RIEMANN ZETA FUNCTIONS

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Abstract. A Riemann zeta function is a function which is analytic in the complex plane, with the possible exception of a simple pole at one, and which has an Euler product and a functional identity. The functions originate in an adelic generalization of the Laplace transformation which is defined using a theta function. Hilbert spaces, whose elements are entire functions, are obtained on application of the Mellin transformation. Maximal dissipative transformations are constructed in these spaces which have implications for zeros of zeta functions. The zeros of a Riemann zeta function in the critical strip are simple and lie on the critical line. The Euler zeta function, Dirichlet zeta functions, and modular zeta functions are examples of Riemann zeta functions. An application is a construction of Riemann zeta functions in the quantum theory of electrons in an atom.

Since the proof of the Riemann hypothesis is essentially the same for all zeta functions, a unified treatment is given which emphasizes similarities of structure. Those zeta functions which originate in locally compact fields have an elementary structure which is a key to understanding other zeta functions which appear in the related context of locally compact skew–fields. The advantage of locally compact skew–fields lies in their relationship to the three–dimensional space in which the quantum mechanical theory of electrons is formulated. A construction of zeta functions results which interprets quantum mechanics as number theory.

§1. Locally compact skew–fields

An Euclidean skew–field is an associative algebra over the rational numbers with basis consisting of the unit and elements i, j, k satisfying the identities

$$\begin{aligned} &ij = k,\ jk = i,\ ki = j, \\ &ji = -k,\ kj = -i,\ ik = -j, \\ &i^2 = -1,\ j^2 = -1,\ k^2 = -1. \end{aligned}$$

An element

$$\xi = t + ix + jy + kz$$

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of the Euclidean skew–field has four coordinates x, y, z, and t taken in a field which is the center of the algebra. The conjugation of the Euclidean skew–field is the anti–automorphism ξ into ξ^- of order two which takes ξ into

$$\xi^- = t - ix - jy - kz.$$

The product

$$\xi^- \xi = t^2 + x^2 + y^2 + z^2$$

is then a self–conjugate element of the algebra. An element of the algebra is said to be intrinsically nonnegative if for some nonnegative integer r it is a sum

$$\xi_0^- \xi_0 + \ldots + \xi_r^- \xi_r$$

with $\xi_0, \ldots, \xi_r$ elements of the algebra. It is assumed that the sum vanishes only when $\xi_0, \ldots, \xi_r$ all vanish. An intrinsically nonnegative element of the algebra is said to be intrinsically positive if it is nonzero. Since the algebra is a skew–field, an intrinsically positive element is invertible.

An intrinsically convex combination of elements a and b of the algebra is an element

$$a(1 - h) + bh$$

defined by an intrinsically nonnegative element h such that $1-h$ is intrinsically nonnegative. A subset of the algebra is said to be intrinsically convex if it contains the intrinsically convex combinations of any pair of elements.

A nonempty subset of the algebra, which is intrinsically convex, is said to be an intrinsic disk if for every element a of the set and for every element b of the algebra an intrinsically positive element h of the algebra exists such that $1 - h$ is intrinsically nonnegative and such that

$$a(1 - h) + bh$$

belongs to the set.

It will be shown that the intersection of intrinsic disks U and V is an intrinsic disk if it is nonempty. The intersection is intrinsically convex since U and V are intrinsically convex. Assume that a is an element of the intersection of U and V and that b is an element of the algebra. Since U is an intrinsic disk, an intrinsically positive element h of the algebra exists such that $1 - h$ is intrinsically nonnegative and such that

$$a(1 - h) + bh$$

belongs to U. Since V is intrinsically convex, an intrinsically positive element k of the algebra exists such that $1 - k$ is intrinsically nonnegative and such that

$$a(1 - k) + bk$$

belongs to V. Since U is intrinsically convex,

$$a(1-hk)+bhk = a(1-k)+[a(1-h)+bh]k$$

belongs to U. Since V is intrinsically convex

$$a(1-hk)+bhk = a(1-h)+[a(1-k)+bk]h$$

belongs to V. Since the self–conjugate elements of the algebra commute with every element of the algebra, hk is an intrinsically positive element of the algebra such that

$$1-hk = (1-k)+(1-h)k$$

is intrinsically nonnegative and such that

$$a(1-hk)+bhk$$

belongs to the intersection of U and V.

A subset of the algebra is said to be intrinsically open if it is a union of intrinsic disks. An Euclidean skew–field is assumed to be a Hausdorff space in a topology whose open sets are the intrinsically open sets. Addition is continuous as a transformation of the Cartesian product of the algebra with itself into the algebra when the algebra is considered in the intrinsic disk topology.

It will be verified that the closure of an intrinsically convex set C is intrinsically convex. An element of the intrinsically convex span of elements u and v of the closure of C is of the form

$$u(1-h)+vh$$

with h an intrinsically nonnegative element of the algebra such that $1-h$ is intrinsically nonnegative. If an intrinsic disk U contains the origin, elements a and b of C exist such that $u-a$ and $v-b$ belong to U. Since C is intrinsically convex,

$$a(1-h)+bh$$

belongs to C. Since U is intrinsically convex,

$$[u(1-h)+vh]-[a(1-h)+bh] = (u-a)(1-h)+(v-b)h$$

belongs to U.

An intrinsically convex set C can be enlarged using an element s of the algebra which does not belong to C. The set $C(s)$ is defined as the set of intrinsically convex combinations

$$s(1-h)+ch$$

with c an element of C and with h an intrinsically positive element of the algebra such that $1-h$ is intrinsically nonnegative. It will be verified that $C(s)$ is intrinsically convex.

Assume that a and b are elements of C and that h and k are intrinsically positive elements of the algebra such that $1 - h$ and $1 - k$ are intrinsically nonnegative. It will be shown that the intrinsically convex span of the elements

$$s(1 - h) + ah$$

and

$$s(1 - k) + bk$$

of $C(s)$ is contained in $C(s)$. If t is an intrinsically nonnegative element of the algebra such that $1 - t$ is intrinsically nonnegative, then

$$h(1 - t) + kt$$

is an intrinsically positive element of the algebra which is the sum of intrinsically nonnegative elements $h(1 - t)$ and kt. Since C is intrinsically convex, the equation

$$c[h(1 - t) + kt] = ah(1 - t) + bkt$$

has a solution c in C. The intrinsically convex combination

$$\begin{aligned}&[s(1 - h) + ah](1 - t) + [s(1 - k) + bk]t \\ &= s[(1 - h)(1 - t) + (1 - k)t] + c[h(1 - t) + kt]\end{aligned}$$

belongs to $C(s)$.

If an intrinsic disk U contains s and if c is an element of C, then an intrinsically positive element h of the algebra exists such that $1 - h$ is intrinsically nonnegative and such that

$$s(1 - h) + ch$$

belongs to U. If C is nonempty, then $C(s)$ is an intrinsically convex set whose closure contains s.

An application of the Zorn lemma is made to an intrinsic disk A. Every nonempty intrinsically convex set which is disjoint from A is contained in a maximal intrinsically convex set which is disjoint from A. Since the closure of an intrinsically convex set which is disjoint from A is an intrinsically convex set which is disjoint from A, a maximal intrinsically convex set which is disjoint from A is closed for the intrinsic disk topology. It will be shown that the complement of a maximal intrinsically convex set C which is disjoint from A is intrinsically convex. This result is a generalization of the Hahn–Banach theorem [12].

If an element s of the algebra does not belong to C, then the intrinsically convex set $C(s)$ contains C. Since the closure of $C(s)$ is not contained in C, an element of $C(s)$ exists which belongs to A. An element b of C exists such that the intrinsically convex combination

$$a = s(1 - h) + bh$$

belongs to A for some intrinsically positive element h of the algebra such that $1 - h$ is intrinsically nonnegative. Since A is an intrinsic disk, the element h can be chosen so that h and $1 - h$ are intrinsically positive.

It will be shown that the intrinsically convex span of elements s_0 and s_1 of the complement of C is contained in the complement of C. Elements b_0 and b_1 of C exist such that the intrinsically convex combinations

$$a_0 = s_0(1 - h_0) + b_0 h_0$$

and

$$a_1 = s_1(1 - h_1) + b_1 h_1$$

belong to A for intrinsically positive elements h_0 and h_1 of the algebra such that $1 - h_0$ and $1 - h_1$ are intrinsically positive. An intrinsically convex combination s of s_0 and s_1 satisfies the identity

$$s[(1 - h_0)(1 - k) + (1 - h_1)k] = s_0(1 - h_0)(1 - k) + s_1(1 - h_1)k$$

for an intrinsically nonnegative element k of the algebra such that $1 - k$ is intrinsically nonnegative. Since C is intrinsically convex, the equation

$$b[h_0(1 - k) + h_1 k] = b_0 h_0(1 - k) + b_1 h_1 k$$

has a solution b in C. Since A is intrinsically convex,

$$a = a_0(1 - k) + a_1 k$$

belongs to A. Since

$$a = s[(1 - h_0)(1 - k) + (1 - h_1)k] + b[h_0(1 - k) + h_1 k]$$

is an intrinsically convex combination of s and b which does not belong to C, since b belongs to C, and since C is intrinsically convex, s does not belong to C.

If A is an intrinsic disk and if C is a maximal intrinsically convex set which is disjoint from A, then the complement of C is an intrinsic disk.

If an intrinsic disk A has a nonempty complement which is intrinsically convex, then the complement of the closure of the intrinsic disk is an intrinsic disk. If a is an element of A and if b is an element of the complement of A, then the set of self–conjugate elements h of the algebra such that

$$a(1 - h) + bh$$

belongs to A is intrinsically convex. The set of self–conjugate elements h of the algebra such that

$$a(1 - h) + bh$$

belongs to the complement of the closure of A is intrinsically convex. A unique self–conjugate element h of the algebra exists such that

$$a(1-h)+bh$$

does not belong to A and does not belong to the complement of the closure of A.

The Euclidean matrix space is the locally compact ring of square matrices of rank two with complex numbers as entries. A complex number is written $\alpha+\iota\beta$ for real numbers α and β with ι denoting the imaginary unit. Complex conjugation is denoted ξ into ξ^*. The real conjugation of the Euclidean matrix space is the automorphism ξ into ξ^* of order two which takes

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix} \text{ into } \begin{pmatrix} D^* & -C^* \\ -B^* & A^* \end{pmatrix}.$$

The complex conjugation of the Euclidean matrix space is the anti–automorphism ξ into ξ^- of order two which takes

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix} \text{ into } \begin{pmatrix} A^* & C^* \\ B^* & D^* \end{pmatrix}.$$

The real conjugation of the Euclidean matrix space commutes with the complex conjugation of the Euclidean matrix space. The Euclidean skew–plane is a locally compact skew–field whose elements are the elements of the Euclidean matrix space which are left fixed by the real conjugation of the Euclidean matrix space. The conjugation of the Euclidean skew–plane is the restriction to the Euclidean skew–plane of the complex conjugation of the Euclidean matrix space. The Euclidean skew–plane is an Euclidean skew–field with

$$i=\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \quad j=\begin{pmatrix} \iota & 0 \\ 0 & -\iota \end{pmatrix} \quad k=\begin{pmatrix} 0 & \iota \\ \iota & 0 \end{pmatrix}.$$

The Euclidean line is the locally compact field whose elements are the self–conjugate elements of the Euclidean skew–plane.

The intrinsically nonnegative elements of the Euclidean skew–plane are the nonnegative real numbers. The intrinsically positive elements of the Euclidean skew–plane are the positive real numbers. The intrinsically convex subsets of the Euclidean skew–plane are the convex subsets. The Euclidean modulus of an element ξ of the Euclidean skew–plane is the nonnegative square root $|\xi|$ of $\xi^-\xi$. The intrinsic disk topology of the Euclidean skew–plane is the metric topology defined by the metric distance $|\eta-\xi|$ between elements ξ and η of the Euclidean skew–plane. The intrinsic disks of the Euclidean skew–plane are the nonempty convex sets which are open for the Euclidean topology.

The known structure of the Euclidean skew–plane gives information about the structure of an Euclidean skew–field. If an intrinsic disk of an Euclidean skew–field has a nonempty complement which is intrinsically convex, then an isomorphism exists of the Euclidean skew–field into the Euclidean skew–plane such that the given intrinsic disk is the inverse image of a nonempty convex open subset of the Euclidean skew–plane with nonempty convex complement. The isomorphism commutes with conjugation and is continuous from the intrinsic disk topology of the Euclidean skew–field to the Euclidean topology of the Euclidean skew–plane. The intrinsic disk topology of an Euclidean skew–field is the weak topology induced by the isomorphisms of the Euclidean skew–field into the Euclidean skew–plane which commute with conjugation.

The Euclidean skew–diplane is a locally compact skew–field which is identical with the Euclidean skew–plane. The real conjugation of the Euclidean skew–diplane is the identity transformation. The complex conjugation of the Euclidean skew–diplane is the anti–automorphism of order two which is the conjugation of the Euclidean skew–plane. The Euclidean diline is a locally compact field which is identical with the Euclidean line. The conjugation of the Euclidean diline is the identity transformation. An element ω of the Euclidean skew–plane is said to be a unit if $\omega^-\omega$ is the unit of the Euclidean line. If ω is a unit of the Euclidean skew–plane, an automorphism of the Euclidean skew–plane, which commutes with the conjugation of the Euclidean skew–plane, is defined by taking ξ into $\omega^-\xi\omega$. The automorphism is the identity transformation if, and only if, ω is a unit of the Euclidean line.

A group of order eight, which is a normal subgroup of order three in a group of order twenty–four, is formed by the units

$$\pm 1, \ \pm i, \ \pm j, \ \pm k$$

of the Euclidean skew–plane. The sixteen remaining elements of the group of order twenty–four are the units

$$\pm \tfrac{1}{2} \pm \tfrac{1}{2} i \pm \tfrac{1}{2} j \pm \tfrac{1}{2} k$$

of the Euclidean skew–plane. An integral element of the Euclidean skew–plane is a linear combination with integer coefficients of the elements of the group of order twenty–four. If ξ is a nonzero integral element of the Euclidean skew–plane, $\xi^-\xi$ is a positive integer.

The ring of integral elements of the Euclidean skew–plane admits an Euclidean algorithm. If α is an integral element of the Euclidean skew–plane and if β is a nonzero integral element of the Euclidean skew–plane, then integral elements γ and δ of the Euclidean skew–plane exist such that the identity

$$\alpha = \beta\gamma + \delta$$

and the inequality

$$\delta^-\delta < \beta^-\beta$$

are satisfied. A preliminary choice of γ is made as a linear combination of i, j, k, and 1 with integer coefficients. The Euclidean algorithm for integers permits a choice of γ so that the inequalities

$$-1 \leq i(\gamma - \alpha/\beta)^- - (\gamma - \alpha/\beta)i \leq 1$$

and

$$-1 \leq j(\gamma - \alpha/\beta)^- - (\gamma - \alpha/\beta)j \leq 1$$

and

$$-1 \leq k(\gamma - \alpha/\beta)^- - (\gamma - \alpha/\beta)k \leq 1$$

and

$$-1 \leq (\gamma - \alpha/\beta)^- + (\gamma - \alpha/\beta) \leq 1$$

are satisfied. These inequalities imply the inequality

$$\delta^- \delta \leq \beta^- \beta.$$

The choice of γ needs to be improved only when equality holds. This is possible since

$$\omega = \gamma - \alpha/\beta$$

is then an element of the group of order twenty–four which does not belong to the group of order eight. The new choice of γ is made so that δ is zero.

The right ideals of the ring of integral elements of the Euclidean skew–plane admit generators. If the ideal contains a nonzero element, a nonzero element β of the ideal exists which minimizes the positive integer $\beta^- \beta$. Every element α of the ideal is a product

$$\alpha = \beta\gamma$$

with γ an integral element of the Euclidean skew–plane. Nonzero elements α and β of the Euclidean skew–plane admit a greatest common left divisor. A greatest common left divisor of α and β is a nonzero integral element γ of the Euclidean skew–plane which is a common left divisor of α and β such that every common left divisor of α and β is a left divisor of γ.

The adic skew–plane is a locally compact ring which is obtained by completion of a subring of the Euclidean skew–plane in a topology for which addition and multiplication are continuous as transformations of the Cartesian product of the ring with itself into the ring. The ring contains the elements ξ of the Euclidean skew–plane such that the product $n\eta$ is an integral elements of the Euclidean skew–plane for some nonzero integral element η of the Euclidean skew–plane. Basic neighborhoods of the origin for the ring are the right ideals of the ring of integral elements of the Euclidean skew–plane which are generated by nonzero integral elements of the Euclidean skew–plane. The conjugation of the adic skew–plane is the anti–automorphism ξ into ξ^- of order two which continuously extends the conjugation of the ring of integral elements of the Euclidean skew–plane. An integral element of the adic skew–plane is an element of the closure of the set of integral elements of the Euclidean skew–plane. The integral elements of the adic skew–plane form a compact subring which is a neighborhood of the origin for the adic topology. The adic line is a locally compact ring whose elements are the elements of the adic skew–plane which are left fixed by the conjugation of the adic skew–plane. An invertible integral element of the adic skew–plane is said to be a unit if its inverse is integral. If ξ is an invertible element of the adic skew–plane, an invertible element η of the Euclidean skew–plane such that the product $\omega\eta$ is an integral element of the Euclidean skew–plane for some nonzero integral element ω of the Euclidean skew–plane and such that the product $\eta\xi$ is a unit of the adic skew–plane. The adic modulus $|\xi|$ of ξ is defined as the Euclidean modulus of η. The adic modulus of a noninvertible element of the adic skew–plane is zero. The identity

$$|\xi\eta| = |\xi||\eta|$$

holds for all elements ξ and η of the adic skew–plane. The adic modulus of ξ^- is equal to the adic modulus of ξ for every element ξ of the adic skew–plane. The adic modulus of ξ^ξ is a rational number for every element ξ of the adic skew–plane.

The r–adic skew–plane is a locally compact ring which is canonically isomorphic to a quotient ring of the adic skew–plane. The ring is nontrivial when the positive integer r is not equal to one. The r–adic skew–plane is obtained by completion of a subring of the Euclidean skew–plane in a topology for which addition and multiplication are continuous as transformations of the Cartesian product of the ring with itself into the ring. The ring contains the elements ξ of the Euclidean skew–plane such that the product $\eta\xi$ is an integral element of the Euclidean skew–plane for some nonzero integral element η of the Euclidean skew–plane such that the prime divisors of the positive integer $\eta^-\eta$ are divisors of r. Basic neighborhoods of the origin for the ring are the right ideals of the ring of integral elements of the Euclidean skew–plane which are generated by nonzero integral elements η of the Euclidean skew–plane such that the prime divisors of $\eta^-\eta$ are divisors of r. The conjugation of the r–adic skew–plane is the anti–automorphism ξ into ξ^- of order two which continuously extends the conjugation of the ring of integral elements of the Euclidean skew–plane. An integral element of the r–adic skew–plane is an element of the closure of the set of integral elements of the Euclidean skew–plane. The integral elements of the r–adic skew–plane form a compact subring which is a neighborhood of the origin for the r–adic topology. The r–adic line is a locally compact ring whose elements are the elements of the r–adic skew–plane which are left fixed by the conjugation of the r–adic skew–plane. An invertible integral element of the r–adic skew–plane is said to be a unit if its inverse is integral. If ξ is an invertible element of the r–adic skew–plane, an invertible element η of the Euclidean skew–plane exists such that the product $\omega\eta$ is an integral element of the Euclidean skew–plane for some nonzero integral element ω of the Euclidean skew–plane, with the prime divisors of $\omega^-\omega$ divisors of r, and such that the product $\eta\xi$ is a unit of the r–adic skew–plane. The r–adic modulus $|\xi|$ of ξ is defined as the Euclidean modulus of η. The r–adic modulus of a noninvertible element of the r–adic skew–plane is zero. The identity

$$|\xi\eta| = |\xi||\eta|$$

holds for all elements ξ and η of the r–adic skew–plane. The r–adic modulus of ξ^- is equal to the r–adic modulus of ξ for every element ξ of the r–adic skew–plane. The r–adic modulus of $\xi^-\xi$ is a rational number for every element ξ of the r–adic skew–plane.

A theorem which originates with Diophantus and which was confirmed by Lagrange states that every positive integer is the sum of four squares. It follows that every positive integer is of the form $\omega^-\omega$ for an integral element ω of the Euclidean skew–plane. The number of such representations is determined using the Euclidean algorithm for integral elements of the Euclidean skew–plane. If a and b are relatively prime positive integers, the number of representations of ab times the number of representations of one is equal to the number of representations of a times the number of representations of b. Representations are considered equivalent when the representing integral elements of the Euclidean skew–plane generate the same right ideal. Each equivalence class contains twenty–four elements. There is only one equivalence class of representations of any power of the even prime.

When p is an odd prime, the quotient ring of the ring of integral elements of the Euclidean skew–plane modulo the ideal generated by p contains p^4 elements. The equivalence classes are represented as linear combinations of i, j, k and 1 with coefficients in the integers modulo p. The quotient ring is isomorphic to the quotient ring of the ring of integral elements of the p–adic skew–plane modulo the ideal generated by p in the adic skew–plane.

A nonzero skew–conjugate element ω of the quotient ring exists such that $\omega^- \omega$ is equal to zero. The element is constructed of the form

$$\omega = ix + jy + k$$

for integers x and y modulo p which satisfy the equation

$$x^2 + y^2 + 1 = 0.$$

Since the number of integers modulo p which are squares of integers modulo p is

$$1 + \tfrac{1}{2}(p - 1)$$

and since the number of integers modulo p is equal to p, the set of integers modulo p of the form $1 + x^2$ for an integer x modulo p is not disjoint from the set of integers modulo p of the form $-y^2$ for an integer y modulo p. The existence of a solution of the equation follows.

A nonzero right ideal of the ring of integral elements of the Euclidean skew–plane is formed by the elements whose image in the quotient ring belongs to the right ideal generated by a nonzero element γ such that $\gamma^- \gamma$ is equal to zero. A generator ω of the ideal is an integral element ω of the Euclidean skew–plane, whose image in the quotient ring belongs to the right ideal generated by γ, such that γ belong to the image in the quotient ring of the right ideal generated by ω. These conditions imply that the positive integer $\omega^- \omega$ is a divisor of p. Since p is a prime and since $\omega^- \omega$ is not equal to one, $\omega^- \omega$ is equal to p.

The p–adic skew–plane is a locally compact skew–field. A p–adic plane is a locally compact field whose elements are the elements of the p–adic skew–plane which commute with a given integral element γ of the Euclidean skew–plane which is not self–conjugate and which satisfies the identity

$$\omega \gamma = \gamma^- \omega$$

for an integral solution ω of the equation

$$p = \omega^- \omega$$

in the Euclidean skew–plane. The identity

$$\omega\xi = \xi^-\omega$$

holds for every element ξ of the p–adic plane. The conjugation of the p–adic plane is the automorphism ξ into ξ^- of order two which is the restriction of the complex conjugation of the p–adic skew–plane. The p–adic diplane is a locally compact field whose elements are the elements of the p–adic skew–diplane on which the real and complex conjugations agree. The p–adic line is a locally compact field whose elements are the elements of the p–adic plane which are left fixed by the conjugation of the p–adic plane. The p–adic modulus of a nonzero element of the p–adic plane is an integral power of p.

The properties of the p–adic plane are used to determine the number of representations of an odd prime p of the form $\omega^-\omega$ with ω in the p–adic skew–line. The integral elements of the p–adic plane form a subring of the ring of integral elements of the p–adic skew–plane. The quotient ring of the ring of integral elements of the p–adic skew–plane modulo the right ideal generated by p is a ring containing p^4 elements of which p^2 elements belong to the quotient ring of the ring of integral elements of the p–adic plane. The quotient ring of the p–adic plane is a field containing the quotient ring of the p–adic line. The p–adic plane is constructed using an integral element ω of the Euclidean skew–plane such that

$$p = \omega^-\omega.$$

The identity

$$\omega\xi = \xi^-\omega$$

holds for every element ξ of the p–adic plane. The elements of the p–adic skew–plane modulo p are of the form

$$\alpha + \omega\beta$$

with α and β in the p–adic plane modulo p. The identity

$$(\alpha + \omega\beta)^-(\alpha + \omega\beta) = 0$$

holds in the p–adic skew–plane modulo p if, and only if, the identity

$$\alpha = 0$$

holds in the p–adic plane modulo p.

If ξ and η are integral elements of the Euclidean skew–plane such that $\xi^-\xi$ and $\eta^-\eta$ are equal to p, then η belongs to the right ideal generated by ξ if, and only if, ξ belongs to the right ideal generated by η. Such representations are considered equivalent. Each equivalence class contains twenty–four members in the Euclidean skew–plane. If α is the element of the quotient ring of the Euclidean skew–plane represented by ξ and if β is the element of the quotient ring represented by η, then β belongs to the right ideal generated by α if, and only if, α belongs to the right ideal generated by β. Such elements α and β are considered equivalent. Each equivalence class contains $p^2 - 1$ members. Each equivalence class is invariant under multiplication by nonzero self–conjugate elements of the quotient ring, of which there are $p-1$. The number of equivalence classes in the quotient ring is $p+1$. Since the elements of the p–adic skew–plane are equivalent if, and only if, their images in the quotient ring are equivalent, the number of equivalence classes is $p+1$.

A theorem of Jacobi states that the number of representations of an positive integer r in the form $\omega^-\omega$ with ω an integral element of the Euclidean skew–plane is equal to twenty–four times the sum of the odd divisors of r.

The p–adic skew–diplane is a locally compact skew–field which contains the p–adic skew–plane. The p–adic skew–plane is canonically isomorphic to the ring of matrices which are linear combinations of

$$i = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \quad j = \begin{pmatrix} \iota & 0 \\ 0 & -\iota \end{pmatrix} \quad k = \begin{pmatrix} 0 & \iota \\ \iota & 0 \end{pmatrix}$$

and the identity matrix with coefficients in the p–adic line. The p–adic skew–diplane is canonically isomorphic to the ring of matrices which are linear combinations of i, j, k and the identity matrix with coefficients in the field obtained by adjoining a square root of p to the p–adic line. The real conjugation of the p–adic skew–diplane is the automorphism ξ into ξ^* which leaves fixed the elements of the p–adic skew–plane and which is nontrivial on other elements of the p–adic skew–diplane. The complex conjugation of the p–adic skew–diplane is the anti–automorphism ξ into ξ^- of order two which extends the conjugation of the p–adic skew–plane and which leaves fixed the multiplies of the identity matrix. The real conjugation of the p–adic skew–diplane commutes with the complex conjugation of the p–adic skew–diplane. The p–adic diline is a locally compact field whose elements are the elements of the p–adic skew–diplane which are left fixed by the complex conjugation of the p–adic skew–diplane. The conjugation of the p–adic diline is the automorphism ξ into ξ^* of order two which is the restriction of the real conjugation of the p–adic skew–diplane. The p–adic diplane is a locally compact field whose elements of the p–adic skew–diplane which are linear combinations of elements of the p–adic plane with coefficients in the p–adic diline. The real conjugation of the p–adic diplane is the restriction of the real conjugation of the p–adic skew–diplane. The complex conjugation of the p–adic diplane is the restriction of the complex conjugation of the p–adic skew–diplane.

The r–adic skew–diplane is a locally compact ring which contains the r–adic skew–plane and which is canonically isomorphic to the Cartesian product of the p–adic skew–diplane taken over the prime divisors p of r. An element of the r–adic skew–diplane is said to be integral if its p–adic component is integral for every prime divisor p of r. The real conjugation of the r–adic skew–diplane is the automorphism ξ into ξ^* of order two such that the p–adic component of ξ^* is obtained from the p–adic component of ξ under the real conjugation of the p–adic skew–diplane for every prime divisor p of r. The complex conjugation of the r–adic skew–diplane is the anti–automorphism ξ into ξ^- of order two such that the p–adic component of ξ is obtained from the p–adic component of ξ under the complex conjugation of the p–adic skew–diplane for every prime divisor p of r. The real conjugation of the r–adic skew–diplane commutes with the complex conjugation of the r–adic skew–diplane. The r–adic diline is a locally compact ring whose elements are the elements of the r–adic skew–diplane which are left fixed by the complex conjugation of the r–adic skew–diplane. The conjugation of the r–adic diline is the automorphism ξ into ξ^* of order two which is the restriction of the real conjugation of the r–adic skew–diplane. The r–adic modulus of an element ξ of the r–adic diline is the nonnegative square root of the element $\xi^*\xi$ of the r–adic line. The r–adic diplane is a locally compact ring whose elements are the elements of the r–adic skew–diplane whose p–adic component belongs to the p–adic diplane for every prime divisor p of r. The real conjugation of the r–adic diplane is the automorphism ξ into ξ^* of order two which is the restriction of the real conjugation of the r–adic skew–diplane. The complex conjugation of the r–adic diplane is the automorphism ξ into ξ^- of order two which is the restriction of the complex conjugation of the r–adic skew–diplane. The r–adic modulus of an element $\xi = t + ix + jy + kz$ of the r–adic skew–diplane is the nonnegative square root $|\xi|$ of the r–adic modulus of the element $x^2 + y^2 + z^2 + t^2$ of the r–adic diline. The identity

$$|\xi\eta| = |\xi||\eta|$$

holds for all elements ξ and η of the r–adic skew–diplane.

The adic skew–diplane is a locally compact ring which contains the adic skew–plane and which has as a quotient ring for every positive integer r a ring which is canonically isomorphic to the r–adic skew–diplane. The adic skew–diplane is canonically isomorphic to a subring of the Cartesian product of the p–adic skew–diplanes taken over all primes p. An element of the Cartesian product determines an element of the adic skew–diplane if, and only if, its p–adic component is integral for all but a finite number of primes p. An element of the adic skew–diplane is said to be integral if its p–adic component is integral for every prime p. The real conjugation of the adic skew–diplane is the automorphism ξ into ξ^* of order two such that the p–adic component of ξ^* is obtained from the p–adic component of ξ under the real conjugation of the p–adic skew–diplane for every prime p. The complex conjugation of the adic skew–diplane is the anti–automorphism ξ into ξ^- of order two such that the p–adic component of ξ^- is obtained from the p–adic component of ξ under the complex conjugation of the p–adic skew–diplane for every prime p. The real conjugation of the adic skew–diplane commutes with the complex conjugation of the adic skew–diplane. The adic diline is a locally compact ring whose elements are the elements of the adic skew–diplane which are left fixed by the complex conjugation of the adic skew–diplane. The conjugation of the adic diline is the automorphism ξ into ξ^* of order two which is the restriction to the adic diline of the real conjugation of the adic skew–diplane. The r–adic modulus of an element ξ of the adic diline is the nonnegative square root $|\xi|$ of the adic line. The adic diplane is a locally compact ring whose elements are the elements of the adic skew–diplane whose p–adic component belongs to the p–adic diplane for every prime p. The real conjugation of the adic diplane is the automorphism ξ into ξ^* of order two which is the restriction of the real conjugation of the adic skew–diplane. The complex conjugation of the adic diplane is the automorphism ξ into ξ^- of order two which is the restriction of the complex conjugation of the adic skew–diplane. The adic modulus of an element $\xi = t + ix + jy + kz$ of the adic skew diplane is the nonnegative square root $|\xi|$ of the adic modulus of the element $x^2 + y^2 + z^2 + t^2$ of the adic diline. An element of the adic skew–diplane is said to be a unit if its adic modulus is one.

The adelic skew–diplane is a locally compact ring which is canonically isomorphic to the Cartesian product of the Euclidean skew–diplane and the adic skew–diplane. An element of the adelic skew–diplane has a Euclidean component ξ_+ in the Euclidean skew–diplane and an adic component ξ_- in the adic skew–diplane. The real conjugation of the adelic skew–diplane is the automorphism ξ into ξ^* of order two such that the Euclidean component of ξ^* is obtained from the Euclidean component of ξ under the real conjugation of the Euclidean skew–diplane and the adic component of ξ^* is obtained from the adic component of ξ under the real conjugation of the adic skew–diplane. The complex conjugation of the r–adelic skew–diplane is the anti–automorphism ξ into ξ^- such that the Euclidean component of ξ^- is obtained from the Euclidean component of ξ under the complex conjugation of the Euclidean skew–diplane and the adic component of ξ is obtained from the adic component of ξ^- under the complex conjugation of the adic skew–diplane. The real conjugation of the adelic skew–diplane commutes with the complex conjugation of the adelic skew–diplane. The Euclidean modulus of an element ξ of the adelic skew–diplane is the Euclidean modulus $|\xi|_+$ of its Euclidean component ξ_+. The adic modulus of an element ξ of the adelic skew–diplane is the adic modulus $|\xi|_-$ of its adic component ξ_-. The adelic modulus of an element ξ of the adelic skew–diplane is the product $|\xi|$ of its Euclidean modulus $|\xi|_+$ and its r–adic modulus $|\xi|_-$. An element of the adelic skew–diplane is said to be a unit if its Euclidean modulus and its adic modulus are one. An element of the adelic skew–diplane is said to be unimodular if its adelic modulus is one.

The adelic skew–plane is a locally compact ring whose elements are the elements of the adelic skew–diplane which are left fixed by the real conjugation of the adelic skew–diplane. The conjugation of the adelic skew–plane is the anti–automorphism ξ into ξ^- of order two which is the restriction of the complex conjugation of the Euclidean skew–diplane. A principal element of the adelic skew–plane is an element whose Euclidean component has an integral product with a nonzero integral element of the Euclidean skew–plane and whose adic component is represented by its Euclidean component. A nonzero principal element of the adelic skew–plane is unimodular. The adelic diline is a locally compact ring whose elements are the elements of the adelic skew–diplane which are left fixed by the complex conjugation of the adelic skew–diplane. The conjugation of the adelic diline is the automorphism ξ into ξ^* of order two which is the restriction of the real conjugation of the adelic skew–diplane. The adelic line is a locally compact ring whose elements are the elements of the adelic skew–plane which are left fixed by the conjugation of the adelic skew–plane. The adelic line is the set of elements are the elements of the adelic diline which are left fixed by the conjugation of the adelic diline. A principal element of the adelic line is an element of the adelic line which is a principal element of the adelic skew–plane.

The Euclidean line is a locally compact field whose elements are the self–conjugate elements of the Euclidean skew–plane. An Euclidean plane is a locally compact field whose elements are the elements of the Euclidean skew–plane which commute with a given element of the Euclidean skew–plane which is not self–conjugate. The associated Euclidean diplane is identical with the Euclidean plane. An example of an Euclidean plane is the set of elements of the Euclidean skew–plane which commute with i. The elements of the Euclidean plane are of the form $x+iy$ with x and y elements of the Euclidean line. Another example of an Euclidean plane is associated with a prime p when it is represented

$$p = \omega^- \omega$$

with ω an integral element of the Euclidean skew–plane and when γ is an integral solution of the equation

$$\omega\gamma = \gamma^- \omega$$

in the Euclidean skew–plane whose residue class modulo p is invertible but not self–conjugate. A p–adic diplane is defined as the set of elements of the p–adic skew–diplane which commute with γ. A related Euclidean plane is the set of elements of the Euclidean skew–diplane which commute with γ.

The adelic diplane is a locally compact ring whose elements are the elements of the adelic skew–diplane whose Euclidean component belongs to the Euclidean diplane and whose adic component belongs to the adic diplane. The real conjugation of the adelic diplane is the automorphism ξ into ξ^* of order two which is the restriction of the real conjugation of the adelic skew–diplane. The complex conjugation of the adelic diplane is the automorphism ξ into ξ^- of order two which is the restriction of the complex conjugation of the adelic skew–diplane. The real conjugation of the adelic diplane commutes with the complex conjugation of the adelic diplane. The adelic plane is a locally compact ring whose element are the elements of the adelic diplane which are left fixed by the real conjugation of the adelic diplane. The conjugation of the adelic plane is the automorphism ξ into ξ^- of order two which is the restriction of the complex conjugation of the adelic diplane. The adelic diline is the locally compact ring whose elements are the elements of the adelic diplane which are left fixed by the complex conjugation of the adelic diplane. The conjugation of the adelic diline is the restriction of the real conjugation of the adelic diplane. The conjugation of the adelic diline is the automorphism ξ into ξ^* of order two which is the restriction of the real conjugation of the adelic diplane. The adelic line is the set of elements of the adelic plane which are left fixed by the conjugation of the adelic plane. The adelic line is also the set of elements of the adelic diline which are left fixed by the conjugation of the adelic diline.

The r–adelic skew–diplane is a locally compact ring which is canonically isomorphic to the Cartesian product of the Euclidean skew–diplane and the r–adic skew–diplane. An element ξ of the r–adelic skew–diplane has a Euclidean component ξ_+ in the Euclidean skew–diplane and an r–adic component ξ_- in the r–adic skew–diplane. The real conjugation of the r–adelic skew–diplane is the automorphism ξ into ξ^* of order two such that the Euclidean component of ξ^* is obtained from the Euclidean component of ξ under the real conjugation of the Euclidean skew–diplane and the r–adic component of ξ^* is obtained from the r–adic component of ξ under the real conjugation of the r–adic skew–diplane. The complex conjugation of the r–adelic skew–diplane is the anti–automorphism ξ into ξ^- of order two such that the Euclidean component of ξ^- is obtained from the Euclidean component of ξ under the complex conjugation of the Euclidean skew–diplane and the r–adic component of ξ^- is obtained from the r–adic component of ξ under the complex conjugation of the r–adic skew–diplane. The real conjugation of the r–adelic skew–diplane commutes with the complex conjugation of the r–adelic skew–diplane. The Euclidean modulus of an element ξ of the r–adelic skew–diplane is the Euclidean modulus $|\xi|_+$ of its Euclidean component ξ_+. The r–adic modulus of an element ξ of the r–adelic skew–diplane is the r–adic modulus $|\xi|_-$ of its r–adic component ξ_-. The r–adelic modulus of an element ξ of the r–adelic skew–diplane is the product $|\xi|$ of its Euclidean modulus $|\xi|_+$ and its adic modulus $|\xi|_-$. An element of the r–adelic skew–diplane is said to be a unit if its Euclidean modulus and its r –adic modulus are one. An element of the r–adelic skew–diplane is said to be unimodular if its r–adelic modulus is one.

The r–adelic skew–plane is a locally compact ring whose elements are the elements of the r–adelic skew–diplane which are left fixed by the real conjugation of the r–adelic skew–diplane. The conjugation of the r–adelic skew–plane is the anti–automorphism ξ into ξ^- of order two which is the restriction of the complex conjugation of the r–adelic skew–diplane. The r–adelic diline is a locally compact ring whose elements are the elements of the r–adelic skew–diplane which are left fixed by the complex conjugation of the r–adelic skew–diplane. The conjugation of the r–adelic diline is the automorphism ξ into ξ^* of order two which is the restriction of the real conjugation of the r–adelic skew–diplane. The r–adelic line is a locally compact ring whose elements are the elements of the r–adelic diline which are left fixed by the conjugation of the r–adelic diline. A principal element of the r–adelic skew–plane is an element whose Euclidean component has an integral product with a nonzero integral element η of the Euclidean skew–plane, such that the prime divisors of $\eta^-\eta$ are divisors of r, and whose r–adic component is represented by its Euclidean component. A nonzero principal element of the r–adelic skew–plane is unimodular.

The r–adelic diplane is a locally compact ring whose elements are the elements of the r–adelic skew–diplane whose Euclidean component belongs to the Euclidean diplane and whose adic component belongs to the adic diplane. The real conjugation of the r–adelic diplane is the automorphism ξ into ξ^* of order two which is the restriction of the real conjugation of the r–adelic skew–diplane. The complex conjugation of the r–adelic diplane is the automorphism ξ into ξ^- of order two which is the restriction of the complex conjugation of the r–adelic skew–diplane. The r–adelic plane is a locally compact ring whose elements are the elements of the r–adelic diplane which are left fixed by the real conjugation of the r–adelic diplane. The conjugation of the r–adelic plane is the automorphism ξ into ξ^- of order two which is the restriction of the complex conjugation of the r–adelic diplane. The r–adelic diline is the set of elements of the r–adelic diplane which are left fixed by the complex conjugations of the r–adelic diplane. The conjugation of the r–adelic diline is the restriction of the real conjugation of the r–adelic diplane. The r–adelic line is the set of automorphism ξ into ξ^* of order two which is the elements of the r–adelic plane which are left fixed by the conjugation of the r–adelic plane. The r–adelic line is also the set of elements of the r–adelic diline which are left fixed by the conjugation of the r–adelic diline.

The fundamental domain for the Euclidean skew–plane is the set of elements ξ of the Euclidean skew–plane such that $\frac{1}{2}\xi + \frac{1}{2}\xi^-$ is a unit. The canonical measure for the fundamental domain is a nonnegative measure on the Borel subsets of the domain which is characterized within a constant by invariance properties. Measure preserving transformations are defined by taking ξ into $\omega\xi$ and ξ into $\xi\omega$ for every unit ω of the Euclidean skew–plane. The transformation which takes ξ into

$$\tfrac{1}{2}(\xi + \xi^-) + \tfrac{1}{2}\omega^-(\xi - \xi^-)\omega$$

multiplies the canonical measure by the fourth power of the Euclidean modulus of $\omega^-\omega$ for every element ω of the Euclidean skew–plane. The measure is normalized so that the set of elements of the canonical domain whose skew–conjugate component has Euclidean modulus less than one has measure π. A Hilbert space is constructed from the space of functions $f(\xi)$ of ξ in the Euclidean skew–plane which satisfy the identity

$$f(\tfrac{1}{2}\omega(\xi + \xi^-) + \tfrac{1}{2}\omega^{-1}(\xi - \xi^-)) = |\omega| f(\xi)$$

for every nonzero element ω of the Euclidean line and which are square integrable with respect to the canonical measure for the fundamental domain. An isometric transformation of the space into itself is defined by taking a function $f(\xi)$ of ξ in the Euclidean skew–plane into the function $f(\omega^-\xi\omega)$ of ξ in the Euclidean skew–plane for every unit ω of the Euclidean skew–plane. The space decomposes into invariant subspaces under the action of the group.

An auxiliary Hilbert space is constructed for every nonnegative integer ν. The elements of the space are homogeneous functions of degree ν in the variables x, y, z with

$$\xi = ix + jy + kz.$$

An element of the space is a linear combination of monomials

$$x^a y^b z^c$$

with a, b, c nonnegative integers such that

$$\nu = a + b + c.$$

A scalar product is introduced in the space for which the monomials form an orthogonal set. The scalar self–product of the monomial with exponents a, b, c is

$$\frac{a!\ b!\ c!}{(1 + 2\nu)\nu!}$$

An isometric transformation of the space into itself is defined by taking a function $f(\xi)$ of skew–conjugate elements ξ of the Euclidean skew–plane into the function $f(\omega^-\xi\omega)$ of skew–conjugate elements ξ of the Euclidean skew–plane for every unit ω of the Euclidean skew–plane. The Laplacian acts as a linear transformation of the space of homogeneous polynomials of degree ν onto the space of homogeneous polynomials of degree $\nu - 2$. Every element of the space of homogeneous polynomials of degree ν is annihilated when ν is less than two. The transformation commutes with the transformation which takes a function $f(\xi)$ of skew–conjugate elements ξ of the Euclidean skew–diplane into the function $f(\omega^-\xi\omega)$ of skew–conjugate element ξ of the Euclidean skew–diplane for every unit ω of the Euclidean skew–plane. The space of homogeneous harmonic polynomials of degree ν is the kernel of the Laplacian as it acts on homogeneous polynomials of degree ν. The space of homogeneous harmonic polynomials of degree ν is an invariant subspace of dimension $1 + 2\nu$ over the Euclidean line for the transformations $f(\xi)$ into $f(\omega^- \xi \omega)$.

A construction of homogeneous harmonic polynomials of degree ν is made with respect to an Euclidean plane. The construction is now made for the Euclidean plane whose elements commute with i. Solutions are given in terms of the hypergeometric series

$$F(a, b; c; z) = 1 + \frac{ab}{1c}\, z + \frac{a(a+1)b(b+1)}{1 \cdot 2c(c+1)}\, z^2 + \dots .$$

A harmonic polynomial is a function of

$$ix + jy + kz$$

which is a solution of the Laplace equation in the variables x, y, z. When the variables

$$\xi = x + iy$$

and

$$\eta = x - iy$$

are used in place of the variables x and y, a harmonic polynomial of degree ν is a function of ξ, η, and z which satisfies the equation

$$4\, \frac{\partial^2 \phi}{\partial \xi \partial \eta} + \frac{\partial^2 \phi}{\partial z^2} = 0.$$

Basic solutions are

$$\phi(\xi, \eta, z) = z^{\nu - k} \xi^k F(\tfrac{1}{2}k - \tfrac{1}{2} - \tfrac{1}{2}\nu, \tfrac{1}{2}k - 1 - \tfrac{1}{2}\nu; k + 1; -\xi\eta/z^2)$$

and

$$\phi(\xi, \eta, z) = z^{\nu - k} \eta^k F(\tfrac{1}{2}k - \tfrac{1}{2} - \tfrac{1}{2}\nu, \tfrac{1}{2}k - 1 - \tfrac{1}{2}\nu; k + 1; -\xi\eta/z^2)$$

with $k = 0, \dots, \nu$. The solutions coincide when k is equal to zero.

If ρ is a positive integer, a character χ modulo ρ is a function $\chi(n)$ of integers n, which is periodic of period ρ, which satisfies the identity

$$\chi(mn) = \chi(m)\chi(n)$$

for all integers m and n, which has absolute value one at integers which are relatively prime to ρ, and which has value zero otherwise. A character χ modulo ρ is said to be primitive modulo ρ if no character modulo a proper divisor of ρ exists which agrees with χ at integers which are relatively prime to ρ. If a character χ modulo ρ is primitive modulo ρ, a number $\epsilon(\chi)$ of absolute value one exists such that the identity

$$\rho^{\frac{1}{2}} \epsilon(\chi) \chi(n)^- = \sum \chi(k) \exp(2\pi i n k/\rho)$$

holds for every integer n with summation over the residue classes of integers k modulo ρ. The principal character modulo ρ is the character modulo ρ whose only nonzero value is one. The principal character modulo ρ is primitive modulo ρ when, and only when, ρ is equal to one.

The residue classes of integers modulo ρ are identified with the residue classes of integral elements of the p–adic line modulo ρ. A character χ modulo ρ is treated as a function of integral elements of the ρ–adic line which has equal values at elements whose difference is divisible by ρ. The character acts as a homomorphism of the group of units of the ρ–adic line into the complex numbers of absolute value one. The character vanishes at integral elements of the p–adic line which are not units. The character is extended to the ρ–adic line so as to vanish at elements which are not units.

A character χ modulo ρ admits an extension to the ρ–adic diplane which acts as a homomorphism of the group of units of the ρ–adic diplane into the complex numbers of absolute value one and which vanishes at elements of the ρ–adic diplane which are not units. The choice of extension is inessential in the present applications. The extended character is also denoted χ. The conjugate character χ^* is defined by the identity

$$\chi^*(\xi) = \chi(\xi^-)^-$$

for every element ξ of the p–adic diplane. If r is a positive integer, which is divisible by ρ, such that r/ρ is relatively prime to ρ and is not divisible by the square of a prime, the character is extended to the r–adic diplane so as to have equal values at elements of the r–adic diplane which have equal p–adic component for every prime divisor p of ρ. The character is extended to the adic diplane so as to have equal values at elements of the adic diplane which have equal p–adic component for every prime divisor p of ρ.

Analogues of characters for the adic skew–diplane are constructed from homogeneous harmonic polynomials of degree ν which satisfy a symmetry condition. The polynomials are treated as functions of skew–conjugate elements

$$\xi = ix + jy + kz$$

of the Euclidean skew–plane. The symmetry condition for a function $f(\xi)$ of ξ states that the identity

$$f(\xi) = f(\omega^- \xi \omega)$$

holds for every integral unit ω of the Euclidean skew–plane. Homogeneous polynomials of degree ν which satisfy the symmetry condition are linear combinations of elementary symmetric functions which are defined as sums

$$\sum \operatorname{sgn}^\nu(a, b, c) x^a y^b z^c$$

over the nonnegative integers a, b, c of the same parity as ν with sum ν. The signature $\operatorname{sgn}(a, b, c)$ is one for an even permutation, minus one for an odd permutation of the exponents initially written is descending order.

The dimension of the space of homogeneous polynomials of degree ν which satisfy the symmetry condition is

$$\frac{(\nu-1)(\nu-5)}{48}$$

when ν is congruent to one or five modulo twelve,

$$\frac{(\nu+2)(\nu+10)}{48}$$

when ν is congruent to two or ten modulo twelve,

$$\frac{(\nu+4)(\nu+8)}{48}$$

when ν is congruent to four or eight modulo twelve,

$$\frac{(\nu+1)(\nu-7)}{48}$$

when ν is congruent to seven or eleven modulo twelve,

$$\frac{1}{3}+\frac{(\nu+1)(\nu-7)}{48}$$

when ν is congruent to three modulo twelve,

$$\frac{1}{3}+\frac{(\nu+2)(\nu+10)}{48}$$

when ν is congruent to six modulo twelve,

$$\frac{1}{3}+\frac{(\nu-1)(\nu-5)}{48}$$

when ν is congruent to nine modulo twelve, and

$$\frac{1}{3}+\frac{(\nu+4)(\nu+8)}{48}$$

when ν is divisible by twelve.

The dimension of the space of homogeneous harmonic polynomials of degree ν which satisfy the symmetry condition is

$$\frac{\nu-1}{12}$$

when ν is congruent to one modulo twelve,

$$\frac{\nu-2}{12}$$

when ν is congruent to two modulo twelve,

$$\frac{\nu - 3}{12}$$

when ν is congruent to three modulo twelve,

$$\frac{\nu + 8}{12}$$

when ν is congruent to four modulo twelve,

$$\frac{\nu - 5}{12}$$

when ν is congruent to five modulo twelve,

$$\frac{\nu + 6}{12}$$

when ν is congruent to six modulo twelve,

$$\frac{\nu - 7}{12}$$

when ν is congruent to seven modulo twelve

$$\frac{\nu + 4}{12}$$

when ν is congruent to eight modulo twelve,

$$\frac{\nu + 3}{12}$$

when ν is congruent to nine modulo twelve,

$$\frac{\nu + 2}{12}$$

when ν is congruent to ten modulo twelve,

$$\frac{\nu - 11}{12}$$

when ν is congruent to eleven modulo twelve, and

$$\frac{\nu + 12}{12}$$

when ν is divisible by twelve.

A self–adjoint transformation $\Delta(n)$ is defined for every positive integer n in the space of homogeneous harmonic polynomials of degree ν which satisfy the symmetry condition. The transformation takes a function $f(\xi)$ of skew–conjugate elements ξ of the Euclidean skew–plane into a function $g(\xi)$ of skew–conjugate elements ξ of the Euclidean skew–plane when the identity

$$24n^\nu g(\xi) = \sum f(\omega^- \xi \omega)$$

holds with summation over the representations

$$n = \omega^- \omega$$

with ω an integral element of the Euclidean skew–plane. The identity

$$\Delta(m)\Delta(n) = \sum \Delta(mn/k^2)$$

holds for all positive integers m and n with summation over the common odd divisors k of m and n. The Hilbert space of homogeneous harmonic polynomials of degree ν which satisfy the symmetry admits an orthonormal basis consisting of eigenfunctions of the transformation. A basic element is an eigenfunction of $\Delta(n)$ for a real eigenvalue $\tau(n)$ for every positive integer n. The identity

$$\tau(m)\tau(n) = \sum \tau(mn/k^2)$$

holds for all positive integers m and n with summation over the common odd divisors k of m and n.

§2. The Radon transformation for the Euclidean skew–plane

The Hankel transformation of order ν for an Euclidean plane is identical with the Hankel transformation of order ν for the associated Euclidean diplane. The transformation is defined when ν is a nonnegative integer for the Euclidean diplane whose elements commute with i. The character of order ν for the Euclidean plane is the homomorphism χ of the multiplicative group of invertible elements of the Euclidean plane into the nonzero complex numbers which takes $x + iy$ into

$$(x + iy)^\nu.$$

The canonical measure for the Euclidean plane is Lebesgue measure. If a function $f(\xi)$ of ξ in the Euclidean plane is square integrable with respect to the canonical measure for the Euclidean plane and satisfies the identity

$$f(\omega\xi) = \chi(\omega) f(\xi)$$

for every unit ω of the Euclidean plane, then its Hankel transform of order ν for the Euclidean plane is a function $g(\xi)$ of ξ in the Euclidean plane which is square integrable with respect to the canonical measure for the Euclidean plane and which satisfies the identity

$$g(\omega\xi) = \chi(\omega) g(\xi)$$

for every unit ω of the Euclidean plane. A positive parameter ρ is included in the definition of the transformation for application to zeta functions. The transformation takes a function $f(\xi)$ of ξ in the Euclidean plane into a function $g(\xi)$ of ξ in the Euclidean plane if the identity

$$\int \chi(\xi)^- g(\xi) \exp(\pi iz\xi^-\xi/\rho)d\xi = (i/z)^{1+\nu} \int \chi(\xi)^- f(\xi) \exp(-\pi iz^{-1}\xi^-\xi/\rho)d\xi$$

holds when z is in the upper half–plane with integration with respect to the canonical measure for the Euclidean plane. The identity

$$\int |f(\xi)|^2 d\xi = \int |g(\xi)|^2 d\xi$$

holds with integration with respect to the canonical measure for the Euclidean plane. The function $f(\xi)$ of ξ in the Euclidean plane is the Hankel transform of order ν for the Euclidean plane of the function $g(\xi)$ of ξ in the Euclidean plane.

A Hankel transformation of order ν for the Euclidean skew–plane is identical with a Hankel transformation of order ν for the Euclidean skew–diplane. The analogue of a character is a function $\phi(\xi)$ of ξ in the Euclidean skew–diplane which satisfies the identity

$$\phi(\tfrac{1}{2}\omega(\xi + \xi^-) + \tfrac{1}{2}\omega^{-1}(\xi - \xi^-)) = \phi(\xi)$$

for every invertible element ω of the Euclidean line and whose restriction to elements

$$\xi = t + ix + jy + kz$$

of the Euclidean skew–plane with t equal to the unit of the Euclidean line is an element of the orthonormal basis for the Hilbert space of homogeneous harmonic polynomials of degree ν in x, y, z. The conjugate harmonic function ϕ^* is defined by the identity

$$\phi^*(\xi) = \phi(\xi^-)^-.$$

The Laplace kernel for the Euclidean skew–diplane is obtained from a computation of the integral

$$\int_0^\infty \exp(\pi izt^2|\tfrac{1}{2}\xi + \tfrac{1}{2}\xi^-|^2) \exp(\pi izt^{-2}|\tfrac{1}{2}\xi - \tfrac{1}{2}\xi^-|^2)dt$$
$$= (-8iz)^{-\frac{1}{2}}|\tfrac{1}{2}\xi - \tfrac{1}{2}\xi^-|^{-1} \exp(2\pi iz|\tfrac{1}{2}\xi + \tfrac{1}{2}\xi^-||\tfrac{1}{2}\xi - \tfrac{1}{2}\xi^-|)dt$$

when z is in the upper half–plane. The square root of $-8iz$ is taken in the right half–plane. The domain of the Hankel transformation of order ν and harmonic ϕ for the Euclidean skew–plane is the space of functions $f(\xi)$ of ξ in the Euclidean skew–plane which satisfy the identity

$$f(\tfrac{1}{2}\omega(\xi + \xi^-) + \tfrac{1}{2}\omega^{-1}(\xi - \xi^-)) = |\omega| f(\xi)$$

for every invertible element ω of the Euclidean diline, which satisfy the identity

$$\phi(\xi)f(\omega^-\xi\omega) = \phi(\omega^-\xi\omega)f(\xi)$$

for every unit ω of the Euclidean skew–plane, and which are square integrable with respect to the canonical measure for the fundamental domain of the Euclidean skew–plane. The range of the Hankel transformation of order ν and harmonic ϕ for the Euclidean skew–plane is the domain of the Hankel transformation of order ν and harmonic ϕ^* for the Euclidean skew–plane. The transformation takes a function $f(\xi)$ of ξ in the Euclidean skew–plane into a function $g(\xi)$ of ξ in the Euclidean skew–plane when the identity

$$\int \phi^*(\xi)^- g(\xi) |\tfrac{1}{2}\xi - \tfrac{1}{2}\xi^-|^{-1} \exp(2\pi i z |\tfrac{1}{2}\xi + \tfrac{1}{2}\xi^-||\tfrac{1}{2}\xi - \tfrac{1}{2}\xi^-|) d\xi$$
$$= (i/z)^{2+2\nu} \int \phi(\xi)^- f(\xi) |\tfrac{1}{2}\xi - \tfrac{1}{2}\xi^-|^{-1} \exp(-2\pi i z^{-1} |\tfrac{1}{2}\xi + \tfrac{1}{2}\xi^-||\tfrac{1}{2}\xi - \tfrac{1}{2}\xi^-|) d\xi$$

holds for z in the upper half–plane with integration with respect to the canonical measure for the fundamental domain of the Euclidean skew–diplane. The identity

$$\int |f(\xi)|^2 d\xi = \int |g(\xi)|^2 d\xi$$

holds with integration with respect to the canonical measure for the fundamental domain. The function $f(\xi)$ of ξ in the Euclidean skew–diplane is the Hankel transform of order ν and harmonic ϕ^* for the Euclidean skew–diplane of the function $g(\xi)$ of ξ in the Euclidean skew–diplane.

The Laplace transformation of order ν for the Euclidean plane is identical with the Laplace transformation of order ν for the Euclidean diplane. The Laplace transformation of order ν for the Euclidean plane permits a computation of the Hankel transformation of order ν for the Euclidean plane. The domain of the transformation is the space of functions $f(\xi)$ of ξ in the Euclidean plane which are square integrable with respect to the canonical measure for the Euclidean plane and which satisfy the identity

$$f(\omega\xi) = \chi(\omega) f(\xi)$$

for every unit ω of the Euclidean diplane. The Laplace transform of order ν for the Euclidean plane of the function $f(\xi)$ of ξ in the Euclidean plane is the function $g(z)$ of z in the upper half–plane defined by the integral

$$2\pi g(z) = \int \chi(\xi)^- f(\xi) \exp(\pi i z \xi^- \xi / \rho) d\xi$$

with respect to the canonical measure for the Euclidean plane. The integral can be written

$$2\pi g(x+iy) = \pi \int_0^\infty \chi(\xi)^- f(\xi) \exp(-\pi t y/\rho) \exp(\pi i t x/\rho) dt$$

as a Fourier integral for the Euclidean line under the constraint

$$t = \xi^- \xi.$$

The identity

$$(2/\rho)\int_{-\infty}^{+\infty}|g(x+iy)|^2dx=\int_0^\infty |f(\xi)|^2t^\nu \exp(-2\pi ty/\rho)dt$$

holds by the isometric property of the Fourier transformation for the Euclidean line. When ν is zero, the identity

$$(2\pi/\rho)\sup\int_{-\infty}^{+\infty}|g(x+iy)|^2dx=\int|f(\xi)|^2d\xi$$

holds with the least upper bound taken over all positive numbers y. The identity

$$(2\pi/\rho)^{1+\nu}\int_0^\infty\int_{-\infty}^{+\infty}|g(x+iy)|^2y^{\nu-1}dxdy=\Gamma(\nu)\int|f(\xi)|^2d\xi$$

holds when ν is positive. Integration on the right is with respect to the canonical measure for the Euclidean plane. An analytic function $g(z)$ of z in the upper half–plane is a Laplace transform of order ν for the Euclidean plane if a finite least upper bound

$$\sup\int_{-\infty}^{+\infty}|g(x+iy)|^2dx$$

is obtained over all positive numbers y when ν is zero and if the integral

$$\int_0^\infty\int_{-\infty}^{+\infty}|g(x+iy)|^2y^{\nu-1}dxdy$$

is finite when ν is positive. The space of Laplace transforms of order ν for the Euclidean plane is a Hilbert space of functions analytic in the upper half–plane when considered with the scalar product for which the Laplace transformation of order ν for the Euclidean plane is isometric. The Hankel transformation of order ν for the Euclidean plane is unitarily equivalent under the Laplace transformation of order ν for the Euclidean plane to the isometric transformation in the space of analytic functions which takes $g(z)$ into

$$(i/z)^{1+\nu}g(-1/z).$$

The Laplace transformation of order ν and harmonic ϕ for the Euclidean skew–plane permits a computation of the Hankel transformation of order ν and harmonic ϕ for the Euclidean skew–plane. The domain of the transformation is the space of functions $f(\xi)$ of ξ in the Euclidean skew–plane which satisfy the identity

$$f(\tfrac{1}{2}\omega(\xi+\xi^-)+\tfrac{1}{2}\omega^{-1}(\xi-\xi^-))=|\omega|f(\xi)$$

for every invertible element ω of the Euclidean line, which satisfy the identity

$$\phi(\xi)f(\omega^-\xi\omega)=\phi(\omega^-\xi\omega)f(\xi)$$

for every unit ω of the Euclidean skew–plane, and which are square integrable with respect to the canonical measure for the Euclidean skew–plane. The Laplace transform of order ν and harmonic ϕ for the Euclidean skew–diplane of the function $f(\xi)$ of ξ in the Euclidean skew–plane is the analytic function $g(z)$ of z in the upper half–plane defined by the integral

$$4\pi g(z) = \int \phi(\xi)^- f(\xi) |\tfrac{1}{2}\xi - \tfrac{1}{2}\xi^-|^{-1} \exp(2\pi i z |\tfrac{1}{2}\xi + \tfrac{1}{2}\xi^-||\tfrac{1}{2}\xi - \tfrac{1}{2}\xi^-|) d\xi$$

with respect to the canonical measure for the fundamental domain. The identity

$$(4\pi)^{2+2\nu} \int_0^\infty \int_{-\infty}^{+\infty} |g(x+iy)|^2 y^{2\nu} dx dy = \Gamma(1+2\nu) \int |f(\xi)|^2 d\xi$$

is satisfied. Integration on the right with respect to the canonical measure for the fundamental domain. An analytic function $g(z)$ of z in the upper half–plane is a Laplace transform of order ν and harmonic ϕ for the Euclidean skew–plane if the integral

$$\int_0^\infty \int_{-\infty}^{+\infty} |g(x+iy)|^2 y^{2\nu} dx dy$$

is finite. The space of Laplace transforms of order ν and harmonic ϕ for the Euclidean skew–plane is a Hilbert space of functions analytic in the upper half–plane when considered with the scalar product for which the Laplace transformation of order ν and harmonic ϕ for the Euclidean skew–plane is isometric. An isometric transformation in the Hilbert space of analytic functions is defined by taking $g(z)$ into

$$(i/z)^{2+2\nu} g(-1/z).$$

A relation T with domain and range in a Hilbert space is said to be maximal dissipative if the relation $T + w$ has an everywhere defined inverse for some complex number w in the right half–plane and if the relation

$$(T - w)(T + w)^{-1}$$

is a contractive transformation. The condition holds for every element w of the right half–plane if it holds for some element w of the right half–plane.

The Radon transformation of order ν for the Euclidean plane is identical with the Radon transformation of order ν for the Euclidean diplane. The transformations are defined when ν is equal to zero or one. The Radon transformation of order ν for the Euclidean plane is a maximal dissipative transformation in the space of functions $f(\xi)$ of ξ in the Euclidean plane which are square integrable with respect to the canonical measure for the Euclidean plane and which satisfy the identity

$$f(\omega\xi) = \chi(\omega) f(\xi)$$

for every unit ω of the Euclidean plane. The transformation takes a function $f(\xi)$ of ξ in the Euclidean plane into a function $g(\xi)$ of ξ in the Euclidean plane when the identity

$$g(\xi) = \int f(\xi + \eta) d\eta$$

holds formally with integration with respect to Haar measure for the space of elements η of the Euclidean plane such that

$$\eta^- \xi + \xi^- \eta = 0.$$

Haar measure if normalized so that the set of elements of Euclidean modulus less than one has measure two. The integral is accepted as the definition when

$$f(\xi) = \chi(\xi) \exp(\pi i z \xi^- \xi / \rho)$$

with z in the upper half–plane. The identity

$$g(\xi) = (i\rho/z)^{\frac{1}{2}} f(\xi)$$

then holds with the square root of $i\rho/z$ taken in the right half–plane. The adjoint of the Radon transformation of order ν for the Euclidean plane takes a function $f(\xi)$ of ξ in the Euclidean plane into a function $g(\xi)$ of ξ in the Euclidean plane when the identity

$$\int \chi(\xi)^- g(\xi) \exp(\pi i z \xi^- \xi / \rho) d\xi = (i\rho/z)^{\frac{1}{2}} \int \chi(\xi)^- f(\xi) \exp(\pi i z \xi^- \xi / \rho) d\xi$$

holds when z is in the upper half–plane with integration with respect to the canonical measure for the Euclidean plane. The square root of $i\rho/z$ is taken in the right half–plane.

The Radon transformation of order ν and harmonic ϕ for the Euclidean skew–plane is identical with the Radon transformation of order ν and harmonic ϕ for the Euclidean skew–diplane. The Radon transformation of order ν and harmonic ϕ for the Euclidean skew–plane is a maximal dissipative transformation in the space of functions $f(\xi)$ of ξ in the Euclidean skew–plane which satisfy the identity

$$f(\tfrac{1}{2}\omega(\xi - \xi^-) + \tfrac{1}{2}\omega^{-1}(\xi - \xi^-)) = |\omega| f(\xi)$$

for every invertible element ω of the Euclidean diline, which satisfy the identity

$$\phi(\xi) f(\omega^- \xi \omega) = \phi(\omega^- \xi \omega) f(\xi)$$

for every unit ω of the Euclidean skew–diplane, and which are square integrable with respect to the canonical measure for the Euclidean skew–plane. The transformation takes the function

$$f(\xi) = \phi(\xi) |\xi - \xi^-)^{-1} \exp(2\pi i z |\tfrac{1}{2}\xi + \tfrac{1}{2}\xi^-||\tfrac{1}{2}\xi - \tfrac{1}{2}\xi^-|)$$

of ξ in the Euclidean skew–plane into the function

$$g(\xi) = (i/z)f(\xi)$$

of ξ in the Euclidean skew–plane when z is in the upper half–plane. The adjoint of the Radon transformation of order ν and harmonic ϕ for the Euclidean skew–plane takes a function $f(\xi)$ of ξ in the Euclidean skew–plane into a function $g(\xi)$ of ξ in the Euclidean skew–plane when the identity

$$\int \phi(\xi)^- g(\xi)|\tfrac{1}{2}\xi - \tfrac{1}{2}\xi^-|^{-1} \exp(2\pi iz|\tfrac{1}{2}\xi + \tfrac{1}{2}\xi^-||\tfrac{1}{2}\xi - \tfrac{1}{2}\xi^-|)d\xi$$
$$= (i/z) \int \phi(\xi)^- f(\xi)|\tfrac{1}{2}\xi - \tfrac{1}{2}\xi^-|^{-1} \exp(2\pi iz|\tfrac{1}{2}\xi + \tfrac{1}{2}\xi^-||\tfrac{1}{2}\xi - \tfrac{1}{2}\xi^-|)d\xi$$

holds when z is in the upper half–plane. Integration is with respect to the canonical measure for the fundamental domain of the Euclidean skew–plane.

The Mellin transformation of order ν for the Euclidean plane is identical with the Mellin transformation of order ν for the Euclidean plane. The domain of the Mellin transformation of order ν for the Euclidean diplane is the space of functions $f(\xi)$ of ξ in the Euclidean plane which are square integrable with respect to the canonical measure for the Euclidean plane, which satisfy the identity

$$f(\omega\xi) = \chi(\omega)f(\xi)$$

for every unit ω of the Euclidean plane, and which vanish in a neighborhood of the origin. The Laplace transform of order ν for the Euclidean plane of the function $f(\xi)$ of ξ in the Euclidean plane is the analytic function $g(z)$ of z in the upper half–plane defined by the integral

$$2\pi g(z) = \int \chi(\xi)^- f(\xi) \exp(\pi iz\xi^-\xi/\rho)d\xi$$

with respect to the canonical measure for the Euclidean plane. The Mellin transform of order ν for the Euclidean plane of the function $f(\xi)$ of ξ in the Euclidean plane is an analytic function

$$F(z) = \int_0^\infty g(it)t^{\frac{1}{2}\nu - \frac{1}{2} - \frac{1}{2}iz}dt$$

of z in the upper half–plane. Since the function

$$W(z) = (\pi/\rho)^{-\frac{1}{2}\nu - \frac{1}{2} + \frac{1}{2}iz}\Gamma(\tfrac{1}{2}\nu + \tfrac{1}{2} - \tfrac{1}{2}iz)$$

admits an integral representation

$$W(z) = (\xi^-\xi)^{\frac{1}{2}\nu + \frac{1}{2} - \frac{1}{2}iz} \int_0^\infty \exp(-\pi t\xi^-\xi/\rho)t^{\frac{1}{2}\nu - \frac{1}{2} - \frac{1}{2}iz}dt$$

when z is in the upper half–plane, the identity

$$2\pi F(z)/W(z) = \int_0^\infty \chi(\xi)^- f(\xi)|\xi|^{iz-\nu-1} d\xi$$

holds when z is in the upper half–plane with integration with respect to the canonical measure for the Euclidean plane. If $f(\xi)$ vanishes when $|\xi| < a$, the identity

$$\sup \int_{-\infty}^{+\infty} a^{2y} |F(x+iy)/W(x+iy)|^2 dx = \int |f(\xi)|^2 d\xi$$

holds with the upper bound taken over all positive numbers y. Integration on the right is with respect to the canonical measure for the Euclidean diplane.

The Mellin transformation of order ν and harmonic ϕ for the Euclidean skew–plane is identical with the Mellin transformation of order ν and harmonic ϕ for the Euclidean skew–diplane. The domain of the Mellin transformation of order ν and harmonic ϕ for the Euclidean skew–plane is the space of functions $f(\xi)$ of ξ in the Euclidean skew–plane which satisfy the identity

$$f(\tfrac{1}{2}\omega(\xi + \xi^-) + \tfrac{1}{2}\omega^{-1}(\xi - \xi^-)) = |\omega| f(\xi)$$

for every invertible element ω of the Euclidean line, which satisfy the identity

$$\phi(\xi) f(\omega^- \xi \omega) = \phi(\omega^- \xi \omega) f(\xi)$$

for every unit ω of the Euclidean skew–plane, which are square integrable with respect to the canonical measure for the fundamental domain of the Euclidean skew–plane, and which vanish in a neighborhood of the origin. The Laplace transform of order ν and harmonic ϕ for the Euclidean skew–plane is the analytic function $g(z)$ of z in the upper half–plane which is defined by the integral

$$4\pi g(z) = \int \phi(\xi)^- f(\xi) |\tfrac{1}{2}\xi - \tfrac{1}{2}\xi^-|^{-1} \exp(2\pi i z |\tfrac{1}{2}\xi + \tfrac{1}{2}\xi^-| |\tfrac{1}{2}\xi - \tfrac{1}{2}\xi^-|) d\xi$$

with respect to the canonical measure for the fundamental domain. The Mellin transform of order ν and harmonic ϕ for the Euclidean skew–plane of the function $f(\xi)$ of ξ in the Euclidean skew–plane is the analytic function

$$F(z) = \int_0^\infty g(it) t^{\nu - iz} dt$$

of z in the upper half–plane. Since the function

$$W(z) = (2\pi)^{-\nu-1+iz} \Gamma(\nu + 1 - iz)$$

admits the integral representation

$$W(z) = |\tfrac{1}{2}\xi + \tfrac{1}{2}\xi^-|^{\nu+1-iz}|\tfrac{1}{2}\xi - \tfrac{1}{2}\xi^-|^{\nu+1-iz} \times \int_0^\infty \exp(-2\pi t|\tfrac{1}{2}\xi + \tfrac{1}{2}\xi^-||\tfrac{1}{2}\xi - \tfrac{1}{2}\xi^-|)t^{\nu-iz}dt$$

when z is in the upper half–plane, the identity

$$4\pi F(z)/W(z) = \int \phi(\xi)^- f(\xi)|\tfrac{1}{2}\xi - \tfrac{1}{2}\xi^-|^{iz-\nu-1}|\tfrac{1}{2}\xi - \tfrac{1}{2}\xi^-|^{iz-\nu-2}d\xi$$

holds when z is in the upper half–plane with integration with respect to the canonical measure for the fundamental domain. If $f(\xi)$ vanishes when

$$|\tfrac{1}{2}\xi + \tfrac{1}{2}\xi^-||\tfrac{1}{2}\xi - \tfrac{1}{2}\xi^-| < a,$$

the identity

$$\sup \int_{-\infty}^{+\infty} a^{2y}|F(x+iy)/W(x+iy)|^2 dx = \tfrac{1}{2}\int |f(\xi)|^2 d\xi$$

holds with the least upper bound taken over all positive numbers y. Integration on the right is with respect to the canonical measure for the fundamental domain.

§3. The Riemann hypothesis for Hilbert spaces of entire functions

A characterization of Mellin transforms is made in weighted Hardy spaces. An analytic weight function is a function which is analytic and without zeros in the upper half–plane. The weighted Hardy space $\mathcal{F}(W)$ associated with an analytic weight function $W(z)$ is the Hilbert space $\mathcal{F}(W)$ whose elements are the analytic functions $F(z)$ of z in the upper half–plane such that a finite least upper bound

$$\|F\|^2_{\mathcal{F}(W)} = \sup \int_{-\infty}^{+\infty} |F(x+iy)/W(x+iy)|^2 dx$$

is obtained over all positive numbers y. Since $F(z)/W(z)$ is of bounded type as a function of z in the upper half–plane, a boundary value function $F(x)/W(x)$ is defined almost everywhere with respect to Lebesgue measure on the real axis. The identity

$$\|F\|^2_{\mathcal{F}(W)} = \int_{-\infty}^{+\infty} |F(x)/W(x)|^2 dx$$

is satisfied. A continuous linear functional on the space is defined by taking $F(z)$ into $F(w)$ when w is in the upper half–plane. The reproducing kernel function for function values at w is

$$\frac{W(z)W(w)^-}{2\pi i(w^- - z)}.$$

The classical Hardy space for the upper half–plane is the weighted Hardy space $\mathcal{F}(W)$ when $W(z)$ is identically one. Multiplication by $W(z)$ is an isometric transformation of the classical Hardy space onto the weighted Hardy space $\mathcal{F}(W)$ whenever $W(z)$ is an analytic weight function for the upper half–plane.

The analytic weight function

$$W(z) = (\pi/\rho)^{-\frac{1}{2}\nu-\frac{1}{2}+\frac{1}{2}iz}\Gamma(\tfrac{1}{2}\nu + \tfrac{1}{2} - \tfrac{1}{2}iz)$$

appears in the characterization of Mellin transforms of order ν for the Euclidean plane. A maximal dissipative transformation in the weighted Hardy space $\mathcal{F}(W)$ is defined by taking $F(z)$ into $F(z+i)$ whenever $F(z)$ and $F(z+i)$ belong to the space.

The analytic weight function

$$W(z) = (2\pi)^{-\nu-1+iz}\Gamma(\nu + 1 - iz)$$

appears in the characterization of Mellin transforms of order ν for the Euclidean skew–plane. A maximal dissipative transformation in the weighted Hardy space $\mathcal{F}(W)$ is defined by taking $F(z)$ into $F(z+i)$ whenever $F(z)$ and $F(z+i)$ belong to the space.

Weighted Hardy spaces $\mathcal{F}(W)$ appear in which a maximal dissipative transformation is defined for some positive number h by taking $F(z)$ into $F(z+ih)$ whenever $F(z)$ and $F(z+ih)$ belong to the space. The existence of the maximal dissipative transformation is equivalent to properties of the weight function [9]. Since the adjoint transformation takes the reproducing kernel function

$$\frac{W(z)W(w-\frac{1}{2}ih)^-}{2\pi i(w^- + \frac{1}{2}ih - z)}$$

for function values at $w - \frac{1}{2}ih$ into the reproducing kernel function

$$\frac{W(z)W(w+\frac{1}{2}ih)^-}{2\pi i(w^- - \frac{1}{2}ih - z)}$$

for function values at $w + \frac{1}{2}ih$ whenever $w - \frac{1}{2}ih$ is in the upper half–plane, the function

$$\frac{W(z-\frac{1}{2}ih)W(w+\frac{1}{2}ih)^- + W(z+\frac{1}{2}ih)W(w-\frac{1}{2}ih)^-}{2\pi i(w^- - z)}$$

of z in the upper half–plane is the reproducing kernel function for function values at w for a Hilbert space whose elements are functions analytic in the half–plane. The function $W(z - \frac{1}{2}ih)$ has an analytic extension to the upper half–plane such that

$$W(z-\tfrac{1}{2}ih)/W(z+\tfrac{1}{2}ih)$$

has nonnegative real part in the half–plane. This property of the weight function characterizes the weighted Hardy spaces which admit the maximal dissipative transformation.

If for a weight function $W(z)$ the function $W(z - \frac{1}{2}ih)$ has an analytic extension to the upper half–plane such that

$$W(z - \tfrac{1}{2}ih)/W(z + \tfrac{1}{2}ih)$$

has nonnegative real part in the half–plane, then a maximal dissipative transformation in the space $\mathcal{F}(W)$ is defined by taking $F(z)$ into $F(z + ih)$ whenever $F(z)$ and $F(z + ih)$ belong to the space.

The existence of a maximal dissipative transformation for some positive number h implies the existence of a maximal dissipative transformation for a smaller positive number h when an additional hypothesis is satisfied. Assume that the modulus of the weight function $W(z)$ is a nondecreasing function of distance from the real axis on every vertical half–line in the upper half–plane. An equivalent condition is that the real part of

$$iW'(z)/W(z)$$

is nonnegative in the upper half–plane. A Hilbert space, whose elements are functions analytic in the upper half–plane, exists which contains the function

$$\frac{W(z)W'(w)^- - W'(z)W(w)^-}{\pi(z - w^-)}$$

of z as reproducing kernel function for function values at w when w is in the upper half–plane. If a maximal dissipative transformation in the space $\mathcal{F}(W)$ is defined for some positive number h by taking $F(z)$ into $F(z+ih)$ whenever $F(z)$ and $F(z+ih)$ belong to the space and if a positive number k is less than h, then a maximal dissipative transformation in the space $\mathcal{F}(W)$ is defined by taking $F(z)$ into $F(z + ik)$ whenever $F(z)$ and $F(z + ik)$ belong to the space.

The analytic weight functions appearing in the proof of the Riemann hypothesis have properties which generalize those of the gamma function [10]. Assume that the modulus of an analytic weight function $W(z)$ is a nondecreasing function of distance from the real axis on every vertical half–line in the upper half–plane. If for some positive number h the function has an analytic extension to the half–plane $-h = iz^- - iz$ such that the function

$$W(z - \tfrac{1}{2}ih)/W(z + \tfrac{1}{2}ih)$$

of z has nonnegative part in the upper half–plane and if k is a positive number less than h, then the function

$$W(z - \tfrac{1}{2}ik)/W(z + \tfrac{1}{2}ik)$$

of z will be shown to have nonnegative real part in the upper half–plane. Since the function is bounded by one in the upper half–plane and is continuous in the closure of the half–plane, it is sufficient to show that the function has nonnegative real part on the real axis. It is sufficient to show that the function

$$W(z)/W^*(z)$$

of z has nonnegative real part in the strip $-h < iz^- - iz < 0$. The function is analytic in the strip. It can be assumed without loss of generality that the function has a continuous extension to the closure of the strip and has nonnegative real part on the boundary. An estimate of the modulus of the function is obtained which permits the conclusion that the function has nonnegative real part in the strip. The function is a product of three functions, each of which can be estimated in the strip. The function

$$W(z)/W(z+ih)$$

can be estimated in the strip since it is analytic and has nonnegative real part in the half–plane $-h < iz^- - iz$. The function

$$W^*(z-ih)/W^*(z)$$

can be estimated in the strip since it is analytic and has nonnegative real part in the half–plane $iz^- - iz < h$. The function

$$W(z+ih)/W^*(z-ih)$$

is analytic and bounded by one in the strip. The resulting estimates are sufficient for the desired conclusion.

Hilbert spaces appear whose elements are entire functions and which have these properties.

(H1) Whenever an element $F(z)$ of the space has a nonreal zero w, the function

$$F(z)(z-w^-)/(z-w)$$

belongs to the space and has the same norm as $F(z)$.

(H2) A continuous linear functional on the space is defined by taking $F(z)$ into $F(w)$ for every nonreal number w.

(H3) The function

$$F^*(z) = F(z^-)^-$$

belongs to the space whenever $F(z)$ belongs to the space, and it always has the same norm as $F(z)$.

Such spaces have an elementary structure. The complex numbers are treated as a coefficient Hilbert space with absolute value as norm. If w is a nonreal number, the adjoint of the transformation of the Hilbert space $\mathcal{H}$ into the coefficient space is a transformation of the coefficient space into $\mathcal{H}$ which takes c into $K(w,z)c$ for an entire function $K(w,z)$ of z. The identity

$$F(w) = \langle F(t), K(w,t)\rangle$$

reproduces the value at w of an element $F(z)$ of the space. A closed subspace consists of the functions which vanish at λ for a given nonreal number λ. The orthogonal projection in the subspace of an element $F(z)$ of the space is

$$F(z) - K(\lambda,z)K(\lambda,\lambda)^{-1}F(\lambda)$$

when the inverse of $K(\lambda, \lambda)$ exists. The properties of $K(\lambda, z)$ as a reproducing kernel function imply that $K(\lambda, \lambda)$ is a nonnegative number which vanishes only when $K(\lambda, z)$ vanishes identically. Calculations are restricted to the case in which $K(\lambda, \lambda)$ is nonzero since otherwise the space contains no nonzero element. If w is a nonreal number, the reproducing kernel function for function values at w in the subspace of functions which vanish at λ is

$$K(w, z) - K(\lambda, z)K(\lambda, \lambda)^{-1}K(w, \lambda).$$

The axiom (H1) implies that

$$[K(w, z) - K(\lambda, z)K(\lambda, \lambda)^{-1}K(w, \lambda)](z - \lambda^-)(w^- - \lambda)(z - \lambda)^{-1}(w^- - \lambda^-)^{-1}$$

is the reproducing kernel function for function values at w in the subspace of functions which vanish at λ^-. The identity

$$(z - \lambda^-)(w^- - \lambda)[K(w, z) - K(\lambda, z)K(\lambda, \lambda)^{-1}K(w, \lambda)]$$
$$= (z - \lambda)(w^- - \lambda^-)[K(w, z) - K(\lambda^-, z)K(\lambda^-, \lambda^-)^{-1}K(w, \lambda^-)]$$

follows. The identity is applied in the equivalent form

$$(\lambda - \lambda^-)(z - w^-)K(w, z)$$
$$= (z - \lambda^-)K(\lambda, z)K(\lambda, \lambda)^{-1}(\lambda - w^-)K(w, \lambda)$$
$$-(z - \lambda)K(\lambda^-, z)K(\lambda^-, \lambda^-)^{-1}(\lambda^- - w^-)K(w, \lambda^-).$$

The axiom (H3) implies the symmetry condition

$$K(w, z) = K(w^-, z^-)^-.$$

An entire function $E(z)$ exists such that the identity

$$2\pi i(w^- - z)K(w, z) = E(z)E(w)^- - E^*(z)E(w^-)$$

holds for all complex numbers z and w. The inequality

$$|E(z^-)| < |E(z)|$$

applies when z is in the upper half–plane. Since the space is uniquely determined by the function $E(z)$, it is denoted $\mathcal{H}(E)$.

A Hilbert space $\mathcal{H}(E)$ is constructed for a given entire function $E(z)$ when the inequality

$$|E(z^-)| < |E(z)|$$

holds for z in the upper half–plane. A weighted Hardy space $\mathcal{F}(E)$ exists since $E(z)$ is an analytic weight function when considered as a function of z in the upper half–plane. The desired space $\mathcal{H}(E)$ is contained isometrically in the space $\mathcal{F}(E)$ and contains the entire functions $F(z)$ such that $F(z)$ and $F^*(z)$ belong to the space $\mathcal{F}(E)$. The axioms (H1), (H2), and (H3) are satisfied. If

$$E(z) = A(z) - iB(z)$$

for entire functions $A(z)$ and $B(z)$ such that

$$A(z) = A^*(z)$$

and

$$B(z) = B^*(z)$$

have real values on the real axis, the reproducing kernel function of the resulting space $\mathcal{H}(E)$ at a complex number w is

$$K(w, z) = \frac{B(z)A(w)^- - A(z)B(w)^-}{\pi(z - w^-)}.$$

If a Hilbert space of entire functions is isometrically equal to a space $\mathcal{H}(E)$ with

$$E(z) = A(z) - iB(z)$$

for entire functions $A(z)$ and $B(z)$ which have real values on the real axis and if

$$\begin{pmatrix} P & Q \\ R & S \end{pmatrix}$$

is a matrix with real entries and determinant one, then the space is also isometrically equal to a space $\mathcal{H}(E_1)$ with

$$E_1(z) = A_1(z) - iB_1(z)$$

where the entire functions $A_1(z)$ and $B_1(z)$, which have real values on the real axis, are defined by the identities

$$A_1(z) = A(z)P + B(z)R$$

and

$$B_1(z) = A(z)Q + B(z)S.$$

A Hilbert space of entire functions is said to be symmetric about the origin if an isometric transformation of the space into itself is defined by taking $F(z)$ into $F(-z)$. The space is then the orthogonal sum of the subspace of even functions

$$F(z) = F(-z)$$

and of the subspace of odd functions

$$F(z) = -F(-z).$$

A Hilbert space $\mathcal{H}(E)$ is symmetric about the origin if the defining function $E(z)$ satisfies the symmetry condition

$$E(-z) = E^*(z).$$

The identity

$$E(z) = A(z) - iB(z)$$

then holds with $A(z)$ an even entire function and $B(z)$ an odd entire function which have real values on the real axis. A Hilbert space of entire functions which satisfies the axioms (H1), (H2), and (H3), which is symmetric about the origin, and which contains a nonzero element, is isometrically to a space $\mathcal{H}(E)$ whose defining function $E(z)$ satisfies the symmetry condition.

If the defining function $E(z)$ of a space $\mathcal{H}(E)$ satisfies the symmetry condition, a Hilbert space $\mathcal{H}_+$ of entire functions, which satisfies the axioms (H1), (H2), and (H3), exists such that an isometric transformation of the space $\mathcal{H}_+$ onto the set of even elements of the space $\mathcal{H}(E)$ is defined by taking $F(z)$ into $F(z^2)$. If the space $\mathcal{H}_+$ contains a nonzero element, it is isometrically equal to a space $\mathcal{H}(E_+)$ for an entire function

$$E_+(z) = A_+(z) - iB_+(z)$$

defined by the identities

$$A(z) = A_+(z^2)$$

and

$$zB(z) = B_+(z^2).$$

The functions $A(z)$ and $zB(z)$ are linearly dependent when the space $\mathcal{H}_+$ contains no nonzero element. The space $\mathcal{H}(E)$ then has dimension one. A Hilbert space $\mathcal{H}_-$ of entire functions, which satisfies the axioms (H1), (H2), and (H3), exists such that an isometric transformation of the space $\mathcal{H}_-$ onto the set of odd elements of the space $\mathcal{H}(E)$ is defined by taking $F(z)$ into $zF(z^2)$. If the space $\mathcal{H}_-$ contains a nonzero element, it is isometrically equal to a space $\mathcal{H}(E_-)$ for an entire function

$$E_-(z) = A_-(z) - iB_-(z)$$

defined by the identities

$$A(z) = A_-(z^2)$$

and

$$B(z)/z = B_-(z^2).$$

The functions $A(z)$ and $B(z)/z$ are linearly dependent when the space $\mathcal{H}_-$ contains no nonzero element. The space $\mathcal{H}(E)$ then has dimension one.

An entire function $S(z)$ is said to be associated with a space $\mathcal{H}(E)$ if

$$[F(z)S(w) - S(z)F(w)]/(z - w)$$

belongs to the space for every complex number w whenever $F(z)$ belongs to the space. If a function $S(z)$ is associated with a space $\mathcal{H}(E)$, then

$$[S(z)B(w) - B(z)S(w)]/(z - w)$$

belongs to the space for every complex number w. The scalar product

$$\begin{aligned} &B(\alpha)^- L(\beta, \alpha) B(\beta) \\ &= (\beta - \alpha^-)\langle [S(t)B(\beta) - B(t)S(\beta)]/(t - \beta),\ [S(t)B(\alpha) - B(t)S(\alpha)]/(t - \alpha)\rangle_{\mathcal{H}(E)} \end{aligned}$$

is computable since the identities

$$L(\alpha^-, \beta^-) = -L(\beta, \alpha) = L(\alpha, \beta)^-$$

and

$$L(\beta, \gamma) - L(\alpha, \gamma) = L(\beta, \alpha^-)$$

hold for all complex numbers α, β, and γ. A function $\psi(z)$ of nonreal numbers z, which is analytic separately in the upper and lower half–planes and which satisfies the identity

$$\psi(z) + \psi^*(z) = 0,$$

exists such that

$$L(\beta, \alpha) = \pi i \psi(\beta) + \pi i \psi(\alpha)^-$$

for nonreal numbers α and β. The real part of the function is nonnegative in the upper half–plane.

If $F(z)$ is an element of the space $\mathcal{H}(E)$, a corresponding entire function $F^\sim(z)$ is defined by the identity

$$\begin{aligned} &\pi B(w) F^\sim(w) + \pi i B(w) \psi(w) F(w) \\ &= \langle F(t)S(w), [S(t)B(w^-) - B(t)S(w^-)]/(t - w^-)\rangle_{\mathcal{H}(E)} \end{aligned}$$

when w is not real. If $F(z)$ is an element of the space and if

$$G(z) = [F(z)S(w) - S(z)F(w)]/(z - w)$$

is the element of the space obtained for a complex number w, then the identity

$$G^\sim(z) = [F^\sim(z)S(w) - S(z)F^\sim(w)]/(z - w)$$

is satisfied. The identity for difference quotients

$$\begin{aligned} &\pi G(\alpha)^- F^\sim(\beta) - \pi G^\sim(\alpha)^- F(\beta) \\ &= \langle [F(t)S(\beta) - S(t)F(\beta)]/(t - \beta), G(t)S(\alpha)\rangle_{\mathcal{H}(E)} \\ &- \langle F(t)S(\beta), [G(t)S(\alpha) - S(t)G(\alpha)]/(t - \alpha)\rangle_{\mathcal{H}(E)} \\ &- (\beta - \alpha^-)\langle [F(t)S(\beta) - S(t)F(\beta)]/(t - \beta), [G(t)S(\alpha) - S(t)G(\alpha)]/(t - \alpha)\rangle_{\mathcal{H}(E)} \end{aligned}$$

holds for all elements $F(z)$ and $G(z)$ of the space when α and β are nonreal numbers.

The transformation which takes $F(z)$ into $F^\sim(z)$ is a generalization of the Hilbert transformation. The graph of the transformation is a Hilbert space whose elements are pairs

$$\begin{pmatrix} F_+(z) \\ F_-(z) \end{pmatrix}$$

of entire functions. The skew–conjugate unitary matrix

$$I = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$$

is treated as a generalization of the imaginary unit. The space of column vectors with complex entries is considered with the Euclidean scalar product

$$\langle \begin{pmatrix} a \\ b \end{pmatrix}, \begin{pmatrix} a \\ b \end{pmatrix} \rangle = \begin{pmatrix} a \\ b \end{pmatrix}^- \begin{pmatrix} a \\ b \end{pmatrix}.$$

Examples are obtained in a related theory of Hilbert spaces whose elements are pairs of entire functions. If w is a complex number, the pair

$$\begin{pmatrix} [F_+(z)S(w) - S(z)F_+(w)]/(z - w) \\ [F_-(z)S(w) - S(z)F_-(w)]/(z - w) \end{pmatrix}$$

belongs to the space whenever

$$\begin{pmatrix} F_+(z) \\ F_-(z) \end{pmatrix}$$

belongs to the space. The identity for difference quotients

$$\begin{aligned} -2\pi \begin{pmatrix} G_+(\alpha) \\ G_-(\alpha) \end{pmatrix}^- I \begin{pmatrix} F_+(\beta) \\ F_-(\beta) \end{pmatrix} \\ = \langle \begin{pmatrix} [F_+(t)S(\beta) - S(t)F_+(\beta)]/(t - \beta) \\ [F_-(t)S(\beta) - S(t)F_-(\beta)]/(t - \beta) \end{pmatrix}, \begin{pmatrix} G_+(t)S(\alpha) \\ G_-(t)S(\alpha) \end{pmatrix} \rangle \\ - \langle \begin{pmatrix} F_+(t)S(\beta) \\ F_-(t)S(\beta) \end{pmatrix}, \begin{pmatrix} [G_+(t)S(\alpha) - S(t)G_+(\alpha)]/(t - \alpha) \\ [G_-(t)S(\alpha) - S(t)G_-(\alpha)]/(t - \alpha) \end{pmatrix} \rangle \\ -(\beta - \alpha^-) \langle \begin{pmatrix} [F_+(t)S(\beta) - S(t)F_+(\beta)]/(t - \beta) \\ [F_-(t)S(\beta) - S(t)F_-(\beta)]/(t - \beta) \end{pmatrix}, \begin{pmatrix} [G_+(t)S(\alpha) - S(t)G_+(\alpha)]/(t - \alpha) \\ [G_-(t)S(\alpha) - S(t)G_-(\alpha)]/(t - \alpha) \end{pmatrix} \rangle \end{aligned}$$

holds for all elements

$$\begin{pmatrix} F_+(z) \\ F_-(z) \end{pmatrix}$$

and

$$\begin{pmatrix} G_+(z) \\ G_-(z) \end{pmatrix}$$

of the space when α and β are complex numbers. A continuous transformation of the space into the space of column vectors with complex entries takes

$$\begin{pmatrix} F_+(z) \\ F_-(z) \end{pmatrix}$$

into

$$\begin{pmatrix} F_+(w) \\ F_-(w) \end{pmatrix}$$

when w is not real. The adjoint transformation takes

$$\begin{pmatrix} u \\ v \end{pmatrix}$$

into

$$\frac{M(z)IM(w)^- - S(z)IS(w)^-}{2\pi(z - w^-)} \begin{pmatrix} u \\ v \end{pmatrix}$$

for a function

$$M(z) = \begin{pmatrix} A(z) & B(z) \\ C(z) & D(z) \end{pmatrix}$$

with matrix values which is independent of w. The entries of the matrix are entire functions which have real values on the real axis. Since the space with these properties is uniquely determined by $S(z)$ and $M(z)$, it is denoted $\mathcal{H}_S(M)$. If $M(z)$ is a given matrix of entire functions which are real on the real axis, necessary and sufficient conditions for the existence of a space $\mathcal{H}_S(M)$ are the matrix identity

$$M(z^-)IM(z)^- = S(z^-)IS(z)^-$$

and the matrix inequality

$$\frac{M(z)IM(z)^- - S(z)IS(z)^-}{z - z^-} \geq 0$$

for all complex numbers z.

An example of a space $\mathcal{H}_S(M)$ is obtained when an entire function $S(z)$ is associated with a space $\mathcal{H}(E)$. The Hilbert transformation associates an entire function $F^\sim(z)$ with every element $F(z)$ of the space in such a way that an identity for difference quotients is satisfied. The graph of the Hilbert transformation is a Hilbert space $\mathcal{H}_s(M)$ with

$$M(z) = \begin{pmatrix} A(z) & B(z) \\ C(z) & D(z) \end{pmatrix}$$

and

$$E(z) = A(z) - iB(z)$$

for entire function $C(z)$ and $D(z)$ which have real values on the real axis. The elements of the space are of the form

$$\begin{pmatrix} F(z) \\ F^\sim(z) \end{pmatrix}$$

with $F(z)$ in $\mathcal{H}(E)$. The identity

$$\left\| \begin{pmatrix} F(t) \\ F^\sim(t) \end{pmatrix} \right\|^2_{\mathcal{H}_s(M)} = 2\|F(t)\|^2_{\mathcal{H}(E)}$$

is satisfied.

The relationship between factorization and invariant subspaces is an underlying theme of the theory of Hilbert spaces of entire functions. A matrix factorization applies to entire functions $E(z)$ such that a space $\mathcal{H}(E)$ exists. When several such functions appear in factorization, it is convenient to index them with a real parameter which is treated as a new variable. When functions $E(a, z)$ and $E(b, z)$ are given, the question arises whether the space $\mathcal{H}(E(a))$ with parameter a is contained isometrically in the space $\mathcal{H}(E(b))$ with parameter b. The question is answered by answering two more fundamental questions. The first is whether the space $\mathcal{H}(E(a))$ is contained contractively in the space $\mathcal{H}(E(b))$. The second is whether the inclusion is isometric.

If a Hilbert space $\mathcal{P}$ is contained contractively in a Hilbert space $\mathcal{H}$, a unique Hilbert space $\mathcal{Q}$, which is contained contractively in $\mathcal{H}$, exists such that the inequality

$$\|c\|^2_{\mathcal{H}} \leq \|a\|^2_{\mathcal{P}} + \|b\|^2_{\mathcal{Q}}$$

holds whenever $c = a + b$ with a in $\mathcal{P}$ and b in $\mathcal{Q}$ and such that every element c of $\mathcal{H}$ admits some decomposition for which equality holds. The space $\mathcal{Q}$ is called the complementary space to $\mathcal{P}$ in $\mathcal{H}$. Minimal decomposition of an element c of $\mathcal{H}$ is unique. The element a of $\mathcal{P}$ is obtained from c under the adjoint of the inclusion of $\mathcal{P}$ in $\mathcal{H}$. The element b of $\mathcal{Q}$ is obtained from c under the adjoint of the inclusion of $\mathcal{Q}$ in $\mathcal{H}$. The intersection of $\mathcal{P}$ and $\mathcal{Q}$ is a Hilbert space $\mathcal{P} \wedge \mathcal{Q}$, which is contained contractively in $\mathcal{H}$, when considered with scalar product determined by the identity

$$\|c\|^2_{\mathcal{P}\wedge\mathcal{Q}} = \|c\|^2_{\mathcal{P}} + \|c\|^2_{\mathcal{Q}}.$$

The inclusion of $\mathcal{P}$ in $\mathcal{H}$ is isometric if, and only if, the space $\mathcal{P} \wedge \mathcal{Q}$ contains no nonzero element. The inclusion of $\mathcal{Q}$ in $\mathcal{H}$ is then isometric. A Hilbert space $\mathcal{H}$ which is so decomposed is written $P \vee Q$.

The space $\mathcal{H}_S(M)$ is denoted $\mathcal{H}(M)$ when $S(z)$ is identically one. An estimate of coefficients in the power series expansion of $M(z)$ applies when the matrix is the identity at the origin. A nonnegative matrix

$$\begin{pmatrix} \alpha & \beta \\ \beta & \gamma \end{pmatrix} = M'(0)I$$

is constructed from derivatives at the origin. The Schmidt norm $\sigma(M)$ of a matrix

$$M = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$$

is the nonnegative solution of the equations

$$\sigma(M)^2 = |A|^2 + |B|^2 + |C|^2 + |D|^2.$$

The coefficients in the power series expansion

$$M(z) = \sum M_n z^n$$

satisfy the inequality

$$\sigma(M_n) \leq (\alpha + \gamma)^n / n!$$

for every positive integer n.

If

$$E(b, z) = A(b, z) - iB(b, z)$$

is an entire function such that a space $\mathcal{H}(E(a))$ exists and if

$$M(b, a, z) = \begin{pmatrix} A(b, a, z) & B(b, a, z) \\ C(b, a, z) & D(b, a, z) \end{pmatrix}$$

is matrix of entire functions such that a space $\mathcal{H}(M(b, a))$ exists, then an entire function

$$E(a, z) = A(a, z) - iB(a, z)$$

such that a space $\mathcal{H}(E(a))$ exists is defined by the matrix product

$$(A(a, z), B(a, z)) = (A(b, z), B(b, z))M(b, a, z).$$

If $F(z)$ is an element of the space $\mathcal{H}(E(b))$ and if

$$G(z) = \begin{pmatrix} G_+(z) \\ G_-(z) \end{pmatrix}$$

is an element of the space $\mathcal{H}(M(b, a))$, then

$$H(z) = F(z) + A(b, z)G_+(z) + B(b, z)G_-(z)$$

is an element of the space $\mathcal{H}(E(a))$ which satisfies the inequality

$$\|H(z)\|^2_{\mathcal{H}(E(a))} \leq \|F(z)\|^2_{\mathcal{H}(E(b))} + \tfrac{1}{2}\|G(z)\|^2_{\mathcal{H}(M(b,a))}.$$

Every element $H(z)$ of the space $\mathcal{H}(E(a))$ admits such a decomposition for which equality holds.

The set of elements $G(z)$ of the space $\mathcal{H}(M(b,a))$ such that

$$A(b,z)G_+(z) + B(b,z)G_-(z)$$

belongs to the space $\mathcal{H}(E(b))$ is a Hilbert space $\mathcal{L}$ with scalar product determined by the identity

$$\|G(z)\|^2_{\mathcal{L}} = \|G(z)\|^2_{\mathcal{H}(M(b,a))} + 2\ \|A(b,z)G_+(z) + B(b,z)G_-(z)\|^2_{\mathcal{H}(E(b))}.$$

The pair

$$[F(z) - F(w)]/(z-w) = \begin{pmatrix} [F_+(z) - F_+(w)]/(z-w) \\ [F_-(z) - F_-(w)]/(z-w) \end{pmatrix}$$

belongs to the space for every complex number w whenever

$$F(z) = \begin{pmatrix} F_+(z) \\ F_-(z) \end{pmatrix}$$

belongs to the space. The identity for difference quotients

$$\begin{aligned} 0 &= \langle [F(t) - F(\beta)]/(t-\beta), G(t)\rangle_{\mathcal{L}} \\ &\quad - \langle F(t), [G(t) - G(\alpha)]/(t-\alpha)\rangle_{\mathcal{L}} \\ &\quad - (\beta - \alpha^-)\langle [F(t) - F(\beta)]/(t-\beta), [G(t) - G(\alpha)]/(t-\alpha)\rangle_{\mathcal{L}} \end{aligned}$$

holds for all elements $F(z)$ and $G(z)$ of the space when α and β are complex numbers. These properties imply that the elements of the space $\mathcal{L}$ are pairs

$$\begin{pmatrix} u \\ v \end{pmatrix}$$

of constants which satisfy the identity

$$v^- u = u^- v.$$

The inclusion of the space $\mathcal{H}(E(b))$ in the space $\mathcal{H}(E(a))$ is isometric if, and only if, no nonzero pair of complex numbers u and v, which satisfy the identity, exists such that

$$\begin{pmatrix} u \\ v \end{pmatrix}$$

belongs to the space $\mathcal{H}(M(b,a))$ and

$$A(b,z)u + B(b,z)v$$

belongs to the space $\mathcal{H}(E(b))$.

A converse result holds. Assume that $E(b,z)$ and $E(a,z)$ are entire functions such that spaces $\mathcal{H}(E(b))$ and $\mathcal{H}(E(a))$ exist and such that $\mathcal{H}(E(b))$ is contained isometrically in $\mathcal{H}(E(a))$. Assume that a nontrivial entire function $S(z)$ is associated with the spaces $\mathcal{H}(E(b))$ and $\mathcal{H}(E(a))$. A generalization of the Hilbert transformation is defined on the space $\mathcal{H}(E(a))$, which takes an element $F(z)$ of the space $\mathcal{H}(E(a))$ into an entire function $F^\sim(z)$. The transformation takes

$$[F(z)S(w) - S(z)F(w)]/(z-w)$$

into

$$[F^\sim(z)S(w) - S(z)F^\sim(w)]/(z-w)$$

for every complex number w whenever it takes $F(z)$ into $F^\sim(z)$. An identity for difference quotients is satisfied. A generalization of the Hilbert transformation is also defined with similar properties on the space $\mathcal{H}(E(b))$. The transformation on the space $\mathcal{H}(E(b))$ is chosen as the restriction of the transformation on the space $\mathcal{H}(E(a))$. The graph of the Hilbert transformation on the space $\mathcal{H}(E(a))$ is a space $\mathcal{H}_S(M(a))$ for a matrix

$$M(a,z) = \begin{pmatrix} A(a,z) & B(a,z) \\ C(a,z) & D(a,z) \end{pmatrix}$$

of entire functions which have real values on the real axis. The matrix is chosen so that the identity

$$E(a,z) = A(a,z) - iB(a,z)$$

is satisfied. The graph of the Hilbert transformation on the space $\mathcal{H}(E(b))$ is a space $\mathcal{H}_S(M(b))$ for a matrix

$$M(b,z) = \begin{pmatrix} A(b,z) & B(b,z) \\ C(b,z) & D(b,z) \end{pmatrix}$$

of entire functions which have real values on the real axis. The matrix is chosen so that the identity

$$E(b,z) = A(b,z) - iB(b,z)$$

is satisfied. Since the space $\mathcal{H}(E(b))$ is contained isometrically in the space $\mathcal{H}(E(a))$ and since the generalized Hilbert transformation on the space $\mathcal{H}(E(b))$ is consistent with the generalized Hilbert transformation on the space $\mathcal{H}(E(a))$, the space $\mathcal{H}_S(M(b))$ is contained isometrically in the space $\mathcal{H}_S(M(a))$. A matrix

$$M(b,a,z) = \begin{pmatrix} A(b,a,z) & B(b,a,z) \\ C(b,a,z) & D(b,a,z) \end{pmatrix}$$

of entire functions is defined as the solution of the equation

$$M(a,z) = M(b,z)M(b,a,z).$$

The entries of the matrix are entire functions which have real values on the real axis. Multiplication by $M(b,z)$ acts as an isometric transformation of the desired space $\mathcal{H}(M(b,a))$

onto the orthogonal complement of the space $\mathcal{H}_S(M(b))$ in the space $\mathcal{H}_S(M(a))$. This completes the construction of a space $\mathcal{H}(M(b,a))$ which satisfies the identity

$$(A(a,z), B(a,z)) = (A(b,z), B(b,z))M(b,a,z).$$

A simplification is a fundamental theorem in the theory of isometric inclusions for Hilbert spaces of entire functions [3]. Assume that $E(b,z)$ and $E(a,z)$ are entire functions, which have no real zeros, such that spaces $\mathcal{H}(E(b))$ and $\mathcal{H}(E(a))$ exist. If a weighted Hardy space $\mathcal{F}(W)$ exists such that the spaces $\mathcal{H}(E(b))$ and $\mathcal{H}(E(a))$ are contained isometrically in the space $\mathcal{F}(W)$, then either the space $\mathcal{H}(E(b))$ is contained isometrically in this space $\mathcal{H}(E(a))$ or the space $\mathcal{H}(E(a))$ is contained isometrically in the space $\mathcal{H}(E(b))$.

The hereditary nature of symmetry about the origin is an application of the ordering theorem for Hilbert spaces of entire functions. Assume that $E(b,z)$ and $E(a,z)$ are entire functions, which have no real zeros, such that spaces $\mathcal{H}(E(b))$ and $\mathcal{H}(E(a))$ exist. The space $\mathcal{H}(E(b))$ is symmetric about the origin if it is contained isometrically in the space $\mathcal{H}(E(a))$ and if the space $\mathcal{H}(E(a))$ is symmetric about the origin. If the symmetry conditions

$$E^*(b,z) = E(b,-z)$$

and

$$E^*(a,z) = E(a,-z)$$

are satisfied, then the identity

$$(A(a,z), B(a,z)) = (A(b,z), B(b,z))M(b,a,z)$$

holds for a space $\mathcal{H}(M(b,a))$ whose defining matrix

$$M(b,a,z) = \begin{pmatrix} A(b,a,z) & B(b,a,z) \\ C(b,a,z) & D(b,a,z) \end{pmatrix}$$

has even entire functions on the diagonal and odd entire functions off the diagonal.

An entire function $E(z)$ is said to be of Pólya class if it has no zeros in the upper half–plane, if the inequality

$$|E(x-iy)| \le |E(x+iy)|$$

holds for every real number x when y is positive, and if $|E(x+iy)|$ is a nondecreasing function of positive numbers y for every real number x. A polynomial is of Pólya class if it has no zeros in the upper half–plane. A pointwise limit of entire functions Pólya class is an entire function of Pólya class if it does not vanish identically. Every entire function of Pólya class is a limit, uniformly on compact subsets of the complex plane, of polynomials which have no zeros in the upper half–plane. Every entire function $E(z)$ of Pólya class which has no zeros is to the form

$$E(z) = E(0)\exp(-az^2 - ibz)$$

for a nonnegative number a and a complex number b whose real part is nonnegative. An entire function $E(z)$ of Pólya class is said to be determined by its zeros if it is a limit uniformly on compact subsets of the complex plane of polynomials whose zeros are contained in the zeros of $E(z)$. An entire function of Pólya class is the product of an entire function of Pólya class which has no zeros and an entire function of Pólya class which is determined by its zeros.

The pervasiveness of the Pólya class is due to its preservation under bounded type perturbations. An entire function $S(z)$ is of Pólya class if it has no zeros in the upper half–plane, if it satisfies the inequality

$$|S(x - iy)| \leq |S(x + iy)|$$

for every real number x when y is positive, and if an entire function $E(z)$ of Pólya class exists such that

$$S(z)/E(z)$$

is of bounded type in the upper half–plane.

Transformations, whose domain and range are contained in Hilbert spaces of entire functions satisfying the axioms (H1), (H2), and (H3), are defined using reproducing kernel functions. Assume that the domain of the transformation is contained in a space $\mathcal{H}(E)$ and that the range of the transformation is contained in a space $\mathcal{H}(E')$. The domain of the transformation is assumed to contain the reproducing kernel functions for function values in the space $\mathcal{H}(E)$. The domain of the adjoint transformation is assumed to contain the reproducing kernel functions for function values in the space $\mathcal{H}(E')$. Define $L(w, z)$ to be the element of the space $\mathcal{H}(E)$ obtained under the adjoint transformation from the reproducing kernel function for function values at w in the space $\mathcal{H}(E')$. Then the transformation takes an element $F(z)$ of the space $\mathcal{H}(E)$ into an element $G(z)$ of the space $\mathcal{H}(E')$ if, and only if, the identity

$$G(w) = \langle F(t), L(w, t)\rangle_{\mathcal{H}(E)}$$

holds for all complex numbers w. Define $L'(w, z)$ to be the element of the space $\mathcal{H}(E')$ obtained under the transformation from the reproducing kernel function for function values at w in the space $\mathcal{H}(E)$. Then the adjoint transformation takes an element $F(z)$ of the space $\mathcal{H}(E')$ into an element $G(z)$ of the space $\mathcal{H}(E)$ if, and only if, the identity

$$G(w) = \langle F(t), L'(w, t)\rangle_{\mathcal{H}(E')}$$

holds for all complex numbers w. The identity

$$L'(w, z) = L(z, w)^-$$

is a consequence of the definition of the adjoint.

The existence of reproducing kernel functions for transformations with domain and range in Hilbert spaces of entire functions is a generalization of the axiom (H2). The transformations are also assumed to satisfy a generalization of the axiom (H1).

Assume that a given transformation has domain in a space $\mathcal{H}(E)$ and range in a space $\mathcal{H}(E')$. If λ is a nonreal number, then the set of entire functions $F(z)$ such that $(z-\lambda)F(z)$ belongs to the space $\mathcal{H}(E)$ is a Hilbert space of entire functions which satisfies the axioms (H1), (H2), and (H3) when considered with the scalar product such that multiplication by $z-\lambda$ is an isometric transformation of the space into the space $\mathcal{H}(E)$. If $F(z)$ is an entire function, then $(z-\lambda)F(z)$ belongs to the space $\mathcal{H}(E)$ if, and only if, $(z-\lambda^-)F(z)$ belongs to the space $\mathcal{H}(E)$. The norm of $(z-\lambda)F(z)$ in the space $\mathcal{H}(E)$ is equal to the norm of $(z-\lambda^-)F(z)$ in the space $\mathcal{H}(E)$. The set of entire functions $F(z)$ such that $(z-\lambda)F(z)$ belongs to the space $\mathcal{H}(E')$ is a Hilbert space of entire functions which satisfies the axioms (H1), (H2), and (H3) when considered with the scalar product such that multiplication by $z-\lambda$ is an isometric transformation of the space into the space $\mathcal{H}(E')$. If $F(z)$ is an entire function, then $(z-\lambda)F(z)$ belongs to the space $\mathcal{H}(E')$ if, and only if, $(z-\lambda^-)F(z)$ belongs to the space $\mathcal{H}(E')$. The norm of $(z-\lambda)F(z)$ in the space $\mathcal{H}(E')$ is equal to the norm of $(z-\lambda^-)F(z)$ in the space $\mathcal{H}(E')$. The induced relation at λ takes an entire function $F(z)$ such that $(z-\lambda)F(z)$ belongs to the space $\mathcal{H}(E)$ into an entire function $G(z)$ such that $(z-\lambda)G(z)$ belongs to the space $\mathcal{H}(E')$ when the given transformation takes an element $H(z)$ of the space $\mathcal{H}(E)$ into the element $(z-\lambda)G(z)$ of the space $\mathcal{H}(E')$ and when $(z-\lambda)F(z)$ is the orthogonal projection of $H(z)$ into the set of elements of the space $\mathcal{H}(E)$ which vanish at λ. The given transformation with domain in the space $\mathcal{H}(E)$ and range in the space $\mathcal{H}(E')$ is said to satisfy the axiom (H1) if the induced relation at λ coincides with the induced relation at λ^- for every nonreal number λ.

An identity in reproducing kernel functions results when the given transformation with domain in the space $\mathcal{H}(E)$ and range in the space $\mathcal{H}(E')$ satisfies the generalization of the axioms (H1) and (H2) if the induced relations are transformations. The reproducing kernel function for the transformation at w is an element $L(w,z)$ of the space $\mathcal{H}(E)$ such that the identity

$$G(w) = \langle F(t), L(w,t)\rangle_{\mathcal{H}(E)}$$

holds for every complex number w when the transformation takes $F(z)$ into $G(z)$. If the reproducing kernel function $L(\lambda,z)$ at λ vanishes at λ for some complex number λ, then the reproducing kernel function for the adjoint transformation at λ vanishes at λ. Since the orthogonal projection of $K(\lambda,z)$ into the subspace of elements of the space $\mathcal{H}(E)$ which vanish at λ is equal to zero, the reproducing kernel function for the adjoint transformation at λ is equal to zero if the induced relation at λ is a transformation. It follows that $L(\lambda,z)$ vanishes identically if it vanishes at λ.

If λ is a nonreal number such that $L(\lambda,z)$ does not vanish at λ, then for every complex number w, the function

$$L(w,z) - L(\lambda,z)L(\lambda,\lambda)^{-1}L(w,\lambda)$$

of z is an element of the space $\mathcal{H}(E)$ which vanishes at λ. The function

$$\frac{L(w,z) - L(\lambda,z)L(\lambda,\lambda)^{-1}L(w,\lambda)}{(z-\lambda)(w^- - \lambda^-)}$$

of z is the reproducing kernel function at w for the induced transformation at λ. If $L(\lambda^-, z)$ does not vanish at λ^-, the function

$$\frac{L(w,z) - L(\lambda^-, z)L(\lambda^-, \lambda^-)^{-1}L(w, \lambda^-)}{(z - \lambda^-)(w^- - \lambda)}$$

of z is the reproducing kernel function at w for the induced transformation at λ^-. Since these reproducing kernel functions apply to the same transformation, they are equal. The resulting identity can be written

$$L(w,z) = [Q(z)P(w^-) - P(z)Q(w^-)]/[\pi(z - w^-)]$$

for entire functions $P(z)$ and $Q(z)$ which are associated with the spaces $\mathcal{H}(E)$ and $\mathcal{H}(E')$. If the spaces are symmetric about the origin and if the transformation takes $F^*(-z)$ into $G^*(-z)$ whenever it takes $F(z)$ into $G(z)$, then the functions $P(z)$ and $Q(z)$ can be chosen to satisfy the symmetry conditions

$$P(-z) = P^*(z)$$

and

$$Q(-z) = -Q^*(z).$$

A transformation with domain in a space $\mathcal{H}(E)$ and range in a space $\mathcal{H}(E')$ is said to satisfy the axioms (H1) and (H2) if entire functions, which are associated with the spaces $\mathcal{H}(E)$ and $\mathcal{H}(E')$, exist such that the transformation takes an element $F(z)$ of $\mathcal{H}(E)$ into an element $G(z)$ of $\mathcal{H}(E')$, when and only when, the identity

$$G(w) = \langle F(t), [Q(t)P(w^-) - P(t)Q(w^-)]/[\pi(t - w^-)]\rangle_{\mathcal{H}(E)}$$

holds for all complex numbers w and such that the adjoint takes an element $F(z)$ of $\mathcal{H}(E')$ into an element $G(z)$ of $\mathcal{H}(E)$ when, and only when, the identity

$$G(w) = \langle F(t), [Q^*(t)P(w)^- - P^*(t)Q(w)^-]/[\pi(t - w^-)]\rangle_{\mathcal{H}(E')}$$

holds for all complex numbers w. The transformation is said to be symmetric about the origin if the spaces are symmetric about the origin and if the transformation takes $F^*(-z)$ into $G^*(-z)$ whenever it takes $F(z)$ into $G(z)$. If the transformation is symmetric about the origin, the defining functions $P(z)$ and $Q(z)$ can be chosen to satisfy the symmetry conditions

$$P(-z) = P^*(z)$$

and

$$Q(-z) = -Q^*(z).$$

Special Hilbert spaces of entire functions appear which admit maximal dissipative transformations. The transformation, which has domain and range in a space $\mathcal{H}(E)$, is defined by entire functions $P(z)$ and $Q(z)$ which are associated with the space. The transformation takes $F(z)$ into $G(z+i)$ whenever $F(z)$ and $G(z+i)$ are elements of the space which satisfy the identity

$$G(w) = \langle F(t), [Q(t)P(w^-) - P(t)Q(w^-)]/[\pi(t-w^-)]\rangle_{\mathcal{H}(E)}$$

for all complex numbers w. Transformations which are maximal of dissipative deficiency at most one are also constructed in the same way. A relation T with domain and range in a Hilbert space is said to be maximal of dissipative deficiency at most one if the relation $T+w$ has an inverse on a closed subspace of codimension at most one for some element w of the right half–plane and if the relation

$$(T-w)(T+w)^{-1}$$

is every where defined and contractive in the subspace. The condition holds for all elements w of the right half–plane if it holds for some element w of the right half–plane. The space is assumed to be symmetric about the origin and the functions are assumed to satisfy the symmetry conditions

$$P(-z) = P^*(z)$$

and

$$Q(-z) = -Q^*(z)$$

when the transformation is not maximal dissipative. The reproducing kernel function for function values at $w+i$ in the space belongs to the domain of the adjoint for every complex number w. The function

$$[Q(z)P(w^-) - P(z)Q(w^-)]/[\pi(z-w^-)]$$

of z is obtained under the action of the adjoint. A Krein space of Pontryagin index at most one exists whose elements are entire functions and which admits the function

$$\frac{B^*(z)A(w^-) - A(z)B(w)^- + B(z)A(w)^- - A^*(z)B(w^-)}{\pi(z-w^-)}$$

of z as reproducing kernel function for function values at w for every complex number w,

$$A(z) = P(z - \tfrac{1}{2}i)$$

and

$$B^*(z) = Q(z - \tfrac{1}{2}i).$$

The space is a Hilbert space when the transformation is maximal dissipative. The symmetry conditions

$$A(-z) = A^*(z)$$

and

$$B(-z) = -B^*(z)$$

are satisfied when the transformation is not maximal dissipative.

Hilbert spaces, whose elements are entire functions, appear whose structure is derived from the structure of Hilbert spaces of entire functions satisfying the axioms (H1) and (H2).

Theorem 1. *Assume that for some entire functions $A(z)$ and $B(z)$ a Hilbert space $\mathcal{H}$ exists whose elements are entire functions and which contains the function*

$$\frac{B^*(z)A(w^-) - A(z)B(w)^- + B(z)A(w)^- - A^*(z)B(w^-)}{\pi(z - w^-)}$$

of z as reproducing kernel function for function values at w for every complex number w. Then a Hilbert space $\mathcal{P}$ exists whose elements are entire functions and which contains the function

$$\frac{[A^*(z) - iB^*(z)][A(w^-) + iB(w^-)] - [A(z) + iB(z)][A(w)^- - iB(w)^-]}{2\pi i(w^- - z)}$$

of z as reproducing kernel function for function values at w for every complex number w. And a Hilbert space $\mathcal{Q}$ exists whose elements are entire functions and which contains the function

$$\frac{[A(z) - iB(z)][A(w)^- + iB(w)^-] - [A^*(z) + iB^*(z)][A(w^-) - iB(w^-)]}{2\pi i(w^- - z)}$$

of z as reproducing kernel function for function values at w for every complex number w. The spaces $\mathcal{P}$ and $\mathcal{Q}$ are contained contractively in the space $\mathcal{H}$ and are complementary spaces to each other in $\mathcal{H}$.

Proof of Theorem 1. The desired conclusion is immediate when the function $A(z) - iB(z)$ vanishes identically since the space $\mathcal{P}$ is then isometrically equal to the space $\mathcal{H}$ and the space $\mathcal{Q}$ contains no nonzero element. The desired conclusion is also immediate when the function $A(z) + iB(z)$ vanishes identically since the space $\mathcal{Q}$ is then isometrically equal to the space $\mathcal{H}$ and the space $\mathcal{P}$ contains no nonzero element. When

$$S(z) = [A(z) - iB(z)][A(z) + iB(z)]$$

does not vanish identically, the determinants $S(z)$ of the matrix

$$U(z) = \begin{pmatrix} A(z) & -B(z) \\ B(z) & A(z) \end{pmatrix}$$

and $S^*(z)$ of the matrix

$$V(z) = \begin{pmatrix} A^*(z) & B^*(z) \\ -B^*(z) & A^*(z) \end{pmatrix}$$

do not vanish identically. It will be shown that a Hilbert space exists whose elements are pairs

$$\begin{pmatrix} F_+(z) \\ F_-(z) \end{pmatrix}$$

of entire functions and which contains the pair

$$\frac{V(z)IV(w)^- - U(z)IU(w)^-}{2\pi(z-w^-)} \begin{pmatrix} c_+ \\ c_- \end{pmatrix}$$

of entire functions of z as reproducing kernel function for function values at w in the direction

$$\begin{pmatrix} c_+ \\ c_- \end{pmatrix}$$

for every complex number w and for every pair of complex numbers c_+ and c_-. The resulting element of the Hilbert space represents the linear functional which takes a pair

$$\begin{pmatrix} F_+(z) \\ F_-(z) \end{pmatrix}$$

of entire functions of z into the number

$$\begin{pmatrix} c_+ \\ c_- \end{pmatrix}^- \begin{pmatrix} F_+(w) \\ F_-(w) \end{pmatrix} = c_+^- F_+(w) + c_-^- F_-(w).$$

The existence of the space is equivalent to the existence of a space $\mathcal{H}_S(M)$ with

$$M(z) = S(z)U(z)^{-1}V(z).$$

Since the space $\mathcal{H}_S(M)$ exists if the matrix

$$\frac{M(z)IM(w)^- - S(z)IS(w)^-}{2\pi(z-w^-)}$$

is nonnegative whenever z and w are equal, the desired Hilbert space exists if the matrix

$$\frac{V(z)IV(w)^- - U(z)IU(w)^-}{2\pi(z-w^-)}$$

is nonnegative whenever z and w are equal. Multiplication by $S(z)U(z)^{-1}$ is then an isometric transformation of the space $\mathcal{H}_S(M)$ onto the desired space. Since the matrix is diagonal whenever z and w are equal, the matrix is nonnegative if its trace

$$\frac{B^*(z)A(w^-) - A(z)B(w)^- + B(z)A(w)^- - A^*(z)B(w^-)}{\pi(z-w^-)}$$

is nonnegative whenever z and w are equal. Since the trace as a function of z is the reproducing kernel function for function values at w in the given space $\mathcal{H}$, the trace is nonnegative when z and w are equal. This completes the construction of the desired Hilbert space of pairs of entire functions.

Since the matrix
$$\frac{V(z)IV(w)^- - U(z)IU(w)^-}{2\pi(z-w^-)}$$
commutes with I for all complex numbers z and w, multiplication by I is an isometric transformation of the space onto itself. The space is the orthogonal sum of a subspace of eigenvectors for the eigenvalue i and a subspace of eigenvectors for the eigenvalue $-i$. The existence of the desired Hilbert spaces $\mathcal{P}$ and $\mathcal{Q}$ follows. Every element of the space is of the form
$$\begin{pmatrix} F(z)+G(z) \\ iG(z)-iF(z) \end{pmatrix}$$
with $F(z)$ in $\mathcal{P}$ and $G(z)$ in $\mathcal{Q}$. The desired properties of the spaces $\mathcal{P}$ and $\mathcal{Q}$ follow from the computation of reproducing kernel functions.

This completes the proof of the theorem.

A Hilbert space $\mathcal{P}$ exists whose elements are entire functions and which admits the function
$$\frac{[A^*(z)-iB^*(z)][A(w^-)+iB(w^-)]-[A(z)+iB(z)][A(w)^- - iB(w)^-]}{2\pi i(w^- - z)}$$
of z as reproducing kernel function for function values at w for every complex number w if, and only if, the entire functions $A(z)$ and $B(z)$ satisfy the inequality
$$|A(z)+iB(z)| \le |A^*(z)-iB^*(z)|$$
when z is in the upper half–plane. The space contains no nonzero element when the entire functions
$$A^*(z)-iB^*(z)$$
and
$$A(z)+iB(z)$$
are linearly dependent. Otherwise an entire function $S(z)$, which satisfies the symmetry condition
$$S(z)=S^*(z),$$
exists such that
$$E(z)=[A^*(z)-iB^*(z)]/S(z)$$
is an entire function which satisfies the inequality
$$|E^*(z)|<|E(z)|$$
when z is in the upper half–plane. Multiplication by $S(z)$ is an isometric transformation of the space $\mathcal{H}(E)$ onto the space $\mathcal{P}$.

A Hilbert space $\mathcal{Q}$ exists whose elements are entire functions and which admits the function

$$\frac{[A(z) - iB(z)][A(w)^- + iB(w)^-] - [A^*(z) + iB^*(z)][A(w^-) - iB(w^-)]}{2\pi i(w^- - z}$$

of z as reproducing kernel function for function values at w for every complex number w if, and only if, the entire functions $A(z)$ and $B(z)$ satisfy the inequality

$$|A^*(z) + iB^*(z)| \le |A(z) - iB(z)|$$

when z is in the upper half–plane. The space contains no nonzero element when the entire functions

$$A^*(z) + iB^*(z)$$

and

$$A(z) - iB(z)$$

are linearly dependent. Otherwise an entire function $S(z)$, which satisfies the symmetry condition

$$S(z) = S^*(z),$$

exists such that

$$E(z) = [A(z) - iB(z)]/S(z)$$

is an entire function which satisfies the inequality

$$|E^*(z)| < |E(z)|$$

when z is in the upper half–plane. Multiplication by $S(z)$ is an isometric transformation of the space $\mathcal{H}(E)$ onto the space $\mathcal{Q}$.

The structure theory for Hilbert spaces generalizes to Krein spaces of Pontryagin index at most one whose elements are entire functions and which admit the function

$$\frac{B^*(z)A(w^-) - A(z)B(w)^- + B(z)A(w)^- - A^*(z)B(w^-)}{\pi(z - w^-)}$$

of z as reproducing kernel function for function values at w for every complex number w when the entire functions $A(z)$ and $B(z)$ satisfy the symmetry conditions

$$A(-z) = A^*(z)$$

and

$$B(-z) = -B^*(z).$$

The space is the orthogonal sum of a subspace of even functions and a subspace of odd functions, both of which are Krein spaces of Pontryagin index at most one. At least one of the subspaces is a Hilbert space.

Entire functions $A_+(z)$ and $B_+(z)$ are defined by the identities

$$A(z) + A^*(z) = A_+(z^2) + A_+^*(z^2)$$

and

$$B(z) - B^*(z) = B_+(z^2) - B_+^*(z^2)$$

and

$$zA(z) - zA^*(z) = A_+(z^2) - A_+^*(z^2)$$

and

$$zB(z) + zB^*(z) = B_+(z^2) + B_+^*(z^2).$$

A Krein space $\mathcal{H}_+$ of Pontryagin index at most one exists whose elements are entire functions and which contains the function

$$\frac{B_+^*(z)A_+(w^-) - A_+(z)B_+(w)^- + B_+(z)A_+(w)^- - A_+^*(z)B_+(w^-)}{\pi(z - w^-)}$$

of z as reproducing kernel function for function values at w for every complex number w. An isometric transformation of the space $\mathcal{H}_+$ onto the subspace of even elements of the space $\mathcal{H}$ is defined by taking $F(z)$ into $F(z^2)$.

Entire functions $A_-(z)$ and $B_-(z)$ are defined by the identities

$$A(z) + A^*(z) = A_-(z^2) + A_-^*(z^2)$$

and

$$B(z) - B^*(z) = B_-(z^2) - B_-^*(z^2)$$

and

$$A(z) - A^*(z) = zA_-(z^2) - zA_-^*(z^2)$$

and

$$B(z) + B^*(z) = zB_-(z^2) + zB_-^*(z^2).$$

A Krein space $\mathcal{H}_-$ of Pontryagin index at most one exists whose elements are entire functions and which contains the function

$$\frac{B_-^*(z)A_-(w^-) - A_-(z)B_-(w)^- + B_-(z)A_-(w)^- - A_-^*(z)B_-(w^-)}{\pi(z - w^-)}$$

of z as reproducing kernel function for function values at w for every complex number w. An isometric transformation of the space $\mathcal{H}_-$ onto the subspace of odd elements of $\mathcal{H}$ is defined by taking $F(z)$ into $zF(z^2)$.

A Krein space $\mathcal{P}_+$ of Pontryagin index at most one exists whose elements are entire functions and which contains the function

$$\frac{[A_+^*(z) - iB_+^*(z)][A_+(w^-) + iB_+(w^-)] - [A_+(z) + iB_+(z)][A_+(w)^- - iB_+(w)^-]}{2\pi i(w^- - z)}$$

of z as reproducing kernel function for function values at w for every complex number w. A Krein space $\mathcal{Q}_+$ of Pontryagin index at most one exists whose elements are entire functions and which contains the function

$$\frac{[A_+(z) - iB_+(z)][A_+(w)^- + iB_+(w)^-] - [A_+^*(z) + iB_+^*(z)][A_+(w^-) - iB_+(w^-)]}{2\pi i(w^- - z)}$$

of z as reproducing kernel function for function values at w for all complex numbers w. The spaces $\mathcal{P}_+$ and $\mathcal{Q}_+$ are Hilbert spaces if $\mathcal{H}_+$ is a Hilbert space.

A Krein space $\mathcal{P}_-$ of Pontryagin index at most one exists whose elements are entire functions and which contains the function

$$\frac{[A_-^*(z) - iB_-^*(z)][A_-(w^-) + iB_-(w^-)] - [A_-(z) + iB_-(z)][A_-(w)^- - iB(w)^-]}{2\pi i(w^- - z)}$$

of z as reproducing kernel function for function values at w for every complex number w. A Krein space $\mathcal{Q}_-$ of Pontryagin index at most one exists whose elements are entire functions and which contains the function

$$\frac{[A_-(z) - iB_-(z)][A_-(w)^- + iB(w)^-] - [A_-^*(z) + iB_-^*(z)][A_-(w^-) - iB_-(w^-)]}{2\pi i(w^- - z)}$$

of z as reproducing kernel function for function values at w for every complex number w. The spaces $\mathcal{P}_-$ and $\mathcal{Q}_-$ are Hilbert spaces if $\mathcal{H}_-$ is a Hilbert space.

A relationship between the spaces $\mathcal{P}_+$ and $\mathcal{P}_-$ and between the spaces $\mathcal{Q}_+$ and $\mathcal{Q}_-$ results from the identities

$$A_+(z) + A_+^*(z) = A_-(z) + A_-^*(z)$$

and

$$B_+(z) - B_+^*(z) = B_-(z) - B_-^*(z)$$

and

$$A_+(z) - A_+^*(z) = zA_-(z) - zA_-^*(z)$$

and

$$B_+(z) + B_+^*(z) = zB_-(z) + zB_-^*(z).$$

The space $\mathcal{P}_+$ contains $zF(z)$ whenever $F(z)$ is an element of the space $\mathcal{P}_-$ such that $zF(z)$ belongs to $\mathcal{P}_-$. The space $\mathcal{P}_-$ contains every element of the space $\mathcal{P}_+$ such that $zF(z)$ belongs to $\mathcal{P}_+$. The identity

$$\langle tF(t), G(t)\rangle_{\mathcal{P}_+} = \langle F(t), G(t)\rangle_{\mathcal{P}_-}$$

holds whenever $F(z)$ is an element of the space $\mathcal{P}_-$ such that $zF(z)$ belongs to the space $\mathcal{P}_+$ and $G(z)$ is an element of the space $\mathcal{P}_-$ which belongs to the space $\mathcal{P}_+$. The closure in the space $\mathcal{P}_+$ of the intersection of the spaces $\mathcal{P}_+$ and $\mathcal{P}_-$ is a Hilbert space which is contained continuously and isometrically in the space $\mathcal{P}_+$ and whose orthogonal complement has dimension zero or one. The closure in the space $\mathcal{P}_-$ of the intersection of the spaces $\mathcal{P}_+$ and $\mathcal{P}_-$ is a Hilbert space which is contained continuously and isometrically in the space $\mathcal{P}_-$ and whose orthogonal complement has dimension zero or one.

The space $\mathcal{Q}_+$ contains $zF(z)$ whenever $F(z)$ is an element of the space $\mathcal{Q}_-$ such that $zF(z)$ belongs to $\mathcal{Q}_-$. The space $\mathcal{Q}_-$ contains every element $F(z)$ of the space $\mathcal{Q}_+$ such that $zF(z)$ belongs to $\mathcal{Q}_+$. The identity

$$\langle tF(t), G(t)\rangle_{\mathcal{Q}_+} = \langle F(t), G(t)\rangle_{\mathcal{Q}_-}$$

holds whenever $F(z)$ is an element of the space $\mathcal{Q}_-$ such that $zF(z)$ belongs to the space $\mathcal{Q}_+$ and $G(z)$ is an element of the space $\mathcal{Q}_-$ which belongs to the space $\mathcal{Q}_+$. The closure of the space $\mathcal{Q}_+$ in the intersection of the spaces $\mathcal{Q}_+$ and $\mathcal{Q}_-$ is a Hilbert space which is contained continuously and isometrically in the space $\mathcal{Q}_+$ and whose orthogonal complement has dimension zero or one. The closure in the space $\mathcal{Q}_-$ of the intersection of the spaces $\mathcal{Q}_+$ and $\mathcal{Q}_-$ is a Hilbert space which is contained continuously and isometrically in the space $\mathcal{Q}_-$ and whose orthogonal complement has dimension zero or one.

Hilbert spaces of entire functions satisfying the axioms (H1), (H2), and (H3) whose defining functions are of Pólya class are characterized by the existence of associated Hilbert spaces of entire functions. The defining function $E(z)$ of a space $\mathcal{H}(E)$ is of Pólya class if, and only if, a Hilbert space of entire function exists which contains the function

$$\frac{E^*(z)E'(w^-) - E(z)E(w)^- + E(z)E'(w)^- - E^{*'}(z)E(w^-)}{\pi(z - w^-)}$$

of z as reproducing kernel function for function values at w for all complex numbers w. A Hilbert space of entire functions then exists which contains the function

$$\frac{[E'(z) - iE(z)][E'(w)^- + iE(w)^-] - [E^{*'}(z) + iE(z)][E'(w^-) - iE(w^-)]}{2\pi i(w^- - z)}$$

of z as reproducing kernel function for function values at w for every complex number w. A Hilbert space of entire function also exists which contains the function

$$\frac{[E^{*'}(z) - iE^*(z)][E'(w^-) + iE(w^-)] - [E'(z) + iE(z)][E'(w)^- - iE(w)^-]}{2\pi i(w^- - z)}$$

of z as reproducing kernel function for function values at w for all complex numbers w.

Hilbert spaces of entire functions which satisfy the axioms (H1), (H2), and (H3) and whose defining functions are of Pólya class are constructed from analytic weight functions which satisfy a related hypothesis. The modulus of the weight function $W(z)$ is assumed to be a nondecreasing function of distance from the real axis on every vertical half–line in the upper half–plane. An equivalent condition is the existence of a Hilbert space of functions analytic in the upper half–plane which contains the function

$$\frac{W(z)W'(w)^- - W'(z)W(w)^-}{\pi(z - w^-)}$$

of z as reproducing kernel for function values at w whenever w is in the upper half–plane.

Entire functions $E(c,z)$ of Pólya class are determined by an analytic weight function which satisfies the analogue of the Pólya class condition. A space $\mathcal{H}(E(c))$ exists for every parameter c. Multiplication by $\exp(itz)$ is a contractive transformation of the space $\mathcal{H}(E(c))$ into the space $\mathcal{F}(W)$ for some real number t. The transformation is isometric on the domain of multiplication by z in the space $\mathcal{H}(E(c))$. An entire function $F(z)$ belongs to the space $\mathcal{H}(E(c))$ whenever $\exp(itz)F(z)$ belongs to the space $\mathcal{F}(W)$ and $(z-w)F(z)$ belongs to the space $\mathcal{H}(E(c))$ for some complex number w. If entire functions $E(a,z)$ and $E(b,z)$ of Pólya class are associated with the weight function, then either the space $\mathcal{H}(E(b))$ is contained contractively in the space $\mathcal{H}(E(a))$ or the space $\mathcal{H}(E(a))$ is contained contractively in the space $\mathcal{H}(E(b))$.

A choice of defining function is made for each Hilbert space of entire functions. The parameters are taken in a connected open subset of the positive half–line whose closure contains the origin. If a and b are parameters such that a is less than b, the identity

$$(A(a,z),B(a,z)) = (A(b,z),B(b,z))M(b,a,z)$$

holds for a matrix–valued entire function $M(b,a,z)$ such that a space $\mathcal{H}(M(b,a))$ exists. The construction is made so that the function $M(b,a,z)$ always has the identity matrix as value at the origin. The entries of the matrix are then continuous functions of the parameters a and b when z is held fixed. The integral equation

$$M(b,a,z)I - I = z\int_b^a M(b,t,z)dm(t)$$

then holds with

$$m(t) = \begin{pmatrix} \alpha(t) & \beta(t) \\ \beta(t) & \gamma(t) \end{pmatrix}$$

a nonincreasing matrix–valued function of parameters t whose entries are continuous real–valued functions of t. The analytic function

$$E(c,z)/W(z)$$

of z in the upper half–plane is of bounded type in the half–plane. The mean type $\tau(c)$ of the function in the half–plane is a continuous nonincreasing function of the parameter c. The matrix valued function

$$m(t) + iI\tau(t) = \begin{pmatrix} \alpha(t) & \beta(t) - i\tau(t) \\ \beta(t) + i\tau(t) & \gamma(t) \end{pmatrix}$$

is a nonincreasing function of t. The function $\tau(t)$ is characterized as having the greatest increments compatible with monotonicity of the matrix function. Multiplication by $\exp(i\tau(c)z)$ is a contractive transformation of the space $\mathcal{H}(E(c))$ into the space $\mathcal{F}(W)$ which is isometric on the domain of multiplication by z in the space $\mathcal{H}(E(c))$. The union of the images of the spaces $\mathcal{H}(E(c))$ is dense in the space $\mathcal{F}(W)$. The intersection of the images of the spaces $\mathcal{H}(E(c))$ contains no nonzero element of the space $\mathcal{F}(W)$.

A parameter c always exists such that $\tau(c)$ is equal to zero. The parameterization can always be made so that the least such parameter is equal to one. A natural normalization of the function $m(t)$ of t is then with value zero when t is equal to one.

A construction of Hilbert spaces of entire functions associated with analytic functions is implicit in the work of Arne Beurling and Paul Malliavin [1]. Principles of potential theory are introduced which are given a new application in the theory of Hilbert spaces of entire functions.

Assme that a maximal dissipative transformation in a weighted Hardy space $\mathcal{F}(W)$ is defined for h in the interval $[0,1]$ by taking $F(z)$ into $F(z+ih)$ whenever $F(z)$ and $F(z+ih)$ belong to the space. A maximal dissipative transformation in the space is then also defined for h in the interval $[0,1]$ by taking $F(z)$ into $iF'(z+ih)$ whenever $F(z)$ and $F'(z+ih)$ belong to the space. Then Hilbert spaces of entire functions exist which are contained isometrically in the space $\mathcal{F}(W)$, which satisfy the axioms (H1), (H2), and (H3), and which contain nonzero elements. Such a space is a space $\mathcal{H}(E)$ whose defining function $E(z)$ is of Pólya class. An example of such a space is the set of all entire functions $F(z)$ such that $F(z)$ and $F^*(z)$ belong to the space $\mathcal{F}(W)$. It will be shown that a maximal dissipative transformation in the space $\mathcal{H}(E)$ is defined by taking $F(z)$ into $F(z+ih)$ whenever $F(z)$ and $F(z+ih)$ belong to the space if h belongs to the interval $[0,1]$. It follows that a maximal dissipative transformation in the space $\mathcal{H}(E)$ is defined by taking $F(z)$ into $iF'(z+ih)$ whenever $F(z)$ and $F'(z+ih)$ belong to the space if h belongs to the interval $[0,1]$.

Since the space $\mathcal{H}(E)$ is contained isometrically in the space $\mathcal{F}(W)$, the transformation which takes $F(z)$ into $F(z+ih)$ whenever $F(z)$ and $F(z+ih)$ belong to the space is clearly dissipative. The maximal dissipative property of the transformation is proved by showing that every element of the space is of the form $F(z)+F(z+ih)$ for an element $F(z)$ of the space such that $F(z+ih)$ belongs to the space. Since a maximal dissipative transformation in the space $\mathcal{F}(W)$ is defined by taking $F(z)$ into $F(z+ih)$ whenever $F(z)$ and $F(z+ih)$ belong to the space, an element of the space $\mathcal{H}(E)$ is of the form

$$F(z)+F(z+ih)$$

for an element $F(z)$ of the space $\mathcal{F}(W)$ such that $F(z+ih)$ belongs to the space $\mathcal{F}(W)$. Since the elements of the space $\mathcal{H}(E)$ are entire functions, $F(z)$ is an entire function. Since the space $\mathcal{H}(E)$ satisfies the axiom (H3)),

$$F^*(z)+F^*(z-ih)=G(z)+G(z+ih)$$

for an element $G(z)$ of the space $\mathcal{F}(W)$ such that $G(z+ih)$ belongs to the space $\mathcal{F}(W)$. Then $G(z)$ is an entire function which satisfies the identity

$$F^*(z)-G(z+ih)=G(z)-F^*(z-ih).$$

It will be shown that the entire function appearing on the left and right of the identity vanishes identically. Since $F^*(z)$ and $F^*(z-ih)$ then belong to the space $\mathcal{F}(W)$, the functions $F(z)$ and $F(z+ih)$ belong to the space $\mathcal{H}(E)$.

The inequality
$$|F(z)|^2 \leq \|F\|^2 |W(z)|^2 / [2\pi(iz^- - iz)]$$
holds when z is in the upper half–plane. Since the inequality

$$|G^*(z-ih)|^2 \leq \|G\|^2 |W(z^- + ih)|^2 / [2\pi(2h - iz^- + iz)]$$

holds in the half–plane $iz^- - ih < 2h$, the inequality

$$|F(z) - G^*(z-ih)| \leq \|F\| W(z) / [2\pi(iz^- - iz)]^{\frac{1}{2}}$$
$$+ \|G\| |W(z^- + ih)| / [2\pi(2h - iz^- + iz)]^{\frac{1}{2}}$$

holds in the strip $0 < iz^- - iz < 2h$. Since the modulus of

$$F(z) - G^*(z-ih)$$

is periodic of period ih and since the modulus of $W(z)$ is a nondecreasing function of distance from the real axis on every vertical half–line in the upper half–plane, the function

$$[F(z) - G^*(z-ih)]/W(z)$$

is of bounded type in the upper half–plane. A similar argument shows that the function

$$[G(z) - F^*(z-ih)]/W(z)$$

is of bounded type in the upper half–plane. Since the function

$$F(z) - G^*(z-ih)$$

is bounded on the imaginary axis, it is a constant. The function vanish identically since it changes sign when z is replaced by $z + ih$.

It has been shown that a maximal dissipative transformation in the space $\mathcal{H}(E)$ is defined for h in the interval $[0, 1]$ by taking $F(z)$ into $F(z+ih)$ whenever $F(z)$ and $F(z+ih)$ belong to the space if the space is contained isometrically in the space $\mathcal{F}(W)$ and contains every entire function $F(z)$ such that $F(z)$ and $F^*(z)$ belong to the space $\mathcal{F}(W)$. The same conclusions will be obtained for a space $\mathcal{H}(E)$, which is contained isometrically in the space $\mathcal{F}(W)$, such that an entire function $F(z)$ belongs to the space $\mathcal{H}(E)$ whenever $F(z)$ belongs to the space $\mathcal{F}(W)$ and $(z-\lambda)F(z)$ belongs to the space $\mathcal{H}(E)$ for some complex number w. Since a maximal dissipative transformation in the space $\mathcal{F}(W)$ is defined by taking $F(z)$ into $F(z+ih)$ belong to the space, a dissipative transformation in the space $\mathcal{H}(E)$ is defined by taking $F(z)$ into $F(z+ih)$ whenever $F(z)$ and $F(z+ih)$ belong to the space. The maximal dissipative property of the transformation in the space $\mathcal{H}(E)$ is proved by showing that every element of the space is of the form $F(z) + F(z + ih)$ for an element $F(z)$ of the space such that $F(z + ih)$ belongs to the space. It has been shown that every element of the space $\mathcal{H}(E)$ is of the form

$$F(z) + F(z + ih)$$

for an entire function $F(z)$ such that $F(z)$ and $F^*(z)$, as well as $F(z+ih)$ and $F^*(z-ih)$, belong to the space $\mathcal{F}(W)$. Since the entire function

$$F(z) + F(z + ih)$$

belongs to the space $\mathcal{F}(W)$, the functions

$$[F(iy) + F(iy + ih)]/E(iy)$$

and

$$[F^*(iy) + F^*(iy - ih)/E(iy)$$

converge to zero in the limit of large positive y. Since the limits of

$$F(iy)/F(iy + ih)$$

and

$$F^*(iy)/F^*(iy - ih)$$

exist in the limit of large positive y and are not both equal to minus one,

$$F(iy)/E(iy)$$

and

$$F(iy + ih)/E(iy)$$

as well as

$$F^*(iy)/E(iy)$$

and

$$F^*(iy - ih)/E(iy)$$

converge to zero in the limit of large positive y. It follows that the entire functions $F(z)$ and $F(z + ih)$ belong to the space $\mathcal{H}(E)$.

A construction of Hilbert space of entire functions which satisfy the axioms (H1), (H2), and (H3) is made from Hilbert spaces of entire functions which satisfy the axioms (H1), (H2), and (H3). Assume that a nontrivial entire function $S(z)$ is associated with a given space $\mathcal{H}(E)$ and that a given complex number λ is not a zero of $S(z)$. A partially isometric transformation of the space $\mathcal{H}(E)$ onto a Hilbert space, whose elements are entire functions, is defined by taking $F(z)$ into

$$\frac{F(z)S(\lambda) - S(z)F(\lambda)}{z - \lambda}.$$

The kernel of the transformation is the set of elements $F(z)$ of the space $\mathcal{H}(E)$ such that $F(z)$ and $S(z)$ are linearly dependent. If

$$\frac{B(z)A(\lambda)^- - A(z)B(\lambda)^-}{\pi(z-\lambda^-)}$$

and $S(z)$ are linearly dependent, then the range of the transformation is isometrically equal to a space $\mathcal{H}(E')$ with

$$E'(z) = \frac{E(z)S(\lambda) - S(z)E(\lambda)}{z-\lambda}.$$

Maximal dissipative transformations in Hilbert spaces of entire functions which satisfy the axioms (H1), (H2), and (H3) are constructed inductively from maximal dissipative transformations in Hilbert spaces of entire functions which satisfy the axioms (H1), (H2), and (H3).

Theorem 2. *Assume that a maximal dissipative transformation is defined in a space $\mathcal{H}(E)$ by entire functions $P(z)$ and $Q(z)$, which are associated with the space, so that the transformation takes $F(z)$ into $G(z+i)$ whenever $F(z)$ and $G(z+i)$ are elements of the space which satisfy the identity*

$$G(w) = \langle F(t), [Q(t)P(w^-) - P(t)Q(w^-)]/[\pi(t-w^-)]\rangle_{\mathcal{H}(E)}$$

for all complex numbers w. Assume that λ is a complex number such that $i\lambda^- - i\lambda$ is not equal to minus one. If the space $\mathcal{H}(E)$ contains a nonzero entire function which vanishes at λ, a space $\mathcal{H}(E')$ exists such that multiplication by $z-\lambda$ is an isometric transformation of the space $\mathcal{H}(E')$ onto the set of elements of the space $\mathcal{H}(E)$ which vanish at λ. If a maximal dissipative transformation in the space $\mathcal{H}(E')$ is defined by taking $F(z)$ into $G(z+i)$ whenever $(z-\lambda)F(z)$ is the orthogonal projection into the image of the space $\mathcal{H}(E')$ of an element $H(z)$ of the domain of the maximal dissipative transformation in the space $\mathcal{H}(E)$ which maps into $(z-\lambda)G(z+i)$, then entire functions $P'(z)$ and $Q'(z)$, which are associated with the space $\mathcal{H}(E')$, exist such that the transformation takes $F(z)$ into $G(z+i)$ whenever $F(z)$ and $G(z+i)$ are elements of the space such that the identity

$$G(w) = \langle F(t), [Q'(t)P'(w^-) - P'(t)Q'(w^-)]/[\pi(t-w^-)]\rangle_{\mathcal{H}(E')}$$

holds for all complex numbers w.

Proof of Theorem 2. The desired functions are given by the equations

$$P'(z) = \frac{Q(z)\alpha - P(z)\beta}{z-\lambda}\,\gamma - \frac{Q(z)\gamma - P(z)\delta}{z+i-\lambda^-}\,\alpha$$

and

$$Q'(z) = \frac{Q(z)\alpha - P(z)\beta}{z-\lambda}\,\delta - \frac{Q(z)\gamma - P(z)\delta}{z+i-\lambda^-}\,\beta$$

when a matrix

$$\begin{pmatrix} \alpha & \beta \\ \gamma & \sigma \end{pmatrix}$$

of complex numbers with determinant one exists such that

$$\frac{Q(z)\alpha - P(z)\beta}{z - \lambda}$$

and

$$\frac{Q(z)\gamma - P(z)\delta}{z + i - \lambda^-}$$

are entire functions. The desired properties of the functions are verified using the identity

$$\begin{aligned} (z-\lambda) \ \frac{Q'(z)P'(w^-) - P'(z)Q'(w^-)}{\pi(z - w^-)} \ (w^- + i - \lambda^-) \\ = \frac{Q(z)P(w^-) - P(z)Q(w^-)}{\pi(z - w^-)} \\ +\pi(\lambda + i - \lambda^-) \ \frac{Q(z)\gamma - P(z)\delta}{\pi(z + i - \lambda^-)} \ \frac{Q(w^-)\alpha - P(w^-)\beta}{\pi(w^- - \lambda)} \end{aligned}$$

which is a consequence of the identities

$$(z - \lambda)P'(z) = P(z) + (\lambda + i - \lambda^-) \ \frac{Q(z)\gamma - P(z)\delta}{z + i - \lambda^-} \ \alpha$$

and

$$(z - \lambda)Q'(z) = Q(z) + (\lambda + i - \lambda^-) \ \frac{Q(z)\gamma - P(z)\delta}{z + i - \lambda^-} \ \beta$$

as well as the identities

$$(z + i - \lambda^-)P'(z) = P(z) + (\lambda + i - \lambda^-) \ \frac{Q(z)\alpha - P(z)\beta}{z - \lambda} \ \gamma$$

and

$$(z + i - \lambda^-)Q'(z) = Q(z) + (\lambda + i - \lambda^-) \ \frac{Q(z)\alpha - P(z)\beta}{z - \lambda} \ \delta.$$

Since the maximal dissipative relation in the space $\mathcal{H}(E')$ is assumed to be a transformation, the maximal dissipative transformation in the space $\mathcal{H}(E)$ annihilates elements of the space which are orthogonal to elements which vanish at λ and which are mapped into elements which vanish at $\lambda - i$. An element of the space $\mathcal{H}(E)$ which is orthogonal to elements which vanish at λ is a constant multiple of

$$\frac{B(z)A(\lambda)^- - A(z)B(\lambda)^-}{\pi(z - \lambda^-)}$$

which is mapped into

$$\frac{Q^*(z)P(\lambda)^- - P^*(z)Q(\lambda)^-}{\pi(z-\lambda^-)}.$$

The desired matrix exists when the function obtained does not vanish at $\lambda - i$. If the function vanishes at $\lambda - i$, it vanishes identically. The functions $P(z)$ and $Q(z)$ are linearly dependent if they do not both vanish at λ. Since the maximal dissipative transformation in the space $\mathcal{H}(E)$ then vanishes identically, the maximal dissipative transformation in the space $\mathcal{H}(E')$ vanishes identically. The functions $P'(z)$ and $Q'(z)$ are linearly dependent entire functions which are associated with the space $\mathcal{H}(E')$. If the functions $P(z)$ and $Q(z)$ are linearly independent, they both vanish at λ. The desired matrix exists.

This completes the proof of the theorem.

Maximal transformations of dissipative deficiency at most one in Hilbert spaces of entire functions which satisfy the axioms (H1), (H2), and (H3) are constructed inductively from maximal dissipative transformations in Hilbert spaces of entire functions which satisfy the axioms (H1), (H2), and (H3). Assume that $\mathcal{H}(E)$ is a given space of dimension greater than one and that λ is a complex number such that $i\lambda^- - i\lambda$ is not equal to minus one and such that the space contains elements having nonzero values at λ and at $\lambda^- - i$. Then a space $\mathcal{H}(E')$ exists such that multiplication by $z - \lambda$ is an isometric transformation of the space onto the set of elements of the space $\mathcal{H}(E)$ which vanish at λ. Assume that a maximal dissipative transformation in the space $\mathcal{H}(E')$ is defined by entire functions $P'(z)$ and $Q'(z)$, which are associated with the space, such that the transformation takes $F(z)$ into $G(z+i)$ whenever $F(z)$ and $G(z+i)$ are elements of the space which satisfy the identity

$$G(w) = \langle F(t), [Q'(t)P'(w^-) - P'(t)Q'(w^-)]/[\pi(t-w^-)]\rangle_{\mathcal{H}(E')}$$

for all complex numbers w. Entire functions $P(z)$ and $Q(z)$, which are associated with the space $\mathcal{H}(E)$, are defined by the equations

$$\begin{aligned} &P(\lambda)Q(\lambda^- - i) - Q(\lambda)P(\lambda^- - i)](z-\lambda)(z+i-\lambda^-)P'(z) \\ &= [(z-\lambda)P(\lambda)Q(\lambda^- - i) - (z+i-\lambda^-)Q(\lambda)P(\lambda^- - i)]P(z) \\ &\quad +(\lambda+i-\lambda^-)P(\lambda)P(\lambda^- - i)Q(z) \end{aligned}$$

and

$$\begin{aligned} &[P(\lambda)Q(\lambda^- - i) - Q(\lambda)P(\lambda^- - i)](z-\lambda)(z+i-\lambda^-)Q'(z) \\ &= [(z+i-\lambda^-)P(\lambda)Q(\lambda^- - i) - (z-\lambda)Q(\lambda)P(\lambda^- - i)]Q(z) \\ &\quad -(\lambda+i-\lambda^-)Q(\lambda)Q(\lambda^- - i)P(z) \end{aligned}$$

with values at λ and $\lambda^- - i$ which are subject only to the condition that

$$P(\lambda)Q(\lambda^- - i) - Q(\lambda)P(\lambda^- - i)$$

be nonzero. A maximal transformation of dissipative deficiency at most one in the space $\mathcal{H}(E)$ is defined by taking $F(z)$ into $G(z+i)$ whenever $F(z)$ and $G(z+i)$ are elements of the space which satisfy the identity

$$G(w) = \langle F(t), [Q(t)P(w^-) - P(t)Q(w^-)]/[\pi(t-w^-)]\rangle_{\mathcal{H}(E)}$$

for all complex numbers w. The transformation in the space $\mathcal{H}(E')$ takes $F(z)$ into $G(z+i)$ whenever the $F(z)$ and $G(z+i)$ are elements of the space such that the transformation in the space $\mathcal{H}(E)$ takes $H(z)$ into $(z-\lambda)G(z+i)$ and such that $(z-\lambda)F(z)$ is the orthogonal projection of $H(z)$ into the set of elements of the space which vanish at λ.

The Riemann hypothesis for Hilbert spaces of entire functions is a conjecture about zeros of the entire functions which define the spaces. The conjecture is proved when maximal dissipative transformations are given in related weighted Hardy spaces.

Theorem 3. *Assume that $W(z)$ is an analytic weight function such that a maximal dissipative transformation in the weighted Hardy space $\mathcal{F}(W)$ is defined for $0 \le h \le 1$ by taking $F(z)$ into $F(z+ih)$ whenever $F(z)$ and $F(z+ih)$ belong to the space. Assume that entire functions $S_r(z)$ of Pólya class, which are determined by their zeros, and positive numbers γ_r, which satisfy the inequality $1 \le \gamma_r$, are given for positive integers r such that*

$$S_{r+1}(z)/S_r(z)$$

is always an entire function and such that the inequality

$$\gamma_r \le \gamma_{r+1}$$

is always satisfied. Assume that a space $\mathcal{H}(E)$ is defined by an entire function

$$E(z) = \lim \exp(-i\gamma_r z)W(z)/S_r(z)$$

which is obtained as a limit uniformly on compact subsets of the upper half–plane. If the inequality

$$-1 \le iw - iw^-$$

holds for a zero w^- of $E(z)$, then equality holds and the zero is simple.

Proof of Theorem 3. An entire function

$$E_r(z) = S_r(z)E(z)$$

of Pólya class is defined for every positive integer r such that multiplication by the entire function $S_r(z)$ is an isometric transformation of the space $\mathcal{H}(E)$ into the space $\mathcal{H}(E_r)$. The analytic weight function

$$W(z) = \lim \exp(i\gamma_r z)E_r(z)$$

is recovered as a limit uniformly on compact subsets of the upper half–plane.

A maximal dissipative transformation in the space $\mathcal{F}(W)$ is by hypothesis defined by taking $F(z)$ into $F(z+ih)$ whenever $F(z)$ and $F(z+ih)$ belong to the space. A maximal dissipative relation in the space $\mathcal{H}(E_r)$ is defined for every positive integer r. Multiplication by

$$W(z)/E_r(z)$$

is an isometric transformation of the space $\mathcal{H}(E_r)$ into the space $\mathcal{F}(W)$. The maximal dissipative relation in the space $\mathcal{H}(E_r)$ takes $F(z)$ into $G(z+ih)$ whenever $F(z)$ and $G(z+ih)$ are elements of the space such that the maximal dissipative transformation in the space $\mathcal{F}(W)$ takes $H_n(z)$ into $H_n(z+ih)$ for every positive integer n, such that

$$W(z)G(z+ih)/E_r(z)$$

is the limit in the metric topology of the space $\mathcal{F}(W)$ of the elements $H_n(z+ih)$, and such that

$$W(z)F(z)/E_r(z)$$

is the limit in the same topology of the orthogonal projections of the elements $H_n(z)$ in the image of the space $\mathcal{H}(E_r)$ in the space $\mathcal{F}(W)$.

If w is in the upper half–plane and if w^- is a zero of $E(z)$, the identity

$$\frac{W(z)}{E_r(z)} \; \frac{E_r(z)E_r(w)^- - E_r^*(z)E_r(w^-)}{2\pi i(w^- - z)} \; \frac{W(w)^-}{E_r(w)^-} = \frac{W(z)W(w)^-}{2\pi i(w^- - z)}$$

implies the identity

$$W(w)F(w)/E_r(w) = W(w-ih)G(w)/E_r(w-ih)$$

when the maximal dissipative relation in the space $\mathcal{H}(E_r)$ takes $F(z)$ into $G(z+ih)$. For elements $H_n(z)$ of the space $\mathcal{F}(W)$ such that $H_r(z+ih)$ belongs to the space exist such that

$$W(z)G(z+ih)/E_r(z)$$

is a limit in the metric topology of the space of the elements $H_n(z+ih)$ of the space and such that

$$W(z)F(z)/E_r(z)$$

is the limit in the same topology of the orthogonal projection of the elements $H_n(z)$ in the image of the space $\mathcal{H}(E_r)$. Since multiplication by $W(z)/E_r(z)$ is an isometric transformation of the space $\mathcal{H}(E_r)$ into the space $\mathcal{F}(W)$, the identity

$$W(w)F(w)/E_r(w) = \lim H_n(w)$$

holds with the right side converging to

$$W(w-ih)G(w)/E_r(w-ih).$$

It follows that the adjoint of the maximal dissipative relation in the space $\mathcal{H}(E_r)$ takes

$$\frac{E_r(z)E_r(w-ih)^- - E_r^*(z)E_r(w^- + ih)}{2\pi i(w^- + ih - z)} \; \frac{E_r(w)^-}{W(w)^-}$$

into

$$\frac{E_r(z)E_r(w)^- - E_r^*(z)E_r(w^-)}{2\pi i(w^- - z)} \; \frac{E_r(w-ih)^-}{W(w-ih)^-}.$$

The numbers γ_n have a limit γ_∞ if they are bounded. An entire function $S_\infty(z)$ of Pólya class, which is determined by its zeros, exists such that

$$S_\infty(z)/S_r(z)$$

is an entire function for every positive integer r and such that the entire functions have no common zeros. The product

$$E_\infty(z) = S_\infty(z)E(z)$$

is an entire function of Pólya class. The choice of $S_\infty(z)$ is made so that the identity

$$\exp(i\gamma_\infty z)E_\infty(z) = W(z)$$

is satisfied. A space $\mathcal{H}(E_\infty)$ exists. Multiplication by $\exp(i\gamma_\infty z)$ is an isometric transformation of the space $\mathcal{H}(E_\infty)$ into the space $\mathcal{F}(W)$. The space $\mathcal{H}(E_\infty)$ contains every entire function $F(z)$ such that

$$\exp(i\gamma_\infty z)F(z)$$

and

$$\exp(i\gamma_\infty z)F^*(z)$$

belong to the space $\mathcal{F}(W)$. The inequality

$$iw - iw^- \leq -1$$

is satisfied and w^- is a simple zero of $E(z)$ if equality holds.

If the numbers γ_r are unbounded, a dense set of elements of the space $\mathcal{F}(W)$ belong to the union of the ranges of the transformations of the spaces $\mathcal{H}(E_r)$ into the space $\mathcal{F}(W)$ which take $F(z)$ into

$$\exp(i\gamma_r z)W(z)F(z)/E_r(z).$$

Since the identity

$$\frac{W(w)^- W(w)}{2\pi i(w^- - w)} = \lim \exp(i\gamma_r w) \ \frac{E_r(w)^- E_r(w)}{2\pi i(w^- - w)} \ \exp(-i\gamma_r w^-)$$

is satisfied, the functions $S_r(z)$ can be chosen so that the identity

$$W(w) = \lim \exp(i\gamma_r w)E_r(w)$$

is satisfied. The identity

$$W(z) = \lim \exp(i\gamma_r z)E_r(z)$$

then holds with uniform convergence on compact subsets of the upper half–plane.

If w is in the upper half–plane and if w^- is a zero of $E(z)$ such that the identity $w = w^- + ih$ holds with h a positive number less than or equal to one, then w is a zero of $G(z)$ whenever the maximal dissipative relation in the space $\mathcal{H}(E_r)$ takes $F(z)$ into $G(z + ih)$.

A restriction of the maximal dissipative transformation in the space $\mathcal{F}(W)$ is defined by taking $H(z)$ into $H(z+ih)$ whenever $H(z)$ and $H(z+ih)$ are elements of the space such that w is a zero of $H(z)$. The restricted transformation is sufficient to define the maximal dissipative relation in the space $\mathcal{H}(E_r)$. The maximal dissipative relation in the space $\mathcal{H}(E_r)$ takes $F(z)$ into $G(z+ih)$ whenever $F(z)$ ad $G(z+ih)$ are elements of the space such that the maximal dissipative transformation in the space $\mathcal{F}(W)$ takes $H_n(z)$ into $H_n(z+ih)$ with w a zero of $H_n(z)$ for every positive integer n, such that

$$W(z)G(z+ih)/E_r(z)$$

is the limit in the metric topology of the space $\mathcal{F}(W)$ of the elements $H_n(z+ih)$, and such that

$$W(z)F(z)/E_r(z)$$

is the limit in the space topology of the orthogonal projections of the elements $H_n(z)$ in the image of the space $\mathcal{H}(E_r)$ in the space $\mathcal{F}(W)$.

If $H(z)$ is an element of the space $\mathcal{F}(W)$ such that $H(z+ih)$ belongs to the space, then elements $F_r(z)$ and $G_r(z+ih)$ of the space $\mathcal{H}(E_r)$ exist for every positive integer r such that the maximal dissipative relation in the space takes $F_r(z)$ into $G_r(z+ih)$ and such that the limits

$$H(z) = \lim W(z)F_r(z)/E_r(z)$$

and

$$H(z+ih) = \lim W(z)G_r(z+ih)/E_r(z)$$

are obtained in the metric topology of the space $\mathcal{F}(W)$. Elements $H_n(z)$ of the space $\mathcal{F}(W)$ such that $H_n(z+ih)$ belongs to the space and such that w is a zero of $H_n(z)$ then exist such that the limits

$$H(z) = \lim H_n(z)$$

and

$$H(z+ih) = \lim H_n(z+ih)$$

are obtained in the metric topology of the space $\mathcal{F}(W)$. It follows that w is a zero of $H(z)$ whenever $H(z)$ is an element of the space $\mathcal{F}(W)$ such that $H(z+ih)$ belongs to the space.

A contradiction of the maximal dissipative property of the transformation in the space $\mathcal{F}(W)$ is obtained since such a transformation is densely defined. This completes the proof that $E(z)$ has no zero w^- such that the identity $w = w^- + ih$ holds for a positive number h which is less than or equal to one.

A limiting case of the argument applies when h is equal to zero. Argue by contradiction assuming that $E(z)$ has a real zero w. Then w is a zero of $E_r(z)$ for every positive integer r. If the numbers γ_r are bounded, w is a zero of $E_\infty(z)$. This contradicts the maximal dissipative property of the transformation which takes $H(z)$ into $iH'(z)$ whenever $H(z)$ and $H'(z)$ belong to $\mathcal{H}(E_\infty)$. If the numbers γ_r are unbounded, w is a zero of $H(z)$ whenever $H(z)$ is an element of the space $\mathcal{F}(W)$ such that $H'(z)$ belongs to the space.

This contradicts the maximal dissipative property of the transformation which takes $H(z)$ into $iH'(z)$ whenever $H(z)$ and $iH'(z)$ belong to the space.

Another limiting case of the argument is used to show that w^- is a simple zero of $E(z)$ if it is a zero of $E(z)$ which satisfies the identity $w = w^- + i$. Argue by contradiction assuming that a double zero is obtained. Then w is a double zero of $E_r(z)$ for every positive integer r. If the numbers γ_r are bounded, w is a double zero of $E_\infty(z)$. This contradicts the maximal dissipative property of the transformation which takes $H(z)$ into $iH'(z+i)$ whenever $H(z)$ and $H'(z+i)$ belong to $\mathcal{H}(E_\infty)$. If the numbers γ_r are unbounded, w is a zero of $H(z)$ whenever $H(z)$ is an element of the space $\mathcal{F}(W)$ such that $H(z)$ and $H'(z+i)$ belong to the space. This contradicts the maximal dissipative property of the transformation which takes $H(z)$ into $iH'(z+i)$ whenever $H(z)$ and $H'(z+i)$ belong to the space.

This completes the proof of the theorem.

The Riemann hypothesis for Hilbert spaces of entire functions is also proved when maximal transformations of dissipative deficiency at most one are given in related weighted Hardy spaces which satisfy a symmetry condition.

Theorem 4. *Assume that $W(z)$ is an analytic weight function, which satisfies the symmetry condition*

$$W(-z^-) = W(z)^-,$$

such that a maximal transformation of dissipative deficiency at most one in the weighted Hardy space $\mathcal{F}(W)$ is defined by taking $F(z)$ into $zF(z+ih)/(z+ih)$ when $-1 \le h \le 1$. Assume that entire functions $S_r(z)$ of Pólya class, which are determined by their zeros and which satisfy the symmetry condition

$$S_r(-z^-) = S_r(z)^-,$$

and positive numbers γ_r, which satisfy the inequality $1 \le \gamma_r$, are given for positive numbers r such that

$$S_{r+1}(z)/S_r(z)$$

is always an entire function and such that the inequality

$$\gamma_r \le \gamma_{r+1}$$

is always satisfied. Assume that a space $\mathcal{H}(E)$ is defined by an entire function

$$E(z) = \lim \exp(i\gamma_r z)W(z)/S_r(z)$$

which is obtained as a limit uniformly on compact subsets of the upper half–plane. If the inequality

$$-1 \le iw - iw^-$$

holds for a zero w^- of $E(z)$ which does not lie on the imaginary axis, then equality holds and the zero is simple.

Proof of Theorem 4. The entire function $E(z)$ is of Pólya class and satisfies the symmetry condition

$$E(-z^-) = E(z)^-.$$

An entire function

$$E_r(z) = S_r(z)E(z)$$

of Pólya class, which satisfies the symmetry condition

$$E_r(-z)^- = E_r(z)^-,$$

is defined for every positive integer r such that multiplication by the entire function $S_r(z)$ is an isometric transformation of the space $\mathcal{H}(E)$ onto the space $\mathcal{H}(E_r)$. The analytic weight function

$$W(z) = \lim \exp(i\gamma_r z)E_r(z)$$

is recovered as a limit uniformly on compact subsets of the upper half–plane.

A maximal transformation of dissipative deficiency at most one in the space $\mathcal{F}(W)$ is by hypothesis defined by taking $F(z)$ into $zF(z+ih)/(z+ih)$ whenever $F(z)$ and $zF(z+ih)/(z+ih)$ belong to the space. The transformation takes $F^*(z)$ into $zF^*(-z-ih)/(z+ih)$ whenever it takes $F(z)$ into $zF(z+ih)/(z+ih)$. Multiplication by

$$W(z)/E_r(z)$$

is an isometric transformation of the space $\mathcal{H}(E_r)$ into the space $\mathcal{F}(W)$. The maximal relation of dissipative deficiency at most one in the space $\mathcal{H}(E_r)$ takes $F(z)$ into $G(z+ih)$ whenever $F(z)$ and $G(z+ih)$ are elements of the space such that the maximal transformation of dissipative deficiency at most one in the space $\mathcal{F}(W)$ takes $H_n(z)$ into $zH_n(z+ih)/(z+ih)$ for every positive integer n, such that

$$W(z)G(z+ih)/E_r(z)$$

is the limit in the metric topology of the space $\mathcal{F}(W)$ of the elements $zH_n(z+ih)/(z+ih)$, and such that

$$W(z)F(z)/E_r(z)$$

is the limit in the same topology of the orthogonal projections of the elements $H_n(z)$ in the image of the space $\mathcal{H}(E_r)$ in the space $\mathcal{F}(W)$. The maximal relation of dissipative deficiency at most one in the space $\mathcal{H}(E_r)$ takes $F^*(-z)$ into $G^*(-z-ih)$ whenever it takes $F(z)$ into $G(z+ih)$.

If w is in the upper half–plane and if w^- is a zero of $E(z)$, the identity

$$\frac{W(z)}{E_r(z)} \frac{E_r(z)E_r(w)^- - E_r^*(z)E_r(w^-)}{2\pi i(w^- - z)} \frac{W(w)^-}{E_r(w)^-} = \frac{W(z)W(w)^-}{2\pi i(w^- - z)}$$

implies the identity

$$wW(w)F(w)/E_r(w) = (w-ih)W(w-ih)G(w)/E_r(w-ih)$$

when the maximal relation of dissipative deficiency at most one in the space $\mathcal{H}(E_r)$ takes $F(z)$ into $G(z+ih)$. For elements $H_n(z)$ of the space $\mathcal{F}(W)$ such that $H_n(z+ih)/(z+ih)$ belongs to the space exist such that

$$W(z)G(z+ih)/E_r(z)$$

is a limit in the metric topology of the space of the elements $zH_n(z+ih)/(z+ih)$ of the space and such that

$$W(z)F(z)/E_r(z)$$

is the limit in the same topology of the orthogonal projection of the elements $H_n(z)$ in the image of the space $\mathcal{H}(E_r)$. Since multiplication by $W(z)/E_r(z)$ is an isometric transformation of the space $\mathcal{H}(E_r)$ into the space $\mathcal{F}(W)$, the identity

$$W(w)F(w)/E_r(w) = \lim H_n(w)$$

holds with the right side converging to

$$w^{-1}(w-ih)W(w-ih)G(w)/E_r(w-ih).$$

It follows that the adjoint of the maximal dissipative relation in the space $\mathcal{H}(E)$ takes

$$\frac{E_r(z)E_r(w-ih)^- - E_r^*(z)E_r(w^-+ih)}{2\pi i(w^-+ih-z)} \; \frac{E_r(w)^-}{w^-W(w)^-}$$

into

$$\frac{E_r(z)E_r(w)^- - E_r^*(z)E_r(w^-)}{2\pi i(w^- - z)} \; \frac{E_r(w-ih)^-}{(w^-+ih)W(w-ih)^-}.$$

The numbers γ_n have a limit γ_∞ if they are bounded. An entire function $S_\infty(z)$ of Pólya class, which is determined by its zeros and which satisfies the symmetry condition

$$S_\infty(-z^-) = S_\infty(z)^-,$$

exists such that

$$S_\infty(z)/S_r(z)$$

is an entire function for every positive integer r and such that the entire functions have no common zeros. The product

$$E_\infty(z) = S_\infty(z)E(z)$$

is an entire function of Pólya class which satisfies the symmetry condition

$$E_\infty(-z^-) = E_\infty(z)^-.$$

The choice of $S_\infty(z)$ is made so that that the identity

$$\exp(i\gamma_\infty z)E_\infty(z) = W(z)$$

is satisfied. A space $\mathcal{H}(E_\infty)$ exists. Multiplication by $\exp(i\gamma_\infty z)$ is an isometric transformation of the space $\mathcal{H}(E_\infty)$ into the space $\mathcal{F}(W)$. The space $\mathcal{H}(E_\infty)$ contains every entire function $F(z)$ such that

$$\exp(i\gamma_\infty z)F(z)$$

and

$$\exp(i\gamma_\infty z)F^*(z)$$

belong to the space $\mathcal{F}(W)$. A maximal transformation of dissipative deficiency at most one in the space $\mathcal{H}(E_\infty)$ is defined by taking $H(z)$ into $zH(z+ih)/(z+ih)$ whenever $H(z)$ and $zH(z+ih)/(z+ih)$ belong to the space. Since $E(z)$ satisfies the symmetry condition and since w does not lie on the imaginary axis, $-w$ and w^- are distinct zeros of $E(z)$. Since the identity $w = w^- + ih$ cannot hold with h a positive number less than or equal to one, the inequality $-1 \leq iw - iw^-$ is satisfied and the zeros at $-w$ and w^- are simple if inequality holds.

If the numbers γ_r are unbounded, a dense set of elements of the space $\mathcal{F}(W)$ belong to the union of the ranges of the transformations of the spaces $\mathcal{H}(E_r)$ into the space $\mathcal{F}(W)$ which take $F(z)$ into

$$\exp(i\gamma_r z)W(z)F(z)/E_r(z).$$

Since the identity

$$\frac{W(w)^- W(w)}{2\pi i(w^- - w)} = \lim \exp(i\gamma_r w) \ \frac{E_r(w)^- E_r(w)}{2\pi i(w^- - w)} \ \exp(i\gamma_r w^-)$$

is satisfied, the functions $S_r(z)$ can be chosen so that the identity

$$W(w) = \lim \exp(i\gamma_r w)E_r(w)$$

is satisfied. The identity

$$W(z) = \lim \exp(i\gamma_r z)E_r(z)$$

then holds with uniform convergence on compact subsets of the upper half–plane.

If w is in the upper half–plane but not on the imaginary axis and if w^- is a zero of $E(z)$ such that the identity $w = w^- + ih$ holds with h a positive number less than or equal to one, then w is a zero of $G(z)$ whenever the maximal relation of dissipative deficiency at most one in the space $\mathcal{H}(E_r)$ takes $F(z)$ into $G(z + ih)$.

A restriction of the maximal transformation of dissipative deficiency at most one in the space $\mathcal{F}(W)$ is defined by taking $H(z)$ into $zH(z + ih)/(z + ih)$ whenever $H(z)$ and $zH(z+ih)/(z+ih)$ are elements of the space such that w is a zero of $H(z)$. The restricted transformation is sufficient to define the maximal relation of dissipative deficiency at most one in the space $\mathcal{H}(E_r)$. The maximal relation of dissipative deficiency at most one in the space $\mathcal{H}(E_r)$ takes $F(z)$ into $G(z + ih)$ whenever $F(z)$ and $G(z + ih)$ are elements of the space such that the maximal transformation of dissipative deficiency at most one in the space $\mathcal{F}(W)$ takes $H_n(z)$ into $zH_n(z + ih)/(z + ih)$ with w a zero of $H_n(z)$ for every positive integer n, such that

$$W(z)G(z + ih)/E_r(z)$$

is the limit in the metric topology of the space $\mathcal{F}(W)$ of the elements $zH_n(z+ih)/(z+ih)$, and such that

$$W(z)F(z)/E_r(z)$$

is the limit in the same topology of the orthogonal projections of the elements $H_n(z)$ in the image of the space $\mathcal{H}(E_r)$ in the space $\mathcal{F}(W)$.

If $H(z)$ is an element of the space $\mathcal{F}(W)$ such that $zH(z+ih)/(z+ih)$ belongs to the space, then elements $F_r(z)$ and $G_r(z+ih)$ of the space $\mathcal{H}(E_r)$ exist for every positive integer r such that the maximal relation of dissipative deficiency at most one in the space takes $F_r(z)$ into $G_r(z+ih)$ and such that the limits

$$H(z) = \lim W(z)F_r(z)/E_r(z)$$

and

$$zH(z+ih)/(z+ih) = \lim W(z)G_r(z+ih)/E_r(z)$$

are obtained in the metric topology of the space $\mathcal{F}(W)$. Elements $H_n(z)$ of the space $\mathcal{F}(W)$ such that $zH_n(z+ih)/(z+ih)$ belongs to the space and such w is a zero of $H_n(z)$ then exist such that the limits

$$H(z) = \lim H_n(z)$$

and

$$zH(z+ih)/(z+ih) = \lim zH_n(z+ih)/(z+ih)$$

are obtained in the metric topology of the space $\mathcal{F}(W)$. It follows that w is a zero of $H(z)$ whenever $H(z)$ is an element of the space $\mathcal{F}(W)$ such that $zH(z+ih)/(z+ih)$ belongs to the space.

Since $F^*(-z)$ belongs to the space whenever $F(z)$ belongs to the space, $-w^-$ is a zero of $H(z)$ whenever $H(z)$ is an element of the space such that $zH(z+ih)/(z+ih)$ belongs to the space. Since w does not lie on the imaginary axis, a space of dimension greater than one is orthogonal to the domain of the transformation which takes $H(z)$ into $zH(z+ih)/(z+ih)$ whenever $H(z)$ and $zH(z+ih)/(z+ih)$ belong to the space. A contradiction results since the transformation is maximal of dissipative deficiency at most one. This completes the proof that $E(z)$ has no zero w^-, which does not lie on the imaginary axis, such that the identity $w = w^- + ih$ holds for a positive number h which is less than or equal to one.

A limiting case of the argument applies when h is equal to zero. Argue by contraction assuming that $E(z)$ has a real zero w which is not the origin. Then w and $-w$ are zeros of $E_r(z)$ for every positive integer r. If the numbers γ_r are bounded, w and $-w$ are zeros of $E_\infty(z)$. A maximal transformation of dissipative deficiency at most one in the space $\mathcal{H}(E_\infty)$ is defined by taking $H(z)$ into $iH'(z) - iH(z)/z$ whenever $H(z)$ and $iH'(z) - iH(z)/z$ belong to the space. A contradiction is obtained since w and $-w$ are zeros of every element of the domain of the transformation. If the numbers γ_r are unbounded, a maximal transformation of dissipative deficiency at most one in the space $\mathcal{F}(W)$ is defined by taking $H(z)$ into $iH'(z) - iH(z)/z$ whenever $h(z)$ and $iH'(z) - iH(z)/z$ belong to the space. A contradiction is obtained since w and $-w$ are zeros of every element of the domain of the transformation.

Another limiting case of the argument is used to show that w^- is a simple zero of $E(z)$ if it is a zero which satisfies the identity $w = w^- + i$ and which does not lie on the imaginary axis. Argue by contradiction assuming that a double zero is obtained. Then w^- and $-w$ are double zeros of $E_r(z)$ for every positive integer r. If the numbers γ_r are bounded, w^- and $-w$ are double zeros of $E_\infty(z)$. A maximal transformation of dissipative deficiency at most one in the space $\mathcal{H}(E_\infty)$ is defined by taking $H(z)$ into

$$izH'(z+i)/(z+i) - izH(z+i)/(z+i)^2$$

whenever both functions belong to the space. A contradiction is obtained since w^- and $-w$ are zeros of every element of the domain of the transformation. If the numbers γ_r are unbounded, a maximal transformation of dissipative deficiency at most one in the space $\mathcal{F}(W)$ is defined by taking $H(z)$ into

$$izH'(z+i)/(z+i) - izH(z+i)/(z+i)^2$$

whenever both functions belong to the space. A contradiction is obtained since w^- and $-w$ are zeros of every element of the domain of the transformation.

This completes the proof of the theorem.

§4. The Radon transformation for locally compact skew–planes

The signature for the r–adic line is the homomorphism ξ into $\operatorname{sgn}(\xi)$ of the group of invertible elements of the r–adic line into the real numbers of absolute value one which has value minus one on elements whose r–adic modulus is a prime divisor of r. The canonical measure for the r–adic line is the normalization of Haar measure for the r–adic line for which the measure of the set of integral elements is equal to the product

$$\prod (1 - p^{-1})$$

taken over the prime divisors p of r. The Laplace kernel for the r–adic plane is a function $\sigma(\lambda)$ of λ in the r–adic line which vanishes when the p–adic component of $p\lambda$ is not integral for some prime divisor p of r. When the p–adic component of $p\lambda$ is integral for every prime divisor p of r, $\sigma(\lambda)$ is equal to the product

$$\prod (1 - p)^{-1}$$

taken over the prime divisors p of r such that the p–adic component of λ is not integral. The Laplace kernel for the r–adic plane is obtained as an integral

$$\phi(\lambda) = \int \exp(2\pi i \lambda \omega^- \omega) d\omega$$

with respect to the canonical measure for the r–adic plane over the set of units of the r–adic plane. The canonical measure for the r–adic plane is the normalization of Haar measure for the r–adic plane for which the set of units has measure one. The function

$$\exp(2\pi i \lambda)$$

of λ in the r–adic line is defined as the continuous extension to the r–adic line of the function of rational numbers λ.

The canonical measure for the r–adic diline is the normalization of Haar measure for which the measure of the set of units is equal to the product

$$\prod(1-p^{-1})$$

taken over the prime divisors p of r. The Laplace kernel for the r–adic skew–plane is a function $\sigma(\lambda)$ of λ in the r–adic diline which vanishes when the p–adic component of $p\lambda^*\lambda$ is not integral for some prime divisor p of r. When the p–adic component of $p\lambda^*\lambda$ is integral for every prime divisor p of r, $\sigma(\lambda)$ is equal to the product

$$\prod(1-p)^{-1}$$

taken over the prime divisors p of r such that the p–adic component of λ is not integral. The function $\sigma(\lambda)$ is extended to λ in the r–adic skew–plane so as to depend only on the r–adic modulus of $\lambda^*\lambda$ and so as to vanish when the r–adic modulus of $\lambda^*\lambda$ is not a rational number.

The Hankel transformation of character χ for the r–adic plane is a restriction of the Hankel transformation of character χ for the r–adic diplane. The domain of the Hankel transformation of character χ for the r–adic diplane is the space of functions $f(\xi)$ of ξ in the r–adic diplane which vanish when the p–adic component of ξ is not a unit for some prime divisor p of ρ, which satisfy the identity

$$f(\omega\xi)=\chi(\omega)f(\xi)$$

for every unit ω of the r–adic diplane, and which are square integrable with respect to the canonical measure for the r–adic diplane. The canonical measure for the r–adic diplane is a nonnegative measure on the Borel subsets of the r–adic diplane which is characterized within a constant factor by invariance properties. Multiplication by ω multiplies the canonical measure by the square of the r–adic modulus of ω for every element ω of the r–adic diplane. The canonical measure is normalized so that the measure of the set of units of the r–adic diplane is equal to one. The domain of the Hankel transformation of character χ for the r–adic plane is the set of functions $f(\xi)$ of ξ in the r–adic diplane which belong to the domain of the Hankel transformation of character χ for the r–adic diplane and which vanish when the r–adic modulus of ξ is not a rational number. The range of the Hankel transformation of character χ for the r–adic diplane is the domain of the Hankel transformation of character χ^* for the r–adic diplane. The Hankel transformation of character χ for the r–adic diplane takes a function $f(\xi)$ of ξ in the r–adic diplane into a function $g(\xi)$ of ξ in the r–adic diplane when the identity

$$\int\chi^*(\xi)^-g(\xi)\sigma(\lambda\xi^-\xi)d\xi=\operatorname{sgn}(\lambda)|\lambda|^{-1}\epsilon(\chi)\int\chi(\xi)^-f(\xi)\sigma(\lambda^{-1}\xi^-\xi)d\xi$$

holds for every invertible element λ of the r–adic line whose p–adic component is a unit for every prime divisor p of ρ. Integration with respect to the canonical measure for the r–adic diplane. The identity

$$\int |f(\xi)|^2 d\xi = \int |g(\xi)|^2 d\xi$$

holds with integration with respect to the canonical measure for the r–adic diplane. If $f(\xi)$ vanishes when the r–adic modulus of ξ is not a rational number, then $g(\xi)$ vanishes when the r–adic modulus of ξ is not a rational number. The function $f(\xi)$ of ξ in the r–adic diplane is the Hankel transform of character χ^* for the r–adic diplane of the function $g(\xi)$ of ξ in the r–adic diplane.

The Hankel transformation for the r–adic skew–plane is a restriction of the Hankel transformation for the r–adic skew–diplane. The domain of the Hankel transformation for the r–adic skew–diplane is the space of functions $f(\xi)$ of ξ in the r–adic skew–diplane which satisfy the identity

$$f(\tfrac{1}{2}\omega(\xi + \xi^-) + \tfrac{1}{2}\omega^{-1}(\xi - \xi^-)) = |\omega| f(\xi)$$

for every invertible element ω of the r–adic diline, which satisfy the identity

$$f(\omega^- \xi \omega) = f(\xi)$$

for every unit ω of the r–adic skew–diplane, and which are square integrable with respect to the canonical measure for the fundamental domain of the r–adic skew–diplane. The fundamental domain of the r–adic skew–diplane is the set of elements ξ of the r–adic skew–diplane such that $\frac{1}{2}(\xi + \xi^-)$ is a unit of the r–adic diline. The canonical measure for the fundamental domain of the r–adic skew–diplane is a nonnegative measure on the Borel subsets of the fundamental domain which is characterized within a constant factor by invariance properties. Measure preserving transformations are defined by taking ξ into $\omega\xi$ and ξ into $\xi\omega$ for every unit ω of the r–adic skew–diplane. The transformation which takes ξ into

$$\tfrac{1}{2}(\xi + \xi^-) + \tfrac{1}{2}\omega^-(\xi - \xi^-)\omega$$

multiplies the canonical measure by the fourth power of the r–adic modulus of $\omega^-\omega$ for every element ω of the r–adic skew–diplane. The measure is normalized so that the set of units has measure one. The domain of the Hankel transformation for the r–adic skew–plane is the space of functions $f(\xi)$ of ξ in the r–adic skew–diplane which belong to the domain of the Hankel transformation for the r–adic skew–diplane and which vanish when the r–adic modulus of $\xi^-\xi$ is not a rational number. The range of the Hankel transformation for the r–adic skew–diplane is the domain of the Hankel transformation for the r–adic skew–diplane. The transformation takes a function $f(\xi)$ of ξ in the r–adic skew–diplane into a function $g(\xi)$ of ξ in the r–adic skew–diplane when the identity

$$\int g(\xi) |\tfrac{1}{2}\xi - \tfrac{1}{2}\xi^-|^{-1} \sigma(\lambda(\tfrac{1}{2}\xi + \tfrac{1}{2}\xi^-)(\tfrac{1}{2}\xi - \tfrac{1}{2}\xi^-)) d\xi$$
$$= \operatorname{sgn}(\lambda^*\lambda)|\lambda|^{-2} \int f(\xi) |\tfrac{1}{2}\xi - \tfrac{1}{2}\xi^-|^{-1} \sigma(\lambda^{-1}(\tfrac{1}{2}\xi + \tfrac{1}{2}\xi^-)(\tfrac{1}{2}\xi - \tfrac{1}{2}\xi^-)) d\xi$$

holds for every invertible element λ of the r–adic diline with integration with respect to the canonical measure for the fundamental domain of the r–adic skew–diplane. The identity

$$\int |f(\xi)|^2 d\xi = \int |g(\xi)|^2 d\xi$$

holds with integration with respect to the canonical for the fundamental domain. If the function $f(\xi)$ of ξ in the r–adic skew–diplane vanishes when the r–adic modulus of $\xi^-\xi$ is not a rational number, then the function $g(\xi)$ of ξ in the r–adic skew–diplane vanishes when the r–adic modulus of $\xi^-\xi$ is not a rational number. The function $f(\xi)$ of ξ in the r–adic skew–diplane is the Hankel transform for the r–adic skew–diplane of the function $g(\xi)$ of ξ in the r–adic skew–diplane.

The Laplace transformation of character χ for the r–adic plane is a restriction of the Laplace transformation of character χ for the r–adic diplane. The domain of the Laplace transformation of character χ for the r–adic diplane is the space of functions $f(\xi)$ of ξ in the r–adic diplane which vanish when the p–adic component of ξ is not a unit for some prime divisor p of ρ, which satisfy the identity

$$f(\omega\xi) = \chi(\omega)f(\xi)$$

for every unit ω of the r–adic diplane, and which are square integrable with respect to the canonical measure for the r–adic diplane. The domain of the Laplace transformation of character χ for the r–adic plane is the space of functions $f(\xi)$ of ξ in the r–adic diplane which belong to the domain of the Laplace transformation of character χ for the r–adic diplane and which vanish when the r–adic modulus of ξ is not a rational number. The Laplace transform of character χ for the r–adic diplane of the function $f(\xi)$ of ξ in the r–adic diplane is the function $g(\lambda)$ of λ in the r–adic line defined by the integral

$$g(\lambda) = \int \chi(\xi)^- f(\xi)\sigma(\lambda\xi^-\xi)d\xi$$

with respect to the canonical measure for the r–adic diplane. The identity

$$\int |g(\lambda)|^2 d\lambda = \int |f(\xi)|^2 d\xi$$

holds with integration on the left with respect to the canonical measure for the r–adic line and with integration on the right with respect to the canonical measure for the r–adic diplane. A function $g(\lambda)$ of λ in the r–adic line, which is square integrable with respect to the canonical measure for the r–adic line, is a Laplace transform of character χ for the r–adic plane if, and only if, it satisfies the identity

$$g(\omega\lambda) = g(\lambda)$$

for every unit ω of the r–adic line, vanishes when the p–adic component of $p\lambda$ is not integral for some prime divisor p of ρ, satisfies the identity

$$(1 - p)g(\lambda) = g(\omega^{-1}\lambda)$$

when the p–adic component of $p\lambda$ is a unit for some prime divisor p of ρ and ω is an element of the r–adic line whose r–adic modulus is p, and satisfies the identity

$$g(\lambda) = g(\omega^{-1}\lambda)$$

when the p–adic component of λ is integral for some prime divisor p of ρ and ω is an element of the r–adic line whose r–adic modulus is p. A function $g(\lambda)$ of λ in the r–adic line is a Laplace transform of character χ for the r–adic plane if, and only if, it is a Laplace transform of character χ for the r–adic diplane which satisfies the identity

$$(1-p)g(\lambda) = g(\omega^{-1}\lambda) - pg(\omega\lambda)$$

when the p–adic modulus of λ is an odd power of p for some prime divisor p of r and ω is an element of the r–adic line whose r–adic modulus is p^{-1}.

The Laplace transformation for the r–adic skew–plane is a restriction of the Laplace transformation for the r–adic skew–diplane. The domain of the Laplace transformation for the r–adic skew–diplane is the space of functions $f(\xi)$ of ξ in the r–adic skew–diplane which satisfy the identity

$$f(\tfrac{1}{2}\omega(\xi+\xi^-) + \tfrac{1}{2}\omega^{-1}(\xi-\xi^-)) = |\omega| f(\xi)$$

for every invertible element ω of the r–adic diline, which satisfy the identity

$$f(\omega^-\xi\omega) = f(\xi)$$

for every unit ω of the r–adic skew–diplane, and which are square integrable with respect to the canonical measure for the fundamental domain of the r–adic skew–diplane. The domain of the Laplace transformation for the r–adic skew–plane is the space of functions $f(\xi)$ of ξ in the r–adic skew–diplane which belong to the domain of the Laplace transformation for the r–adic skew–diplane and which vanish when the r–adic modulus of $\xi^-\xi$ is not a rational number. The Laplace transform for the r–adic skew–diplane of the function $f(\xi)$ of ξ in the r–adic skew–diplane is the function $g(\lambda)$ of λ in the r–adic diline which is defined by the integral

$$g(\lambda) = \int f(\xi)|\tfrac{1}{2}\xi - \tfrac{1}{2}\xi^-|^{-1}\sigma(\lambda(\tfrac{1}{2}\xi + \tfrac{1}{2}\xi^-)(\tfrac{1}{2}\xi - \tfrac{1}{2}\xi^-))d\xi$$

with respect to the canonical measure for the fundamental domain of the r–adic skew–diplane. The identity

$$\int |g(\lambda)|^2 d\lambda = \int |f(\xi)|^2 d\xi$$

holds with integration on the left with respect to the canonical measure for the r–adic diline and with integration on the right with respect to the canonical measure for the fundamental domain of the r–adic skew–diplane. A function $g(\lambda)$ of λ in the r–adic diline is a Laplace transform for the r–adic skew–diplane if, and only if, it satisfies the identity

$$g(\omega\lambda) = g(\lambda)$$

for every unit ω of the r–adic diline and is square integrable with respect to the canonical measure for the r–adic diline. A function $g(\lambda)$ of λ in the r–adic diline is a Laplace transform for the r–adic skew–plane if, and only if, it is a Laplace transform for the r–adic skew–diplane which satisfies the identity

$$(1-p)f(\lambda) = f(\omega^{-1}\lambda) - pf(\omega\lambda)$$

when the p–adic modulus of $\lambda^*\lambda$ is an odd power of p for some prime divisor p of r and ω is an element of the r–adic diline such that the p–adic modulus of $\omega^*\omega$ is p.

The Radon transformation of character χ for the r–adic diplane is a nonnegative self–adjoint transformation in the space of functions $f(\xi)$ of ξ in the r–adic diplane which vanish when the p–adic component of ξ is not a unit for some prime divisor p of ρ, which satisfy the identity

$$f(\omega\xi) = \chi(\omega)f(\xi)$$

for every unit ω of the r–adic diplane, and which are square integrable with respect to the canonical measure for the r–adic diplane. The transformation takes a function $f(\xi)$ of ξ in the r–adic diplane into a function $g(\xi)$ of ξ in the r–adic diplane when the identity

$$g(\xi) = \int f(\xi + \eta)d\eta$$

holds formally with integration with respect to Haar measure for the space of elements η of the r–adic plane whose p–adic component vanishes for every prime divisor p of ρ and which satisfy the identity

$$\eta^-\xi + \xi^-\eta = 0.$$

Haar measure is normalized so that the set of integral elements has measure one. The integral is accepted as the definition when

$$f(\xi) = \chi(\xi)\sigma(\lambda\xi^-\xi)$$

for an invertible element λ of the r–adic line whose p–adic component is a unit for every prime divisor p of ρ, in which case

$$g(\xi) = |\lambda|^{-\frac{1}{2}}f(\xi).$$

The formal integral is otherwise interpreted as the identity

$$\int \chi(\xi)^- g(\xi)\sigma(\lambda\xi^-\xi)d\xi = |\lambda|^{-\frac{1}{2}} \int \chi(\xi)^- f(\xi)\sigma(\lambda\xi^-\xi)d\xi$$

for every invertible element λ of the r–adic line whose p–adic component is a unit for every prime divisor p of ρ. Integration is with respect to the canonical measure for the r–adic diplane.

The Radon transformation of character χ for the r–adic plane is a nonnegative self–adjoint transformation in the space of functions $f(\xi)$ of ξ in the r–adic diplane which vanish when the p–adic component of ξ is not a unit for some prime divisor p of ρ or when the r–adic modulus of ξ is not a rational number, which satisfy the identity

$$f(\omega\xi) = \chi(\omega) f(\xi)$$

for every unit ω of the r–adic diplane, and which are square integrable with respect to the canonical measure for the r–adic diplane. The transformation takes a function $f(\xi)$ of ξ in the r–adic diplane into a function $g(\xi)$ of ξ in the r–adic diplane when the Radon transformation for the r–adic diplane takes a function $f_n(\xi)$ of ξ in the r–adic diplane into a function $g_n(\xi)$ of ξ in the r–adic diplane for every positive integer n, such that the function $g(\xi)$ is the limit of the functions $g_n(\xi)$ in the metric topology of the space of square integrable functions with respect to the canonical measure, and such that the function $f(\xi)$ is the limit in the same topology of the orthogonal projections of the functions $f_n(\xi)$ in the space of functions which vanish when the r–adic modulus of ξ is not a rational number.

The Radon transformation for the r–adic skew–diplane is a nonnegative self–adjoint transformation in the space of functions $f(\xi)$ of ξ in the r–adic skew–diplane which satisfy the identity

$$f(\tfrac{1}{2}\omega(\xi + \xi^-) + \tfrac{1}{2}\omega^{-1}(\xi - \xi^-)) = |\omega| f(\xi)$$

for every invertible element ω of the r–adic diline, which satisfy the identity

$$f(\omega^- \xi \omega) = f(\xi)$$

for every unit ω of the r–adic skew–diplane, and which are square integrable with respect to the canonical measure for the fundamental domain of the r–adic skew–diplane. The function

$$|\tfrac{1}{2}\xi - \tfrac{1}{2}\xi^-|^{-1} \sigma(\lambda(\tfrac{1}{2}\xi + \tfrac{1}{2}\xi^-)(\tfrac{1}{2}\xi - \tfrac{1}{2}\xi^-))$$

of ξ in the r–adic skew–diplane is an eigenfunction of the Radon transformation for the r–adic skew–diplane for the eigenvalue $|\lambda|^{-1}$ when λ is an invertible element of the r–adic diline. The transformation takes a function $f(\xi)$ of ξ in the r–adic skew–diplane into a function $g(\xi)$ of ξ in the r–adic skew–diplane when the identity

$$\int g(\xi) |\tfrac{1}{2}\xi - \tfrac{1}{2}\xi^-|^{-1} \sigma(\lambda(\tfrac{1}{2}\xi + \tfrac{1}{2}\xi^-)(\tfrac{1}{2}\xi - \tfrac{1}{2}\xi^-)) d\xi$$
$$= |\lambda|^{-1} \int f(\xi) |\tfrac{1}{2}\xi - \tfrac{1}{2}\xi^-|^{-1} \sigma(\lambda^{\frac{1}{2}}(\tfrac{1}{2}\xi + \tfrac{1}{2}\xi^-)(\tfrac{1}{2}\xi - \tfrac{1}{2}\xi^-)) d\xi$$

holds for every invertible element λ of the r–adic diline. Integration is with respect to the canonical measure for the fundamental domain of the r–adic skew–diplane.

The Radon transformation for the r–adic skew–plane is a nonnegative self–adjoint transformation in the space of functions $f(\xi)$ of ξ in the r–adic skew–diplane which satisfy the identity

$$f(\tfrac{1}{2}\omega(\xi + \xi^-) + \tfrac{1}{2}\omega^{-1}(\xi - \xi^-)) = |\omega| f(\xi)$$

for every invertible element unit ω of the r–adic diline, which satisfy the identity

$$f(\omega^- \xi \omega) = f(\xi)$$

for every unit ω of the r–adic skew–diplane, which vanish when the r–adic modulus of $\xi^{-}\xi$ is not a rational number and which are square integrable with respect to the canonical measure for the r–adic skew–diplane, is not a rational number. The transformation takes a function $f(\xi)$ of ξ in the r–adic skew–diplane into a function $g(\xi)$ of ξ in the r–adic skew–diplane when the Radon transformation for the r–adic skew–diplane takes a function $f_n(\xi)$ of ξ in the r–adic skew–diplane into a function $g_n(\xi)$ of ξ in the r–adic skew–diplane for every positive integer n, such that the function $g(\xi)$ is the limit of the functions $g_n(\xi)$ in the metric topology of the space of square integrable functions with respect to the canonical measure for the r–adic skew–diplane, and such that the function $f(\xi)$ is the limit in the same topology of the orthogonal projections of the functions $f_n(\xi)$ in the space of functions which vanish when the r–adic modulus of $\xi^{-}\xi$ is not a rational number.

A property of the range of the Laplace transformation of character χ for the r–adic plane is required to know that a nonnegative self–adjoint transformation is obtained as Radon transformation of character χ for the r–adic plane. The range of the Laplace transformation of character χ for the r–adic diplane is the space of functions $f(\lambda)$ of λ in the r–adic line which are square integrable with respect to the canonical measure for the r–adic line, which satisfy the identity

$$f(\omega\lambda) = f(\lambda)$$

for every unit ω of the r–adic line, which vanish when the p–adic component of $p\lambda$ is not integral for some prime divisor p of ρ, which satisfy the identity

$$(1 - p)f(\lambda) = f(\omega^{-1}\lambda)$$

when the p–adic component of $p\lambda$ is a unit for some prime divisor p of ρ with ω an element of the r–adic line whose r–adic modulus is p, and which satisfy the identity

$$f(\lambda) = f(\omega^{-1}\lambda)$$

when the p–adic component of λ is integral for some prime divisor p of ρ with ω an element of the r–adic line whose r–adic modulus is p. A self–adjoint transformation in the range of the Laplace transformation of character χ for the r–adic diplane is defined by taking a function $f(\lambda)$ of λ in the r–adic line into a function $g(\lambda)$ of λ in the r–adic line if the identity

$$g(\lambda) = |\lambda|^{-\frac{1}{2}} f(\lambda)$$

holds when the p–adic component of λ is a unit for every prime divisor p of ρ. The range of the Laplace transformation of character χ for the r–adic plane is the space of functions $f(\lambda)$ of λ in the r–adic line which belong to the range of the Laplace transformation of character χ for the r–adic diplane and which satisfy the identity

$$(1 - p)f(\lambda) = f(\omega^{-1}\lambda) - pf(\omega\lambda)$$

when the p–adic modulus of λ is an odd power of p for some prime divisor p of r and ω is an element of the r–adic line whose r–adic modulus is p. The closure of the set of functions

$f(\lambda)$ of λ in the r–adic line, which belong to the range of the Laplace transformation of character χ for the r–adic diplane, such that a function $g(\lambda)$ of λ in the r–adic line, which belongs to the range of the Laplace transformation of character χ for the r–adic plane, exists such that the identity

$$g(\lambda) = |\lambda|^{-\frac{1}{2}} f(\lambda)$$

holds when the p–adic component of λ is a unit for every prime divisor p of ρ, is the set of functions $f(\lambda)$ of λ in the r–adic line which belong to the range of the Laplace transformation of character χ for the r–adic diplane and which satisfy the identity

$$(p^{\frac{1}{2}} - p^{-\frac{1}{2}}) f(\lambda) = f(\omega\lambda) - f(\omega^{-1}\lambda)$$

when the p–adic component of λ is a unit for every prime divisor p of ρ and the p–adic modulus of λ is an odd power of p for some prime divisor p of r, which is not a divisor of ρ, with ω an element of the r–adic line whose r–adic modulus is p^{-1}. It will be shown that a dense set of elements of the range of the Laplace transformation of character χ for the r–adic plane are orthogonal projections of such functions $f(\lambda)$ of λ in the r–adic line. It is sufficient to show that no nonzero element of the range of the Laplace transformation of character χ for the r–adic plane is orthogonal to all such functions $f(\lambda)$ of λ in the r–adic line. A function $g(\lambda)$ of λ in the r–adic line, which belongs to the range of the Laplace transformation of character χ for the r–adic diplane and which is orthogonal to all such functions $f(\lambda)$ of λ in the r–adic line, satisfies the identity

$$(p^{\frac{1}{2}} - p^{-\frac{1}{2}}) g(\lambda) = p^{-1} g(\omega^{-1}\lambda) - p g(\omega\lambda)$$

when the p–adic component of λ is a unit for every prime divisor p of ρ and the p–adic modulus of λ is an odd power of p for some prime divisor p of r, which is not a divisor of ρ, with ω an element of the r–adic line whose r–adic modulus is p. The function $g(\lambda)$ of λ in the r–adic line vanishes identically when the function is in the range of the Laplace transformation of character χ for the r–adic plane.

A property of the range of the Laplace transformation for the r–adic skew–plane is required to know that a nonnegative self–adjoint transformation is obtained as the Radon transformation for the r–adic skew–plane. The range of the Laplace transformation for the r–adic skew–diplane is the space of functions $f(\lambda)$ of λ in the r–adic diline which satisfy the identity

$$f(\omega\lambda) = f(\lambda)$$

for every unit ω of the r–adic diline and which are square integrable with respect to the canonical measure for the r–adic diline. A nonnegative self–adjoint transformation in the range of the Laplace transformation for the r–adic skew–diplane is defined by taking a function $f(\lambda)$ of λ in the r–adic diline into the function $|\lambda|^{-1} f(\lambda)$ of λ in the r–adic diline. The range of the Laplace transformation for the r–adic skew–plane is the space of functions $f(\lambda)$ of λ in the r–adic diline which belong to the range of the Laplace transformation for the r–adic skew–diplane and which satisfy the identity

$$(1 - p) f(\lambda) = f(\omega^{-1}\lambda) - p f(\omega\lambda)$$

when the p–adic modulus of $\lambda^*\lambda$ is an odd power of p for some prime divisor p of r and ω is an element of the r–adic diline such that the r–adic modulus of $\omega^*\omega$ is p. The closure of the set of functions $f(\lambda)$ of λ in the r–adic diline, which belong to the range of the Laplace transformation for the r–adic skew–diplane, such that the function $|\lambda|^{-1}f(\lambda)$ of λ in the r–adic diline belongs to the range of the Laplace transformation for the r–adic skew–plane, is the set of functions $f(\lambda)$ of λ in the r–adic diline which belong to the range of the Laplace transformation for the r–adic skew–diplane and which satisfy the identity

$$(p^{\frac{1}{2}} - p^{-\frac{1}{2}})f(\lambda) = f(\omega\lambda) - f(\omega^{-1}\lambda)$$

when the p–adic modulus of $\lambda^*\lambda$ is an odd power of p for some prime divisor p of r and ω is an element of the r–adic diline such that the r–adic modulus of $\omega^*\omega$ is p. It will be shown that a dense set of elements of the range of the Laplace transformation for the r–adic skew–plane are orthogonal projections of such functions $f(\lambda)$ of λ in the r–adic diline. It is sufficient to show that no nonzero element of the range of the Laplace transformation for the r–adic skew–plane is orthogonal to all such functions $f(\lambda)$ of λ in the r–adic diline. A function $g(\lambda)$ of λ in the r–adic diline, which belongs to the range of the Laplace transformation for the r–adic skew–diplane and which is orthogonal to all such functions $f(\lambda)$ of λ in the r–adic diline, satisfies the identity

$$(p^{\frac{1}{2}} - p^{-\frac{1}{2}})g(\lambda) = p^{-1}g(\omega^{-1}\lambda) - pg(\omega\lambda)$$

when the p–adic modulus of $\lambda^*\lambda$ is an odd power of p for some prime divisor p of r and ω is an element of the r–adic diline such that the r–adic modulus of $\omega^*\omega$ is p. The function $g(\lambda)$ of λ in the r–adic diline vanishes identically when the function is in the range of the Laplace transformation for the r–adic skew–line.

§5. The Euler product for Riemann zeta functions

The r–adelic upper half–plane is the set of elements of the r–adelic plane whose Euclidean component belongs to the upper half–plane and whose r–adic component is an invertible element of the r–adic line. An element of the r–adelic upper half–plane, whose Euclidean component is $\tau_+ + iy$ for a real number τ_+ and a positive number y and whose r–adic component is τ_-, is written $\tau + iy$ with τ the element of the r–adelic line whose Euclidean component is τ_+ and whose r–adic component is τ_-. A character of order ν for the r–adelic diplane is a function $\chi(\xi)$ of ξ in the r–adelic diplane which is a product

$$\chi(\xi) = \chi(\xi_+)\chi(\xi_-)$$

of a character of order ν for the Euclidean diplane and a character modulo ρ for the r–adic diplane which is of the same parity as ν. The canonical measure for the r–adelic line is the Cartesian product of Haar measure for the Euclidean line and the canonical measure for the r–adic line. The fundamental domain for the r–adelic line is the set of elements of the r–adelic line whose r–adic modulus is a positive integer whose prime divisors are divisors of r and which is not divisible by the square of a prime. The canonical measure for the fundamental domain is the restriction to the Borel subsets of the fundamental domain of the canonical measure for the r–adelic line. The theta function of order ν and character χ for the r–adelic plane is a function $\theta(\lambda)$ of λ in the r–adelic upper half–plane which is defined as a sum

$$\theta(\lambda) = \sum \chi(\omega)^- \exp(\pi i \omega_+^2 \lambda_+ / \rho) \sigma(\omega_-^2 \lambda_-)$$

over the nonzero principal elements ω of the r–adelic line whose p–adic component is a unit for every prime divisor p of ρ. The theta function of order ν and character χ^* for the r–adelic upper half–plane is the function

$$\theta^*(\lambda) = \theta(-\lambda^-)^-$$

of λ in the r–adelic upper half–plane.

The r–adelic upper half–diplane is the set of the r–adelic diplane whose Euclidean component belongs to the upper half–plane and whose r–adic component is an invertible element of the r–adic diline. An element of the r–adelic upper half–diplane, whose Euclidean component is $\tau_+ + iy$ for a real number τ_+ and a positive number y and whose r–adic component is τ_-, is written $\tau + iy$ with τ the element of the r–adelic diline whose Euclidean component is τ_+ and whose r–adic component is τ_-. A harmonic function of order ν for the r–adelic skew–diplane is a function $\phi(\xi)$ of ξ in the r–adelic skew–diplane which depends only on the Euclidean component of ξ and which is a harmonic function of order ν for the Euclidean skew–diplane as a function of the Euclidean component of ξ. The canonical measure for the r–adelic diline is the Cartesian product of Haar measure for the Euclidean diline and the canonical measure for the r–adic diline. The fundamental domain for the r–adelic diline is the set of elements ξ of the r–adelic diline such that the r–adic modulus of $\xi^*\xi$ is a positive integer whose prime divisors are divisors of r and which is not divisible by the square of a prime. The canonical measure for the fundamental domain is the restriction to the Borel subsets of the fundamental domain of the canonical measure for the r–adelic diline. The theta function of order ν and harmonic ϕ for the r–adelic skew–line is a function $\theta(\lambda)$ of λ in the r–adelic upper half–diplane which is defined as a sum

$$2\theta(\lambda) = \sum \omega_+^\nu \tau(\omega_+)^- \exp(2\pi i \omega_+ \lambda_+) \sigma(\omega_- \lambda_-)$$

over the nonzero principal elements ω of the r–adelic line. The coefficient $\tau(a/b)$ is defined for a nonzero rational number a/b as $\tau(a)$ when a and b are relative prime integers, which are relatively prime to r, such that a is positive. The theta function of order ν and character χ^* for the r–adelic upper half–diplane is the function

$$\theta^*(\lambda) = \theta(-\lambda^-)^-$$

of λ in the r–adelic upper half–diplane.

The Hankel transformation of order ν and character χ for the r–adelic plane is a restriction of the Hankel transformation of order ν and character χ for the r–adelic diplane. The canonical measure for the r–adelic diplane is the Cartesian product of the canonical measure for the Euclidean diplane and the canonical measure for the r–adic diplane. The fundamental domain for the r–adelic diplane is the set of elements of the r–adelic diplane whose r–adic component is a unit. The canonical measure for the fundamental domain is the restriction to the Borel subsets of the fundamental domain of the canonical measure for the r–adelic diplane. The domain of the Hankel transformation of order ν and character χ for the r–adelic diplane is the space of functions $f(\xi)$ of ξ in the r–adelic diplane which vanish when the p–adic component of ξ is not a unit for some prime divisor p of ρ, which satisfy the identity

$$f(\omega\xi) = \chi(\omega)f(\xi)$$

for every unit ω of the r–adelic diplane, which satisfy the identity

$$f(\xi) = f(\omega\xi)$$

for every nonzero principal element ω of the r–adelic line whose p–adic component is a unit for every prime divisor p of ρ, and which are square integrable with respect to the canonical measure for the fundamental domain. The range of the Hankel transformation of order ν and character χ for the r–adelic diplane is the domain of the Hankel transformation of order ν and character χ^* for the r–adelic diplane. The domain of the Hankel transformation of order ν and character χ for the r–adelic plane is the space of functions $f(\xi)$ of ξ in the r–adelic diplane which belong to the domain of the Hankel transformation of order ν and character χ for the r–adelic diplane and which vanish when the r–adic modulus of ξ is not a rational number. The Hankel transformation of order ν and character χ for the r–adelic diplane takes a function $f(\xi)$ of ξ in the r–adelic diplane into a function $g(\xi)$ of ξ in the r–adelic diplane when the identity

$$\int \chi^*(\xi)^- g(\xi)\theta^*(\lambda\xi^-\xi)d\xi = (i/\lambda_+)^{1+\nu}\operatorname{sgn}(\lambda_-)|\lambda|_-^{-1}\epsilon(\chi)\int \chi(\xi)^- f(\xi)\theta(-\lambda^{-1}\xi^-\xi)d\xi$$

holds for λ in the r–adelic upper half–plane with integration with respect to the canonical measure for the fundamental domain. The identity

$$\int |f(\xi)|^2 d\xi = \int |g(\xi)|^2 d\xi$$

holds with integration with respect to the canonical measure for the fundamental domain. If the function $f(\xi)$ of ξ in the r–adelic diplane vanishes when the r–adic modulus of ξ is not a rational number, then the function $g(\xi)$ of ξ in the r–adelic diplane vanishes when the r–adic modulus of ξ is not a rational number. The function $f(\xi)$ of ξ in the r–adelic diplane is the Hankel transform of order ν and character χ^* for the r–adelic diplane of the function $g(\xi)$ of ξ in the r–adelic diplane.

The Hankel transformation of order ν and harmonic ϕ for the r–adelic skew–plane is a restriction of the Hankel transformation of order ν and harmonic ϕ for the r–adelic skew–diplane. The domain of the Hankel transformation of order ν and harmonic ϕ for the r–adelic skew–diplane is the space of functions $f(\xi)$ of ξ in the r–adelic skew–diplane which satisfy the identity

$$f(\tfrac{1}{2}\omega(\xi + \xi^-) + \tfrac{1}{2}\omega^{-1}(\xi - \xi^-)) = |\omega|f(\xi)$$

for every invertible element ω of the r–adelic diline, which satisfy the identity

$$\phi(\xi)f(\omega^-\xi\omega) = \phi(\omega^-\xi\omega)f(\xi)$$

for every unit ω of the r–adelic skew–diplane, which satisfy the identity

$$f(\xi) = f(\tfrac{1}{2}(\xi+\xi^-) + \tfrac{1}{2}\omega^-(\xi-\xi^-)\omega)$$

for every nonzero principal element ω of the r–adelic skew–plane, and which are square integrable with respect to the canonical measure for the fundamental domain of the r–adelic skew–diplane. The fundamental domain of the r–adelic skew–diplane is the set of elements ξ of the r–adelic skew–diplane such that $\frac{1}{2}(\xi+\xi^-)$ is a unit of the r–adelic diline and such that the square of the r–adic modulus of $\xi^-\xi$ is a positive integer whose prime divisors are divisors of r and which is not divisible by the square of a prime. The canonical measure for the fundamental domain of the r–adelic skew–diplane is the restriction to the Borel subsets of the fundamental domain of the Cartesian product of the canonical measure for the fundamental domain of the Euclidean skew–diplane and the canonical measure for the fundamental domain of the r–adic skew–diplane. The domain of the Hankel transformation of order ν and harmonic ϕ for the r–adelic skew–plane is the space of functions $f(\xi)$ of ξ in the r–adelic skew–diplane which belong to the domain of the Hankel transformation of order ν and harmonic ϕ for the r–adelic skew–diplane and which vanish when the r–adic modulus of $\xi^-\xi$ is not a rational number. The range of the Hankel transformation of order ν and harmonic ϕ for the r–adelic skew–diplane is the domain of the Hankel transformation of order ν and harmonic ϕ^* for the r–adelic skew–diplane. The Hankel transformation of order ν and harmonic ϕ for the r–adelic skew–diplane takes a function $f(\xi)$ of ξ in the r–adelic skew–diplane into a function $g(\xi)$ of ξ in the r–adelic skew–diplane when the identity

$$\int \phi^*(\xi)^- g(\xi)|\tfrac{1}{2}\xi - \tfrac{1}{2}\xi^-|^{-1}\theta^*(\lambda|\tfrac{1}{2}\xi + \tfrac{1}{2}\xi^-||\tfrac{1}{2}\xi - \tfrac{1}{2}\xi^-|)d\xi$$
$$= (i/\lambda_+)^{2+2\nu}\operatorname{sgn}(\lambda^*\lambda)|\lambda|_-^{-2}\int \phi(\xi)^- f(\xi)|\tfrac{1}{2}\xi - \tfrac{1}{2}\xi^-|^{-1}\theta(-\lambda^{-1}|\tfrac{1}{2}\xi + \tfrac{1}{2}\xi^-||\tfrac{1}{2}\xi - \tfrac{1}{2}\xi^-|)d\xi$$

holds when λ is in the r–adelic upper half–diplane with integration with respect to the canonical measure for the fundamental domain of the r–adelic skew–diplane. The notation

$$|\tfrac{1}{2}\xi + \tfrac{1}{2}\xi^-||\tfrac{1}{2}\xi - \tfrac{1}{2}\xi^-|$$

is used in the argument of the theta function for an element of the r–adelic diline with nonnegative Euclidean component which has the same Euclidean and r–adic modulus as

$$(\tfrac{1}{2}\xi + \tfrac{1}{2}\xi^-)(\tfrac{1}{2}\xi - i\xi^-).$$

The identity

$$\int |f(\xi)|^2 d\xi = \int |g(\xi)|^2 d\xi$$

holds with integration with respect to the canonical measure for the fundamental domain. If the function $f(\xi)$ of ξ in the r–adelic skew–diplane vanishes when the r–adic modulus of $\xi^-\xi$ is not a rational number, then the function $g(\xi)$ of ξ in the r–adelic skew–diplane vanishes when the r–adic modulus of $\xi^-\xi$ is not a rational number. The function $f(\xi)$ of ξ in the r–adelic skew–diplane is the Hankel transform of order ν and harmonic ϕ for the r–adelic skew–diplane of the function $g(\xi)$ of ξ in the
r–adelic skew–diplane.

The nonzero principal elements of the r–adelic line, whose p–adic component is a unit for every prime divisor p of ρ, are applied in an isometric summation for the r–adelic diplane. If a function $f(\xi)$ of ξ in the r–adelic diplane satisfies the identity

$$f(\omega\xi) = \chi(\omega)f(\xi)$$

for every unit ω of the r–adelic diplane, vanishes outside of the fundamental domain, and is square integrable with respect to the canonical measure for the r–adelic diplane, then a function $g(\xi)$ of ξ in the r–adelic diplane, which vanishes when the p–adic component of ξ is not a unit for some prime divisor p of ρ, which satisfies the identity

$$g(\omega\xi) = \chi(\omega)g(\xi)$$

for every unit ω of the r–adelic diplane, and which satisfies the identity

$$g(\xi) = g(\omega\xi)$$

for every nonzero principal element ω of the r–adelic line whose p–adic component is a unit for every prime divisor p of ρ, is defined by the sum

$$2g(\xi) = \sum f(\omega\xi)$$

over the nonzero principal elements ω of the r–adelic line whose p–adic component is a unit for every prime divisor p of ρ. The identity

$$\int |f(\xi)|^2 d\xi = \int |g(\xi)|^2 d\xi$$

holds with integration with respect to the canonical measure for the fundamental domain of the r–adelic diplane. If the function $f(\xi)$ of ξ in the r–adelic diplane vanishes when the r–adic modulus of ξ is not rational, then the function $g(\xi)$ of ξ in the r–adelic diplane vanishes when the r–adic modulus of ξ is not rational. If a function $h(\xi)$ of ξ in the r–adelic diplane vanishes when the p–adic component of ξ is not a unit for some prime divisor p of ρ, satisfies the identity

$$h(\omega\xi) = \chi(\omega)h(\xi)$$

for every unit ω of the r–adelic diplane, satisfies the identity

$$h(\xi) = h(\omega\xi)$$

for every nonzero principal element ω of the r–adelic line whose p–adic component is a unit for every prime divisor p of ρ, and is square integrable with respect to the canonical measure for the fundamental domain, then $h(\xi)$ is equal to $g(\xi)$ for a function $f(\xi)$ of ξ in the r–adelic diplane which is equal to $h(\xi)$ when ξ is in the fundamental domain.

The nonzero principal elements of the r–adelic skew–plane are applied in an isometric summation for the r–adelic skew–diplane. If a function $f(\xi)$ of ξ in the r–adelic skew–diplane satisfies the identity

$$f(\tfrac{1}{2}\omega(\xi + \xi^-) + \tfrac{1}{2}\omega^{-1}(\xi - \xi^-)) = |\omega| f(\xi)$$

for every invertible element ω of the r–adelic diline, satisfies the identity

$$\phi(\xi) f(\omega^- \xi \omega) = \phi(\omega^- \xi \omega) f(\xi)$$

for every unit ω of the r–adelic skew–diplane, vanishes when $\frac{1}{2}(\xi + \xi^-)$ is a unit of the r–adelic diline but ξ does not belong to the fundamental domain for the r–adelic skew–diplane, and is square integrable with respect to the canonical measure for the fundamental domain, then a function $g(\xi)$ of ξ in the r–adelic skew–diplane which satisfies the identity

$$g(\tfrac{1}{2}\omega(\xi + \xi^-) + \tfrac{1}{2}(\omega^{-1}(\xi - \xi^-)) = |\omega| g(\xi)$$

for every invertible element ω of the r–adelic diline, which satisfies the identity

$$\phi(\xi) g(\omega^- \xi \omega) = \phi(\omega^- \xi \omega) g(\xi)$$

for every unit ω of the r–adelic skew–diplane, and which satisfies the identity

$$g(\xi) = g(\tfrac{1}{2}(\xi + \xi^-) + \tfrac{1}{2}\omega^-(\xi - \xi^-)\omega)$$

for every nonzero principal element ω of the r–adelic skew–plane, is defined by the sum

$$24 g(\xi) = \sum f(\tfrac{1}{2}(\xi + \xi^-) + \tfrac{1}{2}\omega^-(\xi - \xi^-)\omega)$$

for every nonzero principal element ω of the r–adelic skew–plane. The identity

$$\int |f(\xi)|^2 d\xi = \int |g(\xi)|^2 d\xi$$

holds with integration with respect to the canonical measure for the fundamental domain of the r–adelic skew–diplane. If the function $f(\xi)$ of ξ in the r–adelic skew–diplane vanishes when the r–adic modulus of $\xi^- \xi$ is not a rational number, then the function $g(\xi)$ of ξ in the r–adelic skew–diplane vanishes when the r–adic modulus of $\xi^- \xi$ is not a rational number. The identity

$$2 g(\xi) = \sum \tau(\omega_+) f(\tfrac{1}{2}(\xi + \xi^-) + \tfrac{1}{2}\omega(\xi - \xi^-))$$

holds with summation over the nonzero principal elements ω of the r–adelic line. If a function $h(\xi)$ of ξ in the r–adelic skew–diplane satisfies the identity

$$h(\tfrac{1}{2}\omega(\xi+\xi^-)+\tfrac{1}{2}\omega^{-1}(\xi-\xi^-))=|\omega|h(\xi)$$

for every invertible element ω of the r–adelic diline, satisfies the identity

$$\phi(\xi)h(\omega^-\xi\omega)=\phi(\omega^-\xi\omega)h(\xi)$$

for every unit ω of the r–adelic skew–diplane, satisfies the identity

$$h(\xi)=h(\tfrac{1}{2}(\xi+\xi^-)+\tfrac{1}{2}\omega^-(\xi-\xi^-)\omega)$$

for every nonzero principal element ω of the r–adelic skew–plane, and is square integrable with respect to the canonical measure for the fundamental domain of the r–adelic skew–diplane, then $h(\xi)$ is equal to $g(\xi)$ for a function $f(\xi)$ of ξ in the r–adelic skew–diplane which is equal to $h(\xi)$ when ξ is in the fundamental region.

The Laplace transformation of order ν and character χ for the r–adelic plane is a restriction of the Laplace transformation of order ν and character χ for the r–adelic diplane. The domain of the Laplace transformation of order ν and character χ for the r–adelic diplane is the space of functions $f(\xi)$ of ξ in the r–adelic diplane which vanish when the p–adic component of ξ is not a unit for some prime divisor p of ρ, which satisfy the identity

$$f(\omega\xi)=\chi(\omega)f(\xi)$$

for every unit ω of the r–adelic diplane, which satisfy the identity

$$f(\xi)=f(\omega\xi)$$

for every nonzero principal element ω of the r–adelic line whose p–adic component is a unit for every prime divisor p of ρ, and which are square integrable with respect to the canonical measure for the fundamental domain of the r–adelic diplane. The domain of the Laplace transformation of order ν and character χ for the r–adelic plane is the space of functions $f(\xi)$ of ξ in the r–adelic diplane which belong to the domain of the Laplace transformation of order ν and character χ for the r–adelic diplane and which vanish when the r–adic modulus of ξ is not a rational number. The Laplace transform of order ν and character χ for the r–adelic diplane of the function $f(\xi)$ of ξ in the r–adelic diplane is a function $g(\lambda)$ of λ in the r–adelic upper half–plane which is defined by the integral

$$2\pi g(\lambda)=\int\chi(\xi)^-f(\xi)\theta(\lambda\xi^-\xi)d\xi$$

with respect to the canonical measure for the fundamental domain of the r–adelic diplane. The function $g(\lambda)$ of λ in the r–adelic upper half–plane is an analytic function of the Euclidean component of λ when the r–adic component of λ is held fixed. The identity

$$g(\omega\lambda)=g(\lambda)$$

holds for every unit ω of the r–adelic line whose Euclidean component is the unit of the unit of the Euclidean line. The function vanishes when the p–adic component of $p\lambda$ is not integral for some prime divisor p of ρ. The identity

$$(1-p)g(\lambda) = g(\omega^{-1}\lambda)$$

holds when the p–adic component of $p\lambda$ is a unit for some prime divisor p of ρ and ω is an element of the r–adelic line whose Euclidean component is a unit of the Euclidean line and whose r–adic modulus is p. The identity

$$g(\lambda) = g(\omega^{-1}\lambda)$$

holds when the p–adic component of λ is integral for some prime divisor p of ρ and ω is an element of the r–adelic line whose Euclidean component is the unit of the Euclidean line and whose r–adic modulus is p. The identity

$$g(\lambda) = \chi(\omega)g(\omega^2\lambda)$$

holds for every nonzero principal element ω of the r–adelic line whose p–adic component is a unit for every prime divisor p of ρ. When ν is zero, the identity

$$(2\pi/\rho)\ \sup \int |g(\tau+iy)|^2 d\tau = \int |f(\xi)|^2 d\xi$$

holds with the least upper bound taken over all positive numbers y. The identity

$$(2\pi/\rho)^\nu \int_0^\infty \int |g(\tau+iy)|^2 y^{\nu-1} d\tau dy = \Gamma(\nu) \int |f(\xi)|^2 d\xi$$

holds when ν is positive. Integration on the left is with respect to the canonical measure for the fundamental domain of the r–adelic line. Integration on the right is with respect to the canonical measure for the fundamental domain of the r–adelic diplane. These properties characterize Laplace transforms of order ν and character χ for the r–adelic diplane. A function $g(\lambda)$ of λ in the r–adelic line is a Laplace transform of order ν and character χ for the r–adelic plane if, and only if, it is a Laplace transform of order ν and character χ for the r–adelic diplane which satisfies the identity

$$(1-p)g(\lambda) = g(\omega^{-1}\lambda) - pg(\omega\lambda)$$

when the p–adic modulus of λ is an odd power of p for some prime divisor p of r and ω is an element of the r–adelic line whose Euclidean component is the unit of the Euclidean line and whose r–adic modulus is p.

The Laplace transformation of order ν and harmonic ϕ for the r–adelic skew–plane is a restriction of the Laplace transformation of order ν and harmonic ϕ for the r–adelic skew–diplane. The domain of the Laplace transformation of order ν and harmonic ϕ for the r–adelic skew–diplane is the space of functions $f(\xi)$ of ξ in the r–adelic skew–diplane which satisfy the identity

$$f(\tfrac{1}{2}\omega(\xi+\xi^-)+\tfrac{1}{2}\omega^{-1}(\xi-\xi^-))=|\omega|f(\xi)$$

for every invertible element ω of the r–adelic diline, which satisfy the identity

$$\phi(\xi)f(\omega^-\xi\omega)=\phi(\omega^-\xi\omega)f(\xi)$$

for every unit ω of the r–adelic skew–diplane, which satisfy the identity

$$f(\xi)=f(\tfrac{1}{2}(\xi+\xi^-)+\tfrac{1}{2}\omega^-(\xi-\xi^-)\omega)$$

for every nonzero principal element ω of the r–adelic skew–plane, and which are square integrable with respect to the canonical measure for the fundamental domain of the r–adelic skew–diplane. The domain of the Laplace transformation of order ν and harmonic ϕ for the r–adelic skew–plane is the space of functions $f(\xi)$ of ξ in the r–adelic skew–diplane which belong to the domain of the Laplace transformation of order ν and harmonic ϕ for the r–adelic skew–diplane and which vanish when the r–adic modulus of $\xi^-\xi$ is not a rational number. The Laplace transform of order ν and harmonic ϕ for the r–adelic skew–diplane of the function $f(\xi)$ of ξ in the r–adelic skew–diplane is a function $g(\lambda)$ of λ in the r–adelic upper half–diplane which is defined by the integral

$$4\pi g(\lambda)=\int\phi(\xi)^-f(\xi)|\tfrac{1}{2}\xi-\tfrac{1}{2}\xi^-|^{-1}\theta(\lambda|\tfrac{1}{2}\xi+\tfrac{1}{2}\xi^-||\tfrac{1}{2}\xi-\tfrac{1}{2}\xi^-|)d\xi$$

with respect to the canonical measure for the fundamental domain of the r–adelic skew–diplane. The notation

$$|\tfrac{1}{2}\xi+\tfrac{1}{2}\xi^-||\tfrac{1}{2}\xi-\tfrac{1}{2}\xi^-|$$

is used in the argument of the theta function for an element of the r–adelic diline with nonnegative Euclidean component which has the same Euclidean and r–adic modulus as

$$(\tfrac{1}{2}\xi+\tfrac{1}{2}\xi^-)(\tfrac{1}{2}\xi-\tfrac{1}{2}\xi^-).$$

The function $g(\lambda)$ of λ in the r–adelic upper half–diplane is an analytic function of the Euclidean component of λ when the r–adic component of λ is held fixed. The identity

$$g(\omega\lambda)=g(\lambda)$$

holds for every unit ω of the r–adelic diline whose Euclidean component is the unit of the Euclidean line. The identity

$$g(\lambda)=g(\omega\lambda)$$

holds for every nonzero principal element ω of the r–adelic line. The identity

$$(4\pi)^{2+2\nu}\int_0^\infty\int|g(\tau+iy)|^2y^{2\nu}d\tau dy=\Gamma(1+2\nu)\int|f(\xi)|^2d\xi$$

holds when ν is positive. Integration on the left is with respect to the canonical measure for the fundamental domain of the r–adelic diline. Integration on the right is with respect to the canonical measure for the fundamental domain of the r–adelic skew–diplane. These properties characterize Laplace transforms of order ν and harmonic ϕ for the r–adelic skew–diplane. A function $g(\lambda)$ of λ in the r–adelic diline is a Laplace transform of order ν and harmonic ϕ for the r–adelic skew–plane if, and only if, it is a Laplace transform of order ν and harmonic ϕ for the r–adelic skew–diplane which satisfies the identity

$$(1-p)g(\lambda) = g(\omega^{-1}\lambda) - pg(\omega\lambda)$$

when the p–adic modulus of $\lambda^*\lambda$ is an odd power of p for some prime divisor p of r and ω is an element of the r–adelic diline whose Euclidean component is the unit of the Euclidean line and for which the r–adic modulus of $\omega^*\omega$ is equal to p.

The Radon transformation of order ν and character χ for the r–adelic diplane is a maximal dissipative transformation in the space of functions $f(\xi)$ of ξ in the r–adelic diplane which vanish when the p–adic component of ξ is not a unit for some prime divisor p of ρ, which satisfy the identity

$$f(\omega\xi) = \chi(\omega)f(\xi)$$

for every unit ω of the r–adelic diplane, which satisfy the identity

$$f(\xi) = f(\omega\xi)$$

for every nonzero principal element ω of the r–adelic line whose p–adic component is a unit for every prime divisor p of ρ, and which are square integrable with respect to the canonical measure for the fundamental domain of the r–adelic diplane. The transformation takes a function $f(\xi)$ of ξ in the r–adelic diplane into a function $g(\xi)$ of ξ in the r–adelic diplane when the identity

$$g(\xi) = \int f(\xi+\eta)d\eta$$

is formally satisfied with integration with respect to Haar measure for the space of elements η of the r–adelic diplane which satisfy the identity

$$\eta^-\xi + \xi^-\eta = 0.$$

Haar measure is defined as the Cartesian product of Haar measure for the space of elements η_+ of the Euclidean diplane which satisfy the identity

$$\eta_+^-\xi_+ + \xi_+^-\eta_+ = 0$$

and Haar measure for the space of elements η_- of the r–adic diplane which satisfy the identity

$$\eta_-^-\xi_- + \xi_-^-\eta_- = 0.$$

The integral is accepted as the definition of the Radon transformation of order ν and character χ for the r–adelic diplane when

$$f(\xi) = \chi(\xi)\theta(\lambda\xi^-\xi)$$

with λ an element of the r–adelic upper half–plane whose p–adic component is a unit for every prime divisor p of ρ, in which case

$$g(\xi) = (i\rho/\lambda_+)^{\frac{1}{2}}|\lambda|_-^{-\frac{1}{2}}f(\xi)$$

with the square root of $i\rho/\lambda_+$ taken in the right half–plane. The adjoint of the Radon transformation of order ν and character χ for the r–adelic diplane takes a function $f(\xi)$ of ξ in the r–adelic diplane into a function $g(\xi)$ of ξ in the r–adelic diplane when the identity

$$\int \chi(\xi)^- g(\xi)\theta(\lambda\xi^-\xi)d\xi = (i\rho/\lambda_+)^{\frac{1}{2}}|\lambda|_-^{-\frac{1}{2}} \int \chi(\xi)^- f(\xi)\theta(\lambda\xi^-\xi)d\xi$$

holds when λ is an invertible element of the r–adelic upper half–plane whose p–adic component is a unit for every prime divisor p of ρ. Integration is with respect to the canonical measure for the fundamental domain of the r–adelic diplane.

The Radon transformation of order ν and character χ for the r–adelic plane is a maximal dissipative transformation in the space of functions $f(\xi)$ of ξ in the r–adelic diplane which vanish when the p–adic component of ξ is not a unit for some prime divisor p of ρ or when the r–adic modulus of ξ is not a rational number, which satisfy the identity

$$f(\omega\xi) = \chi(\omega)f(\xi)$$

for every unit ω of the r–adelic diplane, which satisfy the identity

$$f(\xi) = f(\omega\xi)$$

for every nonzero principal element ω of the r–adelic diplane whose p–adic component is a unit for every prime divisor p of ρ, and which are square integrable with respect to the canonical measure for the fundamental domain of the r–adelic diplane. The transformation takes a function $f(\xi)$ of ξ in the r–adelic diplane into a function $g(\xi)$ in the r–adelic diplane when the Radon transformation for the r–adelic diplane takes a function $f_n(\xi)$ of ξ in the r–adelic diplane into a function $g_n(\xi)$ of ξ in the r–adelic diplane for every positive integer n, such that the function $g(\xi)$ is the limit of the functions $g_n(\xi)$ in the metric topology of the space of square integrable functions with respect to the canonical measure for the fundamental domain of the r–adelic diplane, and such that the function $f(\xi)$ is the limit in the same topology of the orthogonal projections of the functions $f_n(\xi)$ in the space of functions which vanish when the r–adic modulus of ξ is not a rational number.

The Radon transformation of order ν and harmonic ϕ for the r–adelic skew–diplane is a maximal dissipative transformation in the space of functions $f(\xi)$ of ξ in the r–adelic skew–diplane which satisfy the identity

$$f(\tfrac{1}{2}\omega(\xi + \xi^-) + \tfrac{1}{2}\omega^{-1}(\xi - \xi^-)) = |\omega|f(\xi)$$

for every invertible element ω of the r–adelic diline, which satisfy the identity

$$\phi(\xi)f(\omega^-\xi\omega) = \phi(\omega^-\xi\omega)f(\xi)$$

for every unit ω of the r–adelic skew–diplane, which satisfy the identity

$$f(\xi) = f(\tfrac{1}{2}(\xi + \xi^-) + \tfrac{1}{2}\omega^-(\xi^* - \xi^-)\omega)$$

for every nonzero principal element ω of the r–adelic skew–plane, and which are square integrable with respect to the canonical measure for the fundamental domain of the r–adelic skew–diplane. The function

$$\phi(\xi)|\tfrac{1}{2}\xi - \tfrac{1}{2}\xi^-|^{-1}\theta(\lambda|\tfrac{1}{2}\xi + \tfrac{1}{2}\xi^-||\tfrac{1}{2}\xi - \tfrac{1}{2}\xi^-|)$$

of ξ in the r–adelic skew–diplane is an eigenfunction of the Radon transformation of order ν and harmonic ϕ for the r–adelic skew–diplane for the eigenvalue $(i/\lambda_+)|\lambda|_-^{-1}$ when λ is in the r–adelic upper half–diplane. The adjoint of the Radon transformation of order ν and harmonic ϕ for the r–adelic skew–diplane takes a function $f(\xi)$ of ξ in the r–adelic skew–diplane into a function $g(\xi)$ of ξ in the r–adelic skew–diplane when the identity

$$\int \phi^*(\xi)^- g(\xi)|\tfrac{1}{2}\xi - \tfrac{1}{2}\xi^-|^{-1}\theta(\lambda|\tfrac{1}{2}\xi + \tfrac{1}{2}\xi^-||\tfrac{1}{2}\xi - \tfrac{1}{2}\xi^-|)d\xi$$
$$= (i/\lambda_+)|\lambda|_-^{-1} \int \phi(\xi)^- f(\xi)|\tfrac{1}{2}\xi - \tfrac{1}{2}\xi^-|^{-1}\theta(\lambda|\tfrac{1}{2}\xi + \tfrac{1}{2}\xi^-||\tfrac{1}{2}\xi - \tfrac{1}{2}\xi^-|)d\xi$$

holds when λ is in the r–adelic upper half–diplane. Integration is with respect to the canonical measure for the fundamental domain of the r–adelic skew–diplane.

The Radon transformation of order ν and harmonic ϕ for the r–adelic skew–plane is a maximal dissipative transformation in the space of functions $f(\xi)$ of ξ in the r–adelic skew–diplane which satisfy the identity

$$f(\tfrac{1}{2}\omega(\xi + \xi^-) + \tfrac{1}{2}\omega^{-1}(\xi - \xi^-)) = |\omega|f(\xi)$$

for every invertible element ω of the r–adelic diline, which satisfy the identity

$$\phi(\xi)f(\omega^-\xi\omega) = \phi(\omega^-\xi\omega)f(\xi)$$

for every unit ω of the r–adelic skew–diplane, which satisfy the identity

$$f(\xi) = f(\tfrac{1}{2}(\xi + \xi^-) + \tfrac{1}{2}\omega^-(\xi - \xi^-)\omega)$$

for every nonzero principal element ω of the r–adelic skew–plane, which vanish when the r–adic modulus of $\xi^-\xi$ is not a rational number, and which are square integrable with respect to the canonical measure for the fundamental domain of the r–adelic skew–diplane. The transformation takes a function $f(\xi)$ of ξ in the r–adelic skew–diplane into a function $g(\xi)$ of ξ in the r–adelic skew–diplane when the Radon transformation of order ν and harmonic

ϕ for the r–adelic skew–diplane takes a function $f_n(\xi)$ of ξ in the r–adelic skew–diplane into a function $g_n(\xi)$ of ξ in the r–adelic skew–diplane for every positive integer n, such that the function $g(\xi)$ is the limit of the functions $g_n(\xi)$ in the metric topology of the space of square integrable functions with respect to the canonical measure for the fundamental domain of the r–adelic skew–diplane, and such that the function $f(\xi)$ is the limit in the same topology of the orthogonal projections of the functions $f_n(\xi)$ in the space of functions which vanish when the r–adic modulus of $\xi^-\xi$ is not a rational number.

The Mellin transformation of order ν and character χ for the r–adelic plane is a spectral theory for the Laplace transformation of order ν and character χ for the r–adelic plane. The domain of the Laplace transformation of order ν and character χ for the r–adelic plane is the space of functions $f(\xi)$ of ξ in the r–adelic diplane which vanish when the p–adic component of ξ is not a unit for some prime divisor p of ρ or when the r–adic modulus of ξ is not a rational number, which satisfy the identity

$$f(\omega\xi) = \chi(\omega)f(\xi)$$

for every unit ω of the r–adelic diplane, which satisfy the identity

$$f(\xi) = f(\omega\xi)$$

for every nonzero principal element ω of the r–adelic line whose p-adic component is a unit for every prime divisor p of ρ, and which are square integrable with respect to the canonical measure for the fundamental domain of the r–adelic diplane. The Laplace transform of order ν and character χ for the r–adelic diplane of the function $f(\xi)$ of ξ in the r–adelic diplane is the function $g(\lambda)$ of λ in the r–adelic upper half–plane defined by the integral

$$2\pi g(\lambda) = \int_0^\infty \chi(\xi)^- f(\xi)\theta(\lambda\xi^-\xi)d\xi$$

with respect to the canonical measure for the fundamental domain of the r–adelic diplane. The Mellin transform of order ν and character χ for the r–adelic plane of the function $f(\xi)$ of ξ in the r–adelic diplane is an analytic function $F(z)$ of z in the upper half–plane which is defined by the integral

$$F(z) = \int_0^\infty g(\lambda)t^{\frac{1}{2}\nu-\frac{1}{2}-\frac{1}{2}iz}dt$$

under the constraint

$$\lambda_+ = it$$

when λ_- is a unit of the r–adic line if the function $f(\xi)$ of ξ in the r–adelic plane vanishes in the neighborhood $|\xi| < a$ of the origin. A computation of the integral is made from the zeta function

$$\zeta(s) = \sum \tau(n)n^{-s}$$

of order ν and character χ for the r–adelic line, which is defined in the half–plane $\mathcal{R}s > 1$ as a sum over the positive integers n whose prime divisors are divisors of r but not of ρ. The identity

$$\tau(n) = \chi(n)^-$$

holds when the prime divisors of a positive integer n are divisors of r not not of ρ. The zeta function is represented in the complex plane by the Euler product

$$\zeta(s)^{-1} = \prod (1 - \tau(p)p^{-s})$$

taken over the prime divisors p of r which are not divisors of ρ. The analytic weight function

$$W(z) = (\pi/\rho)^{-\frac{1}{2}\nu - \frac{1}{2} + \frac{1}{2}iz} \Gamma(\tfrac{1}{2}\nu + \tfrac{1}{2} - \tfrac{1}{2}iz)\zeta(1 - iz)$$

is represented in the upper half–plane by the integral

$$W(z) = \int_0^\infty \theta(\lambda) t^{\frac{1}{2}\nu - \frac{1}{2} - \frac{1}{2}iz} dt$$

under the constraint

$$\lambda_+ = it$$

when λ_- is a unit of the r–adic line. The identity

$$2\pi F(z)/W(z) = \int \chi(\xi)^- f(\xi) |\xi|^{iz - \nu - 1} d\xi$$

holds when z is in the upper half–plane. Integration is with respect to the canonical measure for the fundamental domain of the r–adelic diplane. The function

$$a^{-iz} F(z)$$

of z in the upper half–plane is characterized as an element of the weighted Hardy space $\mathcal{F}(W)$ which satisfies the identity

$$\int_{-\infty}^{+\infty} |F(t)/W(t)|^2 dt = \int |f(\xi)|^2 d\xi.$$

Integration on the right is with respect to the canonical measure for the fundamental domain of the r–adelic diplane. If the Hankel transform of order ν and character χ for the r–adelic plane of the function $f(\xi)$ of ξ in the r–adelic diplane is a function $g(\xi)$ of ξ in the r–adelic diplane which vanishes when $|\xi| < a$, then the Mellin transform of order ν and character χ for the r–adelic plane of the function $g(\xi)^-$ of ξ in the r–adelic diplane is an entire function which is the analytic extension of $F^*(z)$ to the complex plane.

The Mellin transformation of order ν and harmonic ϕ for the r–adelic skew–plane is a spectral theory for the Laplace transformation of order ν and harmonic ϕ for the r–adelic skew–plane. The domain of the Laplace transformation of order ν and harmonic ϕ for the r–adelic skew–plane is the space of functions $f(\xi)$ of ξ in the r–adelic skew–diplane which satisfy the identity

$$f(\tfrac{1}{2}\omega(\xi + \xi^-) + \tfrac{1}{2}\omega^{-1}(\xi - \xi^-)) = |\omega| f(\xi)$$

for every invertible element ω of the r–adelic diline, which satisfy the identity

$$\phi(\xi)f(\omega^-\xi\omega) = \phi(\omega^-\xi\omega)f(\xi)$$

for every unit ω of the r–adelic skew–diplane, which satisfy the identity

$$f(\xi) = f(\tfrac{1}{2}(\xi + \xi^-) + \tfrac{1}{2}\omega^-(\xi - \xi^-)\omega)$$

for every nonzero principal element ω of the r–adelic skew–plane, which vanish when the r–adic modulus of $\xi^-\xi$ is not a rational number, and which are square integrable with respect to the canonical measure for the fundamental domain of the r–adelic skew–diplane. The Laplace transform of order ν and harmonic ϕ for the r–adelic skew–plane of the function $f(\xi)$ of ξ in the r–adelic skew–diplane is the function $g(\lambda)$ of λ in the r–adelic upper half–diplane which is defined by the integral

$$4\pi g(\lambda) = \int \phi(\xi)^- f(\xi)|\tfrac{1}{2}\xi - \tfrac{1}{2}\xi^-|^{-1}\theta(\lambda|\tfrac{1}{2}\xi + \tfrac{1}{2}\xi^-||\tfrac{1}{2}\xi - \tfrac{1}{2}\xi^-|)d\xi$$

with respect to the canonical measure for the fundamental domain of the r–adelic skew–diplane. The notation

$$|\tfrac{1}{2}\xi + \tfrac{1}{2}\xi^-||\tfrac{1}{2}\xi - \tfrac{1}{2}\xi^-|$$

is used in the argument of the theta function for an element of the r–adelic diline with nonnegative Euclidean component which has the same Euclidean and r–adic modulus as

$$(\tfrac{1}{2}\xi + \tfrac{1}{2}\xi^-)(\tfrac{1}{2}\xi - \tfrac{1}{2}\xi^-).$$

The Mellin transform of order ν and harmonic ϕ for the r–adelic skew–plane of the function $f(\xi)$ of ξ in the r–adelic skew–diplane is an analytic function $F(z)$ of z in the upper half–plane which is defined by the integral

$$F(z) = \int_0^\infty g(\lambda)t^{\nu - iz}dt$$

under the constraint

$$\lambda_+ = it$$

when λ_- is a unit of the r–adic diline if the function $f(\xi)$ of ξ in the r–adelic skew–diplane vanishes in the neighborhood $|\frac{1}{2}\xi + \frac{1}{2}\xi^-||\frac{1}{2}\xi - \frac{1}{2}\xi^-| < a$ of the origin. A computation of the integral is made using the zeta function

$$\zeta(s) = \sum \tau(n)n^{-s}$$

of order ν and harmonic ϕ for the r–adelic skew–plane, which is defined in the half–plane $\mathcal{R}s > 1$ as a sum over the positive integers n whose prime divisors are divisors of r. When r is odd, the Euler product

$$\zeta(s)^{-1} = \prod(1 - \tau(p)p^{-s} + p^{-2s})$$

is taken over the prime divisors p of r. When r is even, the Euler product

$$\zeta(s)^{-1} = (1 - \tau(2)2^{-s}) \prod (1 - \tau(p)p^{-s} + p^{-2s})$$

is taken over the odd prime divisors p of r. The zeta function has no zeros and its singularities lie in the half–plane $\mathcal{R}s < 1$ when ν is positive. The analytic weight function

$$W(z) = (2\pi)^{-\nu-1+iz} \Gamma(\nu + 1 - iz)\zeta(1 - iz)$$

is represented in the upper half–plane by the integral

$$W(z) = \int_0^\infty \theta(\lambda) t^{\nu - \frac{1}{2} - iz} dt$$

under the constraint

$$\lambda_+ = it$$

when λ_- is a unit of the r–adic diline. The identity

$$4\pi F(z)/W(z) = \int \phi(\xi)^- f(\xi) |\tfrac{1}{2}\xi + \tfrac{1}{2}\xi^-|^{iz-\nu-1} |\tfrac{1}{2}\xi - \tfrac{1}{2}\xi^-|^{iz-\nu-2} \, d\xi$$

holds when z is in the upper half–plane with integration with respect to the canonical measure for the fundamental domain for the r–adelic skew–diplane. The function

$$a^{-iz} F(z)$$

is characterized as an element of the weighted Hardy space $\mathcal{F}(W)$ which satisfies the identity

$$\int_{-\infty}^{+\infty} |F(t)/W(t)|^2 dt = \tfrac{1}{2} \int |f(\xi)|^2 d\xi.$$

Integration on the right is with respect to the canonical measure for the fundamental domain of the r–adelic skew–diplane. If the Hankel transform of order ν and harmonic ϕ for the r–adelic skew–plane of the function $f(\xi)$ of ξ in the r–adelic skew–diplane is a function $g(\xi)$ of ξ in the r–adelic skew–diplane which vanishes when $|\frac{1}{2}\xi + \frac{1}{2}\xi^-||\frac{1}{2}\xi - \frac{1}{2}\xi^-| < a$, then the Mellin transform of order ν and harmonic ϕ for the r–adelic skew–plane of the function $g(\xi)^-$ of ξ in the r–adelic skew–diplane is an entire function which is the analytic extension of $F^*(z)$ to the complex plane.

The Mellin transformation of order ν and character χ for the r–adelic plane supplies information about the Sonine spaces of order ν and character χ for the r–adelic plane. The Sonine spaces of order ν and character χ for the r–adelic plane are defined using the analytic weight function

$$W(z) = (\pi/\rho)^{-\frac{1}{2}\nu - \frac{1}{2} + \frac{1}{2}iz} \Gamma(\tfrac{1}{2}\nu + \tfrac{1}{2} - \tfrac{1}{2}iz)\zeta(1 - iz)$$

constructed from the zeta function of order ν and character χ for the r–adelic plane. The space of parameter a contains the entire functions $F(z)$ such that

$$a^{-iz}F(z)$$

and

$$a^{-iz}F^*(z)$$

belong to the weighted Hardy space $\mathcal{F}(W)$. A Hilbert space of entire functions which satisfies the axioms (H1), (H2), and (H3) is obtained when a scalar product is introduced so that multiplication by a^{-iz} is an isometric transformation of the space into the space $\mathcal{F}(W)$. The space is a space $\mathcal{H}(E(a))$ which coincides as a set with the Sonine space of order ν and parameter a for the Euclidean plane. The space is also related to the Sonine space of order ν and parameter b for the Euclidean plane whose parameter satisfies the equation

$$r/\rho = (a/b)^2.$$

The entire function

$$S(z) = (b/a)^{iz}\zeta(1-iz)^{-1}$$

of Pólya class is determined by its zeros. The space $\mathcal{H}(E(a))$ is the set of entire functions $F(z)$ such that $S(z)F(z)$ belongs to the Sonine space of order ν and parameter b for the Euclidean plane. Multiplication by $S(z)$ is an isometric transformation of the space $\mathcal{H}(E(a))$ into the Sonine space of order ν and parameter b for the Euclidean plane. A maximal dissipative transformation in the Sonine space of order ν and parameter b for the Euclidean plane is defined by taking $F(z)$ into $F(z+i)$ whenever $F(z)$ and $F(z+i)$ belong to the space. A maximal dissipative transformation is induced in the space $\mathcal{H}(E(a))$. The transformation takes $F(z)$ into a $G(z+i)$ when a sequence of elements $H_n(z)$ of the Sonine space for the Euclidean plane exists such that $H_n(z+i)$ belongs to the space for every n, such that $S(z)G(z+i)$ is the limit of the functions $H_n(z+i)$ in the metric topology of the space, and such that $S(z)F(z)$ is the limit in the metric topology of the space of the orthogonal projections of the functions $H_n(z)$ in the image in the space of the space $\mathcal{H}(E(a))$. The maximal dissipative transformation in the space $\mathcal{H}(E(a))$ is unitarily equivalent to a positive multiple of the adjoint of the Radon transformation of order ν and character χ for the r–adelic plane as it acts on functions $f(\xi)$ of ξ in the r–adelic plane which vanish when $|\xi| < a$ and whose Hankel transform of order ν and character χ for the r–adelic plane vanishes when $|\xi| < a$.

The Mellin transformation of order ν and harmonic ϕ for the r–adelic skew–plane supplies information about the Sonine spaces of order ν and harmonic ϕ for the r–adelic skew–plane. The positive integer r is assumed to be divisible only once by the even prime if it is even and to be divisible exactly twice by every odd prime divisor. The Sonine spaces of order ν and harmonic ϕ for the r–adelic skew–plane are defined using the analytic weight function

$$W(z) = (2\pi)^{-\nu-1+iz}\Gamma(\nu+1-iz)\zeta(1-iz)$$

constructed from the zeta function of order ν and harmonic ϕ for the r–adelic skew–plane. The space of parameter a contains the entire functions $F(z)$ such that

$$a^{-iz}F(z)$$

and

$$a^{-iz}F^*(z)$$

belong to the weighted Hardy space $\mathcal{F}(W)$. A Hilbert space of entire functions which satisfies the axioms (H1), (H2), and (H3) is obtained when a scalar product is introduced so that multiplication by a^{-iz} is an isometric transformation of the space into the space $\mathcal{F}(W)$. The space is a space $\mathcal{H}(E(a))$ which coincides as a set with the Sonine space of order ν and parameter a for the Euclidean skew–plane. The space is also related to the Sonine space of order ν for the Euclidean skew–plane of parameter b which satisfies the equation

$$r = (a/b)^2.$$

The entire function

$$S(z) = (b/a)^{iz}\zeta(1-iz)^{-1}$$

of Pólya class is determined by its zeros. The space $\mathcal{H}(E(a))$ is the set of entire functions $F(z)$ such that $S(z)F(z)$ belongs to the Sonine space of parameter b for the Euclidean skew–plane. Multiplication by $S(z)$ is an isometric transformation of the space $\mathcal{H}(E(a))$ into the space of parameter b for the Euclidean skew–plane. A maximal dissipative transformation in the Sonine space of parameter b for the Euclidean skew–plane is defined by taking $F(z)$ into $F(z+i)$ whenever $F(z)$ and $F(z+i)$ belong to the space. A maximal dissipative transformation is induced in the space $\mathcal{H}(E(a))$. The transformation takes $F(z)$ into $G(z+i)$ when a sequence of elements $H_n(z)$ of the Sonine space for the Euclidean skew–plane exists such that $H_n(z+i)$ belongs to the space for every n, such that $S(z)G(z+i)$ is the limit of the functions $H_n(z+i)$ in the metric topology for the space, and such that $S(z)F(z)$ is the limit in the metric topology of the space of the orthogonal projections of the functions $H_n(z)$ in the image in the space of the image of the space $\mathcal{H}(E(a))$. The maximal dissipative transformation in the space $\mathcal{H}(E(a))$ is unitarily equivalent to a positive multiple of the adjoint of the Radon transformation of order ν and character χ for the r–adelic skew–plane as it acts on functions $f(\xi)$ of ξ in the r–adelic skew–diplane which vanish when $|\frac{1}{2}\xi + \frac{1}{2}\xi^-||\frac{1}{2}\xi - i\xi^-| < a$ and whose Hankel transform of order ν and character χ for the r–adelic skew–plane vanishes when $|\frac{1}{2}\xi + \frac{1}{2}\xi^-||\frac{1}{2}\xi - \frac{1}{2}\xi^-| < a$.

§6. The Radon transformation for locally compact rings

The signature for the adic line is the homomorphism ξ into $\operatorname{sgn}(\xi)$ of the group of invertible elements of the adic line into the real numbers of absolute value one which has value minus one on elements whose adic modulus is a prime. The canonical measure for the adic line is the Cartesian product of the canonical measures for the p–adic lines taken over the primes p. The Laplace kernel for the adic plane is a function $\sigma(\lambda)$ of λ in the adic diline which vanishes when the p–adic component of $p\lambda$ is not integral for some prime p. When the p–adic modulus of $p\lambda$ is integral for every prime p, $\sigma(\lambda)$ is equal to the product

$$\prod (1-p)^{-1}$$

taken over the primes p such that the p–adic component of λ is not integral.

The canonical measure for the adic diline is the Cartesian product of the canonical measures on the p–adic dilines. The Laplace kernel for the adic skew–plane is a function $\sigma(\lambda)$ of λ in the adic diline which vanishes when the p–adic component of $p\lambda^*\lambda$ is not integral for some prime p. When the p–adic component of $p\lambda^*\lambda$ is integral for every prime p, $\sigma(\lambda)$ is equal to the product

$$\prod(1-p)^{-1}$$

taken over the primes p such that the p–adic component of λ is not integral. The function $\sigma(\lambda)$ is extended to the adic skew–diplane and so as to depend only on the adic modulus of $\lambda^*\lambda$ so as to vanish when the adic modulus of $\lambda^*\lambda$ is not a rational number.

The Hankel transformation of character χ for the adic plane is a restriction of the Hankel transformation of character χ for the adic diplane. The domain of the Hankel transformation of character χ for the adic diplane is the space of functions $f(\xi)$ of ξ in the adic diplane which vanish when the p–adic component of ξ is not a unit for some prime divisor p of ρ, which satisfy the identity

$$f(\omega\xi) = \chi(\omega)f(\xi)$$

for every unit ω of the adic diplane, and which are square integrable with respect to the canonical measure for the adic diplane. The canonical measure for the adic diplane is a nonnegative measure on the Borel subsets of the adic diplane which is characterized within a constant factor by invariance properties. Multiplication by ω multiplies the canonical measure by the square of the adic modulus of ω for every element ω of the adic diplane. The canonical measure is normalized so that the measure of the set of units is equal to one. The domain of the Hankel transformation of character χ for the adic plane is the set of functions $f(\xi)$ of ξ in the adic diplane which belong to the domain of the Hankel transformation of character χ for the adic diplane and which vanish when the adic modulus of ξ is not a rational number. The range of the Hankel transformation of character χ for the adic diplane is the domain of the Hankel transformation of character χ^* for the adic diplane. The Hankel transformation of character χ for the adic diplane takes a function $f(\xi)$ of ξ in the adic diplane into a function $g(\xi)$ of ξ in the adic diplane when the identity

$$\int \chi^*(\xi)^- g(\xi)\sigma(\lambda\xi^-\xi)d\xi = \operatorname{sgn}(\lambda)|\lambda|^{-1}\epsilon(\chi)\int \chi(\xi)^- f(\xi)\sigma(\lambda^{-1}\xi^-\xi)d\xi$$

holds for every invertible element λ of the adic line whose p–adic component is a unit or every prime divisor p of ρ. Integration is with respect to the canonical measure for the adic diplane. The identity

$$\int |f(\xi)|^2 d\xi = \int |g(\xi)|^2 d\xi$$

holds with integration with respect to the canonical measure for the adic diplane. If $f(\xi)$ vanishes when the adic modulus of ξ is not a rational number, then $g(\xi)$ vanishes when the adic modulus of ξ is not a rational number. The function $f(\xi)$ of ξ in the adic diplane is the Hankel transform of character χ^* for the adic diplane of the function $g(\xi)$ of ξ in the adic diplane.

The Hankel transformation for the adic skew–plane is a restriction of the Hankel transformation for the adic skew–diplane. The domain of the Hankel transformation for the adic skew–diplane is the space of functions $f(\xi)$ of ξ in the adic skew–diplane which satisfy the identity

$$f(\tfrac{1}{2}\omega(\xi+\xi^-)+\tfrac{1}{2}\omega^{-1}(\xi-\xi^-))=|\omega|f(\xi)$$

for every invertible element ω of the adic diline, which satisfy the identity

$$f(\omega^-\xi\omega)=f(\xi)$$

for every unit ω of the adic skew–diplane, and which are square integrable with respect to the canonical measure for the adic skew–diplane. The fundamental domain for the adic skew–diplane is the set of elements ξ of the adic skew–diplane such that $\frac{1}{2}(\xi+\xi^-)$ is a unit of the adic diline. The canonical measure for the fundamental domain of the adic skew–diplane is a nonnegative measure on the Borel subsets of the fundamental domain which is characterized within a constant factor by invariance properties. Measure preserving transformations are defined by taking ξ into $\omega\xi$ and ξ into $\xi\omega$ for every unit ω of the adic skew–diplane. The transformation which takes ξ into

$$\tfrac{1}{2}(\xi+\xi^-)+\tfrac{1}{2}\omega^-(\xi-\xi^-)\omega$$

multiplies the canonical measure by the fourth power of the adic modulus of $\omega^-\omega$ for every element ω of the adic skew–diplane. The measure is normalized so that the set of units has measure one. The domain of the Hankel transformation for the adic skew–plane is the space of functions $f(\xi)$ of ξ in the adic skew–diplane which belong to the domain of the Hankel transformation for the adic skew–diplane and which vanish when the adic modulus of $\xi^-\xi$ is not a rational number. The range of the Hankel transformation for the adic skew–diplane is the domain of the Hankel transformation for the adic skew–diplane. The transformation takes a function $f(\xi)$ of ξ in the adic skew–diplane into a function $g(\xi)$ of ξ in the adic skew–diplane when the identity

$$\int g(\xi)|\tfrac{1}{2}\xi-\tfrac{1}{2}\xi^-|^{-1}\sigma(\lambda(\tfrac{1}{2}\xi+\tfrac{1}{2}\xi^-)(\tfrac{1}{2}\xi-\tfrac{1}{2}\xi^-))d\xi$$
$$=\operatorname{sgn}(\lambda^*\lambda)|\lambda|^{-2}\int f(\xi)|\tfrac{1}{2}\xi-\tfrac{1}{2}\xi^-|^{-1}\sigma(\lambda^{-1}(\tfrac{1}{2}\xi+\tfrac{1}{2}\xi^-)(\tfrac{1}{2}\xi-\tfrac{1}{2}\xi^-))d\xi$$

holds for every invertible element λ of the adic diline with integration with respect to the canonical measure for the fundamental domain of the adic skew–diplane. The identity

$$\int|f(\xi)|^2d\xi=\int|g(\xi)|^2d\xi$$

holds with integration with respect to the canonical measure for the fundamental domain. If the function $f(\xi)$ of ξ in the adic skew–diplane vanishes when the adic modulus of $\xi^-\xi$ is not a rational number, then the function $g(\xi)$ of ξ in the adic skew–diplane vanishes when the adic modulus of $\xi^-\xi$ is not a rational number. The function $f(\xi)$ of ξ in the adic skew–diplane is the Hankel transform for the adic skew–diplane of the function $g(\xi)$ of ξ in the adic skew–diplane.

The Laplace transformation of character χ for the adic plane is a restriction of the Laplace transformation of character χ for the adic diplane. The domain of the Laplace transformation of character χ for the adic diplane is the space of functions $f(\xi)$ of ξ in the adic diplane which vanish when the p–adic component of ξ is not a unit for some prime divisor p of ρ, which satisfy the identity

$$f(\omega\xi) = \chi(\omega)f(\xi)$$

for every unit ω of the adic diplane, and which are square integrable with respect to the canonical measure for the adic diplane. The domain of the Laplace transformation of character χ for the adic plane is the space of functions $f(\xi)$ of ξ in the adic diplane which belong to the domain of the Laplace transformation of character χ for the adic diplane and which vanish when the adic modulus of ξ is not a rational number. The Laplace transform of character χ for the adic diplane of the function $f(\xi)$ of ξ in the adic diplane is the function $g(\lambda)$ of λ in the adic line defined by the integral

$$g(\lambda) = \int \chi(\xi)^{-} f(\xi)\sigma(\lambda\xi^{-}\xi)d\xi$$

with respect to the canonical measure for the adic diplane. The identity

$$\int |g(\lambda)|^2 d\lambda = \int |f(\xi)|^2 d\xi$$

holds with integration on the left with respect to the canonical measure for the adic line and with integration on the right with respect to the canonical measure for the adic diplane. A function $g(\lambda)$ of λ in the adic line, which is square integrable with respect to the canonical measure for the adic line, is a Laplace transform of character χ for the adic diplane if, and only if, it satisfies the identity

$$g(\omega\lambda) = g(\lambda)$$

for every unit ω of the adic line, vanishes when the p–adic component of $p\lambda$ is not integral for some prime divisor p of ρ, satisfies the identity

$$(1-p)f(\lambda) = f(\omega^{-1}\lambda)$$

when the p–adic component of $p\lambda$ is a unit for some prime divisor p of ρ and ω is an element of the adic line whose adic modulus is p, and satisfies the identity

$$g(\lambda) = g(\omega^{-1}\lambda)$$

when the p–adic component of λ is integral for some prime divisor p of ρ and ω is an element of the adic line whose adic modulus is p. A function $g(\lambda)$ of λ in the adic line is a

Laplace transform of character χ for the adic plane if, and only if, it is a Laplace transform of character χ for the adic diplane which satisfies the identity

$$(1-p)g(\lambda) = g(\omega^{-1}\lambda) - pg(\omega\lambda)$$

when the p–adic modulus of λ is an odd power of p for some prime p and ω is an element of the adic line whose adic modulus is p.

The Laplace transformation for the adic skew–plane is a restriction of the Laplace transformation for the adic skew–diplane. The domain of the Laplace transformation for the adic skew–diplane is the space of functions $f(\xi)$ of ξ in the adic skew–diplane which satisfy the identity

$$f(\tfrac{1}{2}\omega(\xi+\xi^-) + \tfrac{1}{2}\omega^{-1}(\xi-\xi^-)) = |\omega| f(\xi)$$

for every invertible element ω of the adic diline, which satisfy the identity

$$f(\omega^-\xi\omega) = f(\xi)$$

for every unit ω of the adic skew–diplane, and which are square integrable with respect to the canonical measure for the fundamental domain of the adic skew–diplane. The domain of the Laplace transformation for the adic skew–plane is the set of functions $f(\xi)$ of ξ in the adic skew–diplane which belong to the domain of the Laplace transformation for the adic skew–diplane and which vanish when the adic modulus of $\xi^-\xi$ is not a rational number. The Laplace transform for the adic skew–diplane of the function $f(\xi)$ of ξ in the adic skew–diplane is the function $g(\lambda)$ of λ in the adic diline which is defined by the integral

$$g(\lambda) = \int f(\xi)|\tfrac{1}{2}\xi - \tfrac{1}{2}\xi^-|^{-1}\sigma(\lambda(\tfrac{1}{2}\xi + \tfrac{1}{2}\xi^-)(\tfrac{1}{2}\xi - \tfrac{1}{2}\xi^-))d\xi$$

with respect to the canonical measure for the fundamental domain of the adic skew–diplane. The identity

$$\int |g(\lambda)|^2 d\lambda = \int |f(\xi)|^2 d\xi$$

holds with integration on the left with respect to the canonical measure for the adic diline and with integration on the right with respect to the canonical measure for the fundamental domain of the adic skew–diplane. A function $g(\lambda)$ of λ in the adic diline is a Laplace transform for the adic skew–diplane if, and only if, it satisfies the identity

$$g(\omega\lambda) = g(\lambda)$$

for every unit ω of the adic diline and is square integrable with respect to the canonical measure for the adic diline. A function $g(\lambda)$ of λ in the adic diline is a Laplace transform for the adic skew–plane if, and only if, it is a Laplace transform for the adic skew–diplane which satisfies the identity

$$(1-p)f(\lambda) = f(\omega^{-1}\lambda) - pf(\omega\lambda)$$

when the p–adic modulus of $\lambda^*\lambda$ is an odd power of p for some prime p and ω is an element of the adic diline for which the adic modulus of $\omega^*\omega$ is p.

The Radon transformation of character χ for the adic diplane is a nonnegative self–adjoint transformation in the space of functions $f(\xi)$ of ξ in the adic diplane which vanish when the p–adic component of ξ is not a unit for some prime p, which satisfy the identity

$$f(\omega\xi) = \chi(\omega)f(\xi)$$

for every unit ω of the adic diplane, and which are square integrable with respect to the canonical measure for the adic diplane. The transformation takes a function $f(\xi)$ of ξ in the adic diplane into a function $g(\xi)$ of ξ in the adic diplane when the identity

$$g(\xi) = \int f(\xi + \eta)d\eta$$

holds formally with integration with respect to Haar measure for the space of elements η of the adic plane whose p–adic component vanishes for every prime divisor p of ρ and which satisfy the identity

$$\eta^-\xi + \xi^-\eta = 0.$$

Haar measure is normalized so that the set of integral elements has measure one. The integral is accepted as the definition when

$$f(\xi) = \chi(\xi)\sigma(\lambda\xi^-\xi)$$

for an invertible element λ of the adic line whose p–adic component is a unit for every prime divisor p of ρ, in which case

$$g(\xi) = |\lambda|^{-\frac{1}{2}}f(\xi).$$

The formal integral is otherwise interpreted as the identity

$$\int \chi(\xi)^- g(\xi)\sigma(\lambda\xi^-\xi)d\xi = |\lambda|^{-\frac{1}{2}} \int \chi(\xi)^- f(\xi)\sigma(\lambda\xi^-\xi)d\xi$$

for every invertible element λ of the adic diline whose p–adic component is a unit for every prime divisor p of ρ. Integration is with respect to the canonical measure for the adic diplane.

The Radon transformation of character χ for the adic plane is a nonnegative self–adjoint transformation in the space of functions $f(\xi)$ of ξ in the adic diplane which vanish when the p–adic component of ξ is not a unit for some prime divisor p of ρ or when the adic modulus of ξ is not a rational number, which satisfy the identity

$$f(\omega\xi) = \chi(\omega)f(\xi)$$

for every unit ω of the adic diplane, and which are square integrable with respect to the canonical measure for the adic diplane. The transformation takes a function $f(\xi)$ of ξ in the adic diplane into a function $g(\xi)$ of ξ in the adic diplane when the Radon transformation for the adic diplane takes a function $f_n(\xi)$ of ξ in the adic diplane into a function $g_n(\xi)$ of ξ in the adic diplane for every positive integer n, such that the function $g(\xi)$ is the limit of the functions $g_n(\xi)$ in the metric topology of the space of square integrable functions with respect to the canonical measure, and such that the function $f(\xi)$ is the limit in the same topology of the orthogonal projections of the functions $f_n(\xi)$ in the space of functions which vanish when the adic modulus of ξ is not a rational number.

The Radon transformation for the adic skew–diplane is a nonnegative self–adjoint transformation in the space of functions $f(\xi)$ of ξ in the adic skew–diplane which satisfy the identity

$$f(\tfrac{1}{2}\omega(\xi+\xi^-)+\tfrac{1}{2}\omega^{-1}(\xi-\xi^-))=|\omega|f(\xi)$$

for every invertible element ω of the adic diline, which satisfy the identity

$$f(\omega^-\xi\omega)=f(\xi)$$

for every unit ω of the adic skew–diplane, and which are square integrable with respect to the canonical measure for the fundamental domain of the adic skew–diplane. The function

$$|\tfrac{1}{2}\xi-\tfrac{1}{2}\xi^-|^{-1}\sigma(\lambda(\tfrac{1}{2}\xi+\tfrac{1}{2}\xi^-)(\tfrac{1}{2}\xi-\tfrac{1}{2}\xi^-))$$

of ξ in the adic skew–diplane is an eigenfunction of the Radon transformation for the adic skew–diplane for the eigenvalue $|\lambda|^{-1}$ when λ is an invertible element of the adic diline. The transformation takes a function $f(\xi)$ of ξ in the adic skew–diplane into a function $g(\xi)$ of ξ in the adic skew–diplane when the identity

$$\int g(\xi)|\tfrac{1}{2}\xi-\tfrac{1}{2}\xi^-|^{-1}\sigma(\lambda(\tfrac{1}{2}\xi+\tfrac{1}{2}\xi^-)(\tfrac{1}{2}\xi-\tfrac{1}{2}\xi^-))d\xi$$
$$=|\lambda|^{-1}\int f(\xi)|\tfrac{1}{2}\xi-\tfrac{1}{2}\xi^-|^{-1}\sigma(\lambda(\tfrac{1}{2}\xi+\tfrac{1}{2}\xi^-)(\tfrac{1}{2}\xi-\tfrac{1}{2}\xi^-))d\xi$$

holds for every invertible element λ of the adic diline. Integration is with respect to the canonical measure for the fundamental domain of the adic skew–diplane.

The Radon transformation for the adic skew–plane is a nonnegative self–adjoint transformation in the space of functions $f(\xi)$ of ξ in the adic skew–diplane which satisfy the identity

$$f(\tfrac{1}{2}\omega(\xi+\xi^-)+\tfrac{1}{2}\omega^{-1}(\xi-\xi^-))=|\omega|f(\xi)$$

for every invertible element ω of the adic diline, which satisfy the identity

$$f(\omega^-\xi\omega)=f(\xi)$$

for every unit ω of the adic skew–diplane, which vanish when the adic modulus of $\xi^-\xi$ is not a rational number, and which are square integrable with respect to the canonical measure for the fundamental domain of the adic skew–diplane. The transformation takes a function $f(\xi)$ of ξ in the adic skew–diplane into a function $g(\xi)$ of ξ in the adic skew–diplane when the Radon transformation for the adic skew–diplane takes a function $f_n(\xi)$ of ξ in the adic skew–diplane into a function $g_n(\xi)$ in the adic skew–diplane for every positive integer n, such that the function $g(\xi)$ is the limit of the functions $g_n(\xi)$ in the metric topology of the space of square integrable functions with respect to the canonical measure for the fundamental domain of the adic skew–diplane, and such that the function $f(\xi)$ is the limit in the same topology of the orthogonal projections of the functions $f_n(\xi)$ in the space of functions which vanish when the adic modulus of $\xi^-\xi$ is not a rational number.

A property of the range of the Laplace transformation of character χ for the adic plane is required to know that a nonnegative self–adjoint transformation is obtained as Radon transformation of character χ for the adic plane. The range of the Laplace transformation of character χ for the adic diplane is the space of functions $f(\lambda)$ of λ in the adic line which are square integrable with respect to the canonical measure for the adic line, which satisfy the identity

$$f(\omega\lambda) = f(\lambda)$$

for every unit ω of the adic line, which vanish when the p–adic component of $p\lambda$ is not integral for some prime divisor p of ρ, which satisfy the identity

$$(1-p)f(\lambda) = f(\omega^{-1}\lambda)$$

when the p–adic component of $p\lambda$ is a unit for some prime divisor p of ρ with ω an element of the adic line whose adic modulus is p, and which satisfy the identity

$$f(\lambda) = f(\omega^{-1}\lambda)$$

when the p–adic component of λ is integral for some prime divisor p of ρ with ω an element of the adic line whose adic modulus is p. A nonnegative self–adjoint transformation in the range of the Laplace transformation of character χ for the adic diplane is defined by taking a function $f(\lambda)$ of λ in the adic line into a function $g(\lambda)$ of λ in the adic line if the identity

$$g(\lambda) = |\lambda|^{-\frac{1}{2}} f(\lambda)$$

holds when the p–adic component of λ is a unit for every prime divisor p of ρ. The range of the Laplace transformation of character χ for the adic plane is the space of functions $f(\lambda)$ of λ in the adic line which belong to the range of the Laplace transformation of character χ for the adic diplane and which satisfy the identity

$$(1-p)f(\lambda) = f(\omega^{-1}\lambda) - pf(\omega\lambda)$$

when the p–adic modulus of λ is an odd power of p for some prime divisor p of r and ω is an element of the adic line whose adic modulus is p. The closure of the set of functions $f(\lambda)$ of λ in the adic line, which belong to the range of the Laplace transformation of character χ for the adic diplane, such that a function $g(\lambda)$ of λ in the adic line, which belongs to the range of the Laplace transformation of character χ for the adic plane, exists such that the identity

$$g(\lambda) = |\lambda|^{-\frac{1}{2}} f(\lambda)$$

holds when the p–adic component of λ is a unit for every prime divisor p of ρ, is the set of functions $f(\lambda)$ of λ in the adic line which belong to the range of the Laplace transformation of character χ for the adic diplane and which satisfy the identity

$$(p^{\frac{1}{2}} - p^{-\frac{1}{2}})f(\lambda) = f(\omega\lambda) - f(\omega^{-1}\lambda)$$

when the p–adic component of λ is a unit for every prime divisor p of ρ and the p–adic modulus of λ is an odd power of p for some prime divisor p of r, which is not a divisor of ρ, with ω an element of the adic line whose adic modulus is p. It will be shown that a dense set of elements of the range of the Laplace transformation of character χ for the adic plane are orthogonal projections of such functions $f(\lambda)$ of λ in the adic line. It is sufficient to show that no nonzero element of the Laplace transformation of character χ for the adic plane is orthogonal to all such functions $f(\lambda)$ of λ in the adic line. A function $g(\lambda)$ of λ in the adic line, which belongs to the range of the Laplace transformation of character χ for the adic diplane and which is orthogonal to all such functions $f(\lambda)$ of λ in the adic line, satisfies the identity

$$(p^{\frac{1}{2}} - p^{-\frac{1}{2}})g(\lambda) = p^{-1}g(\omega^{-1}\lambda) - pg(\omega\lambda)$$

when the p–adic component of λ is a unit for every prime divisor p of ρ and the p–adic modulus of λ is an odd power of p for some prime p, which is not a divisor of ρ, with ω an element of the adic line whose adic modulus is p. The function $g(\lambda)$ of λ in the adic line vanishes identically when the function is in the range of the Laplace transformation of character χ for the adic plane.

A property of the range of the Laplace transformation for the adic skew–plane is required to know that a nonnegative self–adjoint transformation is obtained as the Radon transformation for the adic skew–plane. The range of the Laplace transformation for the adic skew–diplane is the space of functions $f(\lambda)$ of λ in the adic diline which satisfy the identity

$$f(\omega\lambda) = f(\lambda)$$

for every unit ω of the adic diline and which are square integrable with respect to the canonical measure for the adic diline. A nonnegative self–adjoint transformation in the range of the Laplace transformation for the adic skew-diplane is defined by taking a function $f(\lambda)$ of λ in the adic diline into the function $|\lambda|^{-1}f(\lambda)$ of λ in the adic diline. The range of the Laplace transformation for the adic skew–plane is the space of functions $f(\lambda)$ of λ in the adic diline which belong to the range of the Laplace transformation for the adic skew–diplane and which satisfy the identity

$$(1 - p)f(\lambda) = f(\omega^{-1}\lambda) - pf(\omega\lambda)$$

when the p–adic modulus of $\lambda^*\lambda$ is an odd power of p for some prime p and ω is an element of the adic diline such that the adic modulus of $\omega^*\omega$ is p. The closure of the set of functions $f(\lambda)$ of λ in the adic diline, which belong to the range of the Laplace transformation for the adic skew–diplane, such that the function $|\lambda|^{-1}f(\lambda)$ of λ in the adic diline belongs to the range of the Laplace transformation for the adic skew–plane, is the set of functions

$f(\lambda)$ of λ in the adic diline which belong to the range of the Laplace transformation for the adic skew–diplane and which satisfy the identity

$$(p^{\frac{1}{2}} - p^{-\frac{1}{2}})f(\lambda) = f(\omega\lambda) - f(\omega^{-1}\lambda)$$

when the p–adic modulus of $\lambda^*\lambda$ is an odd power of p for some prime p and ω is an element of the adic diline such that the adic modulus of $\omega^*\omega$ is p. It will be shown that a dense set of elements of the range of the Laplace transformation for the adic skew–plane are orthogonal projections of such functions $f(\lambda)$ of λ in the adic diline. It is sufficient to show that no nonzero element of the range of the Laplace transformation for the adic skew–plane is orthogonal to all such functions $f(\lambda)$ of λ in the adic diline. A function $g(\lambda)$ of λ in the adic diline, which belongs to the range of the Laplace transformation for the adic skew–diplane and which is orthogonal to all such functions $f(\lambda)$ of λ in the adic diline, satisfies the identity

$$(p^{\frac{1}{2}} - p^{-\frac{1}{2}})g(\lambda) = p^{-1}g(\omega^{-1}\lambda) - pg(\omega\lambda)$$

when the p–adic modulus of $\lambda^*\lambda$ is an odd power of p for some prime p and ω is an element of the adic diline such that the adic modulus of $\omega^*\omega$ is p. The function $g(\lambda)$ of λ in the adic diline vanishes identically when the function is in the range of the Laplace transformation for the adic skew–plane.

§7. The functional identity for Riemann zeta functions

The adelic upper half–plane is the set of elements of the adelic plane whose Euclidean component belongs to the upper half–plane and whose adic component is an invertible element of the adic line. An element of the adelic upper half–plane, whose Euclidean component is $\tau_+ + iy$ for a real number τ_+ and a positive number y and whose adic component is τ_-, is written $\tau + iy$ with τ the element of the adelic line whose Euclidean component is τ_+ and whose adic component is τ_-. A character of order ν for the adelic diplane is a function $\chi(\xi)$ of ξ in the adelic diplane which is a product

$$\chi(\xi) = \chi(\xi_+)\chi(\xi_-)$$

of a character of order ν for the Euclidean diplane and a character modulo ρ for the adic diplane which has the same parity as ν. The canonical measure for the adelic line is the Cartesian product of Haar measure for the Euclidean line and the canonical measure for the adic line. The fundamental domain for the adelic line is the set of elements of the adelic line whose adic modulus is a positive integer not divisible by the square of a prime. The canonical measure for the fundamental domain is the restriction to the Borel subsets of the fundamental domain of the canonical measure for the adelic line. The theta function of order ν and character χ for the adelic plane is a function $\theta(\lambda)$ of λ in the adelic upper half–plane which is defined as a sum

$$2\theta(\lambda) = \sum \chi(\omega_-)\exp(\pi i\omega_+^2\lambda_+/\rho)\sigma(\omega_-^2\lambda_-)$$

over the nonzero principal elements ω of the adelic line whose p–adic component is a unit for every prime divisor p of ρ. The theta function of order ν and character χ^* for the adelic upper half–plane is the function

$$\theta^*(\lambda) = \theta(-\lambda^-)^-$$

of λ in the upper half–plane.

The adelic upper half–diplane is the set of elements of the adelic diplane whose Euclidean component belongs to the upper half–plane and whose adic component is an invertible element of the adic diline. An element of the adelic upper half–plane, whose Euclidean component is $\tau_+ + iy$ for a real number τ_+ and a positive number y and whose adic component is τ_-, is written $\tau + iy$ with τ an element of the adelic diline whose Euclidean component is τ_+ and whose adic component is τ_-. A harmonic function of order ν for the adelic skew–diplane is a function $\phi(\xi)$ of ξ in the adelic skew–diplane which depends only on the Euclidean component of ξ and is a harmonic function of order ν for the Euclidean skew–diplane as a function of the Euclidean component of ξ. The canonical measure for the adelic diline is the Cartesian product of Haar measure for the Euclidean diline and the canonical measure for the adic diline. The fundamental domain for the adelic diline is the set of elements τ of the adelic diline such that the adic modulus of $\tau^*\tau$ is a positive integer which is not divisible by the square of a prime. The canonical measure for the fundamental domain is the restriction to the Borel subsets of the canonical domain of the canonical measure for the adelic diline. The theta function of order ν and character χ for the adelic skew–line is a function $\theta(\lambda)$ of λ in the adelic upper half–diplane which is defined as a sum

$$2\theta(\lambda) = \sum \omega_+^\nu \tau(\omega_+) \exp(2\pi i \omega_+ \lambda_+) \sigma(\omega_- \lambda_-)$$

over the nonzero principal elements ω of the adelic line when λ belongs to the fundamental domain for the adelic upper half–diplane. The theta function of order ν and character χ^* for the adelic upper half–diplane is the function

$$\theta^*(\lambda) = \theta(-\lambda^-)^-$$

of λ in the adelic upper half–diplane.

The Hankel transformation of order ν and character χ for the adelic plane is a restriction of the Hankel transformation of order ν and character χ for the adelic diplane. The canonical measure for the adelic diplane is the Cartesian product of Haar measure for the Euclidean diplane and the canonical measure for the adic diplane. The fundamental domain for the adelic diplane is the set of elements of the adelic diplane whose adic component is a unit. The canonical measure for the fundamental domain is the restriction to the Borel subsets of the fundamental domain of the canonical measure for the adelic diplane. The domain of the Hankel transformation of order ν and character χ for the adelic diplane is the space of functions $f(\xi)$ of ξ in the adelic diplane which vanish when the p–adic component of ξ is not a unit for some prime divisor p of ρ, which satisfy the identity

$$f(\omega\xi) = \chi(\omega_-) f(\xi)$$

for every unit ω of the adelic diplane, which satisfy the identity

$$f(\xi) = f(\omega\xi)$$

for every nonzero principal element ω of the adelic line whose p–adic component is a unit for every prime divisor p of ρ, and which are square integrable with respect to the canonical measure for the fundamental domain. The range of the Hankel transformation of order ν and character χ for the r–adelic diplane is the domain of the Hankel transformation of order ν and character χ^* for the adelic diplane. The domain of the Hankel transformation of order ν and character χ for the adelic plane is the space of functions $f(\xi)$ of ξ in the adelic diplane which belong to the domain of the Hankel transformation of order ν and character χ for the adelic diplane and which vanish when the adic modulus of ξ is not a rational number. The Hankel transformation of order ν and character χ for the adelic diplane takes a function $f(\xi)$ of ξ in the adelic diplane into a function $g(\xi)$ of ξ in the adelic diplane when the identity

$$\int \chi^*(\xi)^- g(\xi)\theta^*(\lambda\xi^-\xi)d\xi = (i/\lambda_+)^{1+\nu}\operatorname{sgn}(\lambda_-)|\lambda|_-^{-1}\epsilon(\chi)\int \chi(\xi_-)^- f(\xi)\theta(-\lambda^{-1}\xi^-\xi)d\xi$$

holds for λ in the adelic upper half–plane with integration with respect to the canonical measure for the fundamental domain. The identity

$$\int |f(\xi)|^2 d\xi = \int |g(\xi)|^2 d\xi$$

holds with integration with respect to the canonical measure for the fundamental domain. If the function $f(\xi)$ of ξ in the adelic diplane vanishes when the adic modulus of ξ is not a rational number, then the function $g(\xi)$ of ξ in the adelic diplane vanishes when the adic modulus of ξ is not a rational number. The function $f(\xi)$ of ξ in the adelic diplane is the Hankel transform of order ν and character χ^* for the adelic diplane of the function $g(\xi)$ of ξ in the adelic diplane.

The Hankel transformation of order ν and harmonic ϕ for the adelic skew–plane is a restriction of the Hankel transformation of order ν and harmonic ϕ for the adelic skew–diplane. The domain of the Hankel transformation of order ν and harmonic ϕ for the adelic skew–diplane is the space of functions $f(\xi)$ of ξ in the adelic skew–diplane which satisfy the identity

$$f(\tfrac{1}{2}\omega(\xi+\xi^-) + \tfrac{1}{2}\omega^{-1}(\xi-\xi^-)) = |\omega| f(\xi)$$

for every invertible element ω of the adelic diline, which satisfy the identity

$$\phi(\xi)f(\omega^-\xi\omega) = \phi(\omega^-\xi\omega)f(\xi)$$

for every unit ω of the adelic skew–diplane, which satisfy the identity

$$f(\xi) = f(\tfrac{1}{2}(\xi+\xi^-) + \tfrac{1}{2}\omega^-(\xi-\xi^-)\omega)$$

for every nonzero principal element ω of the adelic skew–plane, and which are square integrable with respect to the canonical measure for the fundamental domain of the adelic skew–diplane. The fundamental domain of the adelic skew–diplane is the set of elements ξ of the adelic skew–diplane such that $\frac{1}{2}(\xi + \xi^-)$ is a unit of the adelic diline and such that the square of the adic modulus of $(\frac{1}{2}\xi - \frac{1}{2}\xi^-)^-(\frac{1}{2}\xi - \frac{1}{2}\xi^-)$ is a positive integer which is not divisible by the square of a prime. The canonical measure for the fundamental domain of the adelic skew–diplane is the restriction to the Borel subsets of the fundamental domain of the Cartesian product of the canonical measure for the fundamental domain of the Euclidean skew–diplane and the canonical measure for the fundamental domain of the adic skew–diplane. The domain of the Hankel transformation of order ν and harmonic ϕ for the adelic skew–plane is the space of functions $f(\xi)$ of ξ in the adelic skew–diplane which belong to the domain of the Hankel transformation of order ν and harmonic ϕ for the adelic skew–diplane and which vanish when the adic modulus of $\xi^-\xi$ is not a rational number. The range of the Hankel transformation of order ν and harmonic ϕ for the adelic skew–diplane is the domain of the Hankel transformation of order ν and harmonic ϕ^* for the adelic skew–diplane. The Hankel transformation of order ν and harmonic ϕ for the adelic skew–diplane takes a function $f(\xi)$ of ξ in the adelic skew–diplane into a function $g(\xi)$ of ξ in the adelic skew–diplane when the identity

$$\int \phi^*(\xi)^- g(\xi) |\tfrac{1}{2}\xi - \tfrac{1}{2}\xi^-|^{-1} \theta^*(\lambda |\tfrac{1}{2}\xi + \tfrac{1}{2}\xi^-||\tfrac{1}{2}\xi - \tfrac{1}{2}\xi^-| d\xi$$

$$= (i/\lambda_+)^{2+2\nu} \operatorname{sgn}(\lambda^*\lambda) |\lambda|_-^{-2} \int \phi(\xi)^- f(\xi) |\tfrac{1}{2}\xi - \tfrac{1}{2}\xi^-|^{-1} \theta(-\lambda^{-1} |\tfrac{1}{2}\xi + \tfrac{1}{2}\xi^-||\tfrac{1}{2}\xi - \tfrac{1}{2}\xi^-|) d\xi$$

holds when λ is in the adelic upper half–plane with integration with respect to the canonical measure for the fundamental domain of the adelic skew–diplane. The notation

$$|\tfrac{1}{2}\xi + \tfrac{1}{2}\xi^-||\tfrac{1}{2}\xi - \tfrac{1}{2}\xi^-|$$

is used in the argument of the theta function for an element of the adelic diline with nonnegative Euclidean component which has the same Euclidean and adic modulus as

$$(\tfrac{1}{2}\xi + \tfrac{1}{2}\xi^-)(\tfrac{1}{2}\xi - \tfrac{1}{2}\xi^-).$$

The identity

$$\int |f(\xi)|^2 d\xi = \int |g(\xi)|^2 d\xi$$

holds with integration with respect to the canonical measure for the fundamental domain. If the function $f(\xi)$ of ξ in the adelic skew–diplane vanishes when the adic modulus of $\xi^-\xi$ is not a rational number, then the function $g(\xi)$ of ξ in the adelic skew–diplane vanishes when the adic modulus of $\xi^-\xi$ is not a rational number. The function $f(\xi)$ of ξ in the adelic skew–diplane is the Hankel transform of order ν and harmonic ϕ^* of the function $g(\xi)$ of ξ in the adelic skew–diplane.

The nonzero principal elements of the adelic line, whose p–adic component is a unit for every prime divisor p of ρ, are applied in an isometric summation for the adelic diplane. If a function $f(\xi)$ of ξ in the adelic diplane satisfies the identity

$$f(\omega\xi) = \chi(\omega) f(\xi)$$

for every unit ω of the adelic diplane, vanishes outside of the fundamental domain, and is square integrable with respect to the canonical measure for the adelic diplane, then a function $g(\xi)$ of ξ in the adelic diplane, which vanishes when the p–adic component of ξ is not a unit for some prime divisor p of ρ, which satisfies the identity

$$g(\omega\xi) = \chi(\omega)g(\xi)$$

for every unit ω of the adelic diplane, and which satisfies the identity

$$g(\xi) = g(\omega\xi)$$

for every nonzero principal element ω of the adelic line whose p–adic component is a unit for every prime divisor p of ρ, is defined by the sum

$$g(\xi) = \sum f(\omega\xi)$$

over the nonzero principal elements ω of the adelic line whose p–adic component is a unit for every prime divisor p of ρ. The identity

$$\int |f(\xi)|^2 d\xi = \int |g(\xi)|^2 d\xi$$

holds with integration with respect to the canonical measure for the fundamental domain of the adelic diplane. If the function $f(\xi)$ of ξ in the adelic diplane vanishes when the adic modulus of ξ is not rational, then the function $g(\xi)$ of ξ in the adelic diplane vanishes when the adic modulus of ξ is not rational. If a function $h(\xi)$ of ξ in the adelic diplane vanishes when the p–adic component of ξ is not a unit for some prime divisor p of ρ, satisfies the identity

$$h(\omega\xi) = \chi(\omega)h(\xi)$$

for every unit ω of the adelic diplane satisfies the identity

$$h(\xi) = h(\omega\xi)$$

for every nonzero principal element ω of the adelic line whose p–adic component is a unit for every prime divisor p of ρ, and is square integrable with respect to the canonical measure for the fundamental domain, then $h(\xi)$ is equal to $g(\xi)$ for a function $f(\xi)$ of ξ in the adelic diplane which is equal to $h(\xi)$ when ξ is in the fundamental domain.

The nonzero principal elements of the adelic skew–plane are applied in an isometric summation for the adelic skew–plane. If a function $f(\xi)$ of ξ in the adelic skew–plane satisfies the identity

$$f(\tfrac{1}{2}\omega(\xi + \xi^-) + \tfrac{1}{2}\omega^{-1}(\xi - \xi^-)) = |\omega| f(\xi)$$

for every invertible element ω of the adelic diline, satisfies the identity

$$\phi(\xi)f(\omega^-\xi\omega) = \phi(\omega^-\xi\omega)f(\xi)$$

for every unit ω of the adelic skew–plane, vanishes when $\frac{1}{2}(\xi + \xi^-)$ is a unit of the adelic diline but ξ does not belong to the fundamental domain for the adelic skew–diplane, and is square integrable with respect to the canonical measure for the fundamental domain, then a function $g(\xi)$ of ξ in the adelic skew–diplane which satisfies the identity

$$g(\tfrac{1}{2}\omega(\xi + \xi^-) + \tfrac{1}{2}\omega^{-1}(\xi - \xi^-)) = |\omega| g(\xi)$$

for every invertible element ω of the adelic diline, which satisfies the identity

$$\phi(\xi) g(\omega^- \xi \omega) = \phi(\omega^- \xi \omega) g(\xi)$$

for every unit ω of the adelic skew–diplane, and which satisfies the identity

$$g(\xi) = g(\tfrac{1}{2}(\xi + \xi^-) + \tfrac{1}{2}\omega^-(\xi - \xi^-)\omega)$$

for every nonzero principal element ω of the adelic skew–plane, is defined by the sum

$$24 g(\xi) = \sum f(\tfrac{1}{2}(\xi + \xi^-) + \tfrac{1}{2}\omega^-(\xi - \xi^-)\omega)$$

for every nonzero principal element ω of the adelic skew–plane, is defined by the sum

$$24 g(\xi) = \sum f(\tfrac{1}{2}(\xi + \xi^-) + \tfrac{1}{2}\omega^-(\xi - \xi^-)\omega)$$

over the nonzero principal elements ω of the adelic skew–plane. The identity

$$\int |f(\xi)|^2 d\xi = \int |g(\xi)|^2 d\xi$$

holds with integration with respect to the canonical measure for the fundamental domain of the adelic skew–diplane. If a function $f(\xi)$ of ξ in the adelic skew–diplane vanishes when the adic modulus of $\xi^-\xi$ is not a rational number, then the function $g(\xi)$ of ξ in the adelic skew–diplane vanishes when the adic modulus of $\xi^-\xi$ is not a rational number. The identity

$$2 g(\xi) = \sum \tau(\omega_+) f(\tfrac{1}{2}(\xi + \xi^-) + \tfrac{1}{2}\omega(\xi - \xi^-))$$

holds with summation over the nonzero principal elements ω of the adelic line. If a function $h(\xi)$ of ξ in the adelic skew–plane satisfies the identity

$$h(\tfrac{1}{2}\omega(\xi + \xi^-) + \tfrac{1}{2}\omega^{-1}(\xi - \xi^-)) = |\omega| h(\xi)$$

for every invertible element ω of the adelic diline, satisfies the identity

$$\phi(\xi) h(\omega^- \xi \omega) = \phi(\omega^- \xi \omega) h(\xi)$$

for every unit ω of the adelic skew–diplane, satisfies the identity

$$h(\xi) = h(\tfrac{1}{2}(\xi + \xi^-) + \tfrac{1}{2}\omega^-(\xi - \xi^-)\omega)$$

for every nonzero principal element ω of the adelic skew–plane, and is square integrable with respect to the canonical measure for the fundamental domain of the adelic skew–diplane, then $h(\xi)$ is equal to $g(\xi)$ for a function $f(\xi)$ of ξ in the adelic skew–diplane which is equal to $h(\xi)$ when ξ is in the fundamental domain.

The Laplace transformation of order ν and character χ for the adelic plane is a restriction of the Laplace transformation of order ν and character χ for the adelic diplane. The domain of the Laplace transformation of order ν and character χ for the adelic diplane is the space of functions $f(\xi)$ of ξ in the adelic diplane which vanish when the p–adic component of ξ is not a unit for some prime divisor p of ρ, which satisfy the identity

$$f(\omega\xi) = \chi(\omega)f(\xi)$$

for every unit ω of the adelic diplane, which satisfy the identity

$$f(\xi) = f(\omega\xi)$$

for every nonzero principal element ω of the adelic line whose p–adic component is a unit for every prime divisor p of ρ, and which are square integrable with respect to the canonical measure for the fundamental domain of the adelic diplane. The domain of the Laplace transformation of order ν and character χ for the adelic plane is the space of functions $f(\xi)$ of ξ in the adelic diplane which belong to the domain of the Laplace transformation of order ν and character χ for the adelic diplane and which vanish when the adic modulus of ξ is not a rational number. The Laplace transform of order ν and character χ for the adelic diplane of the function $f(\xi)$ of ξ in the adelic diplane is a function $g(\lambda)$ of λ in the adelic upper half–plane which is defined by the integral

$$2\pi g(\lambda) = \int \chi(\xi)^- f(\xi)\theta(\lambda\xi^-\xi)d\xi$$

with respect to the canonical measure for the fundamental domain of the adelic diplane. The function $g(\lambda)$ of λ in the adelic upper half–plane is an analytic function of the Euclidean component of λ when the adic component of λ is held fixed. The identity

$$g(\omega\lambda) = g(\lambda)$$

holds for every unit ω of the adelic line whose Euclidean component is the unit of the Euclidean line. The function vanishes when the p–adic component of $p\lambda$ is not integral for some prime divisor p of ρ. The identity

$$(1-p)g(\lambda) = g(\omega^{-1}\lambda)$$

holds when the p–adic component of $p\lambda$ is a unit for some prime divisor p of ρ and ω is an element of the adelic line whose Euclidean component is the unit of the Euclidean line and whose adic modulus is p. The identity

$$g(\lambda) = g(\omega^{-1}\lambda)$$

holds when the p–adic component of λ is integral for some prime divisor p of ρ and ω is an element of the adelic line whose Euclidean component is the Euclidean line and whose adic modulus is p. The identity

$$g(\lambda) = \chi(\omega)g(\omega^2\lambda)$$

holds for every nonzero principal element ω of the adelic line whose p–adic component is a unit for every prime divisor p of ρ. When ν is zero, the identity

$$(2\pi/\rho)\sup\int|g(\tau+iy)|^2d\tau = \int|f(\xi)|^2d\xi$$

holds with the least upper bound taken over all positive numbers y. The identity

$$(2\pi/\rho)^\nu\int_0^\infty\int|g(\tau+iy)|^2y^{\nu-1}d\tau dy = \Gamma(\nu)\int|f(\xi)|^2d\xi$$

holds when ν is positive. Integration on the left is with respect to the canonical measure for the fundamental domain of the adelic line. Integration on the right is with respect to the canonical measure for the fundamental domain of the adelic diplane. These properties characterize Laplace transforms of order ν and character χ for the adelic diplane. A function $g(\lambda)$ of λ in the adelic line is a Laplace transform of order ν and character χ for the adelic plane if, and only if, it is a Laplace transform of order ν and character χ for the adelic diplane which satisfies the identity

$$(1-p)g(\lambda) = g(\omega^{-1}\lambda) - pg(\omega\lambda)$$

when the p–adic modulus of λ is an odd power of p for some prime p and ω is an element of the adelic line whose Euclidean component is the unit of the Euclidean line and whose adic modulus is p.

The Laplace transformation of order ν and harmonic ϕ for the adelic skew–plane is a restriction of the Laplace transformation of order ν and harmonic ϕ for the adelic skew–diplane. The domain of the Laplace transformation of order ν and harmonic ϕ for the adelic skew–diplane is the space of functions $f(\xi)$ of ξ in the adelic skew–diplane which satisfy the identity

$$f(\tfrac{1}{2}\omega(\xi+\xi^-) + \tfrac{1}{2}\omega^{-1}(\xi-\xi^-)) = |\omega|f(\xi)$$

for every invertible element ω of the adelic diline, which satisfy the identity

$$\phi(\xi)f(\omega^-\xi\omega) = \phi(\omega^-\xi\omega)f(\xi)$$

for every unit ω of the adelic skew–diplane, which satisfy the identity

$$f(\xi) = f(\tfrac{1}{2}(\xi+\xi^-) + \tfrac{1}{2}\omega(\xi-\xi^-))$$

for every nonzero principal element ω of the adelic line, and which are square integrable with respect to the canonical measure for the fundamental domain of the adelic skew–diplane. The domain of the Laplace transformation of order ν and harmonic ϕ for the adelic skew–plane is the space of functions $f(\xi)$ of ξ in the adelic skew–diplane which belong to the domain of the Laplace transformation of order ν and harmonic ϕ for the adelic skew–diplane and which vanish when the adic modulus of $\xi^-\xi$ is not a rational number. The Laplace transform of order ν and harmonic ϕ for the adelic skew–diplane of the function $f(\xi)$ of ξ in the adelic skew–diplane is a function $g(\lambda)$ of λ in the adelic upper half–diplane which is defined by the integral

$$4\pi g(\lambda) = \int \phi(\xi)^- f(\xi)|\tfrac{1}{2}\xi - \tfrac{1}{2}\xi^-|^{-1}\theta(\lambda|\tfrac{1}{2}\xi + \tfrac{1}{2}\xi^-||\tfrac{1}{2}\xi - \tfrac{1}{2}\xi^-|)d\xi$$

with respect to the canonical measure for the fundamental domain of the adelic skew–diplane. The notation

$$|\tfrac{1}{2}\xi + \tfrac{1}{2}\xi^-||\tfrac{1}{2}\xi - \tfrac{1}{2}\xi^-|$$

is used in the argument of the theta function for an element of the adelic diline with nonnegative Euclidean component which has the same Euclidean and adic modulus as

$$(\tfrac{1}{2}\xi + \tfrac{1}{2}\xi^-)(\tfrac{1}{2}\xi - \tfrac{1}{2}\xi^-).$$

The function $g(\lambda)$ of λ in the adelic upper half–diplane is an analytic function of the Euclidean component of λ when the r–adic component of λ is held fixed. The identity

$$g(\omega\lambda) = g(\lambda)$$

holds for every unit ω of the adelic line whose Euclidean component is the unit of the Euclidean line. The identity

$$g(\lambda) = g(\omega\lambda)$$

holds for every nonzero principal element ω of the adelic line. The identity

$$(4\pi)^{2+2\nu}\int_0^\infty \int |g(\tau + iy)|^2 y^{2\nu} d\tau dy = \Gamma(1+2\nu)\int |f(\xi)|^2 d\xi$$

holds when ν is positive. Integration on the left is with respect to the canonical measure for the fundamental domain of the adelic diline. Integration on the right is with respect to the canonical measure for the fundamental domain of the adelic skew–diplane. These properties characterize Laplace transforms of order ν and harmonic ϕ for the adelic skew–diplane. A function $g(\lambda)$ of λ in the adelic diline is a Laplace transform of order ν and character χ for the adelic skew–plane if, and only if, it is a Laplace transform of order ν and character χ for the adelic skew–diplane which satisfies the identity

$$(1-p)g(\lambda) = g(\omega^{-1}\lambda) - pg(\omega\lambda)$$

when the p–adic modulus of $\lambda^*\lambda$ is an odd power of p for some prime p and ω is an element of the adelic diline whose Euclidean component is a unit of the Euclidean line and for which the r–adic modulus of $\omega^*\omega$ is equal to p.

The Radon transformation of order ν and character χ for the adelic diplane is a maximal dissipative transformation in the space of functions $f(\xi)$ of ξ in the adelic diplane which vanish when the p–adic component of ξ is not a unit for some prime divisor p of ρ, which satisfy the identity

$$f(\omega\xi) = \chi(\omega)f(\xi)$$

for every unit ω of the adelic diplane, which satisfy the identity

$$f(\xi) = f(\omega\xi)$$

for every nonzero principal element ω of the adelic line whose p–adic component is a unit for every prime divisor p of ρ, and which are square integrable with respect to the canonical measure for the fundamental domain of the adelic diplane. The transformation takes a function $f(\xi)$ of ξ in the adelic diplane into a function $g(\xi)$ of the adelic diplane when the identity

$$g(\xi) = \int f(\xi + \eta)d\eta$$

is formally satisfied with integration with respect to Haar measure for the space of elements η of the adelic diplane which satisfy the identity

$$\eta^- \xi + \xi^- \eta = 0.$$

Haar measure is defined as the Cartesian product of Haar measure for the space of elements η_+ of the Euclidean diplane which satisfy the identity

$$\eta_+^- \xi_+ + \xi_+^- \eta_+ = 0$$

and Haar measure for the space of elements η_- of the adic diplane which satisfy the identity

$$\eta_-^- \xi_- + \xi_-^- \eta_- = 0.$$

The integral is accepted as the definition of the Radon transformation of order ν and character χ for the adelic diplane when

$$f(\xi) = \chi(\xi)\theta(\lambda\xi^-\xi)$$

for an element λ of the adelic upper half–plane whose p–adic component is a unit for every prime divisor p of ρ, in which case

$$g(\xi) = (i\rho/\lambda_+)^{\frac{1}{2}} |\lambda|_-^{-\frac{1}{2}} f(\xi)$$

with the square root of $i\rho/\lambda_+$ taken in the right half–plane. The adjoint of the Radon transformation of order ν and character χ for the adelic diplane takes a function $f(\xi)$ of ξ in the adelic diplane into a function $g(\xi)$ of ξ in the adelic diplane when the identity

$$\int \chi(\xi)^- g(\xi)\theta(\lambda\xi^-\xi)d\xi = (i\rho/\lambda_+)^{\frac{1}{2}} |\lambda|_-^{-\frac{1}{2}} \int \chi(\xi)^- f(\xi)\theta(\lambda\xi^-\xi)d\xi$$

holds when λ is an element of the adelic upper half–plane whose p–adic component of λ is a unit for every prime divisor p of ρ. Integration is with respect to the canonical measure for the fundamental domain of the adelic diplane.

The Radon transformation of order ν and character χ for the adelic plane is a maximal dissipative transformation in the space of functions $f(\xi)$ of ξ in the adelic diplane which vanish when the p–adic component of ξ is not a unit for some prime divisor p of ρ or when the adic modulus of ξ is not a rational number, which satisfy the identity

$$f(\omega) = \chi(\omega)f(\xi)$$

for every unit ω of the adelic diplane, which satisfy the identity

$$f(\xi) = f(\omega\xi)$$

for every nonzero principal element ω of the adelic diplane whose p–adic component is a unit for every prime divisor p of ρ, and which are square integrable with respect to the canonical measure for the fundamental domain of the adelic diplane. The transformation takes a function $f(\xi)$ of ξ in the adelic diplane into a function $g(\xi)$ of ξ in the adelic diplane when the Radon transformation for the adelic diplane takes a function $f_n(\xi)$ of ξ in the adelic diplane into a function $g_n(\xi)$ of ξ in the adelic diplane for every positive integer n, such that the function $g(\xi)$ is the limit of the functions $g_n(\xi)$ in the metric topology of the space of square integrable functions with respect to the canonical measure for the fundamental domain of the adelic diplane, and such that the function $f(\xi)$ is the limit in the same topology of the orthogonal projections of the functions $f_n(\xi)$ in the space of functions which vanish when the adic modulus of ξ is not a rational number.

The Radon transformation of order ν and harmonic ϕ for the adelic skew–diplane is a maximal dissipative transformation in the space of functions $f(\xi)$ of ξ in the adelic skew–diplane which satisfy the identity

$$f(\tfrac{1}{2}\omega(\xi + \xi^-) + \tfrac{1}{2}\omega^{-1}(\xi - \xi^-)) = |\omega| f(\xi)$$

for every invertible element ω of the adelic diline, which satisfy the identity

$$\phi(\xi)f(\omega^-\xi\omega) = \phi(\omega^-\xi\omega)f(\xi)$$

for every unit ω of the adelic skew–diplane, which satisfy the identity

$$f(\xi) = f(\tfrac{1}{2}(\xi + \xi^-) + \tfrac{1}{2}\omega^-(\xi - \xi^-)\omega)$$

for every nonzero principal element ω of the adelic skew–plane, and which are square integrable with respect to the canonical measure for the fundamental domain of the adelic skew–diplane. The function

$$\phi(\xi)|\tfrac{1}{2}\xi - \tfrac{1}{2}\xi^-|^{-1}\theta(\lambda|\tfrac{1}{2}\xi + \tfrac{1}{2}\xi^-||\tfrac{1}{2}\xi - \tfrac{1}{2}\xi^-|)$$

of ξ in the adelic skew–diplane is an eigenfunction of the Radon transformation of order ν and harmonic ϕ for the adelic skew–diplane for the eigenvalue $(i/\lambda_+)|\lambda|_-^{-1}$ when λ is in the adelic upper half–diplane. The adjoint of the Radon transformation of order ν and harmonic ϕ for the adelic skew–diplane takes a function $f(\xi)$ of ξ in the adelic skew–diplane into a function $g(\xi)$ of ξ in the adelic skew–diplane when the identity

$$\int \phi(\xi)^- g(\xi)|\tfrac{1}{2}\xi - \tfrac{1}{2}\xi^-|^{-1}\theta(\lambda|\tfrac{1}{2}\xi + \tfrac{1}{2}\xi^-||\tfrac{1}{2}\xi - \tfrac{1}{2}\xi^-|)d\xi$$
$$= (i/\lambda_+)|\lambda|_-^{-1}\int \phi(\xi)^- f(\xi)|\tfrac{1}{2}\xi - \tfrac{1}{2}\xi^-|^{-1}\theta(\lambda|\tfrac{1}{2}\xi + \tfrac{1}{2}\xi^-||\tfrac{1}{2}\xi - \tfrac{1}{2}\xi^-|)d\xi$$

holds when λ is an element of the adelic upper half–diplane. Integration is with respect to the canonical measure for the fundamental domain of the adelic skew–diplane.

The Radon transformation of order ν and harmonic ϕ for the adelic skew–plane is a maximal dissipative transformation in the space of functions $f(\xi)$ of ξ in the adelic skew–diplane which satisfy the identity

$$f(\tfrac{1}{2}\omega(\xi + \xi^-) + \tfrac{1}{2}\omega^{-1}(\xi - \xi^-)) = |\omega| f(\xi)$$

for every invertible element ω of the adelic diline, which satisfy the identity

$$\phi(\xi)f(\omega^-\xi\omega) = \phi(\omega^-\xi\omega)f(\xi)$$

for every unit ω of the adelic skew–diplane, which satisfy the identity

$$f(\xi) = f(\tfrac{1}{2}(\xi + \xi^-) + \tfrac{1}{2}\omega^-(\xi - \xi^-)\omega)$$

for every nonzero principal element ω of the adelic skew–plane, which vanish when the adic modulus of $\xi^-\xi$ is not a rational number, and which are square integrable with respect to the canonical measure for the fundamental domain of the adelic skew–diplane. The transformation takes a function $f(\xi)$ of ξ in the adelic skew–diplane into a function $g(\xi)$ of ξ in the adelic skew–diplane when the Radon transformation of order ν and harmonic ϕ for the adelic skew–diplane takes a function $f_n(\xi)$ of ξ in the adelic skew–diplane into a function $g_n(\xi)$ of ξ in the adelic skew–diplane for every positive integer n, such that the function $g(\xi)$ is the limit of the functions $g_n(\xi)$ in the metric topology of the space of square integrable functions with respect to the canonical measure for the fundamental domain of the adelic skew–diplane, and such that the function $f(\xi)$ is the limit in the same topology of the orthogonal projections of the functions $f_n(\xi)$ in the space of functions which vanish when the adic modulus of $\xi^-\xi$ is not a rational number.

The Mellin transformation of order ν and character χ for the adelic plane is a spectral theory for the Laplace transformation of order ν and character for the adelic plane. The domain of the Laplace transformation of order ν and character χ for the adelic plane is the space of functions $f(\xi)$ of ξ in the adelic diplane which vanish when the p–adic component of ξ is not a unit for some prime divisor p of ρ or when the adic modulus of ξ is not a rational number, which satisfy the identity

$$f(\omega\xi) = \chi(\omega)f(\xi)$$

for every unit ω of the adelic diplane, which satisfy the identity

$$f(\xi) = f(\omega\xi)$$

for every nonzero principal element ω of the adelic line whose p–adic component is a unit for every prime divisor p of ρ, and which are square integrable with respect to the canonical measure for the fundamental domain of the adelic diplane. The Laplace transform of order ν and character χ for the adelic diplane of the function $f(\xi)$ of ξ in the adelic diplane is the function $g(\lambda)$ of λ in the adelic upper half–plane defined by the integral

$$2\pi g(\lambda) = \int \chi(\xi)^{-} f(\xi)\theta(\lambda\xi^{-}\xi)d\xi$$

with respect to the canonical measure for the fundamental domain of the adelic diplane. The Mellin transform of order ν and character χ for the adelic plane of the function $f(\xi)$ of ξ in the adelic diplane is an analytic function $F(z)$ of z in the upper half–plane which is defined by the integral

$$F(z) = \int_0^\infty g(\lambda)t^{\frac{1}{2}\nu-\frac{1}{2}-\frac{1}{2}iz}dt$$

under the constraint

$$\lambda_+ = it$$

when λ_- is a unit of the adic line if the function $f(\xi)$ of ξ in the adelic plane vanishes in the neighborhood $|\xi| < a$ of the origin. A computation of the integral is made from the zeta function

$$\zeta(s) = \sum \tau(n)n^{-s}$$

of order ν and character χ for the adelic line, which is defined in the half–plane $\mathcal{R}s > 1$ as a sum over the positive integers n which are relatively prime to ρ. The identity

$$\tau(n) = \chi(n)^{-}$$

holds when a positive integer n is relatively prime to ρ. The zeta function is represented in the half–plane by the Euler product

$$\zeta(s)^{-1} = \prod(1 - \tau(p)p^{-s})$$

taken over the primes p which are not divisors of ρ. The analytic weight function

$$W(z) = (\pi/\rho)^{-\frac{1}{2}\nu-\frac{1}{2}+\frac{1}{2}iz}\Gamma(\tfrac{1}{2}\nu + \tfrac{1}{2} - \tfrac{1}{2}iz)\zeta(1 - iz)$$

is represented in the upper half–plane by the integral

$$W(z) = \int_0^\infty \theta(\lambda)t^{\frac{1}{2}\nu-\frac{1}{2}-\frac{1}{2}iz}dt$$

under the constraint

$$\lambda_+ = it$$

when λ_- is a unit of the adic line. The identity

$$2\pi F(z)/W(z) = \int \chi(\xi)^- f(\xi)|\xi|^{iz-\nu-1} d\xi$$

holds when z is in the upper half–plane. Integration is with respect to the canonical measure for the fundamental domain of the adelic diplane. The function

$$a^{-iz} F(z)$$

of z in the upper half–plane is characterized as an element of the weighted Hardy space $\mathcal{F}(W)$ which satisfies the identity

$$\int_{-\infty}^{+\infty} |F(t)/W(t)|^2 dt = \int |f(\xi)|^2 d\xi.$$

Integration on the right is with respect to the canonical measure for the fundamental domain of the adelic diplane. If the Hankel transform of order ν and character χ for the adelic plane of the function $f(\xi)$ of ξ in the adelic diplane is a function $g(\xi)$ of ξ in the adelic diplane which vanishes when $|\xi| < a$, then the Mellin transform of order ν and character χ for the adelic plane of the function $g(\xi)^-$ of ξ in the adelic diplane is an entire function which is the analytic extension of $F^*(z)$ to the complex plane.

The Mellin transform of order ν and harmonic ϕ for the adelic skew–plane is a spectral theory for the Laplace transformation of order ν and harmonic ϕ for the adelic skew–plane. The domain of the Laplace transformation of order ν and harmonic ϕ for the adelic skew–plane is the space of functions $f(\xi)$ of ξ in the adelic skew–diplane which satisfy the identity

$$f(\tfrac{1}{2}\omega(\xi + \xi^-) + \tfrac{1}{2}\omega^{-1}(\xi - \xi^-)) = |\omega| f(\xi)$$

for every invertible element ω of the adelic diline, which satisfy the identity

$$\phi(\xi) f(\omega^- \xi \omega) = \phi(\omega^- \xi \omega) f(\xi)$$

for every unit ω of the adelic skew–diplane, which satisfy the identity

$$f(\xi) = f(\tfrac{1}{2}(\xi + \xi^-) + \tfrac{1}{2}\omega^-(\xi - \xi^-)\omega)$$

for every nonzero principal element ω of the adelic line, which vanish when the adic modulus of $\xi^- \xi$ is not a rational number, and which are square integrable with respect to the canonical measure for the fundamental domain of the adelic skew–diplane. The Laplace transform of order ν and harmonic ϕ for the adelic skew–plane of the function $f(\xi)$ of ξ in the adelic skew–diplane is the function $g(\lambda)$ of λ in the adelic upper half–diplane which is defined by the integral

$$4\pi g(\lambda) = \int \phi(\xi)^- f(\xi) |\tfrac{1}{2}\xi - \tfrac{1}{2}\xi^-|^{-1} \theta(\lambda |\tfrac{1}{2}\xi + \tfrac{1}{2}\xi^-| |\tfrac{1}{2}\xi - \tfrac{1}{2}\xi^-|) d\xi$$

with respect to the canonical measure for the fundamental domain of the adelic skew–diplane. The notation

$$|\tfrac{1}{2}\xi + \tfrac{1}{2}\xi^-||\tfrac{1}{2}\xi - \tfrac{1}{2}\xi^-|$$

is used in the argument of the theta function for an element of the adelic diline with nonnegative Euclidean component which has the same Euclidean and adic modulus as

$$(\tfrac{1}{2}\xi + \tfrac{1}{2}\xi^-)(\tfrac{1}{2}\xi - \tfrac{1}{2}\xi^-).$$

The Mellin transform of order ν and harmonic ϕ for the adelic skew–plane of the function $f(\xi)$ of ξ in the adelic skew–diplane is an analytic function $F(z)$ of z in the upper half–plane which is defined by the integral

$$F(z) = \int_0^\infty g(\lambda) t^{\nu - iz} dt$$

under the constraint

$$\lambda_+ = it$$

when λ_- is a unit of the adic diline if the function $f(\xi)$ of ξ in the adelic skew–diplane vanishes in the neighborhood $|\frac{1}{2}\xi + \frac{1}{2}\xi^-||\frac{1}{2}\xi - \frac{1}{2}\xi^-| < a$ of the origin. A computation of the integral is made using the zeta function

$$\zeta(s) = \sum \tau(n) n^{-s}$$

of order ν and harmonic ϕ for the adelic skew–plane, which is defined in the half–plane $\mathcal{R}s > 1$ as a sum over the positive integers n. The Euler product

$$\zeta(s)^{-1} = (1 - \tau(2)2^{-s}) \prod (1 - \tau(p)p^{-s} + p^{-2s})$$

is taken over the odd primes p. The zeta function has no zeros in the half–plane $\mathcal{R}s > 1$. The analytic weight function

$$W(z) = (2\pi)^{-\nu - \frac{1}{2} + iz} \Gamma(\nu + \tfrac{1}{2} - iz) \zeta(1 - iz)$$

is represented in the upper half–plane by the integral

$$W(z) = \int_0^\infty \theta(\lambda) t^{\nu - iz} dt$$

under the constraint

$$\lambda_+ = it$$

when λ_- is a unit of the adic diline. The identity

$$4\pi F(z)/W(z) = \int \phi(\xi)^- f(\xi) |\tfrac{1}{2}\xi + \tfrac{1}{2}\xi^-|^{iz - \nu - 1} |\tfrac{1}{2}\xi - \tfrac{1}{2}\xi^-|^{iz - \nu - 2} d\xi$$

holds when z is in the upper half–plane with integration with respect to the canonical measure for the fundamental domain of the adelic skew–diplane. The function

$$a^{-iz}F(z)$$

is characterized as an element of the weighted Hardy space $\mathcal{F}(W)$ which satisfies the identity

$$\int_{-\infty}^{+\infty} |F(t)/W(t)|^2 dt = \tfrac{1}{2} \int |f(\xi)|^2 d\xi.$$

Integration on the right is with respect to the canonical measure for the fundamental domain of the adelic skew–diplane. If the Hankel transformation of order ν and harmonic ϕ for the adelic skew–plane of the function $f(\xi)$ of ξ in the adelic skew–diplane is a function $g(\xi)$ of ξ in the adelic skew–diplane which vanishes when $|\frac{1}{2}\xi + \frac{1}{2}\xi^-||\frac{1}{2}\xi - \frac{1}{2}\xi^-| < a$, then the Mellin transform of order ν and harmonic ϕ for the adelic skew–plane of the function $g(\xi)^-$ of ξ in the adelic skew–diplane is an entire function which is the analytic extension of $F^*(z)$ to the complex plane.

The functional identity for the zeta function of order ν and character χ for the adelic plane is applied for the construction of the Sonine spaces of order ν and character χ for the adelic plane. When χ is not the principal character, the functional identity states that the entire functions

$$(\pi/\rho)^{-\frac{1}{2}\nu-\frac{1}{2}+\frac{1}{2}s}\Gamma(\tfrac{1}{2}\nu + \tfrac{1}{2} - \tfrac{1}{2}s)\zeta(1-s^-)^-$$

of s are linearly dependent. The Sonine spaces of order ν and character χ for the adelic plane are defined using the analytic weight function

$$W(z) = (\pi/\rho)^{-\frac{1}{2}\nu+\frac{1}{2}+\frac{1}{2}iz}\Gamma(\tfrac{1}{2}\nu + \tfrac{1}{2} - \tfrac{1}{2}iz)\zeta(1-iz)$$

constructed from the zeta function of order ν and character χ for the adelic plane. The space of parameter a contains the entire functions $F(z)$ such that

$$a^{-iz}F(z)$$

and

$$a^{-iz}F^*(z)$$

belong to the weighted Hardy space $\mathcal{F}(W)$. A Hilbert space of entire functions which satisfies the axioms (H1), (H2), and (H3) is obtained when a scalar product is introduced so that multiplication by a^{-iz} is an isometric transformation of the space into the space $\mathcal{F}(W)$. The weight function is an entire function of Pólya class such that $W(z-i)$ and $W^*(z)$ are linearly dependent when χ is not the principal character. The space is then a space $\mathcal{H}(E)$ with

$$E(z) = a^{iz}W(z)$$

when a is less than or equal to one.

The Mellin transformation of order ν and character χ for the adelic plane supplies information about the Sonine spaces of order ν and character χ for the adelic plane when χ is not the principal character. The Sonine space of parameter one is a space $\mathcal{H}(E)$ whose defining function

$$E(z) = (\pi/\rho)^{-\frac{1}{2}\nu-\frac{1}{2}+\frac{1}{2}iz}\Gamma(\tfrac{1}{2}\nu + \tfrac{1}{2} - \tfrac{1}{2}iz)\zeta(1 - iz)$$

is constructed from the zeta function of order ν and character χ for the adelic plane. The analytic weight function

$$W(z) = (\pi/\rho)^{-\frac{1}{2}\nu-\frac{1}{2}+\frac{1}{2}iz}\Gamma(\tfrac{1}{2}\nu + \tfrac{1}{2} - \tfrac{1}{2}iz)$$

is used to define the Sonine spaces of order ν for the Euclidean plane. Multiplication by the analytic function

$$\zeta(1 - iz)^{-1}$$

of z in the upper half–plane is an isometric transformation of the space $\mathcal{H}(E)$ into the space $\mathcal{F}(W)$. A maximal dissipative transformation in the weighted Hardy space $\mathcal{F}(W)$ is defined by taking $F(z)$ into $F(z + i)$ whenever $F(z)$ and $F(z + i)$ belong to the space. A maximal dissipative transformation is induced in the space $\mathcal{H}(E)$. The transformation takes $F(z)$ into $G(z + i)$ when a sequence of elements $H_n(z)$ of the weighted Hardy space exists such that $H_n(z + i)$ belongs to the space for every n, such that $\zeta(1 - iz)^{-1}G(z + i)$ is the limit of the functions $H_n(z + i)$ in the metric topology of the space, and such that $\zeta(1 - iz)^{-1}F(z)$ is the limit in the metric topology of the space of the orthogonal projections of the functions $H_n(z)$ in the image in the space of the space $\mathcal{H}(E)$. A closed dissipative relation is constructed in the space $\mathcal{H}(E)$. The maximal dissipative property of the relation is obtained from an approximate construction using properties of entire functions of Pólya class which are determined by their zeros. The existence of a maximal dissipative transformation is an application of the representation of elements of the space $\mathcal{H}(E)$ as Mellin transforms of order ν and character χ for the adelic plane. The maximal dissipative transformation in the space $\mathcal{H}(E)$ is unitarily equivalent to the adjoint of the Radon transformation of order ν and character χ for the adelic plane as it acts on functions $f(\xi)$ of ξ in the adelic plane which vanish when $|\xi| < 1$ and whose Hankel transform of orders ν and character χ for the adelic plane vanishes when $|\xi| < 1$.

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The manuscript will be revised until a full treatment of the applications of the Riemann hypothesis has been obtained. Since questions of research priority may arise concerning the proof of the Riemann hypothesis, a previous draft is preserved by the Department of Mathematics of Purdue University.

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