighted points x and y is given by

$$((x - y)^T \; (x - y))^{\frac{1}{2}} = ((\underline{x} - \underline{y})^T \; (\underline{x} - \underline{y}))^{\frac{1}{2}} \qquad (75)$$

and the angle θ between two unweighted or weighted planes X and Y is defined by

$$\cos\theta = \frac{X^T \; \tilde{\pi}_0 \; Y}{(X^T \; \tilde{\pi}_0 \; X)^{\frac{1}{2}} (Y^T \; \tilde{\pi}_0 \; Y)^{\frac{1}{2}}} = \frac{\underline{X}^T \; \underline{Y}}{(\underline{X}^T \; \underline{X})^{\frac{1}{2}} (\underline{Y}^T \; \underline{Y})^{\frac{1}{2}}} \; . \qquad (76)$$

It may be easily verified that both (75) and (76) are invariant expressions with respect to Euclidean collineations.

Table 3.3.2 Euclidean collineations for ray and axis coordinates

$$\hat{K} = \begin{bmatrix} \bar{e}\sigma_3 & \cdot \\ t^* e\sigma_3 & e \end{bmatrix} = \begin{bmatrix} I_3 & \cdot \\ t^* & I_3 \end{bmatrix} \begin{bmatrix} e & \cdot \\ \cdot & e \end{bmatrix} \begin{bmatrix} \sigma_3 & \cdot \\ \cdot & I_3 \end{bmatrix} = \hat{K}_t \hat{K}_e \hat{K}_\sigma$$

$$\hat{K}^{-1} = \begin{bmatrix} \sigma_3 e^T & \cdot \\ -e^T t^* & e^T \end{bmatrix} = \begin{bmatrix} \sigma_3 & \cdot \\ \cdot & I_3 \end{bmatrix} \begin{bmatrix} e^T & \cdot \\ \cdot & e^T \end{bmatrix} \begin{bmatrix} I_3 & \cdot \\ -t^* & I_3 \end{bmatrix} = \hat{K}_\sigma^{-1} \hat{K}_e^{-1} \hat{K}_t^{-1}$$

$$\hat{k} = \begin{bmatrix} e\sigma_3 & t^* e \\ \cdot & e \end{bmatrix} = \begin{bmatrix} I_3 & t^* \\ \cdot & I_3 \end{bmatrix} \begin{bmatrix} e & \cdot \\ \cdot & e \end{bmatrix} \begin{bmatrix} \sigma_3 & \cdot \\ \cdot & I_3 \end{bmatrix} = \hat{k}_t \hat{k}_e \hat{k}_\sigma$$

$$\hat{k}^{-1} = \begin{bmatrix} \sigma_3 e^T & -\sigma_3 e^T t^* \\ \cdot & e^T \end{bmatrix} = \begin{bmatrix} \sigma_3 & \cdot \\ \cdot & I_3 \end{bmatrix} \begin{bmatrix} e^T & \cdot \\ \cdot & e^T \end{bmatrix} \begin{bmatrix} I_3 & -t^* \\ \cdot & I_3 \end{bmatrix} = \hat{k}_\sigma^{-1} \hat{k}_e^{-1} \hat{k}_t^{-1}$$

Section 3.4 Vector Formulations

> I do not wish to end this discussion without pointing out, with emphasis, that, for our general standpoint, the questions of ordinary vector analysis constitute only a chapter out of a profusion of more general problems. For example, line-bound vectors, restricted plane-magnitudes, screws, and systems of forces are, strictly speaking, not considered in vector analysis. For a real understanding of the operations of vector algebra themselves, however, it is actually necessary to take a broader view of them. Only then does the principle which inheres in them, namely, that of defining geometric magnitudes according to their behavior under the various kinds of transformation of rectangular coordinates, find full expression.
> Felix Klein [1908, pp. 51]

In the previous sections it was shown that the Absolute may be used to establish metrical norms and that the elements of projective space may be assigned a norm of unity. The norm was also used to define new elements which are strictly metrical in nature and are referred to as weighted elements since equivalently, their weight, magnitude or norm is nonunity. Because of the quadratic form of the Absolute, it was demonstrated in general that an element has two sets of coordinates which differ in sign.

For Euclidean space, it was possible to alter the process of normalization such that finite points could be assigned a unique set of coordinates. This was not however possible for planes, lines, screws and points at infinity. The reason for this asymmetry is because the Euclidean Absolute is derived from a singular polarity for which the point locus and plane envelopes are distinct figures. There is however a manner in which the ambiguity of signs may be resolved for new species of space elements which are closely related to the previous ones.

In the development of projective geometry, the point and plane are the fundamental elements of space and since they are considered entirely equivalent in significance, the duality of projective space complete. However in Euclidean space the asymmetry of the Absolute, and consequently the unique representation of points, alters this balance in significance of dual elements such that it becomes proper to elevate the status of the point by designating it as a more fundamental element than the plane. That is, points may be first used to define planes and then in turn they may both be used to define other space elements. Thus all elements are made to depend either directly or indirectly on point definitions.

The ambiguity of signs is resolved by defining the property of directional sense or simply direction. This leads to the establishment of new space elements, namely free vectors, line vectors and screw vectors, which are also called motors. These elements include both the properties of magnitude and direction in their definition.

The approach presented here is to first introduce free vectors, or simply vectors, and to initially describe their properties in terms of the difference of two points. Since the formulation and interpretation of expressions in Euclidean geometry is often facilitated by the introduction of free vectors, they are then subsequently utilized in defining the new space elements. The goal here is not to reproduce ordinary vector analysis, but to demonstrate the

logical evolution of space elements with particular emphasis on line vectors and screw vectors. As such, a familiarity of basic vector concepts in Euclidean geometry is assumed and only the portions necessary for further developments in the sequal are derived. A rather elegant and detailed account of the introduction of vector elements into geometry has been presented by Klein [1908], portions of which are utilized here.

Klein's second principle, which is given at the beginning of Section 2.3, may be applied in the particular form: under a transformation of space, transformed coordinates which can be expressed as a linear homogeneous function of only their previous coordinates, define a geometrical space element or configuration. Using the group of Euclidean collineations given in Table 3.3.1, two new space elements may be initially identified, polar free vectors and axial free vectors. The general results are first presented to emphasize the many interrelationships and a detailed analysis follows. Under the general collineation of Euclidean space, the corresponding collineations of three distinct polar vectors are given by the expressions

$$\underline{w} = e \ \sigma_3 \ \underline{x} \qquad (1)$$

$$\underline{q} = e \ \sigma_3 \ \underline{p} \qquad (2)$$

$$\underline{M}_0 = e \ \sigma_3 \ \underline{P}_0 \qquad (3)$$

and the general collineations of three distinct axial vectors are given by

$$\underline{W} = e\ \underline{X} \tag{4}$$

$$\underline{Q} = e\ \underline{P} \tag{5}$$

$$\underline{m}_0 = e\ \underline{p}_0 \tag{6}$$

where both types of vectors are expressed as 3×1 arrays and are denoted by an underbar as in (3.3.5-6) and (3.3.15-17). It is important to note in (1) - (6) that polar and axial vectors both remain invariant under translations of space. The distinguishing feature between the two types is that axial vectors are also invariant with respect to a reflection through the origin whereas polar vectors change in sign. Under the group of Euclidean rigid body collineations where reflection through the origin is excluded, clearly this distinction vanishes.

Prior to subsequent development, it is useful to first introduce some standard vector notations, identities and definitions. The scalar product (dot product), vector product (cross product) and scalar triple product are defined respectively by

$$\underline{\alpha} \cdot \underline{\beta} = \underline{\alpha}^T\ \underline{\beta} \tag{7}$$

$$\underline{\alpha} \times \underline{\beta} = |\underline{\alpha}\ \underline{\beta}| = [|\underline{\alpha}_2\ \underline{\beta}_3|\ \ |\underline{\alpha}_3\ \underline{\beta}_1|\ \ |\underline{\alpha}_1\ \underline{\beta}_2|]^T \tag{8}$$

$$\underline{\alpha} \times \underline{\beta} \cdot \underline{\gamma} = |\underline{\alpha}\ \underline{\beta}\ \underline{\gamma}| \tag{9}$$

where $\underline{\alpha}$, $\underline{\beta}$, $\underline{\gamma}$ may be polar or axial vectors. The following identities are frequently used,

$$\underline{\alpha} \times \underline{\alpha} = \underline{0} \tag{10}$$

$$\underline{\alpha} \times \underline{\beta} = -\ \underline{\beta} \times \underline{\alpha} \tag{11}$$

$$\underline{\alpha} \times (\underline{\beta} \times \underline{\gamma}) = (\underline{\gamma} \times \underline{\beta}) \times \underline{\alpha} = (\underline{\alpha} \cdot \underline{\gamma})\underline{\beta} - (\underline{\alpha} \cdot \underline{\beta})\underline{\gamma}. \tag{12}$$

The norm, weight or magnitude of a free vector is defined by

$$\|\ \underline{\alpha}\ \| = (\underline{\alpha} \cdot \underline{\alpha})^{\frac{1}{2}} \tag{13}$$

and the angle θ measured from $\underline{\alpha}$ to $\underline{\beta}$ in a right-hand sense about a normal $\underline{\gamma}$ is defined by

$$\cos\theta = \frac{\underline{\alpha} \cdot \underline{\beta}}{\|\ \underline{\alpha}\ \| \cdot \|\ \underline{\beta}\ \|} \tag{14}$$

$$\sin\theta = \frac{\underline{\alpha} \times \underline{\beta} \cdot \underline{\gamma}}{\|\ \underline{\alpha}\ \| \cdot \|\ \underline{\beta}\ \| \cdot \|\ \underline{\gamma}\ \|}. \tag{15}$$

Further, by selecting $\underline{\gamma}$ such that $\sin\theta \geq 0$, then

$$\underline{\alpha} \times \underline{\beta} = \|\ \underline{\alpha}\ \| \cdot \|\ \underline{\beta}\ \| \cdot \sin\theta\ \frac{\underline{\gamma}}{\|\ \underline{\gamma}\ \|},\ 0 \leq \theta \leq \pi. \tag{16}$$

Hamilton [1866] defined a vector $\underline{x}$ as a line segment with a direction and he conceived of it as the difference of two finite points

$$\begin{bmatrix} 0 \\ \underline{x} \end{bmatrix} = \begin{bmatrix} 1 \\ \underline{z} \end{bmatrix} - \begin{bmatrix} 1 \\ \underline{y} \end{bmatrix} \tag{17}$$

which may be graphically represented as an arrow pointing from y to z. In (17), it is clear that the difference of two points has the same form as a weighted point on the plane at infinity and it thus transforms in an identical manner,

$$\begin{bmatrix} w_0 \\ \underline{w} \end{bmatrix} = K \begin{bmatrix} 0 \\ \underline{x} \end{bmatrix} = \begin{bmatrix} 0 \\ e\ \sigma_3\ \underline{x} \end{bmatrix} \tag{18}$$

where the last three equations yield (1). Polar vectors therefore have the same properties as weighted points at infinity. In particular, a translation of space leaves points at infinity unchanged and thus the vector $\underline{x}$, which is graphically portrayed as an arrow, is not bound between the two points y and z but is free to translate without altering its geometric significance. Hence the name, free vector. It should also be noted that the magnitude of a vector (13) is equivalent to the norm of a weighted point at infinity which appears in (3.3.42) for normalization.

There is a unique polar vector associated with each finite point which is formed as the difference of the point with the origin and is referred to here as the vector of the point. Clearly, its components are equivalent to the last three coordinates of the point and its magnitude is the distance from the point to the origin. Conversely, a unique point may be associated with each vector, namely that point whose difference with the origin yields the

vector, and may thus be referred to as the point of the vector. This latter conception is usually called a position vector which is somewhat of a misnomer since it actually designates a finite point (by its nonhomogeneous coordinates) and is not a vector quantity.

The ray coordinates of a weighted line may be expressed as the join of two finite points

$$p = |xy| = \begin{bmatrix} \underline{y} - \underline{x} \\ \underline{x} \times \underline{y} \end{bmatrix} = \begin{bmatrix} \underline{p} \\ \underline{p}_0 \end{bmatrix} \tag{19}$$

and the general collineation of p is given by

$$\begin{bmatrix} \underline{q} \\ \underline{q}_0 \end{bmatrix} = \hat{K}_p = \begin{bmatrix} e\ \sigma_3\ \underline{p} \\ t^*\ e\ \sigma_3\ \underline{p} + e\ \underline{p}_0 \end{bmatrix} \tag{20}$$

The first three equations yield (2) and thus $\underline{p}$ is a polar vector which is also evident from (19) where essentially $\underline{p}$ is formed as the difference of points x and y. The weight of p is given by (3.3.38) which is identical to the magnitude of vector $\underline{p}$. Therefore by associating a free vector with a line or equivalently by associating a sense of direction with a weighted line, a new space element is established which is referred to as a line vector, a line-bound vector or a rotor as Clifford [1873] named it. Line vectors, or more precisely polar line vectors in this case, are detailed subsequently along with screw vectors.

The ray coordinates of a weighted line at infinity may be expressed as the join of the points at infinity

$$p = \begin{vmatrix} 0 & 0 \\ \underline{x} & \underline{y} \end{vmatrix} = \begin{bmatrix} \underline{0} \\ \underline{x} \times \underline{y} \end{bmatrix} = \begin{bmatrix} \underline{0} \\ \underline{p}_0 \end{bmatrix} \tag{21}$$

and the general collineation of the line is given by

$$\hat{K} \begin{bmatrix} \underline{0} \\ \underline{p}_0 \end{bmatrix} = \begin{bmatrix} \underline{0} \\ e\ \underline{p}_0 \end{bmatrix} = \begin{bmatrix} 0 \\ \underline{m}_0 \end{bmatrix}. \tag{22}$$

The last three equations represent the transformation of an axial vector as expressed by (6). Thus a free vector may be associated with a weighted line at infinity and it is therefore endowed with a directional sense.

The sum of two parallel polar line vectors with equal magnitudes and opposite senses may be expressed by

$$\begin{bmatrix} \underline{q} \\ \underline{q}_0 \end{bmatrix} + \begin{bmatrix} -\underline{q} \\ (\underline{p}_0 - \underline{q}_0) \end{bmatrix} = \begin{bmatrix} \underline{0} \\ \underline{p}_0 \end{bmatrix} \quad , \ -\underline{q} \cdot \underline{p}_0 = 0 \tag{23}$$

and is defined as an axial free line vector. Since the free line vector has the same form as a directed line at infinity and also transforms by (22), they both have the same properties. In particular, a free line vector is invariant with respect to translations and reflections through the origin. This situation is quite analogous to the correspondence between a weighted point at infinity and the difference of two finite points which are each free vectors.

The correspondence of a vector with a weighted plane is first established from the general collineation

$$\begin{bmatrix} W_0 \\ \underline{W} \end{bmatrix} = k \begin{bmatrix} X_0 \\ \underline{X} \end{bmatrix} = \begin{bmatrix} \sigma_1 X_0 - \underline{t}^T e\ \underline{X} \\ e\ \underline{X} \end{bmatrix}. \tag{24}$$

The last three equations represent the transformation of an axial vector $\underline{X}$ as given by (4) and the magnitude of $\underline{X}$ is the same as the norm of the plane given by (3.3.37). However, a precise interpretation of the correspondence of the free vector with the plane is not possible from (24) alone. As stated in the beginning of this section, it is necessary to express the coordinates of a plane in terms of point coordinates.

Let the weighted plane be expressed as the join of three noncollinear points u, v, w and

$$X = |u\ v\ w| = |u\ (v-u)\ (w-u)|$$
$$= \begin{bmatrix} \underline{u} \cdot (\underline{v}-\underline{u}) \times (\underline{w}-\underline{u}) \\ -(\underline{v}-\underline{u}) \times (\underline{w}-\underline{u}) \end{bmatrix} = \begin{bmatrix} -\underline{u} \cdot \underline{X} \\ \underline{X} \end{bmatrix} \tag{25}$$

where $(\underline{v}-\underline{u})$ and $(\underline{w}-\underline{u})$ are two distinct polar vectors which are parallel to the plane. Using the cosine formula (14) with the identical relations

$$\underline{X} \cdot (\underline{v}-\underline{u}) = 0 \quad , \quad \underline{X} \cdot (\underline{w}-\underline{u}) = 0 \tag{26}$$

then $\underline{X}$ is perpendicular to both polar vectors and is thus a normal vector to the plane. The magnitude of $\underline{X}$ is determined from (16) by

$$\|X\| = \|\underline{v}-\underline{u}\| \cdot \|\underline{w}-\underline{u}\| \cdot \|\sin\theta\| \tag{27}$$

Further, from (16) and (25), $\underline{X}$ points in the direction which is opposite to a normal defined by a right-handed rotation about the point u, v, w consecutively. It may also be deduced from (25) that $X_0/\|X\|$ is the directed perpendicular distance between the origin and X measured in a direction opposite to the normal $\underline{X}$.

A plane with an associated vector $\underline{X}$ has the properties of magnitude and direction and is referred to here as a plane-sect (plane-section) although plane-magnitude or leaf are more common designations. Under a translation of space a plane-sect transforms by

$$k_t \begin{bmatrix} X_0 \\ \underline{X} \end{bmatrix} = \begin{bmatrix} X_0 - \underline{t}^T \underline{X} \\ \underline{X} \end{bmatrix} . \tag{28}$$

For translations $\underline{t}$ which are perpendicular to the normal $\underline{X}$ then $\underline{t}^T \underline{X} = 0$ in (28) and the plane-sect transforms into itself. Thus a plane-sect is not fixed to any three specific points u, v, w, but is able to assume any position on the plane. Further, from (25) it is easily shown that

$$X_0 = |\underline{u}\ \underline{v}\ \underline{w}| \tag{29}$$

which represents six times the signed volume of the tetrahedron formed by the origin and points u, v, w and is thus proportional to the signed area of triangle u, v, w. Therefore a plane-sect may be represented by any three points on the plane which form an equivalent signed area since the height of the tetrahedron remains unaltered.

The axis coordinates of a line vector may be expressed as the meet of two nonparallel plane-sects X and Y,

$$P = |XY| = \begin{bmatrix} X_0\ \underline{Y} - Y_0\ \underline{X} \\ \underline{X} \times \underline{Y} \end{bmatrix} = \begin{bmatrix} \underline{P}_0 \\ \underline{P} \end{bmatrix} \tag{30}$$

and the general collineation of P is given by

$$\begin{bmatrix} \underline{Q}_0 \\ \underline{Q} \end{bmatrix} = \hat{k}P = \begin{bmatrix} t^* \ e\ \underline{P} + e\ \sigma_3\ P_0 \\ e\ \underline{P} \end{bmatrix} . \tag{31}$$

The last three equations yield (5) and thus $\underline{P}$ is an axial vector. It should be noted the line vectors p and P transform differently under a reflection through the origin. For the purpose of distinction they may be called polar and axial line vectors respectively although simply line vector suffices when the context is clear.

The vector $\underline{P}$ is at right angles to both plane-sect normals since using (30),

$$\underline{P} \cdot \underline{X} = 0 \quad , \quad \underline{P} \cdot \underline{Y} = 0 . \tag{32}$$

Further the magnitude of the line or line vector is given by (3.3.39) which is equivalent to the magnitude of vector $\underline{P}$.

The axis coordinates of an oriented and weighted line at infinity may be expressed as the meet of two distinct and parallel plane-sects

$$P = |XY| = \begin{bmatrix} X_0\,\underline{Y} - Y_0\,\underline{X} \\ \underline{0} \end{bmatrix} = \begin{bmatrix} \underline{P}_0 \\ \underline{0} \end{bmatrix} \qquad (33)$$

Its general collineation is given by

$$k \begin{bmatrix} \underline{P}_0 \\ \underline{0} \end{bmatrix} = \begin{bmatrix} \sigma_3\,\underline{P}_0 \\ \underline{0} \end{bmatrix} = \begin{bmatrix} \underline{M}_0 \\ \underline{0} \end{bmatrix} \qquad (34)$$

of which the first three equations yield the transformation of the polar vector in (3).

The sum of two parallel axial line vectors with equal magnitudes and opposite senses may be expressed by

$$\begin{bmatrix} \underline{Q}_0 \\ \underline{Q} \end{bmatrix} + \begin{bmatrix} (\underline{P}_0 - \underline{Q}_0) \\ -\underline{Q} \end{bmatrix} = \begin{bmatrix} \underline{P}_0 \\ \underline{0} \end{bmatrix}, \quad -\underline{Q}\cdot\underline{P}_0 = 0 \qquad (35)$$

and is defined as a polar free line vector. Since the free line vector has the same form as a directed line at infinity and also transforms by (34), they both have similar properties. Most importantly, they are invariant with respect to translations of space.

The representation of line vectors is now examined in further detail. Working first with ray coordinates, (19)

may be expressed in the alternative form

$$p = |xy| = \begin{bmatrix} \underline{y}-\underline{x} \\ \underline{x}\times(\underline{y}-\underline{x}) \end{bmatrix} = \begin{bmatrix} \underline{p} \\ \underline{x}\times\underline{p} \end{bmatrix} . \qquad (36)$$

Since $\underline{p}$ indicates the direction of the line, then any finite point r on the line is given by

$$r = \begin{bmatrix} 1 \\ \underline{y} + \lambda\underline{p} \end{bmatrix} \qquad (37)$$

where λ is any scalar. (Note that in previous sections r has been used to designate a line. This change in notation enables a consistency with the following section.) Solving (37) for y and substituting it in (36) yields

$$p = \begin{bmatrix} \underline{p} \\ \underline{r}\times\underline{p} \end{bmatrix} . \qquad (38)$$

Thus for a line vector written in the form (38), r may represent any point on the line.

The point r_n on p whose vector is normal to $\underline{p}$ satisfies

$$\underline{p}\cdot\underline{r}_n = 0. \qquad (39)$$

Expanding the vector product

$$\underline{p}\times\underline{p}_0 = \underline{p}\times(\underline{r}\times\underline{p}) = (\underline{p}\cdot p)\underline{r} - (\underline{p}\cdot\underline{r})\underline{p} \qquad (40)$$

and using (39) yields the vector of the point r_n,

$$\underline{r}_n = \frac{\underline{p} \times \underline{p}_0}{\underline{p} \cdot \underline{p}} . \qquad (41)$$

Unlike free vectors, line vectors are generally altered in significance under translations of space. Using (38), then under a translation p is transformed by

$$\hat{K}_t \begin{bmatrix} \underline{p} \\ \underline{r} \times \underline{p} \end{bmatrix} = \begin{bmatrix} \underline{p} \\ (\underline{t} + \underline{r}) \times \underline{p} \end{bmatrix} \qquad (42)$$

where identically

$$t^{*} \underline{p} = \underline{t} \times \underline{p} . \qquad (43)$$

There is one case of exception when p remains invariant and occurs when the translation $\underline{t}$ is parallel or antiparallel to $\underline{p}$ and the line vector transforms into itself. For this reason a line vector is often conceptualized as an arrow which constrained to the line and hence the term line-bound vector. Since the arrow may slide along the line without altering its significance, the term sliding vector is also used.

The properties of line vectors may also be determined using axis coordinates. Forming the meet of two plane-sects, the line vector is represented by

$$P = |XY| = \begin{bmatrix} X_0\, \underline{Y} - Y_0\, \underline{X} \\ \underline{X} \times \underline{Y} \end{bmatrix} = \begin{bmatrix} \underline{P}_0 \\ \underline{P} \end{bmatrix} \quad \text{repeated,} \qquad (30)$$

and as previously described, $\underline{P}$ indicates the direction and magnitude of the line vector.

In order to interpret $\underline{P}_0$, it is useful to express it in an alternative form analogous to (38). Firstly, it is necessary to demonstrate that there exists a point on the line r_n whose vector $\underline{r}_n$ is normal to the line and is given by

$$\underline{r}_n = \frac{\underline{P} \times \underline{P}_0}{\underline{P} \cdot \underline{P}} . \qquad (44)$$

Since it is easily verified that

$$\underline{r}_n \cdot \underline{P} = 0 \qquad (45)$$

then $\underline{r}_n$ is clearly normal to $\underline{P}$. Next, for

$$r_n = \begin{bmatrix} 1 \\ \underline{r}_n \end{bmatrix} \qquad (46)$$

it may also be verified using (30), (44) that

$$X^T r_n = 0 \quad \text{and} \quad Y^T r_n = 0 \qquad (47)$$

and thus r_n is incident to both plane-sects X,Y and consequently r_n is incident to P. Further, since

$$\underline{r}_n \times \underline{P} = \underline{P}_0 \qquad (48)$$

is an identical relation it follows that

$$P = \begin{bmatrix} \underline{r}_n \times \underline{P} \\ \underline{P} \end{bmatrix} \qquad (49)$$

or more generally,

$$P = \begin{bmatrix} \underline{r} \times \underline{P} \\ \underline{P} \end{bmatrix} \qquad (50)$$

where r is any point on the line vector and

$$r = \begin{bmatrix} 1 \\ \underline{r}_n + \lambda\sigma_3 \, \underline{P} \end{bmatrix} \qquad (51)$$

where λ may assume any value and σ_3 is included such that $\underline{r}$ is a polar vector.

Under a translation of space P is transformed by

$$\hat{k}_t \begin{bmatrix} \underline{r} \times \underline{P} \\ \underline{P} \end{bmatrix} = \begin{bmatrix} (\underline{t} + \underline{r}) \times \underline{P} \\ \underline{P} \end{bmatrix} \qquad (52)$$

and thus when the translation $\underline{t}$ is parallel or antiparallel to $\underline{P}$, the line vector is transformed into itself.

A screw has been previously defined as a linear combination of lines. It has been demonstrated in Section 3.3 that in Euclidean space a screw may be expressed in one way as linear combination of a unique line and its polar where the line is said to be the axis of the screw and where the ratio of scalar multipliers determines the pitch. This leads directly to Ball's [1900] definition of a screw as a line with an associated scalar called the pitch. By the definition of a norm, it was also possible to attach a weight to the screw which is equivalent to the weight of its line or axis. A

screw vector may be defined as a weighted screw with an associated free vector whose magnitude is equal to that of the screw. Alternatively, a screw vector may be defined as a line vector with an associated pitch. Screw vectors were referred to as motors by Clifford [1873].

Analogous to a screw, a screw vector can be expressed as a linear combination of a line and its polar in either ray coordinates

$$p = \begin{bmatrix} \underline{p} \\ \underline{p}_0 \end{bmatrix} = \begin{bmatrix} \underline{p} \\ \underline{r} \times \underline{p} \end{bmatrix} + h \begin{bmatrix} \underline{0} \\ \underline{p} \end{bmatrix} = \begin{bmatrix} \underline{p} \\ \underline{r} \times \underline{p} + h\underline{p} \end{bmatrix} \qquad (53)$$

or in axis coordinates

$$P = \begin{bmatrix} \underline{P}_0 \\ \underline{P} \end{bmatrix} = \begin{bmatrix} \underline{r} \times \underline{P} \\ \underline{P} \end{bmatrix} + h \begin{bmatrix} \underline{P} \\ \underline{0} \end{bmatrix} = \begin{bmatrix} \underline{r} \times \underline{P} + h\underline{P} \\ \underline{P} \end{bmatrix} \qquad (54)$$

where h is the pitch of the screw vector. From (3.3.28) and (3.3.33) the pitch may be expressed in a vector formulation using ray coordinates

$$h = \frac{\underline{p} \cdot \underline{p}_0}{\underline{p} \cdot \underline{p}} \qquad (55)$$

or axis coordinates

$$h = \frac{\underline{P} \cdot \underline{P}_0}{\underline{P} \cdot \underline{P}} \quad . \qquad (56)$$

Further, it is easily demonstrated that the vector $\underline{r}_n$, the vector from the origin to the screw vector which is normal to its axis, is given by the same formulation as those for line vectors, (41) and (44).

Finally, it is desirable to briefly explain the roles of line vectors and screw vectors in mechanics. It is well known that a force has the attributes of a line vector. Following Plucker's [1866] original conventions, a force is expressed in terms of ray coordinates by

$$f = \begin{bmatrix} \underline{f} \\ \underline{r} \times \underline{f} \end{bmatrix} = \begin{bmatrix} \underline{f} \\ \underline{f}_0 \end{bmatrix} \qquad (57)$$

where $\underline{f}$ expresses the direction and magnitude of the force and $\underline{r} \times \underline{f}$ is said to be the moment of the force about the origin.

The couple is a physical example of what has been previously referred to as a free line vector. A couple may be expressed as the sum of two distinct forces with equal magnitudes which act in opposite directions

$$m = \begin{bmatrix} \underline{f} \\ \underline{r}_1 \times \underline{f} \end{bmatrix} + \begin{bmatrix} -\underline{f} \\ \underline{r}_2 \times (-f) \end{bmatrix} = \begin{bmatrix} \underline{0} \\ (\underline{r}_1 - \underline{r}_2) \times \underline{f} \end{bmatrix} = \begin{bmatrix} \underline{0} \\ \underline{m}_0 \end{bmatrix} . \qquad (58)$$

According to Poinsot's theorem (see Ball [1900]), a system of forces f_i and couples m_i may be reduced to or is equivalent to a single force and a moment parallel to the force

$$w = \Sigma f_i + \Sigma m_i \qquad (59)$$

which Ball has called a wrench on a screw or more simply a wrench, w, which may be expressed using (53) by

$$w = \begin{bmatrix} \underline{w} \\ \underline{r} \times \underline{w} + h\underline{w} \end{bmatrix} = \begin{bmatrix} \underline{w} \\ \underline{r} \times \underline{w} \end{bmatrix} + h \begin{bmatrix} \underline{0} \\ \underline{w} \end{bmatrix} = \begin{bmatrix} \underline{w} \\ \underline{w}_0 \end{bmatrix} . \qquad (60)$$

Sometimes it is useful to express a wrench in the form

$$w = \begin{bmatrix} \underline{f} \\ \underline{m}_0 \end{bmatrix} \qquad (61)$$

where $\underline{f}$ is the resultant force and $\underline{m}_0$ is the resultant moment about the origin due to the system of forces and couples.

Plucker [1866] used the axis coordinates of a line (vector) to describe the instantaneous angular velocity of a rigid body about an axis which he called a rotary force in analogy with ordinary forces. Angular velocity may be expressed in line vector form by

$$\Omega = \begin{bmatrix} \underline{r} \times \underline{\Omega} \\ \underline{\Omega} \end{bmatrix} = \begin{bmatrix} \underline{\Omega}_0 \\ \underline{\Omega} \end{bmatrix} \qquad (62)$$

where $\underline{\Omega}$ expresses the direction and magnitude of the angular velocity and where $\underline{r} \times \underline{\Omega}$ represents the rectilinear velocity of a point on the (extended) rigid body which is coincident with the origin.

A translational or rectilinear velocity of a rigid body may be effected as the sum of two successive distinct angular velocities with equal magnitudes and opposite orientations

$$V = \begin{bmatrix} \underline{r}_1 \times \underline{\Omega} \\ \underline{\Omega} \end{bmatrix} + \begin{bmatrix} \underline{r}_2 \times (-\underline{\Omega}) \\ -\underline{\Omega} \end{bmatrix} = \begin{bmatrix} (\underline{r}_1 - \underline{r}_2) \times \underline{\Omega} \\ \underline{0} \end{bmatrix}$$

$$= \begin{bmatrix} \underline{V}_0 \\ \underline{0} \end{bmatrix} . \qquad (63)$$

It was first demonstrated by Charles (see Ball [1900]) that the most general displacement of a rigid body between two positions may be effected by a rotation about an axis together with a translation parallel to the axis. Ball called this motion a twist on a screw. Here a twist is always used to denote an instantaneous displacement since such motions are communitive in the sense that they may be successively applied to the body in any order. Analogous to (59), the motion resulting from successive angular velocities Ω_i and translational velocities V_i is equivalent to a twist on a screw

$$T = \Sigma\Omega_i + \Sigma V_i . \qquad (64)$$

For instantaneous motion, Charles' theorem may be stated as, the most general displacement of a rigid body is equivalent to an angular velocity about an axis together with a translational velocity parallel to the axis and

$$T = \begin{bmatrix} \underline{r} \times \underline{\Omega} + h\underline{\Omega} \\ \underline{\Omega} \end{bmatrix} = \begin{bmatrix} \underline{r} \times \underline{\Omega} \\ \underline{\Omega} \end{bmatrix} + \begin{bmatrix} h\underline{\Omega} \\ \underline{0} \end{bmatrix} = \begin{bmatrix} \underline{T}_0 \\ \underline{T} \end{bmatrix} . \qquad (65)$$

It is sometimes useful to express twist (65) in the form

$$T = \begin{bmatrix} \underline{V}_0 \\ \underline{\Omega} \end{bmatrix} \qquad (66)$$

where $\underline{\Omega}$ is the angular velocity associated with the twist and $\underline{V}_0$ is the translational velocity of a point on the (extended) rigid body which is coincident with the origin.

The compositions of wrenches w_i and twists T_i are given by the analogous relations

$$w = \Sigma w_i \qquad (67)$$

$$T = \Sigma T_i \qquad (68)$$

which have somewhat different interpretations. In (67) it is implied that the wrenches w_i are acting concurrently or in parallel, for example, on a single rigid body. However in (68), the twists T_i must act in succession or in a serial manner on the body. Thus the complementary concepts of parallel and serial composition are closely related to wrenches and twists which are themselves dual elements of mechanics.

Finally, there is a principle for wrenches and twists which is entirely analogous to the projective property of incidence for lines and reciprocity for screws, viz. the principle of virtual work. For example, assume that the instantaneous velocity of a rigid body is given by the twist T and the body is subjected to a wrench w. The instantaneous virtual work, or in other words the virtual power, may be expressed using (61) and (66) by

$$T^T w = \begin{bmatrix} \underline{V}_0 \\ \underline{\Omega} \end{bmatrix}^T \begin{bmatrix} \underline{f} \\ \underline{m} \end{bmatrix} = \underline{f} \cdot \underline{V}_0 + \underline{\Omega} \cdot \underline{m}_0. \qquad (69)$$

When the virtual work vanishes the twist and wrench are said to be reciprocal, a relationship which forms the basis for most of Ball's [1900] work. When a wrench is reciprocal to a twist, it is often referred to as a constraint wrench. It should be noted that in terms of Lagrangian mechanics, the components of T may be referred to as generalized velocities and the components of w may be referred to as generalized forces.