The positive integers other than 1 may be divided into two classes, prime numbers (such as $2, 3, 5, 7$) which do not admit of resolution into smaller factors, and composite numbers (such as $4, 6, 8, 9$) which do. The prime numbers derive their peculiar importance from the 'fundamental theorem of arithmetic' that a composite number can be expressed in one and only one way as a product of prime factors. A problem which presents itself at the very threshold of mathematics is the question of the distribution of the primes among the integers. Although the series of prime numbers exhibits great irregularities of detail, the general distribution is found to possess certain features of regularity which can be formulated in precise terms and made the subject of mathematical investigation.

The Distribution of Prime Numbers
A. E. Ingham

DEFINITION: A *composite number* is a positive integer $n \geq 2$ that can be factored as a product of two positive integers

$$n = ab \quad \text{where} \quad a < n \quad \text{and} \quad b < n.$$

A *prime number* is a positive integer $p \geq 2$ that cannot be factored as a product of two positive integers both of which are less than p.

Examples:

$$15 = (3)(5) \quad \text{is composite,} \quad 17 = (?)(?) \quad \text{is prime.}$$
$$99 = (9)(11) \quad \text{is composite,} \quad 101 = (?)(?) \quad \text{is prime.}$$

The first 200 prime numbers are:

$2, 3, 5, 7, 11, 13, 17, 19, 23, 29, 31, 37, 41, 43, 47, 53, 59, 61, 67,$
$71, 73, 79, 83, 89, 97, 101, 103, 107, 109, 113, 127, 131, 137, 139,$
$149, 151, 157, 163, 167, 173, 179, 181, 191, 193, 197, 199, 211, 223,$
$227, 229, 233, 239, 241, 251, 257, 263, 269, 271, 277, 281, 283, 293,$
$307, 311, 313, 317, 331, 337, 347, 349, 353, 359, 367, 373, 379, 383,$
$389, 397, 401, 409, 419, 421, 431, 433, 439, 443, 449, 457, 461, 463,$
$467, 479, 487, 491, 499, 503, 509, 521, 523, 541, 547, 557, 563, 569,$
$571, 577, 587, 593, 599, 601, 607, 613, 617, 619, 631, 641, 643, 647,$
$653, 659, 661, 673, 677, 683, 691, 701, 709, 719, 727, 733, 739, 743,$
$751, 757, 761, 769, 773, 787, 797, 809, 811, 821, 823, 827, 829, 839,$
$853, 857, 859, 863, 877, 881, 883, 887, 907, 911, 919, 929, 937, 941,$
$947, 953, 967, 971, 977, 983, 991, 997, 1009, 1013, 1019, 1021, 1031,$
$1033, 1039, 1049, 1051, 1061, 1063, 1069, 1087, 1091, 1093, 1097,$
$1103, 1109, 1117, 1123, 1129, 1151, 1153, 1163, 1171, 1181, 1187,$
$1193, 1201, 1213, 1217, 1223$

A list of consecutive primes beginning with the 1,000,000,000th prime:

$$
\begin{aligned}
&22801763489, 22801763513, 22801763527, 22801763531,\\
&22801763549, 22801763557, 22801763563, 22801763573,\\
&22801763581, 22801763641, 22801763707, 22801763711,\\
&22801763717, 22801763729, 22801763731, 22801763753,\\
&22801763767, 22801763773, 22801763783, 22801763833,\\
&22801763837, 22801763867, 22801763891, 22801763899,\\
&22801763923, 22801763951, 22801763953, 22801763987,\\
&22801764001, 22801764059, 22801764061, 22801764113,\\
&22801764119, 22801764137, 22801764157, 22801764179,\\
&22801764187, 22801764229, 22801764259, 22801764281,\\
&22801764299, 22801764319, 22801764353, 22801764361,\\
&22801764367, 22801764371, 22801764421, 22801764457,\\
&22801764467, 22801764487, 22801764497, 22801764509,\\
&22801764527, 22801764553, 22801764563, 22801764577,\\
&22801764589, 22801764593, 22801764613, 22801764619,\\
&22801764631, 22801764637, 22801764677, 22801764703,\\
&22801764719, 22801764761, 22801764767, 22801764809,\\
&22801764829, 22801764833, 22801764907, 22801764911
\end{aligned}
$$

Lemma. *Every integer $n \geq 2$ can be written as a product of prime numbers.*

Proof. Suppose that there exist integers $n \geq 2$ which cannot be written as a product of prime numbers. Then there exists a smallest positive integer N that cannot be written as a product of prime numbers. Plainly N cannot be a prime number. Therefore N can be written as

$$N = ab$$

where a and b are positive integers, $a < N$ and $b < N$. Because N is the *smallest* integer which cannot be written as a product of prime numbers, it follows that both a and b can be written as a product of prime numbers. Write

$$a = p_1 p_2 \cdots p_L \quad \text{and} \quad b = q_1 q_2 \cdots q_M,$$

where $p_1, p_2, \ldots p_L, q_1, q_2, \ldots q_M$ are prime numbers. But then

$$N = ab = p_1 p_2 \cdots p_L q_1 q_2 \cdots q_M$$

can be written as a product of prime numbers. This contradicts our assumption and proves that N does not exist.

Theorem. (Proposition 20, Book IX, Euclid's *Elements*) *There are infinitely many prime numbers.*

Proof. Suppose that $p_1, p_2, p_3, \ldots p_N$ is a complete list of all prime numbers. Consider the positive integer

$$Q = p_1 p_2 p_3 \cdots p_N + 1.$$

The number Q is *not* divisible by any of the primes from the list $p_1, p_2, p_3, \ldots p_N$. However, the previous lemma asserts that Q is divisible by primes. Thus $p_1, p_2, p_3, \ldots p_N$ cannot be a complete list of all primes. We have shown that there are infinitely many primes.

Theorem. (Proposition 14, Book IX, Euclid's *Elements*) *Every integer $n \geq 2$ can be written as a product of distinct prime numbers to positive integer powers:*

$$n = p_1^{e_1} p_2^{e_2} p_3^{e_3} \cdots p_M^{e_M}.$$

If we arrange the prime numbers so that

$$p_1 < p_2 < p_3 < \cdots < p_M$$

then this factorization of n is unique.

Example: for which positive integers N is there a solution in integers to the equation

$$x^2 + y^2 = N?$$

Consider the case

$$\begin{aligned} N &= 9434516543457490382976 \\ &= (2^7)(3^4)(13^3)(23^2)(61)(97^2)(1364161). \end{aligned}$$

The primes 3 and 23 are congruent to 3 modulo 4 and occur in the prime factorization to *even* powers. The remaining odd primes are congruent to 1 modulo 4. Hence there exists a solution:

$$1117195560^2 + 97125014376^2 = 9434516543457490382976.$$

Factorizations of some random 40 digit odd numbers

$$6745361009838572658711240095786483534093 =$$
$$(11)(13)(36037)(2644528949)(4949623958733958824O9227)$$

$$4890336472118759447609025497813340687565 =$$
$$(5)(185108553725329325 9)(528374985779988873307)$$

$$7122439681436658709008712092833427694087 =$$
$$(13)(547879975495127593000670160987186745699)$$

$$1376984620976357254477698887145677120943 =$$
$$(3^2)(79)(739)(2620686374327182653560612161006867)$$

$$2239857340097561206835547760281439788563 =$$
$$(7)(17)(31)(6709)(835609)(10830557248336587736O111207)$$

$$8870316472658671128566490971476091026583 =$$
$$(1283161736324741807)(6912859245682632111769)$$

$$7638709128564768562498876391028472565789 =$$
$$(157)(919421)(8958617510299)(5906971927125005263)$$

Factorizations of some non-random 40 digit odd numbers

$1111111111111111111111111111111111111111 =$

$(11)(41)(73)(101)(137)(271)(3541)(9091)(27961)$

$(1676321)(5964848081)$

$1000000000000000000000000000000000000001 =$

$(7)(11)(13^2)(157)(859)(6397)(216451)(1058313049)$

(388847808493)

$4333333333333333333333333333333333333333 =$

$(7)(443)(2741)(509813458079196226533335545139413)$

$5555555555555555555599999999999999999999 =$

$(11)(41)(101)(271)(3541)(9091)(27961)$

(500000000000000000009)

$4444444444444444444455555555555555555555 =$

$(3)(5)(11)(41)(101)(149)(271)(3541)(9091)$

$(27961)(413087)(16046383667)$

$8888888888888888888888888888888888888887 =$

$(1013)(65025411611)(134944417265323481201339809)$

Problems about prime numbers

1. In a letter from Christian Goldbach to Euler dated June 7, 1742, he states:

   "Es scheinet ... dass eine jede Zahl, die grösser als 2, ein aggregatum trium numerorum primorum sei."

   (It seems that every number larger than 2 is a sum of three prime numbers.)

   On June 30, 1742, Euler replied to Goldbach stating that:

   "Dass ... ein jeder numerus par eine summa duorum primorum sey, halte ich für ein ganz gewisses theorema, ungeachtet ich dasselbe necht demonstriren kann."

   (... every even integer is a sum of two primes. I regard this as a completely certain theorem, although I cannot prove it.)

2. The twin prime problem: do there exist infinitely many primes p such that $p+2$ is also a prime? Two of the largest known twin primes are

$$(697,053,813)2^{16352} - 1 \quad \text{and} \quad (697,053,813)2^{16352} + 1.$$

3. Do there exist infinitely many primes of the form N^2+1, where N is an integer?

4. Define $\pi(x)$ to be the number of primes less than or equal to the positive real number x. Is there a "formula" for $\pi(x)$ in terms of simpler functions? What is a good approximation to $\pi(x)$?

Great Moments in Prime Number Theory:
L. Euler (1707–1783)

In this definition s is a real variable. If $1 < s$ then Euler discovered that the convergent infinite series

$$1 + \frac{1}{2^s} + \frac{1}{3^s} + \frac{1}{4^s} + \frac{1}{5^s} + \frac{1}{6^s} + \frac{1}{7^s} + \frac{1}{8^s} + \frac{1}{9^s} \cdots,$$

is equal to the convergent infinite product over prime numbers

$$\begin{aligned} = &\Big(1 + \frac{1}{2^s} + \frac{1}{2^{2s}} + \frac{1}{2^{3s}} + \frac{1}{2^{4s}} \cdots\Big) \\ &\Big(1 + \frac{1}{3^s} + \frac{1}{3^{2s}} + \frac{1}{3^{3s}} + \frac{1}{3^{4s}} \cdots\Big) \\ &\Big(1 + \frac{1}{5^s} + \frac{1}{5^{2s}} + \frac{1}{5^{3s}} + \frac{1}{5^{4s}} \cdots\Big) \\ &\Big(1 + \frac{1}{7^s} + \frac{1}{7^{2s}} + \frac{1}{7^{3s}} + \frac{1}{7^{4s}} \cdots\Big) \\ &\Big(1 + \frac{1}{11^s} + \frac{1}{11^{2s}} + \frac{1}{11^{3s}} + \frac{1}{11^{4s}} \cdots\Big) \\ &\cdots \end{aligned}$$

Alternatively:

$$\sum_{n=1}^{\infty} \frac{1}{n^s} = \prod_p \Big(1 - \frac{1}{p^s}\Big)^{-1} \quad \text{if} \quad 1 < s.$$

Euler used this identity to show that

$$\sum_{p \text{ prime}} \frac{1}{p} = \infty.$$

It follows that there are infinitely many prime numbers.

Great Moments in Prime Number Theory: A. M. Legendre (1752–1833) and C. F. Gauss (1777–1855)

In a paper of 1808 Legendre claimed that $\pi(x)$ was approximately equal to

$$\frac{x}{\log x - B}$$

where B is a numerical constant. Using tables of primes up to about $x = 400,000$ he suggested the value $B = 1.08366$.

As early as 1792 or 1793 Gauss recorded in a book of tables that the logarithmic integral

$$\mathrm{li}(x) = \lim_{\epsilon \to 0+} \left\{ \int_0^{1-\epsilon} + \int_{1+\epsilon}^{x} \right\} \frac{1}{\log t}\ dt$$

was a good approximation to $\pi(x)$. However, Gauss only reported his observations in 1849 in a letter to the astronomer Encke.

Evidently both Legendre and Gauss believed that

$$\frac{\pi(x)}{\mathrm{li}(x)} \to 1, \quad \text{or} \quad \frac{\pi(x)\log x}{x} \to 1, \quad \text{as} \quad x \to \infty.$$

It turns out that $\mathrm{li}(x)$ is a much better approximation to $\pi(x)$ in the sense that

$$\pi(x) - \mathrm{li}(x)$$

is relative small when x is large.

Great Moments in Prime Number Theory:
G. L. Dirichlet (1805–1859)

Let $1 \leq a < q$ be relatively prime integers and consider the *arithmetic progression*

$$A(a, q) = \{a, q + a, 2q + a, 3q + a, 4q + a, \dots\}.$$

For example, we have

$$A(3, 7) = \{3, 10, 17, 24, 31, 38, 45, 52, 59, \dots\}.$$

In 1837 Dirichlet proved that such an arithmetic progression contains infinitely many prime numbers. He made use of identities of the form

$$L(s, \chi) = \sum_{n=1}^{\infty} \frac{\chi(n)}{n^s} = \prod_p \Big(1 - \frac{\chi(p)}{p^s}\Big)^{-1} \quad \text{where} \quad 1 < s.$$

By considering the behavior of these functions as $s \to 1+$, Dirichlet was able to prove that

$$\sum_{\substack{p \in A(a,q) \\ p \text{ prime}}} \frac{1}{p} = \infty.$$

Great Moments in Prime Number Theory: P. L. Chebyshev (1821–1894)

Instead of working directly with the prime counting function

$$\pi(x) = \sum_{p \le x} 1,$$

Chebyshev introduced a *weighted* prime counting function

$$\psi(x) = \sum_{p^m \le x} \log p,$$

where the sum is over prime powers. For example,

$$\begin{aligned}
\psi(30) &= \log 2 + \log 3 + \log 2 + \log 5 + \log 7 \\
&\quad + \log 2 + \log 3 + \log 11 + \log 13 \\
&\quad\quad + \log 2 + \log 17 + \log 19 + \log 23 \\
&\quad\quad\quad + \log 5 + \log 3 + \log 29 \\
&= 28.4765\ldots.
\end{aligned}$$

Chebyshev observed that

$$\pi(x) \approx \mathrm{li}(x) \quad \text{if and only if} \quad \psi(x) \approx x.$$

The function $\psi(x)$ satisfies the fundamental identity

$$\sum_{1\leq m\leq x} \psi(x/m) = \sum_{1\leq n\leq x} \log n$$

for all positive real numbers x. Using this identity Chebyshev proved that

$$(0.921)x \leq \psi(x) \leq (1.105)x \quad \text{for all large } x,$$

and from this he obtained the estimate

$$(0.89)\,\mathrm{li}(x) \leq \pi(x) \leq (1.11)\,\mathrm{li}(x) \quad \text{for all large } x.$$

Chebyshev also proved that *if* the limit

$$\lim_{x\to\infty} \frac{\pi(x)}{\mathrm{li}(x)} \quad \text{exits,}$$

then the value of the limit is 1.

A Very Great Moment in Prime Number Theory: B. Riemann (1826–1866)

Riemann wrote one paper of 8 pages on number theory, *Ueber die Anzahl der Primzahlen unter einer gegebenen Grösse*, published in 1859. In this paper he considered the function

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s} = \prod_p \Big(1 - \frac{1}{p^s}\Big)^{-1},$$

but now Riemann assumed that $s = \sigma + it$ is a *complex variable* and observed that the series and product converge to the same analytic function in the half plane $1 < \sigma$. In his paper Riemann briefly sketched proofs of his new discoveries about the function $\zeta(s)$:

1. The function $\zeta(s)$ has an analytic continuation to the complex plane with a simple pole at $s = 1$.
2. The function $\zeta(s)$ satisfies a functional equation that relates its value at s with its value at $1 - s$.
3. The function $\zeta(s)$ has trivial zeros at $s = -2, -4, -6, \ldots$ and infinitely many nontrivial zeros in the *critical strip*:

$$\{s = \sigma + it : 0 \le \sigma \le 1\}.$$

He also gave an estimate for the number of nontrivial zeros $\rho = \beta + i\gamma$ which satisfy $0 < \gamma \le T$. If this number is denoted by $N(T)$ then Riemann asserted that

$$N(T) \approx \frac{T}{2\pi} \log\Big(\frac{T}{2\pi}\Big) - \frac{T}{2\pi}.$$

4. Riemann defined a weighted prime counting function

$$\Pi(x) = \sum_{p^m \le x} \frac{1}{m} \quad \text{where } 1 < x,$$

and asserted that $\Pi(x)$ has the following *explicit formula*:

$$\Pi(x) = \mathrm{li}(x) - \sum_{\rho} \mathrm{li}(x^{\rho}) - \log 2 + \int_x^{\infty} \frac{dx}{(x^2-1)x\log x}.$$

A complete proof was given by H. von Mangoldt in 1895. A simpler explicit formula is

$$\psi(x) = x - \sum_{\rho} \frac{x^{\rho}}{\rho} - \log 2\pi - \frac{1}{2}\log(1-x^{-2}).$$

In both formulas the summation is over the set of nontrivial zeros $\rho = \beta + i\gamma$ of $\zeta(s)$.

5. Riemann then considers the number of *real* zeros of $\zeta(1/2 + it)$ and writes:

Man findet nun in der That etwa so viel reelle Wurzeln innerhalb dieser Grenzen, und es ist sehr wahrscheinlich, dass alle Wurzeln reell sind. Hiervon wäre allerdings ein strenger Beweis zu wünschen; ich habe indess die Aufsuchung desselben nach einigen flüchtigen vergeblichen Versuchen vorläufig bei Seite gelassen, da er für den nächsten Zweck meiner Untersuchung entbehrlich schien.

(One finds in fact about this many real zeros within these bounds, and it is very probable, that all zeros are real. One would certainly like to have a rigorous proof of this, but I have set aside the search for such a proof after a few hasty attempts, because it is not required for the current objective of my investigation.)

The first 20 zeros of the Riemann zeta-function:

$$\frac{1}{2} + i(14.134725142\ldots)$$
$$\frac{1}{2} + i(21.022039639\ldots)$$
$$\frac{1}{2} + i(25.010857580\ldots)$$
$$\frac{1}{2} + i(30.424876126\ldots)$$
$$\frac{1}{2} + i(32.935061588\ldots)$$
$$\frac{1}{2} + i(37.586178159\ldots)$$
$$\frac{1}{2} + i(40.918719012\ldots)$$
$$\frac{1}{2} + i(43.327073281\ldots)$$
$$\frac{1}{2} + i(48.005150881\ldots)$$
$$\frac{1}{2} + i(49.773832478\ldots)$$
$$\frac{1}{2} + i(52.970321478\ldots)$$
$$\frac{1}{2} + i(56.446247697\ldots)$$
$$\frac{1}{2} + i(59.347044003\ldots)$$
$$\frac{1}{2} + i(60.831778525\ldots)$$
$$\frac{1}{2} + i(65.112544048\ldots)$$
$$\frac{1}{2} + i(67.079810529\ldots)$$
$$\frac{1}{2} + i(69.546401711\ldots)$$
$$\frac{1}{2} + i(72.067157674\ldots)$$
$$\frac{1}{2} + i(75.704690699\ldots)$$
$$\frac{1}{2} + i(77.144840069\ldots)$$

Zeros of the Riemann zeta-function, number $10^{12}+1$ through number $10^{12}+20$:

$$\tfrac{1}{2}+i(267,653,395,648.8475231278\ldots)$$
$$\tfrac{1}{2}+i(267,653,395,649.3623669687\ldots)$$
$$\tfrac{1}{2}+i(267,653,395,649.6816309165\ldots)$$
$$\tfrac{1}{2}+i(267,653,395,649.8619899438\ldots)$$
$$\tfrac{1}{2}+i(267,653,395,650.1576654790\ldots)$$
$$\tfrac{1}{2}+i(267,653,395,650.4342666839\ldots)$$
$$\tfrac{1}{2}+i(267,653,395,650.5808912999\ldots)$$
$$\tfrac{1}{2}+i(267,653,395,650.8344795320\ldots)$$
$$\tfrac{1}{2}+i(267,653,395,651.0584820011\ldots)$$
$$\tfrac{1}{2}+i(267,653,395,651.3570661334\ldots)$$
$$\tfrac{1}{2}+i(267,653,395,651.7017825037\ldots)$$
$$\tfrac{1}{2}+i(267,653,395,651.9049020916\ldots)$$
$$\tfrac{1}{2}+i(267,653,395,652.0387527712\ldots)$$
$$\tfrac{1}{2}+i(267,653,395,652.3029254562\ldots)$$
$$\tfrac{1}{2}+i(267,653,395,652.7891109498\ldots)$$
$$\tfrac{1}{2}+i(267,653,395,652.9646206541\ldots)$$
$$\tfrac{1}{2}+i(267,653,395,653.1790233335\ldots)$$
$$\tfrac{1}{2}+i(267,653,395,653.2277635657\ldots)$$
$$\tfrac{1}{2}+i(267,653,395,653.5120490433\ldots)$$
$$\tfrac{1}{2}+i(267,653,395,653.8376801741\ldots)$$
$$\tfrac{1}{2}+i(267,653,395,654.0654230031\ldots)$$

Zeros of the Riemann zeta-function, number $10^{22}+1$ through number $10^{22}+20$:

$$\tfrac{1}{2}+i(1,370,919,909,931,995,308,226.68016095\ldots)$$
$$\tfrac{1}{2}+i(1,370,919,909,931,995,308,226.77659152\ldots)$$
$$\tfrac{1}{2}+i(1,370,919,909,931,995,308,226.94593324\ldots)$$
$$\tfrac{1}{2}+i(1,370,919,909,931,995,308,227.16707942\ldots)$$
$$\tfrac{1}{2}+i(1,370,919,909,931,995,308,227.28945453\ldots)$$
$$\tfrac{1}{2}+i(1,370,919,909,931,995,308,227.45742387\ldots)$$
$$\tfrac{1}{2}+i(1,370,919,909,931,995,308,227.55600131\ldots)$$
$$\tfrac{1}{2}+i(1,370,919,909,931,995,308,227.71882545\ldots)$$
$$\tfrac{1}{2}+i(1,370,919,909,931,995,308,227.80388039\ldots)$$
$$\tfrac{1}{2}+i(1,370,919,909,931,995,308,227.98526449\ldots)$$
$$\tfrac{1}{2}+i(1,370,919,909,931,995,308,228.12614756\ldots)$$
$$\tfrac{1}{2}+i(1,370,919,909,931,995,308,228.25881586\ldots)$$
$$\tfrac{1}{2}+i(1,370,919,909,931,995,308,228.38744704\ldots)$$
$$\tfrac{1}{2}+i(1,370,919,909,931,995,308,228.53069105\ldots)$$
$$\tfrac{1}{2}+i(1,370,919,909,931,995,308,228.72920315\ldots)$$
$$\tfrac{1}{2}+i(1,370,919,909,931,995,308,228.82462962\ldots)$$
$$\tfrac{1}{2}+i(1,370,919,909,931,995,308,228.94497148\ldots)$$
$$\tfrac{1}{2}+i(1,370,919,909,931,995,308,229.12603661\ldots)$$
$$\tfrac{1}{2}+i(1,370,919,909,931,995,308,229.25487358\ldots)$$
$$\tfrac{1}{2}+i(1,370,919,909,931,995,308,229.31039517\ldots)$$

Statements equivalent to the Riemann Hypothesis

1. All nontrivial zeros $\rho = \beta + i\gamma$ of the function $\zeta(s)$ have $\beta = 1/2$.
2. For every $\epsilon > 0$ there is a positive constant $C(\epsilon)$ such that

$$|\operatorname{li}(x) - \pi(x)| \leq C(\epsilon) x^{1/2+\epsilon}.$$

3. For every $\epsilon > 0$ there is a positive constant $C(\epsilon)$ such that

$$|x - \psi(x)| \leq C(\epsilon) x^{1/2+\epsilon}.$$

4. Define the *Möbius function* $\mu(n)$ for positive integers n by

$$\mu(n) = \begin{cases} (-1)^m & \text{if } n \text{ is a product of } m \text{ distinct primes} \\ 0 & \text{if } n \text{ is divisible by the square of a prime.} \end{cases}$$

Then the infinite series

$$\lim_{N \to \infty} \sum_{n=1}^{N} \frac{\mu(n)}{n^\sigma}$$

converges for all $\sigma > 1/2$.

Great Moments in Prime Number Theory:

In 1896, working independently, Jacques Hadamard (1865–1963) and Charles-Jean de la Vallée Poussin (1866–1962) established the crucial fact that

$$\zeta(1+it) \neq 0 \quad \text{for all real numbers } t.$$

From this they obtained a proof of the Prime Number Theorem:

$$\lim_{x\to\infty} \frac{\pi(x)}{\mathrm{li}(x)} = 1.$$

In 1914 J. E. Littlewood (1885–1977) proved that the difference

$$\mathrm{li}(x) - \pi(x)$$

is both positive and negative for arbitrarily large values of x. In 1955 S. Skews showed that the first change of sign in $\mathrm{li}(x) - \pi(x)$ occurs for a value of x that is smaller than

$$10^{10^{10^{1000}}}.$$

Using more information about the zeros of $\zeta(s)$ we now know that the first sign change occurs for a value of x that is smaller than

$$10^{371}.$$

Great Moments in Prime Number Theory:

In 1942 A. Selberg (1917–) proved that a positive proportion of the nontrivial zeros of the Riemann zeta-function have real part equal to 1/2. More precisely, if $N(T)$ is the number of zeros $\rho = \beta + i\gamma$ with $0 < \gamma \leq T$ and $N_0(T)$ is number of zeros $\rho = 1/2 + i\gamma$ with $0 < \gamma \leq T$, then Selberg proved that there exists a positive constant c such that

$$cN(T) \leq N_0(T) \quad \text{for all large values of } T.$$

In 1948 A. Selberg and Paul Erdös (1913–1996) found elementary proofs of the Prime Number Theorem. Their arguments do not use the Riemann zeta-function, complex analysis or Fourier analysis.

In 1970 H. L. Montgomery showed that if the Riemann Hypothesis is true, then the imaginary parts of the nontrivial zeros of the Riemann zeta-function have (in a certain restricted sense) the same pair correlation function as the eigenvalues of a random complex Hermitian or a random unitary matrix. Montgomery also formulated a conjecture, called the *pair correlation conjecture*, which would significantly strengthen his result. This conjecture is consistent with the view that the nontrivial zeros of $\zeta(1/2 + it)$ are the eigenvalues of an unbounded linear operator–as yet undiscovered–acting on a certain Hilbert space.