Choquet expectation and Peng's $g-$expectation

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Abstract

In this paper, we consider two ways to generalize the mathematical expectation of a random variable, the Choquet expectation and Peng's $g-$expectation. An open question has been, for what class of expectation do these two definitions coincide? In this paper, we provide a necessary and sufficient condition which proves that the only expectation which lies in both classes is the traditional linear expectation. This settles another open question about whether Choquet expectation may be used to obtain Monte Carlo-like solution of nonlinear PDE: It cannot, except for some very special cases.

Keywords: backward stochastic differential equation(BSDE), g-expectation, representation theorem of g-expectation, Choquet-expectation

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1 Introduction

The concept of expectation is clearly very important in probability theory. Expectation is usually defined via

$$E\xi = \int_{-\infty}^{\infty} x dF(x)$$

where $F(x) := P(\xi \leq x)$ is the distribution of random variable ξ with respect to the probability measure P. Alternatively, the expectation $E\xi$ can be written as

$$E\xi = \int_{-\infty}^{0} [P(\xi \geq t) - 1]dt + \int_{0}^{+\infty} P(\xi \geq t)dt \tag{0.1}$$

which implies the relation between mathematical expectation and probability measure. One of properties of mathematical expectation is its linearity, that is: for given random variables ξ and η

$$E(\xi + \eta) = E\xi + E\eta. \tag{0.2}$$

Which is equivalent to the additivity of probability measure, i.e.

$$P(A + B) = P(A) + P(B), \quad \text{if } A \cap B = \emptyset. \tag{0.3}$$

From this viewpoint, we sometimes call mathematical expectation (resp. probability measure) linear mathematical expectation (resp. linear probability measure). It is easy to define conditional expectation using the additivity of mathematical expectations, that is the conditional expectation η of a random variable ξ under a given $\sigma-$field $\mathcal{F}$ is a $\mathcal{F}$-measurable random variable such that

$$E\xi I_A = E\eta I_A, \qquad \forall A \in \mathcal{F}. \tag{0.4}$$

It is well known that linear mathematical expectation is a powerful tool for scientists to deal with stochastic phenomena. However, scientists also find that there are many uncertain phenomena which are not easily modelled using linear mathematical expectations. Economists have found that linear mathematical expectations result in the Allais paradox and the Ellsberg paradox, see Allais (1953) and Ellsberg (1961); Physicists find some uncertain phenomena in physics cannot be well explained by linear mathematical expectations, see Feynman (1963). How to deal with uncertain phenomena which cannot be well explained by linear mathematical expectations? Some scientists try to use non-linear mathematical expectations. A natural question is: How to define non-linear mathematical expectations? Choquet (1953) extended the probability measure P in (0.1) to a nonlinear probability measure V (also called the capacity) and obtained the following definition $C(\xi)$ of nonlinear mathematical expectations (called the Choquet expectation):

$$C(\xi) := \int_{-\infty}^{0} [V(\xi \geq t) - 1]dt + \int_{0}^{+\infty} V(\xi \geq t)dt.$$

Because V no longer has property (0.3), the above Choquet expectation $C(\xi)$ usually no longer has property (0.2).

Choquet expectations have many applications in statistics, economics, finance and physics. Unfortunately, scientists also find that it is difficult to define conditional Choquet expectation in term of Choquet expectations. Many papers study Choquet expectation and its applications see for example Anger (1977); Dellacherie (1970); Dow (1994); Graf (1980); Sarin and Wakker (1998); Schmeidler (1989); Wakker (2001); Wasserman (1990) and their references therein. Peng (1997,1999) introduced a kind of nonlinear expectation (he calls it the g-expectation) via a kind of nonlinear backward stochastic differential equation (BSDE for short). One of characteristics of g-expectations is that using g-expectations, it is easy to define conditional expectations in the same way as in (0.4). The application of g-expectations in economics can be found in Chen and Epstein (2002). A open question raised by Peng is: What is the relation between Choquet expectation and Pens's $g-$expectation? Does there exist a capacity such that g-expectation can be represented by a Choquet expectation? An earlier work by Chen and Sulem (2001) shows that the answer is yes for certain special random variables. In this paper we shall further study this question and obtain a necessary and sufficient condition for this open question. This settles another open question about whether Choquet expectation may be used to obtain Monte-Carlo-like solution of nonlinear PDE: It cannot, except for some very special cases.

2 Notations and Lemmas

In this section we shall briefly introduce the concepts of Choquet expectation and g-expectation. For convenience we include some related lemmas that we shall use in this paper for reader's.

Capacity and Choquet expectation: We now introduce briefly the concepts of capacity and Choquet expectation.

DEFINITION 1 *(1) Random variables ξ and η are called comonotonic if*

$$[\xi(\omega)-\xi(\omega^{'})][\eta(\omega)-\eta(\omega^{'})]\geq 0, \qquad \forall\omega,\omega^{'}\in\Omega.$$

(2) **(Comonotonic Additivity)** *A real functional F on $L^2(\Omega,\mathcal{F},P)$ is called comonotonic additive if*

$$F(\xi+\eta)=F(\xi)+F(\eta) \quad \textit{whenever } \xi \textit{ and } \eta \textit{ are comonotonic.}$$

(3) A set function $V:\mathcal{F}\longrightarrow[0,1]$ is called a capacity if

(i) $V(\emptyset)=0, V(\Omega)=1;$

(ii) $\forall A\subseteq B, V(A)\leq V(B).$

(iii) $A_n\uparrow A$, $V(A_n)\uparrow V(A), n\to\infty.$

(4) Let V be a capacity and $\xi \in L^2(\Omega, \mathcal{F}, P)$,denote $C(\xi)$ by

$$C(\xi) := \int_{-\infty}^{0} (V(\xi \geq t) - 1)dt + \int_{0}^{\infty} V(\xi \geq t)dt.$$

We call $C(\xi)$ the Choquet expectation of ξ with respect to capacity V.

Dellacherie (1970) showed that comonotonic additivity is a necessary condition for a functional to be represented by a Choquet expectation.

BSDEs and g-expectation: Let $(\Omega, \mathcal{F}, \mathcal{P})$ be a probability space with filtration $(\mathcal{F}_s)_{s\geq 0}$, $(W_s)_{s\geq 0}$ be a standard d-Brownian motion. For ease of exposition, we assume $d = 1$. The results of this paper can be easily extended to the case $d > 1$. Suppose that $(\mathcal{F}_s)$ is the σ-filtration generated by $(W_s)_{s\geq 0}$, i.e.

$$\mathcal{F}_s = \sigma\{W_r; 0 \leq r \leq s\}.$$

Let $T > 0$, $\mathcal{F}_T = \mathcal{F}$ and $g = g(y, z, t) : \mathbf{R} \times \mathbf{R}^d \times [0, T] \longrightarrow \mathbf{R}$ be a function satisfying

(H.1) $\forall (y, z) \in \mathbf{R} \times \mathbf{R}^d$, $g(y, z, t)$ is continuous in t and $\int_0^T g^2(0, 0, t)dt < \infty$;

(H.2) g is uniformly Lipschitz continuous in (y, z), i.e. there exists a constant $C > 0$ such that $\forall y, y' \in \mathbf{R}$, $z, z' \in \mathbf{R}^d$, $|g(y, z, t) - g(y', z', t)| \leq C(|y - y'| + |z - z'|)$;

(H.3) $g(y, 0, t) \equiv 0, \forall (y, t) \in \mathbf{R} \times [0, T]$.

Let $\mathcal{M}(0, T, \mathbf{R}^n)$ be the set of all $\mathbf{R}^n$-valued, $\mathcal{F}_t-$adapted processes $\{v_t\}$ with

$$E \int_0^T |v_t|^2 dt < \infty;$$

For each $t \in [0, T]$, let $L^2(\Omega, \mathcal{F}_t, P)$ be the set of all $\mathcal{F}_t-$ measurable random variables.

Pardoux and Peng (1990) considered the following backward stochastic differential equation:

$$y_t = \xi + \int_t^T g(y_s, z_s, s)ds - \int_t^T z_s dW_s \tag{1}$$

and showed the following result:

Lemma 1 *Suppose that g satisfies (H.1),(H.2) and (H.3) and $\xi \in L^2(\Omega, \mathcal{F}, P)$. Then BSDE (1) has a unique solution $(y, z) \in \mathcal{M}(0, T; \boldsymbol{R}) \times \mathcal{M}(0, T; \boldsymbol{R}^d)$.*

Using the solution of BSDE (1), Peng (1997) introduced the concept of g-expectation via BSDE (1).

Definition 2 *Suppose g satisfies (H.1)(H.2) and (H.3), given $\xi \in L^2(\Omega, \mathcal{F}, P)$, let (y, z) be the solution of BSDE(1), we denote Peng's g-expectation of ξ by $\mathcal{E}_g[\xi]$ and define it*

$$\mathcal{E}_g[\xi] := y_0$$

From the definition of g-expectation, Peng (1997) introduced the concept of conditional g-expectation:

LEMMA 2 *For any $\xi \in L^2(\Omega, \mathcal{F}, P)$, there exists unique $\eta \in L^2(\Omega, \mathcal{F}_t, P)$ such that*

$$\mathcal{E}_g[I_A \xi] = \mathcal{E}_g[I_A \eta], \quad \forall A \in \mathcal{F}_t,$$

we call η the conditional $g-$expectation of ξ and write η as $\mathcal{E}_g[\xi|\mathcal{F}_t]$. Moreover, $\mathcal{E}_g[\xi|\mathcal{F}_t]$ is the value of the solution $\{y_t\}$ of BSDE (1) at time t. That is

$$\mathcal{E}_g[\xi|\mathcal{F}_t] = y_t.$$

$g-$expectation $\mathcal{E}_g[.]$ preserves many of properties of the classical mathematical expectations, although it does not preserve linearity, see Peng (1997) and Briand et.al.(2000) for details, for example:

LEMMA 3 *(1) If c is a constant, then $\mathcal{E}_g[c] = c$;*

(2) If $\xi_1 \geq \xi_2$, then $\mathcal{E}_g[\xi_1] \geq \mathcal{E}_g[\xi_2]$;

(3) $\mathcal{E}_g[\mathcal{E}_g[\xi|\mathcal{F}_t]] = \mathcal{E}_g[\xi]$;

(4) If ξ is $\mathcal{F}_t-$measurable, then $\mathcal{E}_g[\xi|\mathcal{F}_t] = \xi$.

(5) If $g(y, z, t)$ is deterministic and ξ is independent of $\mathcal{F}_t$, then $\mathcal{E}_g[\xi|\mathcal{F}_t] = \mathcal{E}_g[\xi]$.

From the definition of $g-$expectation, it is natural to define $g-$probability:

DEFINITION 3 *For given $A \in \mathcal{F}$, denote $P_g(A)$ by*

$$P_g(A) = \mathcal{E}_g[I_A],$$

we call $P_g(A)$ the $g-$probability of A. Obviously, $P_g(\cdot)$ is a capacity.

To simplify notation, we sometimes rewrite g-expectation $\mathcal{E}_g[\cdot]$, conditional g-expectation $\mathcal{E}_g[\cdot|\mathcal{F}_t]$ and g-probability $P_g(\cdot)$ as $\mathcal{E}_\mu[.], \mathcal{E}_\mu[.|\mathcal{F}_t], P_\mu(.)$,respectively, if $g(t, z) = \mu_t|z|$.

REMARK 1 *(1) g-expectation and conditional g-expectation depend on the choice of the function g, if g is nonlinear, then g-expectation is usually also nonlinear.*

(2) If $g = 0$, set conditional expectation $E[\cdot|\mathcal{F}_t]$ on the both sides of BSDE (1), then $y_t = \mathcal{E}_g[\xi|\mathcal{F}_t] = E[\xi|\mathcal{F}_t]$, $y_0 = \mathcal{E}_g[\xi] = E[\xi]$, which implies another explanation for mathematical expectation: Conditional mathematical expectations actually are the solution of a simple BSDE and mathematical expectation is the value of this solution at time $t = 0$.

The following is an example of g-expectations:

EXAMPLE 1 *If $\xi \in L^2(\Omega, \mathcal{F}, P)$ and $g(y,z,t) = \mu_t|z|$, where $\mu := \{\mu_t\}$ is a continuous function on $[0,T]$, then*

(i) *g-expectation $\mathcal{E}_\mu[\xi]$:*

$$\mathcal{E}_\mu[\xi] = \begin{cases} E\xi, & \mu = 0; \\ \inf_{Q\in\mathcal{P}} \ E_Q[\xi], & \mu < 0; \\ \sup_{Q\in\mathcal{P}} \ E_Q[\xi], & \mu > 0. \end{cases}$$

(ii) *Conditional g-expectation:*

$$\mathcal{E}_\mu[\xi|\mathcal{F}_t] = \begin{cases} E[\xi|\mathcal{F}_t], & \mu = 0; \\ \operatorname{ess\,inf}_{Q\in\mathcal{P}} \ E_Q[\xi|\mathcal{F}_t], & \mu < 0; \\ \operatorname{ess\,sup}_{Q\in\mathcal{P}} \ E_Q[\xi|\mathcal{F}_t], & \mu > 0. \end{cases}$$

(iii) *g-probability (capacity): $\forall A \in \mathcal{F}$,*

$$P_\mu(A) = \begin{cases} P(A), & \mu = 0; \\ \inf_{Q\in\mathcal{P}} \ Q(A), & \mu < 0; \\ \sup_{Q\in\mathcal{P}} \ Q(A), & \mu > 0, \end{cases}$$

where $\mathcal{P} := \left\{ Q^v : \frac{dQ^v}{dP} := e^{-\frac{1}{2}\int_0^T |v_s|^2 ds + \int_0^T v_s dW_s}, |v_t| \leq |\mu_t|, a.e.\ t \in [0,T] \right\}.$

The following are two key lemmas that we shall use in the next section:

Lemma 4 is from Briand et.al.(2000). We rewrite it in the following form. Lemma 5 is from Peng (1997):

LEMMA 4 *Suppose $\{X_t\}$ is of the following process:*

$$dX_t = a_t dt + b_t dB_t$$

where a and b are two continuous, bounded adapted processes. Then

$$\lim_{s\to t} \frac{\mathcal{E}_g[X_s|\mathcal{F}_t] - EX_s}{s-t} = g(a_t, b_t, t),$$

where the limit is in the sense of $L^2(\Omega, \mathcal{F}_t, P)$.

LEMMA 5 *If g is convex (**resp.** concave) in (y,z), then for any $\xi, \eta \in L^2(\Omega, \mathcal{F}, P)$,*

$$\mathcal{E}_g[\xi + \eta|\mathcal{F}_t] \leq \quad (\textbf{resp.} \geq) \quad \mathcal{E}_g[\xi|\mathcal{F}_t] + \mathcal{E}_g[\eta|\mathcal{F}_t], \qquad t \in [0,T].$$

3 Main Result

The main result in this paper is the following Theorem:

THEOREM 1 *Suppose that g satisfies (H.1),(H.2) and (H.3), then the necessary and sufficient condition for $\mathcal{E}_g[\xi]$ to be represented by a Choquet expectation for any $\xi \in L^2(\Omega, \mathcal{F}, P)$ is that g be linear in z, i.e. there exists a continuous function $\nu(t)$ such that*

$$g(y,z,t) = \nu(t)z.$$

The strategy of the proof is the following. First, we shall show that if $\mathcal{E}_g[\cdot]$ can be represented by a Choquet expectation for all random variables with the form $y + zW_T$, then g is of the form $g(y,z,t) = \mu_t|z| + \nu_t z$. Second, we further show if $g-$expectation can be represented by a Choquet expectation for all random variables with the form $I_{[W_T \geq 1]}$ and $I_{[1 \leq W_T \leq 1]}$, then $\mu_t = 0$.

Lemma 6 is the first step. The first part of Lemma 6 shows the uniqueness of capacity:

LEMMA 6 *If there exists a capacity V such that $\mathcal{E}_g[\xi]$ can be represented by a Choquet expectation for any $\xi \in L^2(\Omega, \mathcal{F}, P)$, then*

(i) $V(A) = P_g(A), \quad \forall A \in \mathcal{F}$;

(ii) There exist two continuous functions μ_t, $\nu(t)$ on $[0,T]$ such that g is of the form:

$$g(y,z,t) = \mu_t|z| + \nu(t)z.$$

PROOF. **The Proof of (i):** Let $C(\xi)$ be the Choquet expectation of ξ with respect to V, if

$$\mathcal{E}_g[\xi] = C(\xi), \forall \xi \in L^2(\Omega, \mathcal{F}, P),$$

particularly, for any $A \in \mathcal{F}$, let us choose $\xi = I_A$, thus $\mathcal{E}_g[I_A] = C(I_A)$, but by the definition of Choquet expectation, $C(I_A) = V(A)$ and $P_g(A) = \mathcal{E}_g[I_A]$ completing the proof of (i).

The Proof of (ii): If $\mathcal{E}_g[\cdot]$ can be represented by a Choquet expectation, by Dellacherie's Theorem in Dellacherie (1970), then $\mathcal{E}_g[\cdot]$ is comonotonic additive, that is

$$\mathcal{E}_g[\xi + \eta] = \mathcal{E}_g[\xi] + \mathcal{E}_g[\eta], \quad \text{whenever } \xi \text{ and } \eta \text{ are comonotonic.} \tag{2}$$

Choosing constants (y_1, z_1, t),$(y_2, z_2, t) \in \mathbf{R}^2 \times [0,T]$ such that $z_1 z_2 \geq 0$. For any $\tau \in [t,T]$, denote $\xi = y_1 + z_1(W_\tau - W_t)$ and $\eta = y_2 + z_2(W_\tau - W_t)$.

It is easy to check that ξ and η are comonotonic and independent of $\mathcal{F}_t$. Note that g is deterministic and y_i and z_i $(i = 1, 2)$ are constants. Applying Lemma 3(5),

$$\mathcal{E}_g[\xi|\mathcal{F}_t] = \mathcal{E}_g[\xi], \quad \mathcal{E}_g[\eta|\mathcal{F}_t] = \mathcal{E}_g[\eta], \quad \mathcal{E}_g[\xi + \eta|\mathcal{F}_t] = \mathcal{E}_g[\xi + \eta],$$

this with (2) implies

$$\frac{\mathcal{E}_g[\xi+\eta|\mathcal{F}_t]-E[\xi+\eta|\mathcal{F}_t]}{\tau-t}=\frac{\mathcal{E}_g[\xi|\mathcal{F}_t]-E[\xi|\mathcal{F}_t]}{\tau-t}+\frac{\mathcal{E}_g[\eta|\mathcal{F}_t]-E[\eta|\mathcal{F}_t]}{\tau-t}. \qquad (3)$$

Let $\tau\to t$ on the both sides of (3), by Lemma 4, we obtain

$$g(y_1+y_2,z_1+z_2,t)=g(y_1,z_1,t)+g(y_2,z_2,t),\forall z_1z_2\geq 0,\quad y_1,y_2\in R, \qquad (4)$$

which implies that g is linear with respect to y in $\mathbf{R}$ and z in $\mathbf{R}_+$ (or $\mathbf{R}_-$).

Thus, for any $(y,z,t)\in\mathbf{R}^2\times[0,T]$, note that $g(y,0,t)=0$ in (H.3) and applying Equality(4)

$$\begin{aligned}
g(y,z,t) &= g\left(y+0,zI_{[z\geq 0]}+zI_{[z\leq 0]},t\right)\\
&= g\left(y,zI_{[z\geq 0]},t\right)+g\left(0,zI_{[z\leq 0]},t\right)\\
&= g\left(y+0,0+zI_{[z\geq 0]},t\right)+g\left(0,-(-z)I_{[z\leq 0]},t\right)\\
&= g(y,0,t)+g\left(0,zI_{[z\geq 0]},t\right)+g\left(0,-(-z)I_{[z\leq 0]},t\right)\\
&= g(0,1,t)zI_{[z\geq 0]}-g(0,-1,t)zI_{[z\leq 0]}\\
&= g(0,1,t)z^+ + g(0,-1,t)(-z)^+\\
&= g(0,1,t)\tfrac{|z|+z}{2}+g(0,-1,t)\tfrac{|z|-z}{2}\\
&= \tfrac{g(0,1,t)+g(0,-1,t)}{2}|z|+\tfrac{g(0,1,t)-g(0,-1,t)}{2}z.
\end{aligned}$$

The second equality is because of $zI_{[z\geq 0]}\cdot zI_{[z\leq 0]}=0$. Where $z^+=\max\{z,0\}$.

Set $\mu_t:=\frac{g(0,1,t)+g(0,-1,t)}{2}$ and $\nu(t):=\frac{g(0,1,t)-g(0,-1,t)}{2}$ to complete the proof.

Next, we shall show that $\mu_t=0$ for $t\in[0,T]$. We need the following lemmas. Lemma 7 is a special case of the Comonotonic Theorem in Chen et.al. (2001):

LEMMA 7 *Suppose Φ is a function such that $\Phi(W_T)\in L^2(\Omega,\mathcal{F},P)$. Let (y_t,z_t) be the solution of*

$$y_t=\Phi(W_T)+\int_t^T\mu_t|z_s|ds-\int_t^T z_s dW_s \qquad (5)$$

where μ_t is a continuous function on $[0,T]$.

(i) If Φ is increasing, then $z_t\geq 0$, a.e. $t\in[0,T]$;

(ii) If Φ is decreasing, then $z_t\leq 0$, a.e. $t\in[0,T]$.

PROOF: For the reader's convenience, we sketch the proof.

For $\epsilon>0$, let $g_\epsilon(z,t)=\mu_t\sqrt{z^2+\epsilon}$, then g_ϵ is a smooth C^2-function and $g_\epsilon\to\mu_t|z|$ as $\epsilon\to 0$.

Let $\{y_s^{\epsilon,t,x},z_s^{\epsilon,t,x}\}_{(0\leq s\leq T)}$ be the solution of the BSDE:

$$y_s=\Phi(W_T-W_t+x)+\int_s^T\mu_r\sqrt{z_r^2+\epsilon}dr-\int_s^T z_r dW_r,\quad 0\leq s\leq T,$$

and $\{y_s^{t,x}, z_s^{t,x}\}_{(0\leq s\leq T)}$ be the solution of the BSDE:

$$y_s = \Phi(W_T - W_t + x) + \int_s^T \mu_r |z_r| dr - \int_s^T z_r dW_r, \quad 0 \leq s \leq T.$$

By the convergence theorem of BSDE, see Proposition 2.1 in El. Karouni et.al. (1997),

$$\left\{y_s^{\epsilon,t,x}, z_s^{\epsilon,t,x}\right\}_{(0\leq s\leq T)} \rightarrow \left\{y_s^{t,x}, z_s^{t,x}\right\}_{(0\leq s\leq T)}, \quad \text{as } \epsilon \rightarrow 0$$

in the sense of $\mathcal{M}(0,T;\mathbf{R}) \times \mathcal{M}(0,T;R^d)$.

Moreover, if we choose $x = 0$, $t = 0$ in $\{y_s^{t,x}, z_s^{t,x}\}$, then $\{y_s^{0,0}, z_s^{0,0}\}$ is the solution of BSDE (5). Thus if we can show for each $t \in [0,T]$, $z_s^{\epsilon,t,x} \geq 0$, a.e. $s \in [0,T]$, by the convergence theorem of BSDE in El. Karouni et.al. (1997), we have $z_s^{t,x} \geq 0$, thus $z_t = z_s^{0,0} \geq 0$.

Without loss of generality, we assume that Φ is a smooth C^2-function, otherwise, we can choose a sequence of smooth C^2-functions Φ_ϵ such that $\Phi_\epsilon \rightarrow \Phi$, as $\epsilon \rightarrow 0$.

Let $u_\epsilon(t,x) = y_t^{\epsilon,t,x}$, by the general Feynman-Kac formula, see Proposition 4.3 in El. Karouni et.al. (1997) or Ma et.al. (1994), u_ϵ is the solution of PDE:

$$\begin{cases} \frac{\partial u_\epsilon}{\partial t} + \frac{1}{2}\frac{\partial^2 u_\epsilon}{\partial x^2} + g_\epsilon(x,t) = 0 \\ u_\epsilon(T,x) = \Phi(x), 0 \leq t \leq T. \end{cases}$$

Moreover,

$$z_s^{\epsilon,t,x} = \frac{\partial u_\epsilon(s, W_s - W_t)}{\partial x}.$$

On the other hand, by the comparison theorem of PDE, $u_\epsilon(t,x)$ is increasing in x if Φ is increasing, thus

$$\frac{\partial u_\epsilon(t,x)}{\partial x} \geq 0.$$

Which implies that $z_s^{\epsilon,t,x} \geq 0, s \geq 0$. Let $\epsilon \rightarrow 0$ and $(x,t) = (0,0)$, we obtain (i). Similarly, we can obtain (ii) if Φ is decreasing. The proof is complete.

Furthermore, we can prove:

LEMMA 8 *Let μ_t be a continuous function on $[0,T]$ and (y,z) be the solution of BSDE:*

$$y_t = \xi + \int_t^T \mu_s |z_s| ds - \int_t^T z_s dW_s. \tag{6}$$

(i) If $\xi = I_{[W_T \geq 1]}$, then $z_t > 0$, $\forall t \in [0,T]$;

(ii) If $\xi = I_{[2 \geq W_T \geq 1]}$, then $P \times \lambda\left(\{(\omega,t) : z_t(\omega) < 0\}\right) > 0$

where λ is Lebesgue measure on $[0,T]$ and $P \times \lambda$ is the product of the probability measure P and the Lebesgue measure λ.

PROOF: (i) Note that the indicator function $I_{[x\geq 1]}$ is increasing, by Lemma 7, $z_t \geq 0, a.e. \quad t \in [0,T]$, thus the BSDE (6) is actually a linear BSDE:

$$y_t = I_{[W_T\geq 1]} + \int_t^T \mu_s z_s ds - \int_t^T z_s dW_s.$$

Let

$$\overline{W}_t = W_t - \int_0^t \mu_s ds,$$

then

$$y_t = I_{[W_T\geq 1]} - \int_t^T z_s d\overline{W}_s. \tag{7}$$

Let Q be the probability measure defined by

$$\frac{dQ}{dP} = \exp[-\frac{1}{2}\int_0^T \mu_s^2 ds + \int_0^T \mu_s dW_s].$$

By Girsanov's lemma, $(\overline{W}_t)_{0\leq t\leq T}$ is a $Q-$Brownian motion.

Set conditional expectation $E_Q[.|\mathcal{F}_t]$ on both sides of BSDE (7), by Markov property,

$$\begin{aligned} y_t &= E_Q[I_{[W_T\geq 1]}|\mathcal{F}_t] \\ &= E_Q[I_{[\overline{W}_T\geq 1-\int_0^T \mu_s ds]}|\mathcal{F}_t] \\ &= E_Q[I_{[\overline{W}_T-\overline{W}_t\geq 1-\int_0^T \mu_s ds-\overline{W}_t]}|\mathcal{F}_t] \\ &= E_Q[I_{[\overline{W}_T-\overline{W}_t\geq 1-\int_0^T \mu_s ds-\overline{W}_t]}|\sigma(W_t)]. \end{aligned}$$

Note that $\sigma(W_s; s\leq t) = \sigma(\overline{W}_s; s\leq t)$ because μ_t is deterministic, thus

$$y_t = E_Q[I_{[\overline{W}_T-\overline{W}_t\geq 1-\int_0^T \mu_s ds-\overline{W}_t]}|\sigma(\overline{W}_t)].$$

Since $\overline{W}_T - \overline{W}_t$ and $\overline{W}_t$ are independent. We have

$$y_t = E_Q[I_{[\overline{W}_T-\overline{W}_t\geq 1-\int_0^T \mu_s ds-h]}]\Big|_{h=\overline{W}_t}.$$

But $\overline{W}_T - \overline{W}_t \sim N(0, T-t)$, therefore,

$$y_t = \int_{1-\int_0^T \mu_s ds-h}^{\infty} \varphi(x)dx\Big|_{h=\overline{W}_t},$$

where $\varphi(x)$ is the density function of the normal distribution $N(0, T-t)$.

By the relation between y_t and z_t, see Corollary 4.1 in El. Karouni et.al. (1997), we have

$$z_t = \frac{\partial y_t}{\partial h}\Big|_{h=\overline{W}_t} = \varphi(1 - \int_0^T \mu_s ds - \overline{W}_t) > 0,$$

that is $z_t > 0, a.e. \quad t \in [0,T]$.

(ii) For given $\xi = I_{[2\geq W_T\geq 1]}$, we assume (ii) is not true, then $z_t \geq 0, a.e.$, which implies that BSDE (6) actually is a linear BSDE:

$$y_t = I_{[2\geq W_T\geq 1]} + \int_t^T \mu_s z_s ds - \int_t^T z_s dW_s.$$

That is

$$y_t = I_{[2\geq W_T\geq 1]} - \int_t^T z_s d\overline{W}_s. \tag{8}$$

Where $\overline{W}_t = W_t - \int_0^t \mu_s ds$.

As in (i), let

$$\frac{dQ}{dP} = \exp\left[-\frac{1}{2}\int_0^T \mu_s^2 ds + \int_0^T \mu_s dW_s\right].$$

Applying Girsanov's lemma again, $(\overline{W}_t)_{0\leq t\leq T}$ is a $Q-$Brownian motion.

Set conditional expectation $E_Q[.|\mathcal{F}_t]$ on both sides of BSDE (8). Note that the fact that $\sigma(W_s; s\leq t) = \sigma(\overline{W}_s; s\leq t)$,

$$\begin{aligned} y_t &= E_Q[I_{[2\geq W_T\geq 1]}|\mathcal{F}_t] \\ &= E_Q[I_{[2-\int_0^T \mu_s ds-\overline{W}_t\geq \overline{W}_T-\overline{W}_t\geq 1-\int_0^T \mu_s ds-\overline{W}_t]}|\mathcal{F}_t] \\ &= E_Q[I_{[2-\int_0^T \mu_s ds-\overline{W}_t\geq \overline{W}_T-\overline{W}_t\geq 1-\int_0^T \mu_s ds-\overline{W}_t]}|\sigma(\overline{W}_t)] \\ &= E_Q[I_{[2-\int_0^T \mu_s ds-h\geq \overline{W}_T-\overline{W}_t\geq 1-\int_0^T \mu_s ds-h]}]\Big|_{h=\overline{W}_t} \end{aligned}.$$

Since $\overline{W}_T - \overline{W}_t \sim N(0, T-t)$,

$$y_t = \int_{1-\int_0^T \mu_s ds-h}^{2-\int_0^T \mu_s ds-h} \varphi(x)dx\Big|_{h=\overline{W}_t}.$$

Therefore, applying the relation between y_t and z_t again,

$$\begin{aligned} z_t &= \frac{\partial y_t}{\partial h}\Big|_{h=\overline{W}_t} \\ &= \varphi(1-\textstyle\int_0^T \mu_s ds - \overline{W}_t) - \varphi(2-\int_0^T \mu_s ds - \overline{W}_t) \\ &= \frac{1}{\sqrt{2\pi(T-t)}}\exp[-\frac{(1-\int_0^T \mu_s ds-\overline{W}_t)^2}{2(T-t)}] - \frac{1}{\sqrt{2\pi(T-t)}}\exp[-\frac{(2-\int_0^T \mu_s ds-\overline{W}_t)^2}{2(T-t)}]. \end{aligned}$$

However, it is easy to check that

$$z_t > 0, \quad t\in[0,T], \text{when} \quad \overline{W}_t < \tfrac{3}{2} - \textstyle\int_0^T \mu_s ds;$$

$$z_t < 0, \quad t\in[0,T], \text{when} \quad \overline{W}_t > \tfrac{3}{2} - \textstyle\int_0^T \mu_s ds,$$

which implies

$$P(z_t > 0) > 0, \qquad P(z_t < 0) > 0, \quad a.e. \forall t\in[0,T],$$

thus $P\times\lambda((\omega,t): z_t(\omega) < 0) > 0$. We obtain a contradiction. The proof is complete.

Let $L^2_+(\Omega,\mathcal{F},P)$ (**resp.** $L^2_-(\Omega,\mathcal{F},P)$) be the set of all nonnegative (**resp.** nonpositive) random variables in $L^2(\Omega,\mathcal{F},P)$.

LEMMA 9 *Suppose that g is a convex (or concave) function, if $\mathcal{E}_g[\cdot]$ is comonotonic additive on $L^2_+(\Omega,\mathcal{F},P)$(or $L^2_-(\Omega,\mathcal{F},P)$). Then $\mathcal{E}_g[.|\mathcal{F}_t]$ is also comonotonic additive on $L^2_+(\Omega,\mathcal{F},P)$(or $L^2_-(\Omega,\mathcal{F},P)$) for any $t\in[0,T]$.*

PROOF: We show the above result on $L^2_+(\Omega,\mathcal{F},P)$, the result on $L^2_-(\Omega,\mathcal{F},P)$ can be proved in the same way.

Because $\mathcal{E}_g[\cdot]$ is comonotonic additive on $L^2_+(\Omega,\mathcal{F},P)$, then $\forall\xi,\eta\in L^2_+(\Omega,\mathcal{F},P)$, we have

$$\mathcal{E}_g[\xi+\eta]=\mathcal{E}_g[\xi]+\mathcal{E}_g[\eta],\qquad \text{whenever } \xi \text{ and } \eta \text{ are comonotonic.} \tag{9}$$

We now show $\forall t\in[0,T]$,

$$\mathcal{E}_g[\xi+\eta|\mathcal{F}_t]=\mathcal{E}_g[\xi|\mathcal{F}_t]+\mathcal{E}_g[\eta|\mathcal{F}_t],\qquad \text{whenever } \xi \text{ and } \eta \text{ are comonotonic.} \tag{10}$$

(1) First, we assume that g is a convex function, by Lemma 5,

$$\mathcal{E}_g[\xi+\eta|\mathcal{F}_t]\le\mathcal{E}_g[\xi|\mathcal{F}_t]+\mathcal{E}_g[\eta|\mathcal{F}_t],\forall t\in[0,T].$$

If Eq (10) is false, then there exists $t\in[0,T]$ such that

$$P\left(\omega:\mathcal{E}_g[\xi+\eta|\mathcal{F}_t]<\mathcal{E}_g[\xi|\mathcal{F}_t]+\mathcal{E}_g[\eta|\mathcal{F}_t]\right)>0.$$

Let

$$A=\{\omega:\mathcal{E}_g[\xi+\eta|\mathcal{F}_t]<\mathcal{E}_g[\xi|\mathcal{F}_t]+\mathcal{E}_g[\eta|\mathcal{F}_t]\}.$$

Obviously $A\in\mathcal{F}$, and

$$I_A\mathcal{E}_g[\xi+\eta|\mathcal{F}_t]<I_A\mathcal{E}_g[\xi|\mathcal{F}_t]+I_A\mathcal{E}_g[\eta|\mathcal{F}_t].$$

Set g-expectation $\mathcal{E}_g[\cdot]$ on both sides of the above inequality. By Peng's strict comparison theorem, see Peng (1997), we have

$$\mathcal{E}_g\left[I_A\mathcal{E}_g[\xi+\eta|\mathcal{F}_t]\right]<\mathcal{E}_g\left\{I_A\mathcal{E}_g[\xi|\mathcal{F}_t]+I_A\mathcal{E}_g[\eta|\mathcal{F}_t]\right\}. \tag{11}$$

Observing the above inequality, applying the convexity of g again to the right hand side of (11), applying Lemma 3(3),

$$\mathcal{E}_g\left\{I_A\mathcal{E}_g[\xi|\mathcal{F}_t]+I_A\mathcal{E}_g[\eta|\mathcal{F}_t]\right\}\le\mathcal{E}_g[I_A\mathcal{E}_g[\xi|\mathcal{F}_t]]+\mathcal{E}_g[I_A\mathcal{E}_g[\eta|\mathcal{F}_t]]=\mathcal{E}_g[I_A\xi]+\mathcal{E}_g[I_A\eta].$$

But the left side of (11) is

$$\mathcal{E}_g[I_A\mathcal{E}_g[\xi+\eta|\mathcal{F}_t]]=\mathcal{E}_g[I_A\xi+I_A\eta].$$

Thus

$$\mathcal{E}_g[I_A\xi+I_A\eta]<\mathcal{E}_g[I_A\xi]+\mathcal{E}_g[I_A\eta]. \tag{12}$$

Furthermore, since ξ and η are positive and comonotonic, obviously $I_A\xi, I_A\eta$ also is positive and comonotonic, by the assumption that $\mathcal{E}_g[\cdot]$ is comonotonic additive, and Dellacherie's Theorem (1970),

$$\mathcal{E}_g[I_A\xi + I_A\eta] = \mathcal{E}_g[I_A\xi] + \mathcal{E}_g[I_A\eta]. \tag{13}$$

(12) contradicts (13), thus

$$\mathcal{E}_g[\xi + \eta|\mathcal{F}_t] = \mathcal{E}_g[\xi|\mathcal{F}_t] + \mathcal{E}_g[\eta|\mathcal{F}_t], \quad \forall t \in [0, T].$$

(2) Second, if g is concave, then by Lemma 5

$$\mathcal{E}_g[\xi + \eta|\mathcal{F}_t] \geq \mathcal{E}_g[\xi|\mathcal{F}_t] + \mathcal{E}_g[\eta|\mathcal{F}_t], \forall t \in [0, T].$$

The rest can be proved in a fashion similar to result (i).

Combining Dellacherie's Theorem (1970) and Lemma 9, we immediately obtain

Corollary 1 *Under the assumption of Lemma 9. Assume $\xi \in L^2_+(\Omega, \mathcal{F}, P)$(or $L^2_-(\Omega, \mathcal{F}, P)$), if $\mathcal{E}_g[\xi]$ can be represented by Choquet expectations, then for each $t \in [0, T]$, $\mathcal{E}_g[\xi|\mathcal{F}_t]$ can also be represented by a Choquet expectation.*

We now study the case where g is of the form $g(t, y, z) = \mu_t|z|$. Obviously, if $\mu_t \geq 0, t \in [0, T]$, then g is convex and if $\mu_t \leq 0, t \in [0, T]$, g is concave.

Lemma 10 *Let $\mu_t \neq 0$ be a continuous function on $[0, T]$ and $g(z, t) = \mu_t|z|$, then there exists a random variable $\xi \in L^2(\Omega, \mathcal{F}, P)$ such that $\mathcal{E}_\mu[\xi]$ cannot be represented by Choquet expectation.*

Proof: Assume the result of this lemma is false, then $\mathcal{E}_\mu[\xi]$ can be represented by a Choquet expectations for any $\xi \in L^2(\Omega, \mathcal{F}, P)$. By Dellachere's Theorem (1970), $\mathcal{E}_\mu[.]$ is comonotonic additive on $L^2(\Omega, \mathcal{F}, P)$.

We now choose two special random variables $\xi_1 = I_{[W_T \geq 1]}$ and $\xi_2 = I_{[2 \geq W_T \geq 1]}$. Let $(y^i, z^i), i = 1, 2$ be the solutions of the following BSDEs corresponding to ξ_1 and ξ_2 respectively:

$$y_t = \xi_i + \int_t^T \mu_s|z_s|ds - \int_t^T z_s dW_s, \quad i = 1, 2.$$

And $(\overline{y}_t, \overline{z}_t)$ be the solution of BSDE:

$$\overline{y}_t = (\xi_1 + \xi_2) + \int_t^T \mu_s|\overline{z}_s|ds - \int_t^T \overline{z}_s dW_s$$

then $y^1_t = \mathcal{E}_\mu[\xi_1|\mathcal{F}_t]$, $y^2_t = \mathcal{E}_\mu[\xi_2|\mathcal{F}_t]$ and $\overline{y}_t = \mathcal{E}_\mu[\xi_1 + \xi_2|\mathcal{F}_t]$.

It is easy to show that ξ_1, ξ_2 are positive and comonotonic, by Lemma 9, $\mathcal{E}_\mu[.|\mathcal{F}_t]$ is also comonotonic additive with respect to ξ_1, ξ_2, that is

$$\mathcal{E}_\mu[\xi_1 + \xi_2|\mathcal{F}_t] = \mathcal{E}_\mu[\xi_1|\mathcal{F}_t] + \mathcal{E}_\mu[\xi_2|\mathcal{F}_t], \quad \forall t \in [0,T].$$

Which can be written anther form, that is

$$\overline{y}_t = y_t^1 + y_t^2, \quad \forall t \in [0,T]. \tag{14}$$

Let $< X, W >$ be the finite variation process generated by the semi-martingale X and Brownian motion W, then from (14)

$$< \overline{y}, W >_t = < y^1 + y^2, W >_t = < y^1, W >_t + < y^2, W >_t, \quad \forall t \in [0,T],$$

but

$$\overline{z}_t = \frac{< \overline{y}, W >_t}{dt}, \quad z_t^1 = \frac{< y^1, W >_t}{dt}, \quad z_t^2 = \frac{< y^2, W >_t}{dt}.$$

Thus

$$\overline{z}_t = z_t^1 + z_t^2, \text{a.e.} \quad t \in [0,T].$$

Applying the above inequality to Eq(14), note that Eq (14) can be rewritten as

$$(\xi_1 + \xi_2) + \int_t^T \mu_s|\overline{z}_s|ds - \int_t^T \overline{z}_s dW_s = \sum_{i=1}^{2} \left(\xi_i + \int_t^T \mu_s |z_s^i| ds - \int_t^T z_s^i dW_s \right).$$

We can obtain

$$\mu_t|z_t^1 + z_t^2| = \mu_t|z_t^1| + \mu_t|z_t^2|, \qquad a.e. \quad t \in [0,T].$$

Since $\mu_t \neq 0$, therefore

$$|z_t^1 + z_t^2| = |z_t^1| + |z_t^2|, \qquad a.e. \quad t \in [0,T]. \tag{15}$$

Obviously, equality (15) is true if and only if $z_t^1 z_t^2 \geq 0$.

However from Lemma 8, we know $z_t^1 > 0, a.e. \quad t \in [0,T]$ and

$$P \times \lambda((\omega, t) : z_t^2(\omega) < 0) > 0.$$

Thus $P \times \lambda((\omega, t) : z_t^1(\omega) z_t^2(\omega) < 0) > 0$, which implies

$$P \times \lambda((\omega, t) : |z_t^1(\omega) + z_t^2(\omega)| < |z_t^1(\omega)| + |z_t^2(\omega)|) > 0$$

which contradicts (15). The lemma is complete.

From the above proof, applying strictly comparison theorem of BSDE, Peng (1997), we have

Corollary 2 *If $\mu_t \neq 0, \forall t \in [0,T]$. Let $\xi_1 = I_{[W_T \geq 1]}$ and $\xi_2 = I_{[2 \geq W_T \geq 1]}$, obviously ξ_1 and ξ_2 are comonotonic, but $\mathcal{E}_\mu[\xi_1 + \xi_2] < \mathcal{E}_\mu[\xi_1] + \mathcal{E}_\mu[\xi_2]$.*

We now prove our main theorem:

The proof of Theorem 1 :
Sufficiency: If $g(y,z,t) = \nu_t z$, for any $\xi \in L^2(\Omega, \mathcal{F}, P)$, let us consider BSDE

$$y_t = \xi + \int_t^T \nu_s z_s ds - \int_t^T z_s dW_s.$$

Let $\overline{W}_t = W_t - \int_0^t \nu_s ds$, then

$$y_t = \xi - \int_t^T z_s d\overline{W}_s.$$

By Girsanov's Lemma, $(\overline{W}_t)_{0\le t\le T}$ is a $Q-$ Brownian motion under Q denoted by

$$\frac{dQ}{dP} = \exp[-\frac{1}{2}\int_0^T v_s^2 ds + \int_0^T v_s dW_s].$$

Thus

$$\mathcal{E}_g[\xi|\mathcal{F}_t] = E_Q[\xi|\mathcal{F}_t], \qquad \mathcal{E}_g[\xi] = E_Q[\xi].$$

Which implies $g-$expectation is a classical mathematical expectation. Obviously the classical mathematical expectation can be represented by Choquet expectations. So the sufficiency proof is complete.

Necessary: For any $\xi \in L^2(\Omega, \mathcal{F}, P)$, by Lemma 6(ii), there exist two continuous functions on $[0,T]$ such that

$$g(y,z,t) = \mu_t|z| + \nu(t)z.$$

Without loss of generality, we assume $\nu(t) = 0, t \in [0,T]$, otherwise, by Girsanov's Lemma, we can rewrite the BSDE

$$y_t = \xi + \int_t^T (\mu_s|z_s| + \nu_s z_s)ds - \int_t^T z_s dW_s$$

as

$$y_t = \xi + \int_t^T \mu_s|z_s|ds - \int_t^T z_s d\overline{W}_s \tag{16}$$

where $\overline{W}_s := W_s - \int_0^s \nu(r)dr$, $(\overline{W}_t)_{0\le t\le T}$ is a $Q-$ Brownian motion under Q denoted by

$$\frac{dQ}{dP} = \exp[-\frac{1}{2}\int_0^T \nu_s^2 ds + \int_0^T \nu_s dW_s].$$

We can consider our question on the probability space $(\Omega, \mathcal{F}, Q)$.

Assume $\mu(t) \not\equiv 0$, then there exists t_0 such that $\mu(t_0) \neq 0$. Without loss of generality, we assume $\mu(t_0) > 0$.

Since μ_t is continuous, then there exists a region of t_0, say $[\overline{t}, \overline{T}] \subset [0,T]$ such that $\forall t \in [\overline{t}, \overline{T}], \mu(t) > 0$.

Let $\xi_1 = I_{[W_{\overline{T}} - W_{\overline{t}} \geq 1]}$ and $\xi_2 = I_{[2 \geq W_{\overline{T}} - W_{\overline{t}} \geq 1]}$. Obviously, ξ_1 and ξ_2 are comonotonic.

We now show that

$$\mathcal{E}_\mu[\xi_1 + \xi_2] < \mathcal{E}_\mu[\xi_1] + \mathcal{E}_\mu[\xi_2],$$

which implies that $\mathcal{E}_\mu[\cdot]$ is not comonotonic additive for comonotonic random variables ξ and η.

Let $\overline{W}_s = \overline{W}_{\bar{t}+s} - \overline{W}_{\bar{t}}$, then $\{\overline{W}_s : 0 \leq s \leq \overline{T} - \overline{t}\}$ is $(\mathcal{F}_s^{'})$ Brownian motion, where

$$\mathcal{F}_s^{'} = \sigma\{\overline{W}_r : 0 \leq r \leq s\} = \sigma\{W_{\bar{t}+r} - W_{\bar{t}} : 0 \leq r \leq s\}.$$

Using the above notation, ξ_1 and ξ_2 can be rewritten as $\xi_1 = I_{[\overline{W}_{\overline{T}-\bar{t}} \geq 1]}$ and $\xi_2 = I_{[2 \geq \overline{W}_{\overline{T}-\bar{t}} \geq 1]}$.

For the given ξ_1 and ξ_2, let $a_t = \mu_{t+\bar{t}}$ and (Y^i, Z^i) be the solutions of the following BSDEs with terminal value ξ_1 and ξ_2 respectively on $[0, \overline{T} - \overline{t}]$:

$$Y_t^i = \xi_i + \int_t^{\overline{T}-\bar{t}} a_s|Z_s^i|ds - \int_t^{\overline{T}-\bar{t}} Z_s^i d\overline{W}_s, \quad t \in [0, \overline{T} - \overline{t}], \quad i = 1, 2 \tag{17}$$

and $(\overline{Y}, \overline{Z})$ be the solution of the BSDE:

$$\overline{Y}_t = \xi_1 + \xi_2 + \int_t^{\overline{T}-\bar{t}} a_s|\overline{Z}_s^i|ds - \int_t^{\overline{T}-\bar{t}} \overline{Z}_s^i d\overline{W}_s, \quad t \in [0, \overline{T} - \overline{t}]. \tag{18}$$

Since $a_t = \mu_{\bar{t}+t} \neq 0, \forall t \in [0, \overline{T} - \overline{t}]$, by Corollary 2,

$$\overline{Y}_t < \overline{Y}_t^1 + \overline{Y}_t^2, \quad t \in [0, \overline{T} - \overline{t}]. \tag{19}$$

On the other hand, for the given ξ_1 and ξ_2, consider the BSDE on $[0, T]$:

$$y_t^i = \xi_i + \int_t^T \mu_s|z_s^i|ds - \int_t^T z_s^i dW_s, \quad i = 1, 2, \quad t \in [0, T], \tag{20}$$

and

$$\overline{y}_t = \xi_1 + \xi_2 + \int_t^T \mu_s|\overline{z}_s^i|ds - \int_t^T \overline{z}_s^i dW_s, \quad t \in [0, T]. \tag{21}$$

Comparing (17) with(20) and (18) with (21),

$$Y_t^i = y_t^i, \qquad i = 1, 2; \qquad \overline{Y}_t = \overline{y}_t, \quad t \in [0, \overline{T} - \overline{t}].$$

But $y_t^1 = \mathcal{E}_\mu[\xi_1|\mathcal{F}_t]$, $y_t^2 = \mathcal{E}_\mu[\xi_2|\mathcal{F}_t]$ and $\overline{y}_t = \mathcal{E}_\mu[\xi_1 + \xi_2|\mathcal{F}_t]$.

Thus

$$Y_0^i = \mathcal{E}_\mu[\xi_i], \quad i = 1, 2, \quad \overline{Y}_0 = \mathcal{E}_\mu[\xi_1 + \xi_2].$$

Applying inequality (19),

$$\mathcal{E}_\mu[\xi_1 + \xi_2] < \mathcal{E}_\mu[\xi_1] + \mathcal{E}_\mu[\xi_2].$$

which contradicts the comonotonic additivity of $\mathcal{E}_g[.]$. Thus $\mu(t) = 0, \forall t \in [0, T]$. The proof is complete.

An interesting application of Theorem 1 is:

COROLLARY 3 *Suppose $\mu \neq 0$ and let $\mathcal{E}_\mu[\cdot]$ be the maximal (minimal) expectations defined in Example 1, then maximal (minimal) expectations cannot be represented by a Choquet expectation for all random variables in $L^2(\Omega, \mathcal{F}, P)$.*

REMARK 2 *(1) In the proofs of Lemma 6(ii) and Theorem 1, we only use the random variables with the form $y + zW_t$ and $I_{[W\in(a,b)]}$, thus Lemma 6(ii) and Theorem 1 actually imply that if and only if g is linear in z that $g-$expectation can be represented by Choquet expectation for all all random variables with the form $f(W_T) \in L^2(\Omega, \mathcal{F}, P)$.*

(2) Because g-expectation depends on the choice of g, if g is nonlinear in z, by Theorem 1, then $g-$expectation is not a Choquet expectation.

(3) It is well understood that mathematical expectation is linear in the sense of

$$E(\xi + \eta) = E\xi + E\eta, \quad \forall\, \xi, \eta \in L^2(\Omega, \mathcal{F}, P).$$

For Choquet expectation, the above equality is still true for Choquet expectation when ξ and η are comonotonic. However, for $g-$expectation, if g is nonlinear, the above additivity no longer holds even for comonotonic random variables. From this viewpoint, our result implies Peng's $g-$expectation usually is more nonlinear than Choquet expectation.

4 Feynman-Kac Formula and Choquet Expectation

Let u be the solution of partial differential equation (PDE)

$$\begin{cases} \frac{\partial u(t,x)}{\partial t} = \frac{1}{2}\frac{\partial^2 u(t,x)}{\partial x^2} \\ u(0,x) = f(x), \quad t \geq 0, \quad x \in R. \end{cases} \tag{22}$$

By the famous Feynman-Kac formula, there exists a probability measure such that the solution $u(t,x)$ of PDE (22) can be represented by mathematical expectation:

$$u(t,x) = Ef(W_t + x) \tag{23}$$

where $\{W_t\}$ is a standard Brownian motion and f is a bounded function.

Formula (23) make it possible to solve linear PDE using Monte Carlo methods (the Limit Law Theorem for additive probabilities).

We consider the following example of a nonlinear PDE. Let u be the solution of PDE:

$$\begin{cases} \frac{\partial u(t,x)}{\partial t} = \frac{1}{2}\frac{\partial^2 u(t,x)}{\partial x^2} + g(u, \frac{\partial u(t,x)}{\partial x}) \\ u(0,x) = f(x), \quad t \geq 0. \end{cases} \tag{24}$$

where g is the function satisfying (H.1), (H.2) and (H.3) in Section 2.

If there exists a capacity such that the solution of PDE (24) can be represented by a Choquet expectation, then applying the Limit Law Theorem for non-additive probabilities in Marinacci(1999) and Dow (1994) would suggest a Monte Carlo-like method could be used to solve non-linear PDE(24). Unfortunately, our result shows that this is not generally possible.

THEOREM 2 *In (23), if $g(y,z)$ is nonlinear in z, then there is no a capacity such that for any bounded function f, the solution $u(t,x)$ of PDE(24) can be represented by a Choquet expectation.*

PROOF: Let $\{W_t\}$ be a Brownian motion, by the general Feynman-Kac formula, see for example El. Karouni et.al. (1997) or Ma et.al.(1994), $u(t,x)$ can be represented by $g-$expectation, i.e.

$$u(t,x) = \mathcal{E}_g[f(W_t + x)]$$

Applying Theorem 1 and Remark 2(1), the proof of this theorem is complete.

REMARK 3 *If g is nonlinear in z, the solution of nonlinear PDE(24) cannot be represented by a Choquet expectation for all bounded functions f. However for some special f and g, g-expectation can still be represented by a Choquet expectation. The following is an example:*

EXAMPLE 2 *Suppose μ is a constant, let $g(z) = \mu|z|$ and $\mathcal{E}_g[\cdot]$ be the related g-expectation. Obviously g is nonlinear, but $\mathcal{E}_g[W_T]$ still can represented by a Choquet expectation, that is*

$$\mathcal{E}_g[W_T] = \int_{-\infty}^{0} (P_g(W_T \geq r) - 1)\,dr + \int_0^\infty (P_g(W_T \geq r)dr.$$

PROOF: By the definition of $g-$expectation, $\mathcal{E}_g[W_T]$ is the value of the solution $\{y_t\}$ of the following BSDE at time $t = 0$:

$$y_t = W_T + \int_t^T \mu|z_s|ds - \int_t^T z_s dW_s \tag{25}$$

and $P_g(W_T \geq r) = \mathcal{E}_g[I_{[W_T \geq r]}]$ is the value of the solution $\{Y_t\}$ of the following BSDE at time $t = 0$:

$$Y_t = I_{[W_T \geq r]} + \int_t^T \mu|Z_s|ds - \int_t^T Z_s dW_s. \tag{26}$$

Obviously, $\{y_t, z_t\} = \{W_t + \mu(T-t), 1\}$ is the solution of BSDE (25), thus $\mathcal{E}_g[W_T] = y_0 = \mu T$.

On the other hand, for BSDE (26), since $I_{[x\geq r]}$ is an increasing function, by Lemma 8(i), $Z_t > 0$, thus BSDE (26) actually is linear.

Solving linear BSDE (26) using Girsanov's lemma, we have

$$P_g(W_T \geq r) = \mathcal{E}_g[I_{(W_T\geq r]}] = Y_0 = E_Q[I_{(W_T\geq r]}] = Q(W_T \geq r),$$

where Q is a probability measure defined by

$$\frac{dQ}{dP} = e^{-\frac{1}{2}\mu^2 T + \mu W_T}.$$

This implies by the definition of mathematical expectation,

$$\int_{-\infty}^{0} (P_g(W_T \geq r) - 1)\, dr + \int_{0}^{\infty} (P_g(W_T \geq r) dr = E_Q[W_T].$$

Furthermore, by the following simple calculation, we have $E_Q[W_T] = \mu T$.

In fact, let $x_t = e^{-\frac{1}{2}\mu^2 t + \mu W_t}$, then $\{x_t\}$ is the solution of SDE:

$$x_t = 1 + \mu \int_0^t x_s dW_s$$

and $Ex_t = 1$.

$$\begin{array}{rcl} E_Q[W_T] & = & E[W_T e^{-\frac{1}{2}\mu^2 T + \mu W_T}] \\ & = & E[W_T(1 + \mu \int_0^T x_s dW_s] \\ & = & \mu E[W_T \int_0^T x_s dW_s] \\ & = & \mu E[\int_0^T x_s ds] \\ & = & \mu \int_0^T E x_s ds \\ & = & \mu T. \end{array}$$

The proof is complete.

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