14.10. Lines of Curvature, Geodesic Torsion, Asymptotic Lines

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Given a surface X, certain curves on the surface play a special role, for example, the curves corresponding to the directions in which the curvature is maximum or minimum.

Definition 14.10.1 Given a surface X, a *line of curvature* is a curve $C\colon t \mapsto X(u(t), v(t))$ on X defined on some open interval I, and having the property that for every $t \in I$, the tangent vector $C'(t)$ is collinear with one of the principal directions at $X(u(t), v(t))$.

Note that we are assuming that no point on a line of curvature is either a planar point or an umbilical point, since principal directions are undefined as such points.

The differential equation defining lines of curvature can be found as follows:

Remember from lemma 14.8.2 of Section 14.8 that the principal directions are the eigenvectors of $d\mathbf{N}_{(u,v)}$.

Therefore, we can find the differential equation defining the lines of curvature by eliminating κ from the two equations from the proof of lemma 14.8.2:

$$\frac{MF - LG}{EG - F^2}u' + \frac{NF - MG}{EG - F^2}v' = -\kappa u',$$
$$\frac{LF - ME}{EG - F^2}u' + \frac{MF - NE}{EG - F^2}v' = -\kappa v'.$$

It is not hard to show that the resulting equation can be written as

$$\det\begin{pmatrix} (v')^2 & -u'v' & (u')^2 \\ E & F & G \\ L & M & N \end{pmatrix} = 0.$$

From the above equation, we see that the u-lines and the v-lines are the lines of curvatures iff $F = M = 0$.

Generally, this differential equation does not have closed-form solutions.

There is another notion which is useful in understanding lines of curvature, the geodesic torsion.

Let $C: s \mapsto X(u(s), v(s))$ be a curve on X assumed to be parameterized by arc length, and let $X(u(0), v(0))$ be a point on the surface X, and assume that this point is neither a planar point nor an umbilic, so that the principal directions are defined.

We can define the orthonormal frame $(\overrightarrow{e_1}, \overrightarrow{e_2}, \mathbf{N})$, known as the *Darboux frame*, where $\overrightarrow{e_1}$ and $\overrightarrow{e_2}$ are unit vectors corresponding to the principal directions, $\mathbf{N}$ is the normal to the surface at $X(u(0), v(0))$, and $\mathbf{N} = \overrightarrow{e_1} \times \overrightarrow{e_2}$.

It is interesting to study the quantity $\dfrac{d\mathbf{N}_{(u,v)}}{ds}(0)$.

If $\overrightarrow{t} = C'(0)$ is the unit tangent vector at $X(u(0), v(0))$, we have another orthonormal frame considered in Section 14.4, namely $(\overrightarrow{t}, \overrightarrow{n_g}, \mathbf{N})$, where $\overrightarrow{n_g} = \mathbf{N} \times \overrightarrow{t}$, and if φ is the angle between $\overrightarrow{e_1}$ and $\overrightarrow{t}$ we have

$$\begin{aligned} \overrightarrow{t} &= \cos\varphi\, \overrightarrow{e_1} + \sin\varphi\, \overrightarrow{e_2}, \\ \overrightarrow{n_g} &= -\sin\varphi\, \overrightarrow{e_1} + \cos\varphi\, \overrightarrow{e_2}. \end{aligned}$$

Lemma 14.10.2 *Given a curve $C: s \mapsto X(u(s), v(s))$ parameterized by arc length on a surface X, we have*

$$\frac{d\mathbf{N}_{(u,v)}}{ds}(0) = -\kappa_N \overrightarrow{t} + \tau_g \overrightarrow{n_g},$$

where κ_N is the normal curvature, and where the geodesic torsion τ_g is given by

$$\tau_g = (\kappa_1 - \kappa_2) \sin\varphi \cos\varphi.$$

From the formula

$$\tau_g = (\kappa_1 - \kappa_2) \sin\varphi \cos\varphi,$$

since φ is the angle between the tangent vector to the curve C and a principal direction, it is clear that the lines of curvatures are characterized by the fact that $\tau_g = 0$.

One will also observe that orthogonal curves have opposite geodesic torsions (same absolute value and opposite signs).

If $\overrightarrow{n}$ is the principal normal, τ is the torsion of C at $X(u(0), v(0))$, and θ is the angle between $\mathbf{N}$ and $\overrightarrow{n}$ so that $\cos\theta = \mathbf{N} \cdot \overrightarrow{n}$, we claim that

$$\tau_g = \tau - \frac{d\theta}{ds},$$

which is often known as *Bonnet's formula*.

Lemma 14.10.3 *Given a curve $C: s \mapsto X(u(s), v(s))$ parameterized by arc length on a surface X, the geodesic torsion τ_g is given by*

$$\tau_g = \tau - \frac{d\theta}{ds} = (\kappa_1 - \kappa_2) \sin\varphi \cos\varphi,$$

where τ is the torsion of C at $X(u(0), v(0))$, and θ is the angle between $\mathbf{N}$ and the principal normal $\overrightarrow{n}$ to C at $s = 0$.

Note that the geodesic torsion only depends on the tangent of curves C. Also, for a curve for which $\theta = 0$, we have $\tau_g = \tau$.

Such a curve is also characterized by the fact that the geodesic curvature κ_g is null.

As we will see shortly, such curves are called geodesics, which explains the name geodesic torsion for τ_g.

Lemma 14.10.3 can be used to give a quick proof of a beautiful theorem of Dupin (1813).

Dupin's theorem has to do with families of surfaces forming a triply orthogonal system.

Given some open subset U of $\mathbb{E}^3$, three families $\mathcal{F}_1, \mathcal{F}_2, \mathcal{F}_3$ of surfaces form a *triply orthogonal system* for U, if for every point $p \in U$, there is a unique surface from each family $\mathcal{F}_i$ passing through p, where $i = 1, 2, 3$, and any two of these surfaces intersect orthogonally along their curve of intersection.

Theorem 14.10.4 *The surfaces of a triply orthogonal system intersect each other along lines of curvature.*

A nice application of theorem 14.10.4 is that it is possible to find the lines of curvature on an ellipsoid.

Indeed, a system of confocal quadrics is triply orthogonal! (see Berger and Gostiaux [?], Chapter 10, Sections 10.2.2.3, 10.4.9.5, and 10.6.8.3, and Hilbert and Cohn-Vossen [?], Chapter 4, Section 28).

We now turn briefly to asymptotic lines. Recall that asymptotic directions are only defined at points where $K < 0$, and at such points, they correspond to the directions for which the normal curvature κ_N is null.

Definition 14.10.5 Given a surface X, an *asymptotic line* is a curve $C\colon t \mapsto X(u(t), v(t))$ on X defined on some open interval I where $K < 0$, and having the property that for every $t \in I$, the tangent vector $C'(t)$ is collinear with one of the asymptotic directions at $X(u(t), v(t))$.

The differential equation defining asymptotic lines is easily found since it expresses the fact that the normal curvature is null:

$$L(u')^2 + 2M(u'v') + N(v')^2 = 0.$$

Such an equation generally does not have closed-form solutions.

Note that the u-lines and the v-lines are asymptotic lines iff $L = N = 0$ (and $F \neq 0$).

Perseverant readers are welcome to compute E, F, G, L, M, N for the Enneper surface:

$$x = u - \frac{u^3}{3} + uv^2$$
$$y = v - \frac{v^3}{3} + u^2 v$$
$$z = u^2 - v^2.$$

Then, they will be able to find closed-form solutions for the lines of curvatures and the asymptotic lines.

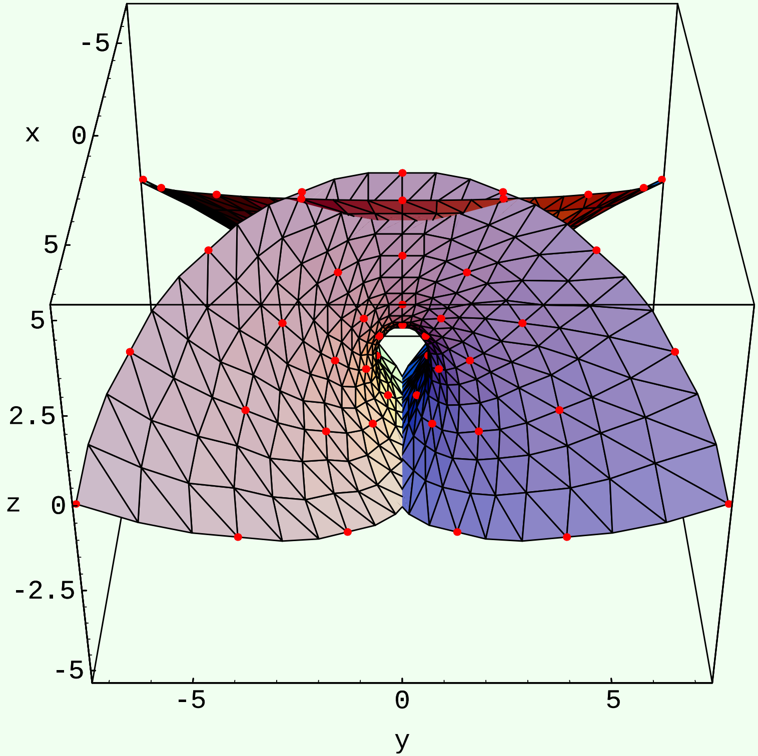

Figure 14.6: the Enneper surface

Parabolic lines are defined by the equation

$$LN - M^2 = 0,$$

where $L^2 + M^2 + N^2 > 0$.

In general, the locus of parabolic points consists of several curves and points.

For fun, the reader should look at Klein's experiment as described in Hilbert and Cohn-Vossen [**?**], Chapter IV, Section 29, page 197.

We now turn briefly to geodesics.