Some Holomorphic Functions connected with the Collatz Problem

Toshio URATA

Department of Mathematics Education, Aichi University of Education, Kariya 448-8542, Japan

1 Collatz problem

In this paper, we consider some holomorphic functions connected with the Collatz problem. The Collatz problem(or conjecture) is well known under the name $3n+1$ problem:
Take any positive integer n. If n is even, replace it by $\frac{n}{2}$; if n is odd, replace it by $3n+1$. Show that after finitely many such steps, this process reaches the number 1.

We bring odd numbers into focus. For non-zero integers, we define a function ϕ as follows:
For a non-zero even integer n, $\phi(n)$ is a unique odd integer m such that

$$n = 2^k m \quad (\exists k = 1, 2, \cdots),$$

for an odd integer n, $\phi(n)$ is a unique odd integer m such that

$$3n + 1 = 2^k m \quad (\exists k = 1, 2, \cdots).$$

Let $O\boldsymbol{Z}$ be the set of all odd integers. Then the above function maps

$$\phi : \ \boldsymbol{Z} \setminus \{0\} \quad \rightarrow \quad O\boldsymbol{Z} \ (\ \subset \ \boldsymbol{Z} \setminus \{0\}),$$

and we call the restriction (to $O\boldsymbol{Z}$)

$$\phi : \ O\boldsymbol{Z} \quad \rightarrow \quad O\boldsymbol{Z}$$

the *Collatz function*.
The Collatz problem asserts that, for each positive odd integer n, we have a positive integer k such that $\phi^k(n) = 1$.

Now, we can easily compute the inverse image of the Collatz function $\phi : \ O\boldsymbol{Z} \quad \rightarrow \quad O\boldsymbol{Z}$.

Fact 1 For $\forall k \in \boldsymbol{Z}$, we have

$$\begin{cases} \phi^{-1}(6k+1) = \{4^n(8k+1) + 4^{n-1} + \cdots + 4 + 1 \in O\boldsymbol{Z} \mid n = 0, 1, 2, \cdots\} \\ \phi^{-1}(6k+3) = \emptyset \\ \phi^{-1}(6k+5) = \{4^n(4k+3) + 4^{n-1} + \cdots + 4 + 1 \in O\boldsymbol{Z} \mid n = 0, 1, 2, \cdots\}. \end{cases}$$

2 Some holomorphic functions on $\boldsymbol{C}$

Here, we construct some holomorphic functions which agree on all positive odd integers with the Collatz function ϕ. We define a meromorphic function

$$G(z) := \sum_{m=0}^{\infty} \phi(2m+1)\Big(\frac{1}{(z-2m-1)^2} - \frac{1}{(2m+1)^2}\Big) \quad \text{in } \boldsymbol{C},$$

and a holomorphic function

$$F(z) := \Big(\frac{4}{\pi^2}\cos^2\frac{\pi z}{2}\Big)G(z) \quad \text{on } \boldsymbol{C}.$$

From Fact 1, we see easily that, for all $z \in \boldsymbol{C}$,

$$\begin{aligned} G(z) &= \sum_{k=0}^{\infty}\sum_{n=0}^{\infty}(6k+1)\Big(\frac{1}{(z-(4^n(8k+1)+4^{n-1}+\cdots+4+1))^2} \\ &\qquad - \frac{1}{(4^n(8k+1)+4^{n-1}+\cdots+4+1)^2}\Big) \\ &\quad + \sum_{k=0}^{\infty}\sum_{n=0}^{\infty}(6k+5)\Big(\frac{1}{(z-(4^n(4k+3)+4^{n-1}+\cdots+4+1))^2} \\ &\qquad - \frac{1}{(4^n(4k+3)+4^{n-1}+\cdots+4+1)^2}\Big). \end{aligned}$$

This meromorphic function $G(z)$ has the Laurent expansion

$$\begin{aligned} G(z) = \frac{\phi(2n+1)}{(z-2n-1)^2} - \frac{\phi(2n+1)}{(2n+1)^2} + \sum_{n\neq m\geq 0}\phi(2m+1)\Big(\frac{1}{4(m-n)^2} - \frac{1}{(2m+1)^2}\Big) \\ + \sum_{k\geq 1}\frac{(k+1)}{2^{k+2}}\Big(\sum_{n\neq m\geq 0}\frac{\phi(2m+1)}{(m-n)^{k+2}}\Big)(z-2n-1)^k \end{aligned}$$

around $2n+1$ in $\boldsymbol{C}$ $(n = 0, 1, \cdots)$, and

$$F(z) = \frac{2}{\pi^2}(1-\cos\pi(z-2n-1))G(z) \quad \text{on } \boldsymbol{C}.$$

Fact 2 For the entire function $F(z)$, we have the following:

(1) $F(2n+1) = \phi(2n+1)$ for each non-negative integer n, and
$F(2n+1) = 0$ for each negative integer n.

(2) $F'(2n+1) = 0$ for each integer n.

Now, we also define

$$K_p(z) := \frac{4^p}{\pi^{2p}} \cos^{2p} \frac{\pi z}{2} \sum_{n=0}^{\infty} \frac{\phi(2n+1)}{(z-2n-1)^{2p}} \qquad \text{on } \boldsymbol{C} \qquad (p = 2, 3, \cdots)$$

and

$$L_p(z) := -\frac{4^p}{\pi^{2p+1}} \cos^{2p} \frac{\pi z}{2} \sin \pi z \sum_{n=0}^{\infty} \frac{\phi(2n+1)}{(z-2n-1)^{2p+1}} \qquad \text{on } \boldsymbol{C} \qquad (p = 1, 2, \cdots).$$

Fact 3 The following identities hold:

(1) $K_p(2n+1) = L_p(2n+1) = L_1(2n+1) = \phi(2n+1) \qquad (n = 0, 1, 2, \cdots,\ p = 2, 3, \cdots)$, and
$K_p(2n+1) = L_p(2n+1) = L_1(2n+1) = 0 \qquad (n = -1, -2, \cdots,\ p = 2, 3, \cdots)$.

(2) $K_p'(2n+1) = L_p'(2n+1) = L_1'(2n+1) = 0 \qquad (n \in \boldsymbol{Z},\ p = 2, 3, \cdots)$.

And we have

$$\begin{cases} (\sin \pi z) F'(z) - \pi(\cos \pi z - 1) F(z) = 2\pi L_1(z) \\ \\ (1 + \cos \pi z) F''(z) + 2\pi(\sin \pi z) F'(z) + (2 - \cos \pi z)\pi^2 F(z) = 3\pi^2 K_2(z) \end{cases} \qquad (\forall z \in \boldsymbol{C}).$$

We note that the function $y(z) = \cos \pi z + 1$ on $\boldsymbol{C}$ satisfies the differential equations

$$\begin{cases} (\sin \pi z) y'(z) - \pi(\cos \pi z - 1) y(z) = 0 \\ \\ (1 + \cos \pi z) y''(z) + 2\pi(\sin \pi z) y'(z) + (2 - \cos \pi z)\pi^2 y(z) = 0. \end{cases}$$

Further we have, for all $z \in \boldsymbol{C}$,

$$\begin{cases} (\sin \pi z) K_p'(z) - p\pi(\cos \pi z - 1) K_p(z) = 2p\pi L_p(z) \qquad (p = 2, 3, \cdots) \\ \\ (\sin \pi z) L_p'(z) - \pi((p+1)\cos \pi z - p) L_p(z) = -\frac{(2p+1)\pi}{2}(\cos \pi z - 1) K_{p+1}(z) \\ \qquad (p = 1, 2, \cdots). \end{cases}$$

The function $y(z) = (\cos \pi z + 1)^p$ on $\boldsymbol{C}$ satisfies the differential equations

$$\begin{cases} (\sin \pi z) y'(z) - p\pi(\cos \pi z - 1) y(z) = 0 \\ \\ (\sin \pi z) y''(z) - \pi(p \cos \pi z - p + 1) y'(z) = 0 \qquad (p = 1, 2, \cdots). \end{cases}$$

3 Attractive fixed points and the Fatou set

Here we state some propositions concerning the Fatou set and the Julia set of the entire function $F(z)$.

First, for $\forall\ x < 0$ in $\boldsymbol{R}$ we find that

$$\begin{aligned}
0 > G(x) &= \sum_{m=0}^{\infty} \phi(4m+1)\Big(\frac{1}{(x-4m-1)^2} - \frac{1}{(4m+1)^2}\Big) \\
&+ \sum_{m=0}^{\infty} \phi(4m+3)\Big(\frac{1}{(x-4m-3)^2} - \frac{1}{(4m+3)^2}\Big) \\
&> \frac{1}{(1-x)^2} - 1 + \frac{5}{(3-x)^2} - \frac{5}{3^2} + \frac{1}{(5-x)^2} - \frac{1}{5^2} + \frac{11}{(7-x)^2} - \frac{11}{7^2} \\
&\quad + \frac{7}{(9-x)^2} - \frac{7}{9^2} + \frac{17}{(11-x)^2} - \frac{17}{11^2} \\
&\quad\quad -\frac{3}{16}\log\frac{(9-x)}{9} - \frac{7}{36} + \frac{7}{4(9-x)} \\
&\quad\quad\quad -\frac{3}{8}\log\frac{(11-x)}{11} - \frac{17}{44} + \frac{17}{4(11-x)},
\end{aligned}$$

because

$$\begin{aligned}
0 > &\sum_{m=3}^{\infty} \phi(4m+1)\Big(\frac{1}{(x-4m-1)^2} - \frac{1}{(4m+1)^2}\Big) \\
&+ \sum_{m=3}^{\infty} \phi(4m+3)\Big(\frac{1}{(x-4m-3)^2} - \frac{1}{(4m+3)^2}\Big) \\
&> \sum_{m=3}^{\infty} (3m+1)\Big(\frac{1}{(x-4m-1)^2} - \frac{1}{(4m+1)^2}\Big) \\
&\quad + \sum_{m=3}^{\infty} (6m+5)\Big(\frac{1}{(x-4m-3)^2} - \frac{1}{(4m+3)^2}\Big) \\
&\quad > \int_2^{\infty} \Big(\frac{3t+1}{(x-4t-1)^2} - \frac{3t+1}{(4t+1)^2}\Big)\,dt \\
&\quad\quad + \int_2^{\infty} \Big(\frac{6t+5}{(x-4t-3)^2} - \frac{6t+5}{(4t+3)^2}\Big)\,dt \\
&\quad\quad\quad = -\frac{3}{16}\log\frac{(9-x)}{9} - \frac{7}{36} + \frac{7}{4(9-x)} \\
&\quad\quad\quad\quad -\frac{3}{8}\log\frac{(11-x)}{11} - \frac{17}{44} + \frac{17}{4(11-x)}.
\end{aligned}$$

Fact 4 We have the following:

(1) $F(x) > 0$ for $\forall\ x > 0$ in $\boldsymbol{R}$.

(2) $$0 \ \geq \ F(x) \ \geq \ \frac{4}{\pi^2}\Big(\cos^2\frac{\pi x}{2}\Big)\Big(\frac{1}{(1-x)^2} - 1 + \frac{5}{(3-x)^2} - \frac{5}{3^2} + \frac{1}{(5-x)^2} - \frac{1}{5^2}$$
$$+ \frac{11}{(7-x)^2} - \frac{11}{7^2} + \frac{7}{(9-x)^2} - \frac{7}{9^2} + \frac{17}{(11-x)^2} - \frac{17}{11^2}$$
$$- \frac{3}{16}\log\frac{(9-x)}{9} - \frac{7}{36} + \frac{7}{4(9-x)}$$
$$-\frac{3}{8}\log\frac{(11-x)}{11} - \frac{17}{44} + \frac{17}{4(11-x)}\Big)$$
for $\forall \ \ x \leq 0$ in $\boldsymbol{R}$.

The Taylor expansion of $G(z)$ around 0 in $\boldsymbol{C}$ is

$$G(z) = \textstyle\sum_{k\geq 1}(k+1)\Big(\textstyle\sum_{m\geq 0}\frac{\phi(2m+1)}{(2m+1)^{k+2}}\Big)z^k$$
$$= \ 2\Big(\textstyle\sum_{m\geq 0}\frac{\phi(2m+1)}{(2m+1)^3}\Big)z + 3\Big(\sum_{m\geq 0}\frac{\phi(2m+1)}{(2m+1)^4}\Big)z^2 + \cdots,$$

and

$$F(z) = \frac{2}{\pi^2}(1+\cos\pi z)G(z) \quad \text{on } \boldsymbol{C}.$$

Proposition 1 *For the entire function $F(z)$, we have the following:*

(1) *$z=0$ is a repelling fixed point of $F(z)$ such that $1.024 < F'(0) < 1.07$, where*

$$F'(0) = \frac{8}{\pi^2}\sum_{n=0}^{\infty}\frac{\phi(2n+1)}{(2n+1)^3}$$
$$= \frac{8}{\pi^2}\sum_{k=0}^{\infty}\sum_{n=0}^{\infty}\frac{6k+1}{(4^n(8k+1)+4^{n-1}+\cdots+4+1)^3}$$
$$+\frac{8}{\pi^2}\sum_{k=0}^{\infty}\sum_{n=0}^{\infty}\frac{6k+5}{(4^n(4k+3)+4^{n-1}+\cdots+4+1)^3}$$
($= 1.043\cdots$ *derived by computer.*)

(2) *$z=1$ is a superattractive (namely $F'(1)=0$) fixed point of $F(z)$.*

(3) *There exists an attractive fixed point z_0 ($\in \boldsymbol{R}$) of $F(z)$ such that $-\frac{1}{20} < z_0 < 0$.*

Further, around $2n+1$ in $\boldsymbol{C}$ $(n = 0, 1, \cdots)$ we have

$$F(z) \ = \ \phi(2n+1)$$
$$+ \ 2\phi(2n+1)\sum_{m\geq 1}(-1)^m\Big(\frac{1}{(2m+1)!} + \frac{1}{(2n+1)^2\pi^2(2m)!}\Big)\pi^{2m}(z-2n-1)^{2m}$$
$$+ \ \frac{2}{\pi^2}(1-\cos\pi(z-2n-1))\sum_{n\neq m\geq 0}\phi(2m+1)\Big(\frac{1}{(z-2m-1)^2} - \frac{1}{(2m+1)^2}\Big).$$

From the above expression, for $\forall n = 0, 1, 2, \cdots$ we can compute that

$$|F(z) - F(2n+1)| < 6(2n+1)\pi^2(z - 2n - 1)^2 \qquad \text{if} \ \ |z - 2n - 1| < \frac{1}{\pi}.$$

Proposition 2 *Every positive odd integer is in the Fatou set $F(F)$ of the entire function $F(z)$. Moreover, for $\forall n = 0, 1, 2, \cdots$ we have*

$$\left\{ z \in \boldsymbol{C} \ \middle| \ |z - 2n - 1| < \frac{1}{12\pi^2(2n+1)} \right\} \quad \subset \quad F(F).$$

Fact 5 From Fact 4 (2) we have

(1) $0 \geq F(x) \geq x + 1$ for $\forall\, x \leq -1$ in $\boldsymbol{R}$.

(2) The composite F^n of F satisfies

$0 \geq \ F^n(x) \ \geq -1$ if $0 \geq \ x \ \geq -n-1$ in $\boldsymbol{R}$ $(\forall n = 1, 2, \cdots)$.

Proposition 3 *Every negative odd integer is in the Julia set $J(F)$ of the entire function $F(z)$. Moreover, we have*

$$J(F) \cap (-\infty, 0] \ = \ \cup_{n>0} F^{-n}(0) \cap (-\infty, 0].$$

By THEOREM 3.1 of [1], we have the following:

Proposition 4 *Every component of the Fatou set of $F(z)$ is simply connected.*

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