Species and Variations on the Theme of Species

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What are Discrete Structures?

The modern solution, ask Google:

Trees, graphs, functions, relations, permutations, cycles, lists, lattices, posets, automata, finite geometries, finite groups, . . .

These are structures of various “kind”, “sort”, or (as we will say) *species*.

Epistemological considerations

Following a slow (and forced) evolution in the history of mathematics, the modern notion of function (due to Dirichlet, 1837) has been made independent of any actual description format.

In the same spirit, it is natural to formalize the notion of "Species of Structures" to make it independent of any description format. This is why a functorial approach naturally comes into play. However we also want formulas.

Species of Structures

Let $\mathbb{B}$ be the category of finite sets with bijections. A *Species (of structures)* is simply a functor

$$F : \mathbb{B} \longrightarrow \mathbb{B}.$$

For a finite set A, an element $s \in F[A]$ is a *structures* of species F on A. We also say that s is a F-structure on A.

For a bijection $\varphi : A \to B$, we further say that

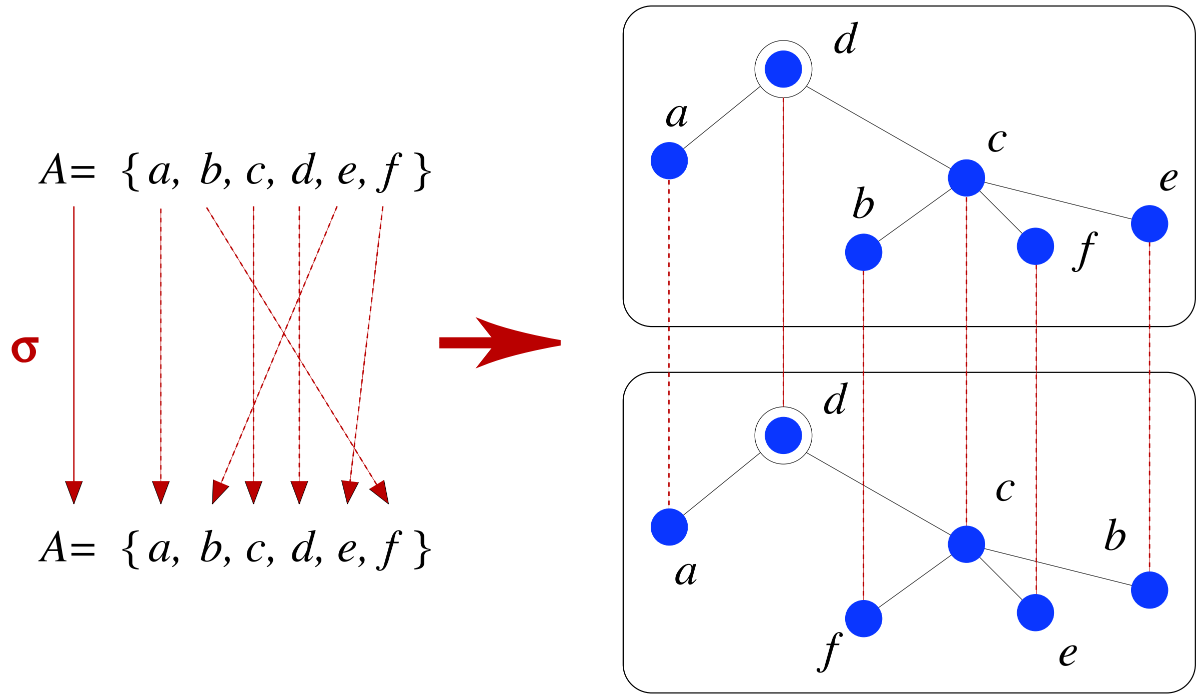
$$F[\varphi] : F[A] \longrightarrow F[B]$$

is the *transport* of F-structures along φ.

Examples of species

1. $\mathcal{P}[A]$. Structures $\pi \in \mathcal{P}[A]$ are *partitions* of A.
2. A^k. Structures are *k-tuples* of elements of A.
3. $\mathcal{G}[A] = \wp[A \times A]$. Structures are *directed graphs* with vertex set A.
4. $\mathcal{S}[A] = \{\ \sigma \mid \sigma : A \xrightarrow{\sim} A,\ \text{bijection}\ \}$. We sometime write S_A for the corresponding *permutation group*.
5. $\mathrm{End}[A] = \{\ f \mid f : A \to A\ \}$. These are the *endofunctions* on A.

Transport of structures along a bijection



Species

=

Action of permutation groups

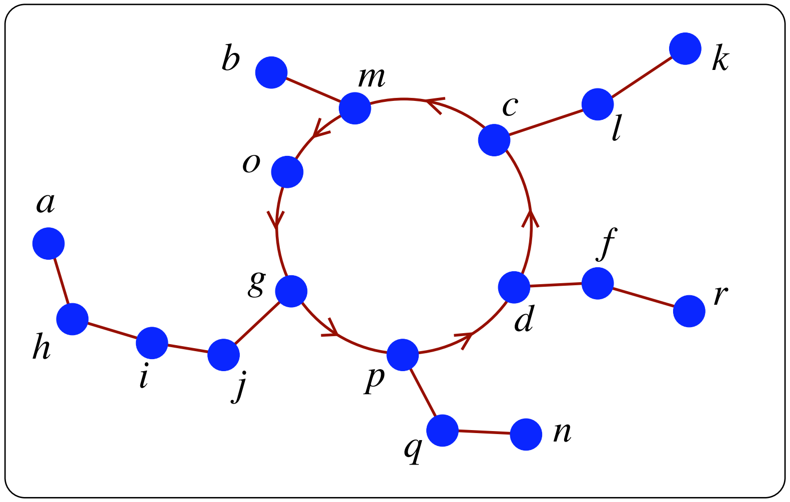
$$S_A \times F[A] \longrightarrow F[A]$$

Orbits are *type* of structures. In other words, a type is an equivalence class ($\in F[A]/\sim$) for the relation

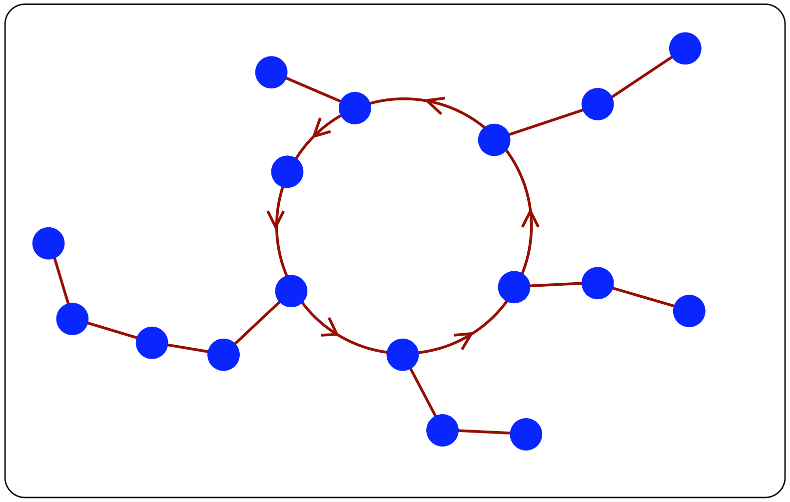
$$s \sim t \quad \Longleftrightarrow \quad F[\sigma](s) = t \quad \text{for some } \sigma \in S_A.$$

We then say that s and t are *isomorphic structures*.

Structures



Type of structures



Enumerative combinatorics

"Enumerative combinatorics is concerned with counting the number of elements of a finite set S." (R.P. Stanley)

He adds the caveat that elements of S will usually have some simple combinatorial definition.

In our setup this translates to counting elements of $F[A]$.

Observe that $\#F[A] = \#F[B]$, whenever A and B have the same number of elements.

Generating series

For each species F we associate the *(exponential) generating series*

$$F(x) := \sum_{n\geq 0} f_n \frac{x^n}{n!},$$

where f_n is the number of elements of $F[A]$, for any A with n elements.

Aim: "Find a simple expression (or identity) for $F(x)$."

Direct computation of associated series

1. X the species *singletons*, whose series is x.
2. E the species *set*, whose series is $\exp(x)$.
3. $\mathcal{S}$ the species *permutations*, whose series is $\dfrac{1}{1-x}$.
4. $\mathcal{E}$ the species *elements*, whose series is $x\,\exp(x)$.
5. $\mathcal{C}$ the species *cyclic permutations*, whose series is

$$\mathcal{C}(x) = \log \frac{1}{1-x}$$

The calculus of species

We introduce operations on species: $F + G$, $F \cdot G$, $F \circ G$, F', ..., so that

$$
\begin{aligned}
(F + G)(x) &= F(x) + G(x) \\
(F \cdot G)(x) &= F(x)\, G(x), \\
(F \circ G)(x) &= F(G(x)), \\
F'(x) &= \frac{d}{dx} F(x), \\
&\vdots
\end{aligned}
$$

Equality of species

We simply write $F = G$, whenever there exists a invertible natural transformation from F to G.

In other words, for any A there is a *natural bijection* between F-structures on A, and G-structures on A. Thus we can make sense of identities such as

$$F \cdot (G + H) = F \cdot G + F \cdot H$$

or

$$(F \cdot G)' = F' \cdot G + F \cdot G'$$

Definition of operations

1. $(F + G)[A] := F[A] + G[A]$, with "$+$" (and "$\Sigma$") denoting disjoint union.

2. $(F \cdot G)[A] := \sum_{B+C=A} F[B] \times G[C]$.

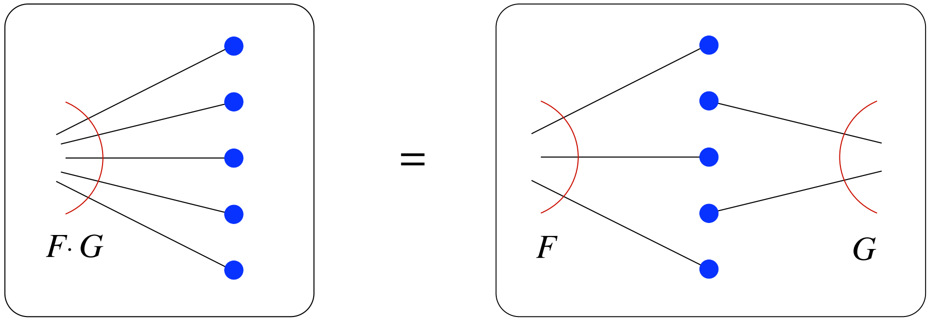
3. When $G[\emptyset] = \emptyset$, $\quad (F(G)$ same as $F \circ G)$.

$$(F \circ G)[A] := \sum_{\pi \in \mathcal{P}[A]} F[\pi] \times \prod_{B \in \pi} G[B].$$

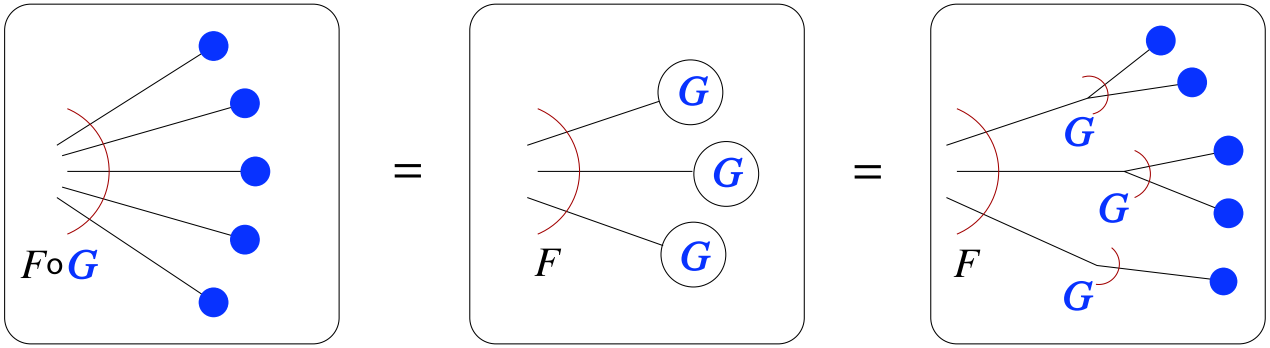
4. $F'[A] := F[A + \{*\}]$.

Generic structures for operations

2. Product:



3. Substitution:



Using the calculus of species

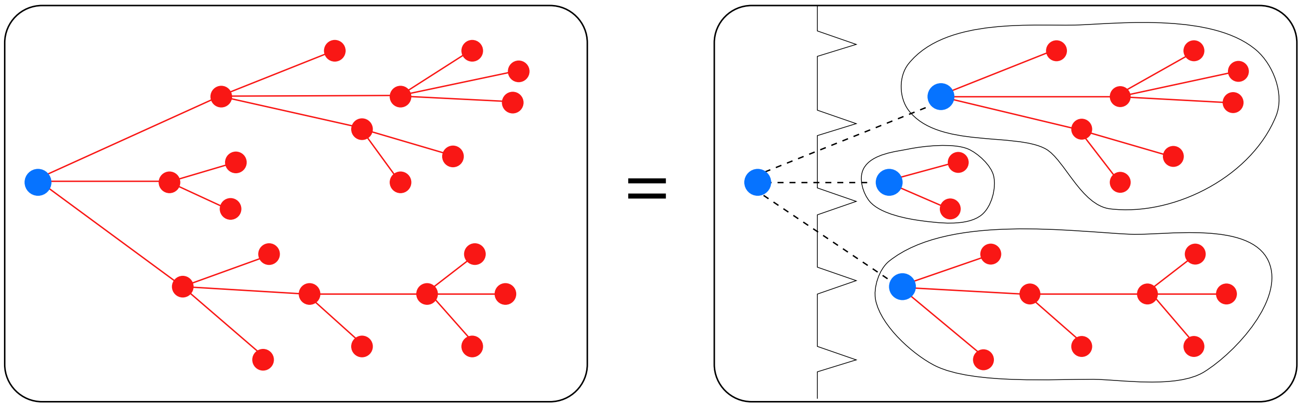
We translate combinatorial decompositions into algebraic identities on species. In particular, this allows implicit definitions.

For example, the equalities

$$\mathcal{S} = E(\mathcal{C}), \qquad \mathcal{A} = X \cdot E(\mathcal{A}),$$
$$E = 1 + E^{+}, \quad \mathcal{P} = E(E^{+}).$$

are almost self evident if one “reads them out loud” in the right manner.

Rooted trees



$$\mathcal{A} = X \cdot E(\mathcal{A}).$$

This can be interpreted various ways.

First variations

It is straightforward to pass to the notion of species in several sorts

$$F : \mathbb{B}^k \longrightarrow \mathbb{B},$$

even with parameters

$$F : \mathbb{B}^k \longrightarrow \mathbb{B}_R,$$

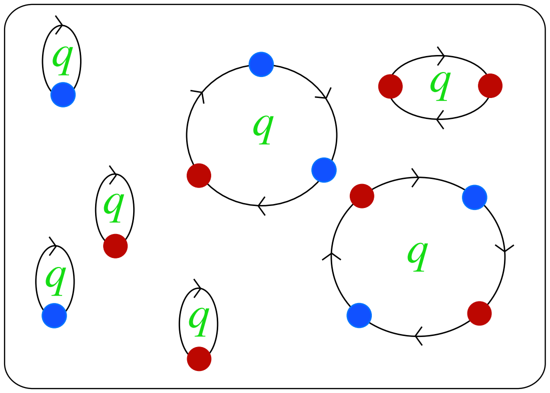
where $\mathbb{B}_R$ is the category of R-weighted finite sets, for R a ring of formal power series.

Remark. *Our criteria for introducing a variation will be to satisfy a clear combinatorial needs.*

Illustration

The species $\mathcal{S}_w(X, Y) = E(q\,\mathcal{C}(X + Y))$ of permutations σ of two sort of elements, weighted by $q^{c(\sigma)}$ (where $c(\sigma)$ is the number of cycles of σ), has generating series

$$\mathcal{S}_w(x, y) = \left(\frac{1}{1 - (x + y)}\right)^q$$



Second variation: tensorial species

A *tensorial species* is a functor

$$F : \mathbb{B} \longrightarrow \mathbb{V},$$

where $\mathbb{V}$ is the category of (finite dimensional) vector spaces over $\mathbb{C}$. This corresponds "essentially" to a family of *linear representations*

$$\mathcal{S}_n \times F[n] \longrightarrow F[n], \qquad (n \geq 0).$$

For example, we can have

$$\mathrm{sign}[A] := \mathbb{C} \quad \text{and} \quad \mathrm{sign}[\sigma](z) := (-1)^{\ell(\sigma)} z.$$

Tensorial operations

1. $(F + G)[A] := F[A] \oplus G[A]$.

2. $(F \cdot G)[A] := \bigoplus_{B+C=A} F[B] \otimes G[C]$.

3. When $G[\emptyset] = \emptyset$, $\quad (F(G) = F \circ G)$.

$$(F \circ G)[A] := \bigoplus_{\pi \in \mathcal{P}[A]} F[\pi] \otimes \bigotimes_{B \in \pi} G[B].$$

4. $F'[A] := F[A + \{*\}]$.

From species to tensorial species

Denote $\mathbb{C}A$ the free vector space generated by a finite set A. To any species F, we naturally associate a tensorial species by setting

$$(\mathbb{C}F)[A] := \mathbb{C}\,F[A].$$

Clearly we get an *operation preserving functor* $\mathbb{C}(-)$, from the category $\mathbb{B}^{\mathbb{B}}$ of species, to the category $\mathbb{V}^{\mathbb{B}}$ of tensorial species. We sometime omit the $\mathbb{C}$ to simply denote F the resulting tensorial species.

Frobenius transform of the character of a tensorial species

As usual, the *character* of F is defined, for $\sigma \in \mathcal{S}_n$, as

$$\chi_F(\sigma) = \text{Trace } F[\sigma].$$

Let $d_k = d_k(\sigma)$ denote the *number of cycles of size* k in the decomposition of σ in disjoint cycles. Then we set

$$Z_F := \sum_{n \geq 0} \frac{1}{n!} \sum_{\sigma \in \mathcal{S}_n} \chi_F(\sigma)\, p_1^{d_1} p_2^{d_2} p_3^{d_3} \cdots.$$

One can consider the p_k's to be independent variables.

Cycle index series

If F is in $\mathbb{B}^{\mathbb{B}}$, then we can make sense of Z_F by considering F as a tensorial species. In this case, for $\sigma \in S_n$, we have

$$\chi_F(\sigma) = \#\{\, s \in F[n] \mid F[\sigma](s) = s\},$$

and Z_F is traditionally called the *cycle index series* of F.

Specializations of Z_F

There are many interesting specializations of Z_F. One example is

$$Z_F(x, 0, 0, \ldots) = \sum_{n \geq 0} \dim(F[n]) \frac{x^n}{n!}.$$

Another, in the case when F is in $\mathbb{B}^{\mathbb{B}}$, is

$$Z_F(x, x^2, x^3, \ldots) = \sum_{n \geq 0} \#(F[n]/\sim)\, x^n.$$

This is essentially Pólya's Theory.

The calculus of Z_F

We have

$$\begin{aligned} Z_{F+G} &= Z_F + Z_G \\ Z_{F\cdot G} &= Z_F\, Z_G, \\ Z_{F\circ G} &= Z_F \circ Z_G, \\ Z_{F'} &= \frac{d}{dp_1} Z_F, \end{aligned}$$

where

$$Z_F \circ Z_G := (Z_F \mid_{p_k \leftarrow g_k}), \qquad \text{with} \qquad g_k = (Z_G \mid_{p_j \leftarrow p_{kj}})$$

Sample of formulas

1. $Z_E = \exp(\sum_{k\geq 1} p_k/k)$.
2. $Z_{\text{sign}} = \exp(\sum_{k\geq 1}(-1)^{k-1} p_k/k)$.
3. $Z_{\mathcal{S}} = \displaystyle\prod_{k\geq 1} \frac{1}{1-p_k}$.
4. $Z_{\mathcal{C}} = \displaystyle\sum_{k\geq 1} \frac{\varphi(k)}{k} \log \frac{1}{1-p_k}$.

Polynomial functors

A functor

$$P : \mathbb{V} \longrightarrow \mathbb{V}$$

is said to be *polynomial*, if

$$P : \mathrm{Hom}(X, Y) \longrightarrow \mathrm{Hom}(X, Y), \quad X, Y \in \mathbb{V},$$

is a polynomial in the following sense. For all $f_i : X \to Y$,

$$P(\lambda_1 f_1 + \ldots + \lambda_r f_r)$$

is a polynomial function in the λ_k's with coefficients in $\mathrm{Hom}(P(X), P(Y))$ (depending on the f_k's).

Polynomial functor = polynomial tensorial species

This means that $P : \mathbb{V} \to \mathbb{V}$ is polynomial if and only if it can be written as

$$P(X) = \bigoplus_{n \geq 0} F[n] \underset{\mathbb{C}\mathcal{S}_n}{\otimes} X^{\otimes n}$$

for some polynomial (finite support) tensorial species F.

We say that P is *homogeneous* of degree n if

$$P(\lambda_1 f_1 + \ldots + \lambda_r f_r)$$

is homogeneous of degree n.

A glimpse of algebraic combinatorics

A representation

$$\rho : GL_m \longrightarrow GL_k$$

is said to be *polynomial* if the entries of $\rho(M)$ are polynomials in the entries of M. They are classified by polynomial functors. Moreover, "irreducible" polynomial representations (that are homogeneous of degree n) correspond to "irreducible" representations of S_n.

Back to Z_F

Let $(\mathbf{t})$ denote the diagonal matrix with entries $\mathbf{t} = t_1, t_2, \ldots, t_m$. For

$$P(X) = \bigoplus_{n \geq 0} F[n] \underset{\mathbb{C}\mathcal{S}_n}{\otimes} X^{\otimes n},$$

we have

$$\text{Trace } P((\mathbf{t})) = Z_F,$$

with $p_k = t_1^k + t_2^k + \ldots + t_m^k$. The Z_F are symmetric polynomials in the t_i's. Irreducible representations correspond to Schur polynomials.

Other variations

Let $\mathbb{A}$ be a groupoid with a comonoidal structure on the corresponding free additive category, with nice features.

In other words, we want to have a "good" notion of *dissection* for objects in $\mathbb{A}$.

Let also $\mathbb{K}$ be a (semi-)ring (two monoidal structures). Then consider functors

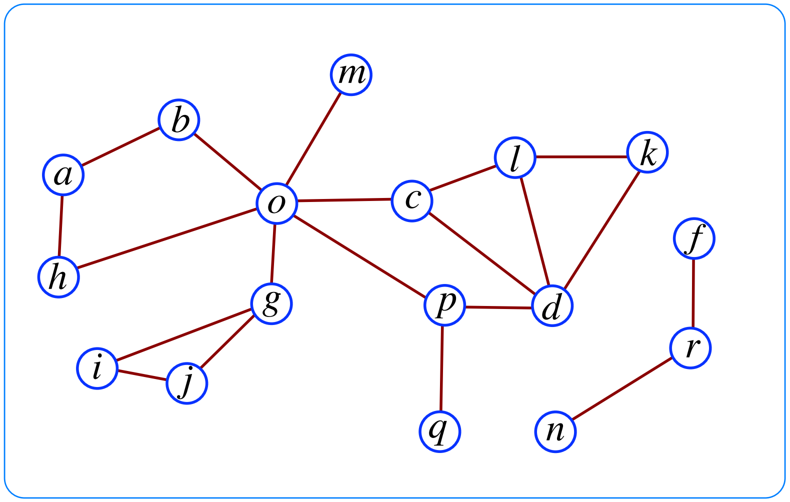
$$F : \mathbb{A} \longrightarrow \mathbb{K},$$

with operations (at least the sum and product) as before.

Illustration: graphical species

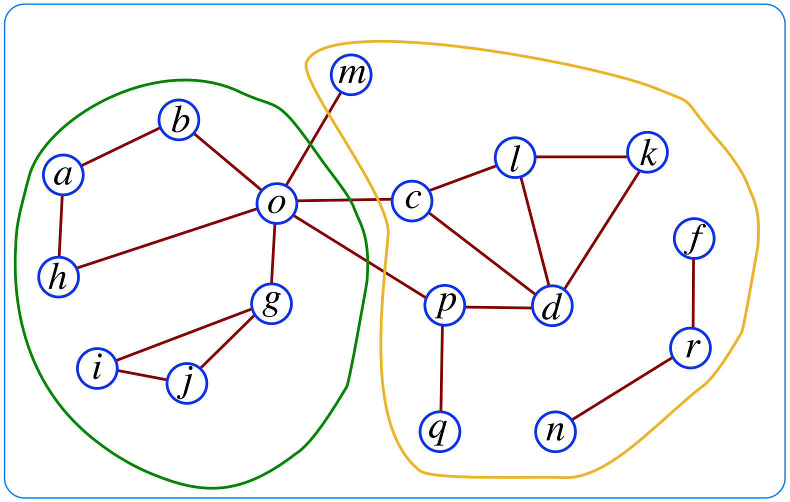
Let $\mathbb{G}$ be the category of graphs on totally ordered finite sets, with isomorphisms as arrows.

An object A of $\mathbb{G}$:



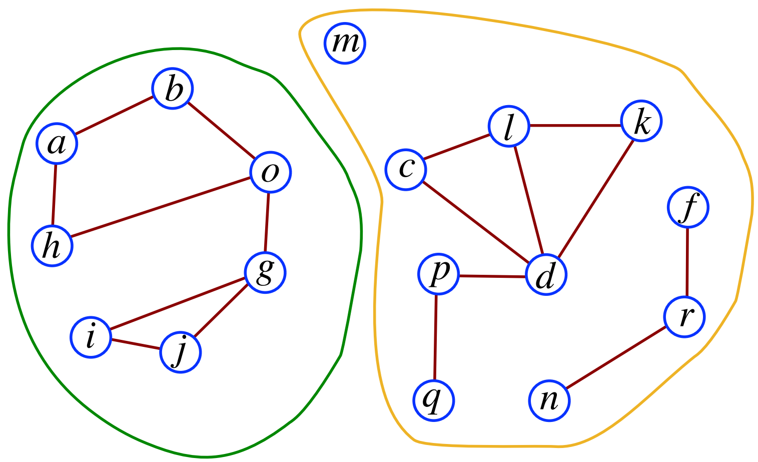
Dissection of graphs

Selecting a subset (and its complement):



Dissection of graphs

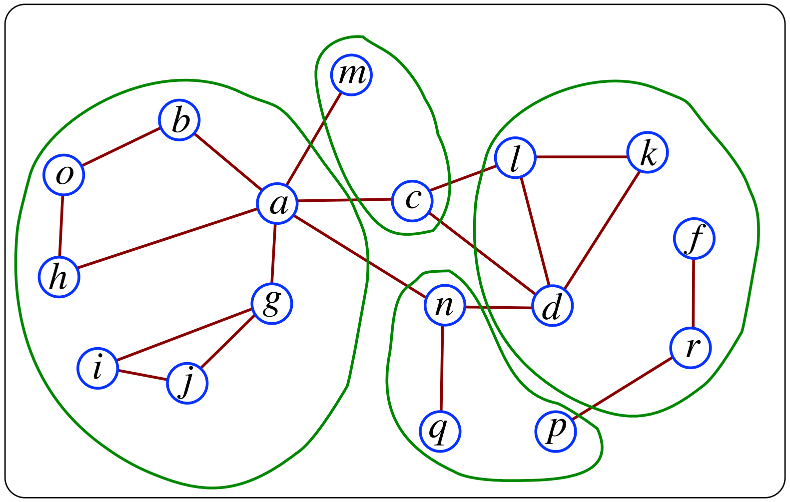
The resulting dissection, (B, C):



obtained by removing connections between B and C.

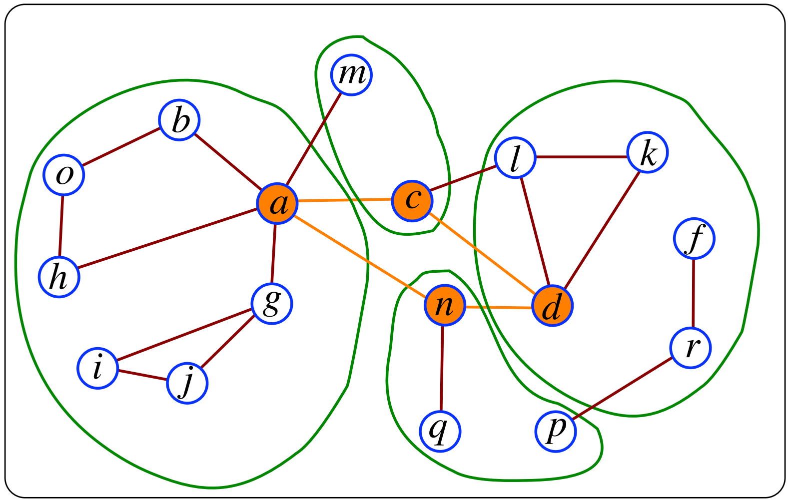
Partitions of graphs

A partition π of a graph:



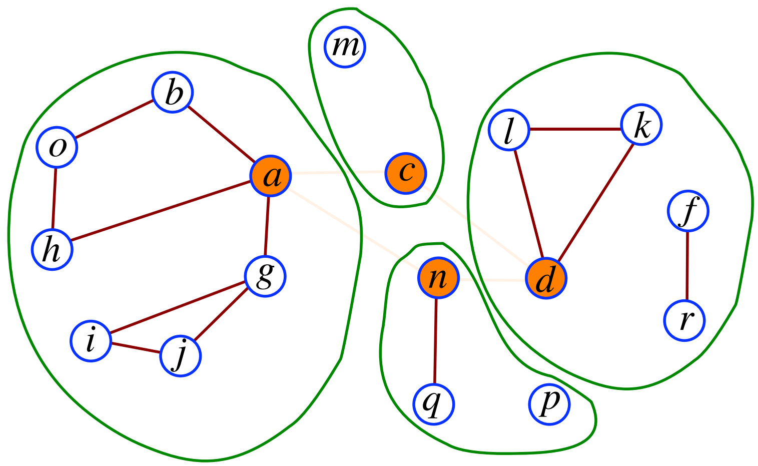
Partitions of graphs

Minimal elements of each block:



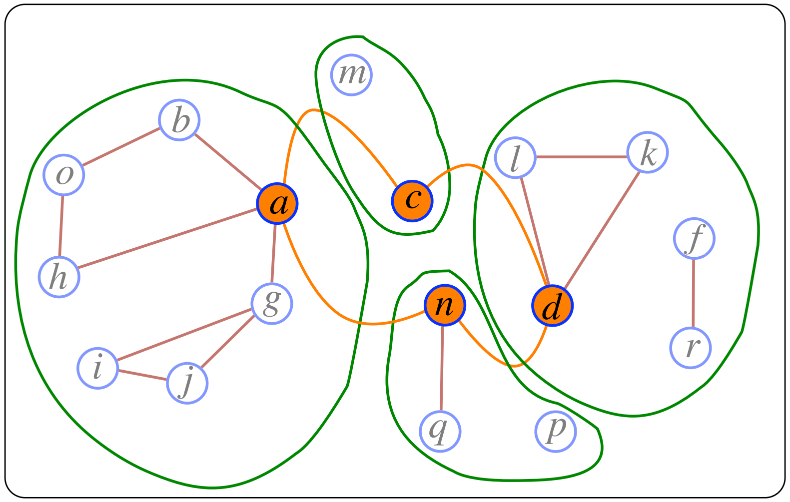
Partitions of graphs

Block graphs $B \in \pi$:



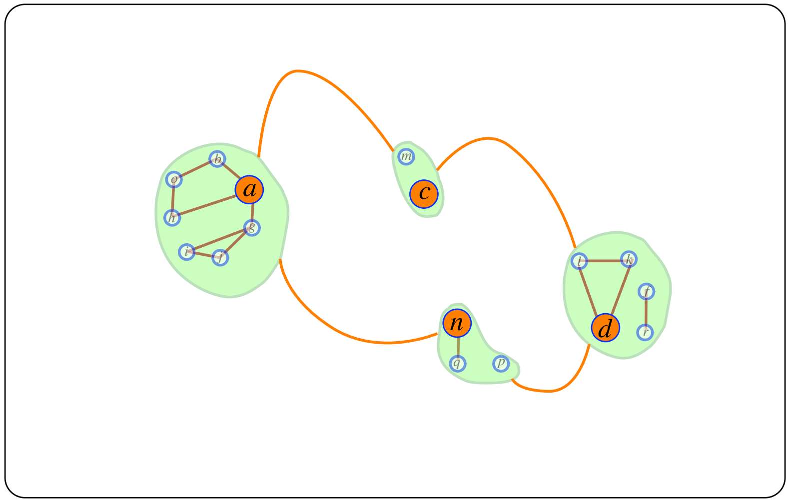
Partitions of graphs

Quotient graph π :



Partitions of graphs

Quotient graph π :



The vertices are the blocks of the partition.

Graphical generating series

For each graphical species F we associate the *(graphical) generating series*

$$F(x) := \sum_{n \geq 0} \sum_{A} f_A \frac{x^n}{2^{\binom{n}{2}} n!},$$

where f_A is the number of elements of $F[A]$, for graphs A on the vertex set $\{1, 2, \ldots, n\}$.

Aim: "Find a simple expression (or identity) for $F(x)$."

Definition of operations

1. $(F + G)[A] := F[A] + G[A]$, with "+" denoting disjoint union.

2. $(F \cdot G)[A] := \sum_{(B,C)} F[B] \times G[C]$.

3. When $G[\emptyset] = \emptyset$, $\qquad (F(G) = F \circ G)$.

$$(F \circ G)[A] := \sum_{\pi} F[\pi] \times \prod_{B \in \pi} G[B].$$

A theorem

For graphical species F and G, we have

$$\begin{aligned}(F+G)(x) &= F(x)+G(x)\\ (F\cdot G)(x) &= F(x)\,G(x),\\ (F\circ G)(x) &= F(G(x)).\end{aligned}$$

Corollary. *Positive integer graphical generating series are closed under product and substitution.*

Other variants

1. Linear species.
2. Permutational species.
3. Partitionnal species.
4. Species with values in the category of G-sets.
5. Species with values in the category of varieties over a finite field.

A sample of recent work involving the use of species

1. M. Guţă and H. Maassen, *Symmetric Hilbert spaces arising from species of structures*, Math. Zeit., 2002.
2. S. Mahajan, *Symplectic Operad Geometry and Graph Homology*, math.QA/0211464, 2002.
3. A. Henderson, *Representations of Wreath Products on Cohomology of De Concini-Procesi Compactifications*, Int. Math. Res., 2004.
4. M. Fiore, N. Gambino, and M. Hyland, *Generalised Species of Structures and Analytic Functors: Cartesian Closed Structure*, Preprint, 2004.