We consider a more general situation in which a curve is traced by a point z on a regular polygonal disk with n sides rolling around another regular polygonal disk with m sides.

Generalized Cyclogons

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A cycloid is the curve traced out by a point on the circumference of a circular disk that rolls without slipping along a straight line. It consists of a periodic sequence of congruent arches resting on the line. If the point is rigidly attached to the disk but not on the circumference it traces out a curtate cycloid if the tracing point lies inside the disk, and a prolate cycloid if it lies outside the disk. Figure 1 shows an example of each type.

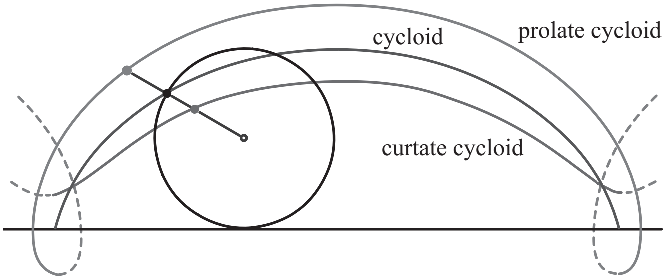

Figure 1. A cycloid, a curtate cycloid, and a prolate cycloid traced out by a point on a rolling disk.

If the rolling disk is replaced by a regular polygon, each vertex traces out a curve we call a cyclogon. In [1] the authors determined by elementary means the area of the cyclogon, that is, the area A of the region under one arch of a cyclogon. We showed that A is equal to the area P of the rolling polygon plus twice the area C of the disk that circumscribes the polygon:

$$A = P + 2C. \tag{1}$$

In the limiting case, when the number of polygonal edges increases without bound, the cyclogon becomes a cycloid, P approaches C, and the cycloidal area is three times the area of the rolling disk.

Motivated by an effort to better understand why the term $2C$ appears in (1), we considered the more general area problem for curtate and prolate cyclogons and found a result that is surprisingly simple. The general formula can be written symbolically as follows:

$$A = P + C + C_z, \tag{2}$$

where P denotes the area of the rolling polygon, C is the area of the disk that circumscribes the polygon, and C_z is the area of a disk whose radius is the distance from the center of the rolling disk to the tracing point z. When z is on the circumference of the rolling disk, we have $C_z = C$ and we get (1). In the limiting case when P approaches C, Eq. (2) gives a known result $A = 2C + C_z$.

Many curves related to cycloids can be obtained by rolling a circular disk around a fixed circular disk (instead of along a line). A point on the circumference of the rolling disk generates an epicycloid if the rolling disk is outside the fixed disk, and a hypocycloid if it is inside. Epicycloids were used by Apollonius around 200 B.C. and by Ptolemy around 200 A.D. to describe the apparent motion of planets. When the tracing point is not on the circumference of the rolling disk, it traces out a trochoid: an epitrochoid if the rolling disk is outside the fixed disk, and a hypotrochoid if it is inside.

We consider a more general situation in which a curve is traced by a point z on a regular polygonal disk with n sides rolling around another regular polygonal disk with m sides. The edges of the two regular polygons are assumed to have the same length. A point z attached rigidly to the n-gon traces out an arch consisting of n circular arcs before repeating the pattern periodically. We call this curve a trochogon—an epitrochogon if the n-gon rolls outside the m-gon, and a hypotrochogon if it rolls inside the m-gon. The trochogon is curtate if z is inside the n-gon, and prolate (with loops) if z is outside the n-gon. If z is at a vertex it traces an epicyclogon or a hypocyclogon. Figure 2 shows a curtate epitrochogon obtained by rolling a square ($n = 4$) outside a 24-gon ($m = 24$).

The main result of this article is a simple and elegant formula for the area of the region between a general trochogonal arch and the fixed polygon. We call this the area of the tro-

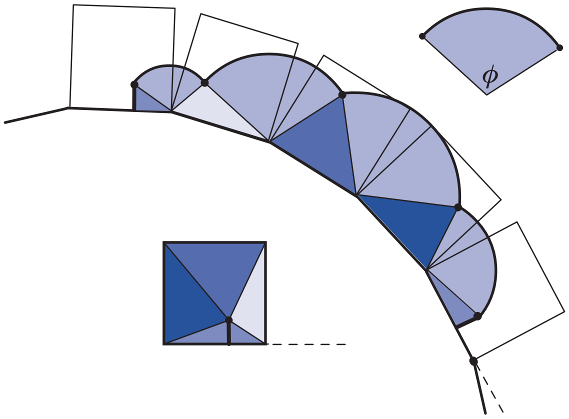

Figure 2. A curtate epitrochogonal arch traced by a point inside a square rolling outside a regular 24-gon.

chogonal arch. It is given by

$$A = P + \left(1 \pm \frac{n}{m}\right)\left(C_z + C\right), \tag{3}$$

with the plus sign for an epitrochogon and the minus sign for a hypotrochogon. If we let m tend to ∞, the fixed m-gon becomes a straight line and we obtain (2) as a limiting case of (3). All results in this paper are obtained without using integral calculus.

A proof of (3) is outlined in the next section after which we discuss a number of special cases.

Area of a Trochogonal Arch

Figure 2 displays the essential features required for treating a general regular n-gon rolling outside a regular m-gon. In Figure 2, the tracing point z is inside the square, and the arch it generates consists of four circular sectors and five triangles, shown shaded. The lower portion of Figure 2 shows how the five triangular pieces fill the square. Because of periodicity, the first and last right triangles outside the 24-gon together have the same area as the bottom triangle in the lower part of Figure 2. So area A is equal to area P, the area of the rolling square, plus the sum of the areas of the four circular sectors.

In the general case of a regular n-gon rolling outside a regular m-gon, the tracing point z attached to the n-gon generates an arch consisting of n circular sectors together with a set of triangles that provide a dissection of the n-gon. So the area A of any trochogonal arch is equal to that of the rolling n-gon P, plus the areas of n circular sectors, the kth sector having area $\frac{1}{2}\phi r_k^2$, where ϕ is the common angle (in radians) subtended by each sector and $r_1, \ldots, r_n$, are the radii of the sectors. Radius r_k is the distance from the tracing point z to the kth vertex of the rolling polygon. Thus, we have

$$A = P + \frac{1}{2}\phi \sum_{k=1}^{n} r_k^2. \tag{4}$$

It is easy to see that $\phi = 2\pi/n + 2\pi/m$, the sum of two exterior angles, so (4) becomes

$$A = P + \left(1 + \frac{n}{m}\right)\frac{\pi}{n}\sum_{k=1}^{n} r_k^2. \tag{5}$$

Now we use a result on sums of squares derived in [2]. In complex number notation, it states that if $z_1, z_2, \ldots, z_n$ lie on a circle of radius r with center at the origin 0, and if the centroid of these points is also at 0, then for any point z in the same plane we have

$$\sum_{k=1}^{n} \left|z - z_k\right|^2 = n\left(|z|^2 + r^2\right). \tag{6}$$

Applying (6) with $r_k = |z - z_k|$ we find

$$\frac{\pi}{n}\sum_{k=1}^{n} r_k^2 = \pi\,|z|^2 + \pi\,r^2 = C_z + C,$$

which, when used in (5) gives the following formula for the area of an epitrochogonal arch:

$$A = P + \left(1 + \frac{n}{m}\right)\left(C_z + C\right). \tag{7}$$

Incidentally, if the rolling n-gon rolls inside the m-gon, the same analysis shows that the area formula for a hypotrochogonal arch is

$$A = P + \left(1 - \frac{n}{m}\right)\left(C_z + C\right), \tag{8}$$

so (7) and (8) together can be combined to give (3).

Applications

We can obtain the limiting case of a circle of radius r rolling around a fixed circle of radius R if we let both n and m tend to ∞ in such a way that their ratio $n/m \to r/R$. Then the limiting case of (3) becomes

$$A = C + \left(1 \pm \frac{r}{R}\right)\left(C_z + C\right). \tag{9}$$

This gives the area of one arch of the classical epitrochoid and hypotrochoid without the use of calculus.

The authors could not find this general result in the literature except for the limiting case $R \to \infty$ and some special cases in which the tracing point z is at a vertex.

Tracing Point at a Vertex

Return now to (3) and take the tracing point z at a vertex of the rolling n-gon. Then the areas C_z and C of the disks are equal, and (3) gives the area of one arch of an epi- or hypocyclogon:

$$A = P + 2\left(1 \pm \frac{n}{m}\right)C. \tag{10}$$

In the limiting case when both n and m tend to ∞ in such a way that $n/m \to r/R$, (10) gives us a known result for the area of one arch of the classical epicycloid or hypocycloid:

$$A = \left(3 \pm 2\frac{r}{R}\right)C. \tag{11}$$

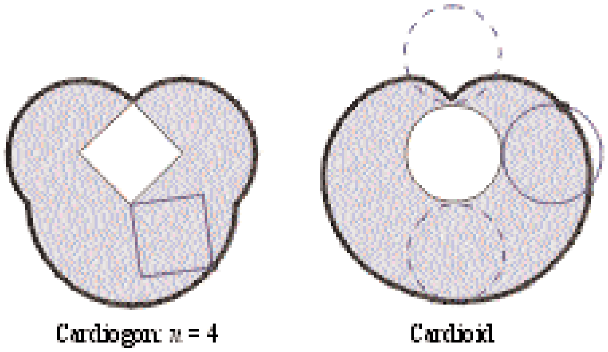

Figure 3. A cardiogon traced by the vertex of an n-gon rolling outside an n-gon. The cardiogon becomes a cardioid as $n \to \infty$.

A special case of (10) is the cardiogon (Figure 3)—an epicyclogon with $n/m = 1$,

$$A = P + 4C. \tag{12}$$

When $n \to \infty$, then $P \to C$, the tracing curve becomes a cardioid, and (12) or (11) give us $A = 5C$. This implies a classical result that the area of the region bounded by a cardioid is equal to $6C$, because the cardioidal arch, of area $5C$, together with the inner disk of area C, fill the cardioid with total area of $6C$.

Another special case of (10) is the nephrogon (Figure 4)—an epicyclogon with $n/m = 1/2$, which gives

$$A = P + 3C \tag{13}$$

When $n \to \infty$ both (13) and (11) give $A = 4C$, for the area of one arch of a nephroid. The nephroid itself encloses two such arches, each of area $4C$, plus the inner disk of area $4C$, giving another proof that a nephroid encloses a region of area of $12C$.

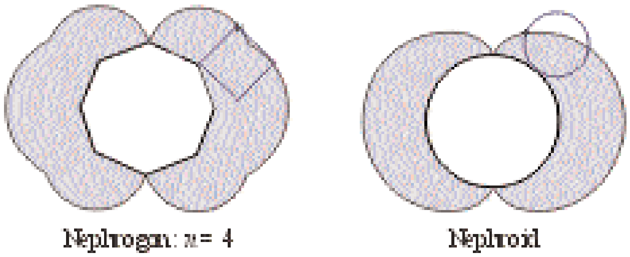

Figure 4. A nephrogon traced by the vertex of an n-gon rolling outside a $2n$-gon. The nephrogon becomes a nephroid as $n \to \infty$.

A related result is the astrogon (Figure 5)—a hypocyclogon with $n/m = 1/4$. Eq. (10) gives

$$A = P + \frac{3}{2}C. \tag{14}$$

When $n \to \infty$, both (14) and (11) give $A = 5/2\,C$ for an astroid, which is a hypocycloid with four cusps ($r/R = 1/4$). The four arches between the hypocycloid and the outer circle (of area $16C$) have a total area of $4A = 10C$, so the region inside the astroid has area $6C$, another classical result obtained without calculus.

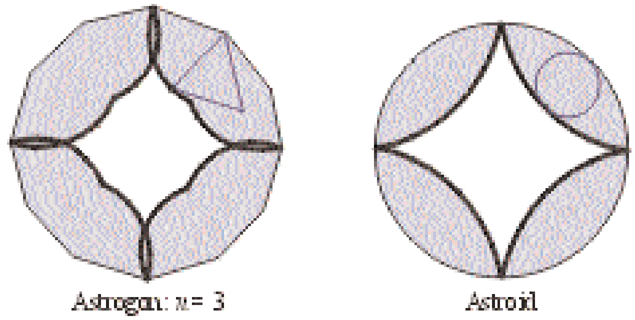

Figure 5. An astrogon traced by the vertex of an n-gon rolling inside a $4n$-gon. The astrogon becomes an astroid as $n \to \infty$.

Another special case of interest is the deltogon (Figure 6)—a hypocyclogon with $n/m = 1/3$. Eq. (10) gives

$$A = P + \frac{4}{3}C, \tag{15}$$

and when $n \to \infty$ (15) and (11) give $A = 7/3\,C$ for the deltoid, which is a hypocycloid with three cusps ($r/R = 1/3$). The three arches between the deltoid and the fixed circle have a total area of $3A = 7C$, the fixed circle has area $9C$, so the region inside the deltoid has area $2C$, another known result.

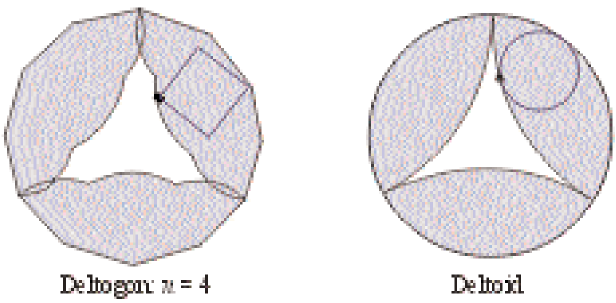

Figure 6. A deltogon traced by the vertex of an n-gon rolling inside a $3n$-gon. The deltogon becomes a deltoid as $n \to \infty$.

A somewhat suprising example is what we call a diamogon—a hypocyclogon with $n/m = 1/2$. The curve is traced by a point z at a vertex of an n-gon rolling inside a $2n$-gon. When the n-gon makes one circuit around the inside of the $2n$-gon, it traces out two curves each consisting of $n - 1$ circular arcs situated symmetrically about a diameter of the $2n$-gon. Examples with $n = 3$ and $n = 4$, are shown in Figure 7.

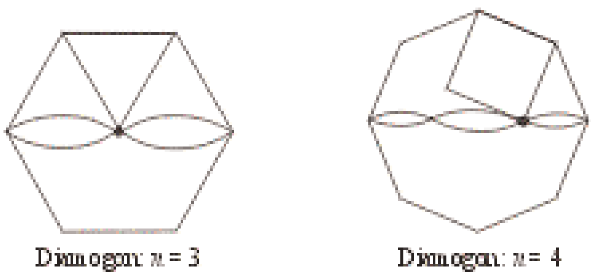

Figure 7. Diamogons traced by a vertex of an n-gon rolling inside a $2n$-gon.

Using (8) we find that the area of one arch of the diamogon is $A = P + C$ because $C_z = C$. The two arches between the diamogon and the outer polygon have area $2A = 2(P + C)$. In the limiting case when $n \to \infty$ this becomes $2A = 4C$. But $4C$ is the area of the fixed circular disk, which means that the area of the region common to the two diamogons tends to zero. In other words, when $n \to \infty$ the diamogon turns into a diameter of the fixed circle traced twice.

Tracing Point not at a Vertex

We conclude with an example of a hypotrochogon traced out by a point z not at a vertex of the n-gon. We consider $n/m = 1/2$ and call the hypotrochogon an ellipsogon because the limiting case $n \to \infty$ gives an ellipse. Figure 8 shows an example of a square rolling inside an octagon with the tracing point z inside the square. In this case the ellipsogon traces out two arches, each consisting of four circular arcs.

In the limiting case $n \to \infty$, (9) shows that the area of one arch is given by $A = C + \frac{1}{2}(C_z + C)$, so the two arches fill out a region of area $2A = 3C + C_z$. The limiting configuration of the ellipsogon is an ellipse enclosing an area equal to $4C - 2A = C - C_z$. If the radius of the inner circle is r and if the distance from z to the center of the inner circle is s then $C - C_z = \pi(r^2 - s^2) = \pi(r + s)(r - s)$. The distances $r + s$ and $r - s$ are the lengths of the semiaxes $a = r + s$ and $b = r - s$ of the ellipse, so we get $C - C_z = \pi ab$ the usual formula for the area of an ellipse.

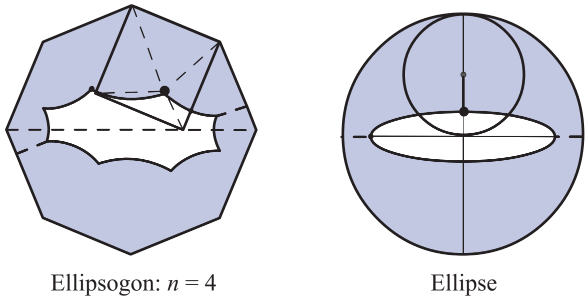

Figure 8. An ellipsogon traced by a point inside an n-gon rolling inside a $2n$-gon. The ellipsogon becomes an ellipse as $n \to \infty$.

The point z also traces an ellipsogon if it is outside the rolling n-gon. If the point z is inside or outside the rolling n-gon and then moves toward a vertex, the ellipsogon becomes a diamogon which, in turn, becomes a diameter as $n \to \infty$. ■

References

1. Tom M. Apostol and Mamikon A. Mnatsakanian, Cycloidal Areas Without Calculus, *Math Horizons*, Sept. 1999, p. 14.
2. Tom M. Apostol and Mamikon A. Mnatsakanian, Sums of Squares of Distances, *Math Horizons*, Nov. 2001, pp. 21–22.