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Erdős Distance Problem for Convex Metrics

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by

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Abstract of the Dissertation

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This dissertation was inspired by the Erdős Distance Problem, which asks: What is the minimum number of distinct distances determined by n distinct points in the plane? Erdős posed this question in 1946 and conjectured that $f(n) = \min_{|E|=n} |\{||x-y||_2 : x, y \in E\}| = \Omega(n^{1-\varepsilon})$ and showed in [17] that this bound is attained by the $\sqrt{n} \times \sqrt{n}$ lattice. The best lower bound to date is $O(n^{19/22})$, due to Katz and Tardos [20]. In this work we ask how the Erdős distance problem changes when distance is defined by different convex metrics. More formally, given a convex metric K, we hope to find good upper and lower bounds on $f_K(n) = \min_{|E|=n} |\{||x-y||_K : x, y \in E\}|$.

In terms of the lower bound, we obtained the following result for all homogeneous, translation-invariant convex metrics. Its proof follows the general structure of Erdős's original proof in [17], with modifications to account for certain difficult cases.

Theorem 1. *For every convex asymmetric metric K and any set P of n points in the plane, there are $\Omega(\sqrt{n})$ distinct distances determined by them. In fact, $f_K(n) \geq \sqrt{\frac{n+2}{12}} - \frac{1}{2}$.*

This bound is asymptotically tight for symmetric metrics K that are convex, but not strictly convex, i.e. $f_K(n) \approx n^{1/2}$. However, for strictly convex metrics we aimed to mimic Székely's method from [50] where he showed that in the Euclidean case $f(n) = \Omega(n^{4/5})$. While we were not able to extend his argument to work for all strictly convex metrics, we did extend it to a class of metrics which includes the L_p metric.

Theorem 2. *Given a set E of n points in the plane, one of them determines $\gtrsim n^{4/5}$ distinct distances from the others if our metric K is strictly convex, has only finitely many axes of symmetry, and all pairs of non-linear bisectors intersect in at most c_o points, for some constant c_o.*

In fact, the argument follows rather nicely, with a few notable obstacles. Foremost, Székely's argument relies heavily on the fact that the bisector of two points in the Euclidean metric is a line. Bisectors in arbitrary metrics may behave in many different ways and this is what prevents us from extending Theorem 2 to all strictly convex metrics. In the case of the L_p metric we are able to resolve the issue of how often bisectors intersect by using a theorem from algebraic geometry, see [24, ch. 1].

CHAPTER 1

Introduction

1.1 Statement of problem

The first paper discussing distance problems was published in 1946, where Erdős introduced a conjecture now widely known as the Erdős distance problem. This initial paper [17] has inspired many subsequent papers, some aimed at establishing the conjecture and others, like this thesis, devoted to variations on the problem. Here we consider the problem, to be stated below, for metrics other than the Euclidean metric, and we briefly summarize earlier work on the Erdős distance problem. Notations, definitions and conventions appear in section 1.3.

The Erdős distance problem asks: Given a set E of n points in the plane, what is the minimum number of distinct distances determined by them? In other words, we want to find $f_2(n) := \min\limits_{|E|=n} |\Delta_2(E)|$ where $\Delta_2(E) = \{||x-y||_2 : x, y \in E\}$ and $||x-y||_2$ denotes Euclidean distance. Erdős showed that $\sqrt{n} \lesssim f_2(n) \lesssim n/\sqrt{\log n}$ and conjectured that $f_2(n) = \Omega(n^{1-\epsilon})$ for all $\epsilon > 0$ [17]. The proof Erdős used to obtain his lower bound on $f_2(n)$ is very general and we will look at it in detail later. He showed that his upper bound is attained by the $\sqrt{n} \times \sqrt{n}$ lattice.

Given that Erdős's lower bound differed so greatly from the conjectured bound of $n^{1-\epsilon}$, it is not surprising that the progress on this problem has been slow and sporadic. Here we provide a history of some of the highlights. In 1952, Moser [41] made the next advance on the lower bound, showing that $f(n) = \Omega(n^{\frac{2}{3}})$.

Chung [12] improved this to $\Omega(n^{\frac{5}{7}})$ in 1984 and Chung, Szemerdi and Trotter [14] improved it to $\Omega(n^{\frac{4}{5}}/(\log n)^C)$ in 1992, for a large constant C. All three of these advancements were achieved through making (increasingly complicated) geometric observations. Then in 1995 Székely [50] showed that given n points in the plane, one of them determines at least $cn^{4/5}$ distinct distances. Székely only improved on the previous result by a logarithmic factor, but it was a major breakthrough since he found a very short proof using incidence theory. Furthermore, he also showed something slightly stronger; namely, that all the different distances come from a single point. This improved upon a result of Clarkson, Edelsbrunner, Guibas, Sharir and Welzl [15] where it was shown that some point determines $\Omega(n^{3/4})$ different distances from the others. Building on Székely's method, Solymosi and Tóth [45] showed that $f_2(n) = \Omega(n^{6/7})$ in 2001. Shortly thereafter Tardos [53] improved this to $\Omega(n^{\frac{4e}{5e-1}-\epsilon}) = \Omega(n^{.8635\cdots})$, which Katz and Tardos [20] improved to $\Omega(n^{.8636\cdots})$ It has been noted, however, that the current methods cannot do better than $\Omega(n^{8/9})$.

Here we address how the Erdős distance problem changes when we substitute the Euclidean metric by other convex metrics. Given a metric K we will look at $f_K(n) = \min_{|E|=n} |\Delta_K(E)|$, where $\Delta_K(E) = \{||x-y||_K : x, y \in E\}$. Since we will often look at the L_p metrics $(1 < p < \infty)$ we will write $f_p(n)$ instead of $f_{L_p}(n)$ for convenience. To see that things can behave quite differently in different metrics we need only look at the L_∞ metric. In chapter 2 we will show that we can modify Erdős's original proof to get that $f_\infty(n) = \Omega(\sqrt{n})$. However, if we again consider the $\sqrt{n} \times \sqrt{n}$ lattice, it is easy to see that there are only $\sqrt{n}-1$ different L_∞ distances and thus $f_\infty(n) \approx \sqrt{n}$.

We will also address the following natural questions: Do all convex metrics with a line segment act like the L_∞ metric? And do all strictly convex metrics behave like the Euclidean metric? The aforementioned result of Clarkson, et al. [15] does work for all convex metrics, so we have some reason to hope that this second question can be answered positively. Agarwal, Nevo, Pach, Pinchasi, Sharir and Smorodinsky recently showed in [1] that $f_K(n) = \Omega(n^{(7/9)}/\kappa_s(n))$ for all sufficiently well-behaved strictly convex metrics K, where $\kappa_s(n)$ depends on K. Here we improve this result for a class of strictly convex metrics, by employing Székely's methods from [50].

1.2 Summary of new results

This work will be roughly organized by the historic milestones associated with the Erdős distance problem. We begin in chapter 2 with the $n^{1/2}$ theory, starting with Erdős's original proof and showing how it extends. In this chapter we will also discuss metrics for which $f_K(n) \approx \sqrt{n}$. In chapter 3 we explore the $n^{2/3}$ theory, and again we will begin by looking at Moser's original argument and use it to prove a minor result. Also, in this chapter we briefly discuss on the results after Moser's $n^{2/3}$ proof and before Székely's $n^{4/5}$ argument. Before going on to Székely's breakthrough $n^{4/5}$ argument, we will need to cover some graph theory; chapter 4 will be devoted to establishing notation and providing the necessary theorems from this field. Chapter 5 will cover the $n^{4/5}$ theory and will contain the majority of our results. We begin this chapter by providing a rather detailed proof of Székely's argument and then show how we can modify it to work for a class of strictly convex metrics, including the L_p metric, for $1 < p < \infty$. In chapter 6 we discuss the $n^{6/7}$ theory of Solymosi and Tóth and observations on generalizing this method. Here we will also give a brief account of the most recent work on the Erdős distance problem. And finally, in chapter 7 we will survey other problems in this area and discuss them in relation to the question of changing the underlying metric.

The main results of this work appear in chapters 2 and 5. In chapter 2 we show that all convex metrics K have $f_K(n) \gtrsim n^{1/2}$. This result has often been cited as true, but there does not appear to be a formal proof in the literature. More surprising perhaps is that we give a universal constant for all asymmetric metrics K. The vast majority of our results appear in chapter 5, wherein we show that $f_K(n) \gtrsim n^{4/5}$ for "potato" metrics. These are metrics where pairs of bisectors (the set of points which are equidistant from two points in our set) can intersect in only finitely many points. We further extend this bound to work for metrics that act like potato metrics except for admitting finitely many axes of symmetry. We are able to show that the L_p metric satisfies these hypotheses. Table 1.1 shows how these results relate to some of the previously known theorems. The final section of this chapter does include some new theorems, which are rather easy corollaries to known results. They are included for completeness and to provide intuition on asymmetric metrics. Chapters 3, 4, 6, and 7 primarily serve as background, with the exception of one new result in section 3.2. Otherwise new results in these chapters are easy extensions of known theorems that are modified to our setting when necessary. In particular, chapters 3 and 7 are exclusively expository in nature.

1.3 Notations and conventions

Throughout we will use the notation $X \lesssim Y$ to mean that there exists a constant c such that $X \leq cY$ and similarly for $X \gtrsim Y$. When we write $X \approx Y$ we mean that both $X \lesssim Y$ and $X \gtrsim Y$. We use the notation $f(n) = O(g(n))$ and $f(n) = \Omega(g(n))$ in the standard way to mean that there exist constants c and k such that $0 \leq f(n) \leq g(n)$ and $0 \leq cg(n) \leq f(n)$ for all $n \geq k$, respectively.

Metric K	Bounds on $f_K(n)$	Reference
All convex K	$f_K(n) \gtrsim n^{1/2}$	Theorem 2.1.2.6
Symmetric K with line segment	$f_K(n) \approx n^{1/2}$	Example 2.2.2.1
All strictly convex K	$f_K(n) \gtrsim n^{3/4}$	See Clarkson, et al. [15]
"Well behaved" strictly convex K	$f_K(n) \gtrsim n^{7/9}/\kappa_s(n)$	See Agarwal, et al. [1]
"Potato" metrics	$f_K(n) \gtrsim n^{4/5}$	Theorem 5.2.5.1
L_p metric $1 < p < \infty$	$f_p(n) \gtrsim n^{4/5}$	Theorem 5.4.5.4
Euclidean, or L_2, metric	$f_2(n) \gtrsim n^{.8636\ldots}$	See Katz and Tardos [20]

Table 1.1: Table of results

Similarly, we use the notation $f(n) = o(g(n))$ to mean that $0 \leq f(n) < cg(n)$ for all $n \geq k$. Constants never depend on n.

Part of the reason for using the above notation is that we really only care about the asymptotic behavior of $f_K(n)$ in terms of powers of n. Therefore we always consider n to be quite large and make little effort to lower the constants in the theorems. We never state explicitly how large n must be.

We use the word *metric* to mean a distance function K possessing the usual properties: positivity, symmetry and the triangle inequality. Throughout this we will only work with convex metrics that are homogeneous and translation-invariant; in other words, those that are defined by a single convex shape. However, in the following we often consider *asymmetric metrics* for which only positivity and the triangle inequality hold. Therefore, it is not always the case that $||x-y||_K = ||y-x||_K$ and so we will define $||x-y||_K = \min_{\mathbb{R}}\{r \mid y \in C_K(x,r)\}$ where $C_K(x,r)$ is the (K)-circle about x of radius r.

It is reasonable to ask whether the triangle inequality holds for all convex asymmetric metrics. In his dissertation [36] L. Ma notes that for any convex metric K, $||x-y||_K = \frac{||x-y||_2}{||x-v||_2}$ where v is the unique point on the ray $\overrightarrow{xy}$ that hits $C_K(x,1)$. With this observation in hand, it is a straightforward exercise to show that the triangle inequality does indeed hold for any convex distance function. However, equality can hold in the triangle inequality for 3 points that are not collinear in other metrics; see [36] for precise conditions.

Finally, we are always working in the plane unless explicitly noted. Many of the theorems and bounds are quite different in higher dimensions (which we discuss briefly in section 7.4).

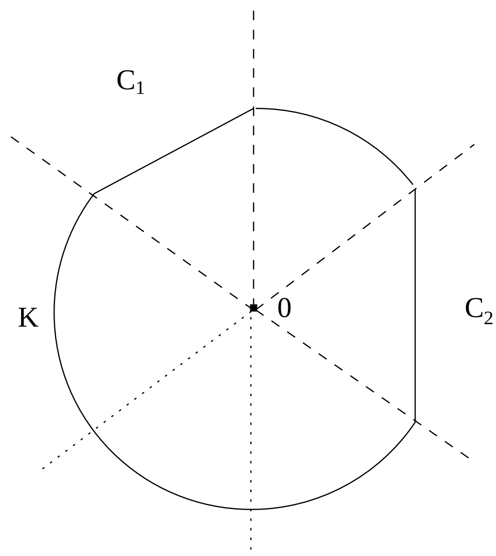

Figure 1.1: Essentially Euclidean metric

1.4 Initial observations

Here we make some observations regarding a specific subclass of asymmetric metrics that are formed by taking slices off the Euclidean ball, such that no two of them are parallel. We acknowledge ahead of time that this is a highly specialized class of metrics, but it serves as a model case. The main result here does not fit in the general framework, but it is a simple corollary to the EDP and it provides intuition for asymmetric metrics. The upcoming result can be extended to more other metrics and we will mention these generalizations at the end of the section.

The class of sliced Euclidean metrics we discuss are those metrics that have the following property: for all $x \in \mathbb{R}^2$ at least one of $||x||_K$ or $||-x||_K$ equals the Euclidean distance $||x||_2 = ||-x||_2$. We call these *essentially Euclidean* metrics. In other words, from the origin the metric looks Euclidean either forwards or backwards. See Figure 1.1. The reader should note that this property precludes K from being symmetric, otherwise we would have two parallel slices. This should be compared with Example 2.2.2.1.

Our first observation is that whenever we get a lower bound on the number of distances measured from some point, we get the same bound for essentially Euclidean metrics. Intuitively this is because either measuring *from* this particular point or *to* it we can use Euclidean portions of such metrics. This means that in the following theorem we may take $\alpha = 19/22$ due to the result of Katz and Tardos [20] mentioned earlier.

Theorem 1.4.1.1. *Let K be an essentially Euclidean metric and let E be a set of n points in the plane. If $f_2(n) \gtrsim n^\alpha$, then $f_K(n) \gtrsim n^\alpha$ as well.*

Proof. Since for any two points x and y in E, either the distance $||x-y||_K$ or the distance $||y-x||_K$ equals the Euclidean distance $||x-y||_2 = ||y-x||_2$ we have that:

$$|\Delta_K(E)| \geq |\Delta_2(E)|.$$

□

It should be clear that we can generalize this theorem to any metric K that is sliced in the same way. For example, looking at Table 1.1 we can get a similar result for *essentially potato* metrics. More generally, the result of Clarkson et al. [15] says that for any strictly convex asymmetric, we get that *essentially strictly convex* metrics determine $\Omega(n^{3/4})$ distinct distances.

Furthermore, from the proof it is clear that we did not utilize the fact that the non-Euclidean portions were straight lines. In fact, we can change the metric here arbitrarily, as long as we preserve the property that at least one of $||x||_K$ or

$|| - x||_K$ measures Euclidean distance for all x in $C_K(O,1)$. So, theorem 1.4.1.1 does indeed extend to a much broader class of metrics.

CHAPTER 2

The $n^{1/2}$ theory

2.1 Erdős's argument

Theorem 2.1.2.1 (Erdős [17]). *Given a set E of n points in the plane there are $\Omega(n^{1/2})$ distinct Euclidean distances determined by them.*

Proof. Take a point P_1 on the convex hull of E. Draw in the t semi-circles around P_1 that capture the remaining $n-1$ points of E. By the pigeonhole principle one of the semi-circles C has at least $(n-1)/t$ points on it. Let P_2 be the left most point on C inside the convex hull. Then each of the distances from P_2 to the remaining $\frac{n-1}{t}-1$ points is distinct. See Figure 2.1. Thus $f_2(n) \geq \max\{t,(n-1)/t\}$, which is minimized when $t = \sqrt{n-3/4}-1/2$. □

To see that Erdos's method of proof works for any strictly convex (asymmetric) metric K we need the following lemma, which we will use repeatedly throughout this work. See [31], [36] for two different proofs.

Lemma 2.1.2.2. *Given a strictly convex asymmetric metric K, two (K)-circles $C_K(x,r)$ and $C_K(y,s)$ intersect in at most 2 points, assuming that $x \neq y$ and $r \neq s$.*

The only thing that requires verification when extending Theorem 2.1.2.1 to all strictly convex asymmetric metrics is that we get $\gtrsim n/t$ distances when

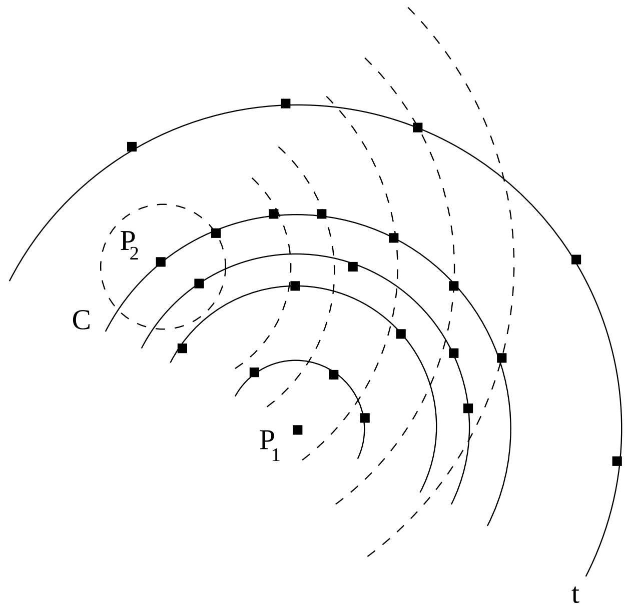

Figure 2.1: Erdős's proof that $f_2(n) \gtrsim n^{1/2}$

measuring from P_2 on C, where C is a (K-)circle about P_1 that contains $\gtrsim n/t$ points. We know, however, from Lemma 2.1.2.2 that C can intersect at most twice with any (K-)circle drawn about P_2. Thus we get at least $\frac{n-1}{2t} - \frac{1}{2}$ distinct distances measured from P_2 and we are done.

Therefore it remains to show that the result holds for metrics K that are convex but not strictly convex. Clearly the above proof will not work verbatim, as it can happen that some circle about P_2 will intersect infinitely often with C and allow for many repeated distances. In the case that the unit ball has only finitely many flat edges, it has been noted [28] that we can obtain the result in the following way. If there are m line segments in the unit ball, then there are at most m arcs between the line segments and we know that one of the edges or arcs has at least $(n-1)/2mt$ points on it. Then we proceed as above to get that $f_K(n) \geq \max\{t, \frac{n-1}{2mt} - 1\}$. Note that the constant will worsen as m grows, but for n large enough we get that $f_K(n) = \Omega(\sqrt{n})$.

This does not address the case of metrics K with infinitely many flat edges in their unit ball. Using the above method we can easily say that $f_K(n) = \Omega(n^{1/3})$, as there could be up to $\sqrt{n/t}$ points on $\sqrt{n/t}$ sides of C and thus $f(n) \gtrsim \max\{t, \sqrt{n/t}\}$. In fact we can achieve $\Omega(\sqrt{n})$ for these metrics as well by using the same general framework. The proof we give works for all convex asymmetric metrics. Furthermore, the reader will note that we obtain a universal constant for all asymmetric metrics. Thus in the case of many-sided polygons our result gives a much better constant than the aforementioned methods. We will need the following geometric lemmas. The first is closely related to Lemma 2.1.2.2. Again, see [31] and [36] for two different proofs.

Lemma 2.1.2.3. *Let K be a convex asymmetric metric and let $C = C_K(1,0)$. Suppose $\alpha > 0$ and $x \neq 0 \in \mathbb{R}^2$. Then $C \cap (\alpha C + x)$ cannot contain more than two maximal line segments* [1]. *In fact, $C \cap (\alpha C + x)$ either contains two maximal line segments, one maximal line segment and up to one point, or at most two points.*

Before going on we state the following geometric observation as a lemma.

Lemma 2.1.2.4. *If a closed convex curve C contains two different maximal line segments L_1 and L_2, then they must lie on different lines.*

Proof. We proceed by contradiction. Suppose without loss of generality that both L_1 and L_2 lie on the x-axis and assume that some point P inside C is in the upper half plane. By maximality, we can further assume that L_1 lies strictly to the left of the origin and L_2 lies strictly to the right of the origin. By convexity and the fact that C is closed, each line segment connecting two points on or inside C must

[1] *We call a line segment of a curve* maximal *if there is no other line segment in the curve that contains it.*

itself lie on or inside C. Thus we get a triangular piece of the upper half plane that must be on or inside C. The three vertices are: the leftmost endpoint of L_1, the rightmost endpoint of L_2, and P. Then since L_1 and L_2 are maximal line segments, the curve connecting the rightmost endpoint of L_1 and the leftmost endpoint of L_2 must go into the lower half plane. Take some point P_- on this portion of C that has negative y-coordinate. Consider the line connecting P_- to the leftmost endpoint of L_1. By convexity this line must be inside or on the curve C, since both endpoints are were taken to be on C. But this contradicts that fact that L_1 was on C, so we are done. □

We are actually concerned with a much more specialized situation than in the previous two lemmas; specifically, we need to know how C intersects with a (K)-circle C' whose origin is on C. In this setting we can conclude that if $C \cap C'$ contains a line segment, then it occurs when corresponding line segments overlap. We state this formally in the following lemma.

Lemma 2.1.2.5. *Let K be a convex asymmetric metric and let $C := C_K(0,1)$. Given $\alpha > 0$ and $x \in C$ and let $C' = \alpha C + x$. Suppose $C \cap C'$ contains a line segment L_{seg}. If L_{seg} is contained in the maximal line segment $L \subset C$, then L_{seg} is also contained in $L' := \{\alpha L + x\} \subset C'$.*

Proof. By Lemma 2.1.2.3 it is clear that if two (K)-circles intersect infinitely often, then it must occur when two parallel line segments of C and C' overlap. And by Lemma 2.1.2.4 we know that there are at most two maximal line segments of C' that are parallel to L, one of which is L' the other we will refer to as the antipodal line segment L''. We aim to show that $L'' \cap L = \emptyset$.

Let ℓ be the line through x that has the same slope as L. We note that x cannot lie on L, otherwise the origin of C would be on its own boundary,

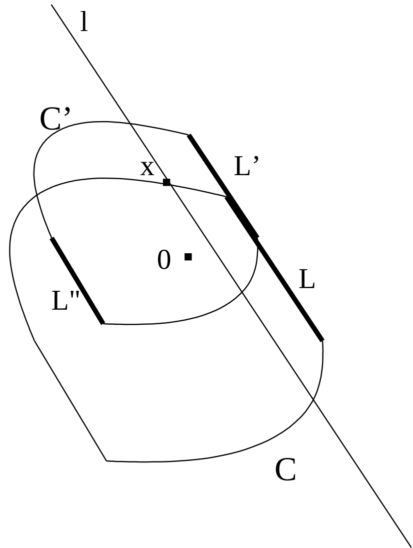

Figure 2.2: Intersecting line segments

contradicting positivity. Thus, we can assume without loss of generality that L is to the right of ℓ (or above it if ℓ is horizontal). Then, since $C' = \alpha C + x$, L' will be to the right of (or above) ℓ. And so if L'' exists, it will be to the left of (or below) ℓ by the mean value theorem. See Figure 2.2. Since L and L'' are on opposite sides of ℓ we get that $L'' \cap L = \emptyset$ as desired. Thus, it must be the case that L_{seg} is contained in $L \cap L'$. □

With these lemmas in hand, we may state the main theorem of this section.

Theorem 2.1.2.6. *For every convex asymmetric metric K, and any set P of n points in the plane, there are $\Omega(\sqrt{n})$ distinct (K)-distances determined by them. In fact, for all such K we have $f_K(n) \geq \sqrt{\frac{n+2}{12}} - \frac{1}{2}$.*

Proof. Take any point $P_1 \in P$. Draw in the (K-)circles around P_1 that capture the remaining $n-1$ points of P. Suppose there are t such circles. By the pigeonhole principle one of these (K-)circles C has at least $(n-1)/t$ points on it.

Let the points N, E, W and S be points on the circle C that are furthest north, east, west and south. Take some representative if there is more than one such point. One of the four arcs of C created by these points must contain at least $(n-1)/4t$ points. Without loss of generality, suppose this occurs on the closed arc from W to N, and call it C_+. Note that all the tangent lines to points on C_+ have positive slope (where they exist) and their slopes decrease from infinity to zero as we pass clockwise from W to N. We are done if we can show that there are $\gtrsim n/t$ distinct distances determined by the points of P on C_+ by using Erdős's argument from Theorem 2.1.2.1.

Let W' be the southernmost point of P on C_+, and similarly let N' be the easternmost point of P on C_+. Let $C'_+ \subset C_+$ denote the closed arc from W' to N'. Note that $W' \neq N'$ otherwise $t \approx n$ and we are done. Consider the line $\overline{W'N'}$, which by construction is southwest of all points of P on C'_+. If $\overline{W'N'}$ has slope of 0 or infinity then C'_+ must be contained in a line since W' and N' mark the westernmost and northernmost points of C_+. In this case we are done, as there are at least $\frac{n-1}{4t} - 1$ different distances when measuring from either W' or N'. Thus we may assume that the slope of $\overline{W'N'}$ is strictly between 0 and ∞. Now, let l_O be the line that is perpendicular to $\overline{W'N'}$ going through the origin of C. Clearly l_O has strictly negative slope and it hits C in precisely two places, and at most once in C'_+. We must consider two different possibilities and show that in both cases we get at least $\frac{n-1}{12t}$ distinct distances measured between points of P on C'_+.

Case 1: Suppose that l_O does not intersect with C'_+. Let $l_{W'}$ and $l_{N'}$ be the lines perpendicular to $\overline{W'N'}$ that go through the points W' and N' respectively. Since l_O does not intersect C'_+, either the origin O of C is above $l_{N'}$ or below $l_{W'}$.

Suppose O lies above the line $l_{N'}$. Then draw in all the (K)-circles around the point W' that go through all the points of P on C'_+. We claim that each circle about W' intersects with C'_+ in only one point. Take one such circle around W' and call it $C_{W'}$. Since, by construction, $C_{W'}$ intersects C at least once above the line $\overline{W'N'}$, we have by the mean value theorem that they also intersect once below the line $\overline{W'N'}$, which is disjoint from the arc C'_+. Thus, by Lemma 2.1.2.3, we know that the only way that $C'_+ \subset C$ and $C_{W'}$ can intersect more than once is if they meet up in parallel line segments. Furthermore, since $C_{W'}$ is centered on C we are in position to apply Lemma 2.1.2.5, so we know that corresponding parallel line segments must overlap.

However, by construction, if we have some line segment L on C'_+ then the corresponding line segment L' on $C_{W'}$ (using the notation of Lemma 2.1.2.5) is below the line $l_{W'}$. Therefore corresponding line segments can never meet and so they intersect at most twice. Thus $C_{W'}$ can intersect C'_+ at most once, and similarly for all circles around W'. We can conclude that there are at least $(n-1)/4t - 1$ distances determined from W'. The case where O is below $l_{W'}$ follows similarly.

Case 2: Next consider the case where l_O intersects the arc C'_+. It may be the case that l_O passes through a line segment L on C_+. Let L_l and L_r denote the leftmost and rightmost points of L. If the L contains at least $\frac{n-1}{12t}$ points then we are done, as there are at least $\frac{n-1}{12t} - 1$ distinct distances when measuring from the point closest to L_l on L. If not, then either the arc from W' to L_l or the arc from L_r to N' contains at least $\frac{n-1}{12t}$ points (where we let $L_l = L_r = \{l_O \cap C'_+\}$ if l_O does not hit a line segment).

If the portion of C'_+ from W' to L_l has at least $\frac{n-1}{12t}$ points on it, then call it C''_+ and let $l_{W'}$ be the line that is perpendicular to $\overline{W'N'}$ going through W'. Draw in all the (K)-circles around W' that go through all the points of P on

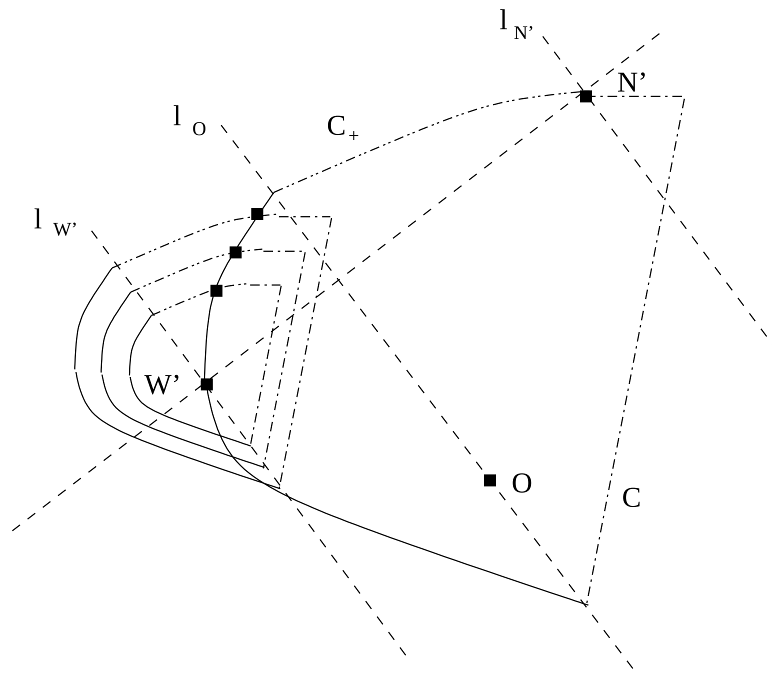

Figure 2.3: (K)-circles around W'

C''_+. See Figure 2.3. As above, each of these circles can intersect with C''_+ in at most one point by construction. Thus we have shown that there are at least $(n-1)/12t - 1$ distances measured from W'. The case where at least $\frac{n-1}{12t}$ points are on the portion of C'_+ between L_r and N' follows in the same way, and so we are done in this case as well.

Following Erdos's method of proof from Theorem 2.1.2.1, we get that $f_K(n) \geq \max\{t, ((n-1)/12t) - 1\}$, which is minimized when $t = \sqrt{\frac{2+n}{12}} - \frac{1}{2}$. □

Remarks 2.1.2.7. We make some remarks regarding the constant in Theorem 2.1.2.6. While the proof gives that $f_K(n) \geq \sqrt{\frac{n+2}{12}} - \frac{1}{2}$ for all asymmetric metrics, we can improve the constant in certain cases. Henceforth, we only consider the highest order term; for example, $\sqrt{n/12}$ in Theorem 2.1.2.6.

If distance is defined by any regular polygon P (including those with arbitrarily many sides), then we know that $f_P(n) \gtrsim \sqrt{n/4}$. We modify the proof in the following way. By considering the slopes of the tangent lines we find that we only need to measure from a point P on the extreme end of the quarter arc that contains at least $n/4t$ points on it. The only concern is that there may be a flat edge that is horizontal or vertical (of course, this happens when $n = 4k$ for some integer k) and then we may get a lot of repeated distances when measuring from the ends. Some simple trigonometric calculations, however, show that this cannot happen, except in the case of a square, where we already have $n/4t$ by the methods mentioned after Theorem 2.1.2.1.

As far as examples go for metrics which are convex but not strictly convex, the lattice shows that there can be as few as $\sqrt{n} - 1$ distinct distances, but we know of no better examples. There are metrics for which we can place n/t points on its unit circle and there are no fewer than $n/4t$ distances when measuring from any point on it (the square with evenly spaced points on it is one such metric). This does not seem to translate to an example where there are as few as $\sqrt{n/4}$ distances, though it does show that we would have to use a new method of proof to improve the constant further.

We now know that for all convex asymmetric metrics K we have $\sqrt{\frac{n+2}{12}} - \frac{1}{2} \leq f_K(n) \leq 2n$. The upper bound can be seen by looking at evenly spaced points on a line, where we get the improved upper bound of n for symmetric metrics. Having these broad bounds, we now aim to show which properties on the metric give us improved bounds, and for which metrics these bounds are tight.

2.2 When is $n^{1/2}$ tight?

We have already noted that for the L_∞ metric (and thus similarly for the L_1 metric) $f_\infty(n) \approx \sqrt{n}$. In fact, given any metric K that is convex but not strictly convex, we can construct an example showing that $f_K(n) \lesssim \sqrt{n}$. Comparing this result with Theorems 1.4.1.1 it should be clear that we are relying heavily on the symmetric property in the following.

Example 2.2.2.1. We may assume without loss of generality that the line segment in the unit ball of K is parallel to the x-axis. And by symmetry, we know that there are in fact two line segments, one in the lower and one in the upper half plane, each parallel to the x-axis. Let $(x_L, 1)$ and $(x_R, 1)$ be the left and right endpoints, respectively, of the line segment in the upper half plane. And let ℓ be the line connecting the origin to $(\frac{x_R - x_L}{2}, 1)$. Define $d := ||x_R - x_L||_K$.

Let the first point of our set E be the origin. Then place two points on the line $y = 1$ centered about $(\frac{x_R - x_L}{2}, 1)$ with (K-)distance $\epsilon > 0$ from each other, where ϵ is a small real number to be determined later. In general, for $h \in \mathbb{Z}$ $(1 \leq h \leq \sqrt{n})$ we will place $h+1$ points at height $y = h$ spaced ϵ from one anther and centered about the point where line ℓ intersects $y = h$. See Figure 2.4. Clearly we have $\frac{(\sqrt{n})(\sqrt{n}+1)}{2} \approx n$ points. And, regardless of ϵ there are at least $\sqrt{n}$ distinct distances, namely those measured from the origin: $1, 2, \ldots, \sqrt{n}$. Also we have the $\sqrt{n}$ distances obtained by looking at the distances measured horizontally along the line $y = \sqrt{n}$: $\epsilon, 2\epsilon, \ldots, \sqrt{n}\epsilon$.

We claim that if $\epsilon = \frac{L}{2\sqrt{n}}$, then the distances mentioned above are the only ones obtained and therefore $f_K(n) \approx \sqrt{n}$. The idea is that if ϵ is made small enough, then E fits in a very narrow cone and so whenever a point at height h is captured in a line segment of (K-)circle that has been drawn about another

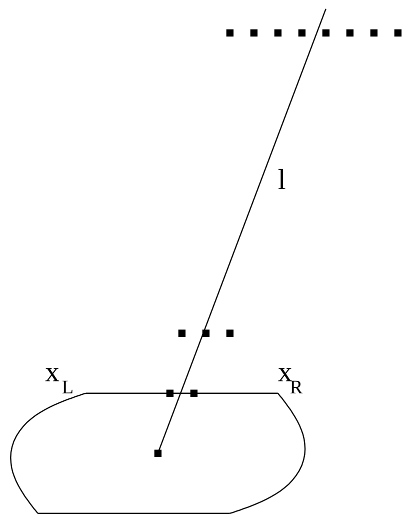

Figure 2.4: Lattice example

point, then all the points on the line $y = h$ are also covered by the same line segment. Now, let us show this claim formally.

We want to verify that we only get the $2\sqrt{n}$ distinct distances mentioned above when measuring from any point of E. We will be done if we can show that by putting the (K-)circles of radius $1, 2, \dots, \sqrt{n}, \epsilon, 2\epsilon, \dots, \sqrt{n}\epsilon$ around any point in our set we cover all of E. It should be clear from construction that the (K-)circles of radius $k\epsilon$, $1 \leq k \leq \sqrt{n}$ about some point will hit all the points that lie on the same horizontal line as that point. Now we need to show that the (K-)circles of radius $1, 2, \dots, \sqrt{n}$ cover the rest of the points.

To see this, first consider a (K-)circle of radius 1 centered about the point where the lines ℓ and $y = \sqrt{n}$ intersect, and call it B. The flat edge on the bottom of this circle clearly captures all the points of E on $y = \sqrt{n} - 1$, has length d, and its midpoint rests on the line ℓ. But if we shift B horizontally in order to be centered at any of the points of E on $y = \sqrt{n}$, we still capture all the points of E on $y = \sqrt{n} - 1$. This is because by construction the furthest we will shift B

is by $d/4$. It is easy to check that the situation only improves as the circles get bigger. The rest of the cases follow similarly and the construction is complete.

Remarks 2.2.2.2. We conjecture that this construction can by modified to work for any asymmetric metric that has two parallel line segments in its unit ball whenever the origin is inside the convex hull of the two parallel edges (by linear transformation). Considering the methods of proof used in section 1.4 one can see that it is not sufficient to have two parallel line segments.

In regards to the question of when $n^{1/2}$ is tight, Konyagin and Łaba investigate polygonal metrics in [23]. They show that for any polygon with only finitely many sides, where the slopes of the sides are algebraic (in some coordinate system), there is a well-distributed set of n points such that $f_K(n) \approx n^{1/2}$. In particular, this class of metrics includes all regular polygons. In fact, in both [23] and [31] they show that this is also the necessary condition on a polygonal norm in order for $f_K(n) \approx n^{1/2}$, when restricted to well-distributed point sets, which we discuss in section 7.3.

CHAPTER 3

From $n^{1/2}$ to $n^{4/5}$ theory

We start this chapter by going over Moser's $n^{2/3}$ argument from 1952. We then use this method of proof to get a result on a class of sliced Euclidean metrics. The final section is a historical summary of the progress made on the Erdős distance problem after Moser's $n^{2/3}$ argument and before Székely's $n^{4/5}$ argument. We do not give complete proofs in this final section, as we do not use these methods.

3.1 Moser's argument

We will sketch Moser's original proof [41] that $f_2(n) = \Omega(n^{2/3})$, using the following two lemmas.

Lemma 3.1.3.1. *Let E be a set of n points in the plane. Given two points P and Q in E, there must exist $\gtrsim \sqrt{n}$ distinct distances from either P or Q.*

Proof. Draw in the circles centered about P and Q that go through the remaining $n-2$ points of E. Suppose this results in p and q circles about the points P and Q, respectively. These circles can intersect at most $2pq$ times and by construction the remaining points of E lie in these intersections. Therefore, $2pq \geq n-2$ and we have our result. $\square$

Note that Lemma 3.1.3.1 also shows that $f_2(n) = \Omega(\sqrt{n})$ and with an improved constant over the proof given by Erdos. However, this argument can only be extended to work for other strictly convex (asymmetric) metrics and has no hope when the the unit ball contains flat edges since it relies on the fact that two unit balls intersect as most twice. So, although the constant is slightly better here, Erdős's proof is more general in some sense. This is the first time we see curvature being used in the Erdős distance problem.

Lemma 3.1.3.2. *Consider an annulus about the origin with inner radius r and outer radius $r+1$. Given $0 < \epsilon \leq 1$, let $P = (r+\epsilon, 0)$ and let Q and R be two points in the annulus and in the first quadrant such that $||P-Q||_2 = ||P-R||_2$. Then $||Q-R||_2 < 2$.*

Intuitively this lemma is easy to understand. All it is saying is that a circle of radius at most $r\sqrt{2}$ centered in this annulus of width 1 will have 'small' intersection with the annulus; in other words, it will be a transversal crossing. Drawing several pictures should convince the reader. See [41] for the detailed proof. With these lemmas in hand we can proceed with Moser's theorem.

Theorem 3.1.3.3 (Moser [41]). *Given n points in the plane there are $\Omega(n^{2/3})$ distinct Euclidean distances.*

Sketch of proof. Let P and Q be two points such that $||P-Q||_2$ is minimal over all pairs of points in E, and name their midpoint O. We may rescale so that $||P-Q||_2 = 2$. One of the half planes determined by the line connecting P and Q contains at least half the remaining points. We consider only these points for the rest of the proof. Now construct a series of semicircles around the point O with radii $1, 2, 3, ...$ Take a number $s \in \mathbb{R}$ such that $1 \leq s \leq n$ and consider two cases: either some annulus contains at least s points, or none of them have as many as s points.

In the first case, Lemma 3.1.3.2 gives us that there must be at least $s/2$ distinct distances amongst the points in the annulus containing s points. This is because we made the annuli small enough in comparison to the minimal distance between all pairs of points.

On the other hand, it may be the case that all of the annuli contain fewer than s points. Here we must further reduce the problem. Break the annuli into three classes by taking every third annulus starting with either the first, second or third. By the pigeonhole principle, one of these classes will contain at least a third of the remaining points, and we will restrict ourselves to this subset of points for the remainder of this case. It is easy to check that all the distances from P or Q to the points in one annulus will differ from all distances from P or Q to the points in another annulus. Suppose there are n_i points in the ith annulus. Then by Lemma 3.1.3.1 we have $f_2(n) \geq \sqrt{n_1} + \sqrt{n_2} + ... + \sqrt{n_t}$ and by construction we have that $\frac{n-2}{6} \leq n_1 + n_2 + ... + n_t$. Indeed for obtaining a lower bound, we may take $n_1 = n_2 = ... = n_t = \frac{n-2}{6t}$, which gives that $f_2(n) \gtrsim \sqrt{nt}$. But by hypothesis we have that $t \geq \frac{n-2}{6s}$ and so $f_2(n) \gtrsim \frac{n}{\sqrt{s}}$.

Combining the two cases we get that $f_2(n) \gtrsim \min\{s, \frac{n}{\sqrt{s}}\}$, which is maximized when $s = n^{2/3}$. □

Remarks 3.1.3.4. We have already noted that Lemma 3.1.3.1 does work for all strictly convex metrics; it is unclear, however, if Lemma 3.1.3.2 works for all strictly convex (asymmetric) metrics. Furthermore, it is not immediately clear that we can create the annuli for non-Euclidean metrics that break into classes as above.

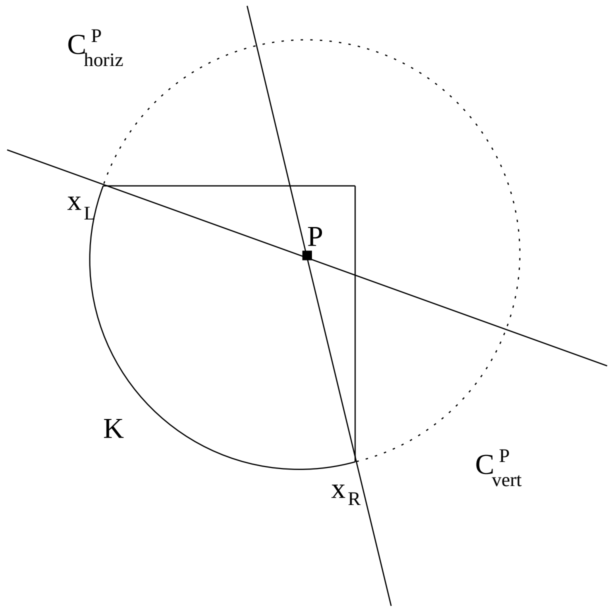

Figure 3.1: Baseball metric

3.2 Baseball metrics

Here look at a second subclass of sliced Euclidean metric (see section 1.4). This type, which we call *baseball* metrics, are Euclidean balls with two slices taken off, one horizontal and one vertical, neither of which eliminates the origin. See Figure 3.1. Furthermore, we require the slices to be long enough so as to create a corner. Otherwise, a simple calculation shows that we are back in the essentially Euclidean case, for which we get much stronger results (see Theorem 1.4.1.1). After stating the main result, we discuss generalizations.

Theorem 3.2.3.1. *Let $I(n,m)$ [1] denote the number of incidences between n points and m circles in the plane. If $I(n,m) \lesssim n^{\alpha}m^{\beta}$, then for a set E of n points in the plane and any baseball metric K, we have $f_K(n) \gtrsim \min\{n^{\frac{4-\alpha-\beta}{4+\beta}}, n^{2/3}\}$.*

[1] *For a more detailed discussion of the quantity $I(n,m)$ see chapter 4. In section 7.2 we discuss bounds on $I(n,m)$ in further detail.*

Before beginning the proof let us establish some notation. Take some point P and consider $C_K(P,1)$. Let X_L and X_R be the left and right ends of the one curved arc in $C_K(P,1)$. Consider the lines $\overline{PX_L}$ and $\overline{PX_R}$, which create two cones; one we call C^P_{horiz} and contains only a horizontal portion of K and the other we call C^P_{vert} and it captures only a vertical portion of K. See Figure 3.1.

Proof. We may assume, without loss of generality, that K is orientated as in figure 3.1; meaning that the horizontal and vertical slices are on the northern and eastern sides, respectively.

Using the general framework from Moser's argument in Theorem 3.1.3.3 we start by taking the two closest points P and Q (using K-distance). By scaling we may assume that this minimal distance equals 2. Notice that the minimal distance will actually be equal to Euclidean distance by the shape of K. Let H_+ denote the portion of the plane above (or to the right of) the line $\overline{PQ}$, and analogously define H_-.

Consider the following 4 closed regions: $C_{horiz} := C^P_{horiz} \bigcap C^Q_{horiz}$, $C_{vert} := C^P_{vert} \bigcap C^Q_{vert}$, $H'_+ := \{H_+ \setminus \{C_{horiz} \bigcup C_{vert}\}\}$, and $H_- := \{H_- \setminus \{C_{horiz} \bigcup C_{vert}\}\}$. By the pigeonhole principle there must be $n/4$ points in at least on of these regions.

It is not too hard to see that if there are at least $n/4$ points of E in either H'_+ or H'_- then we may apply Theorem 3.1.3.3. This is because we are in the essentially Euclidean case here and so we can almost always "see" Euclidean distances one way or another. There are two things we need to check before proceeding with this case.

First, we need to be careful when measuring between the s points within an annuli (Lemma 3.1.3.2). In the case of H'_+, we can simply measure from the northwesternmost point and get $s-1$ distinct K-distances. Similarly, in H'_-, we measure from the southeasternmost point. It is okay that in our application of Lemma 3.1.3.2 we obtain K-distances and that in our application of Lemma 3.1.3.1 we get Euclidean distances, since the cases are disjoint.

Second, we need to address the points of E that lie in the strips formed by the cones that lie in H'_+ and H'_-. See Figure 3.2. We concern ourselves with these points for the following reason. Take x in one of these strips. Then at least two of $||x-P||_K$,$||x-Q||_K$, $||P-x||_K$, or $||Q-x||_K$ may not equal the Euclidean difference. However, an elementary calculation shows that given any annulus of Euclidean radius 1 there can be at most 2 points of $E \bigcap H'_+$ in the two strips (one in each). This is because the strip cannot have K-width greater than 1, the annulus has radius 1 and the minimal distance between points is 2. Therefore, we remove at most two points from each annulus and proceed.

In H'_- it is now clear that we can apply Moser's argument, since we are simply in the Euclidean case. In H'_+, we refer back to the methods used in Theorem 1.4.1.1. Since we have removed the at most two problematic points in each annulus, we have that $||x-P||_K = ||x-P||_2$ and $||x-Q||_K = ||x-Q||_2$, for any point x in $H'_+ \bigcap E$. Therefore, we have $\Omega(n^{2/3})$ distinct K-distances in both of these cases.

Otherwise, we may assume without loss of generality that there are at least $n/4$ points in C_{horiz}. Consider the K-circles about P that hit all points of $E \bigcap C_{horiz}$. Every point lies on a horizontal portion of these circles. Similarly, all the points of $E \bigcap C_{horiz}$ lie on vertical or horizontal portions of the K-circles drawn about the southwesternmost point P_{SW} of this set. Suppose the maximum number of distances from either P or P_{SW} is t. Then by construction, all the points of E in C_{horiz} lie in this lattice of $t \times t$ lines. See Figure 3.3.

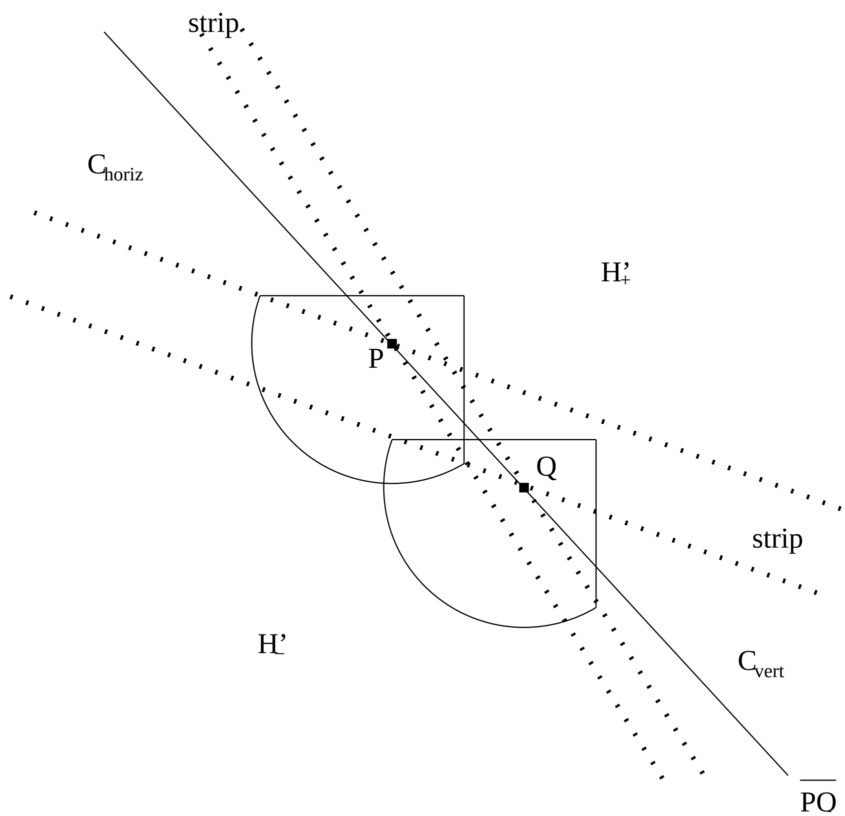

Figure 3.2: Strips and regions

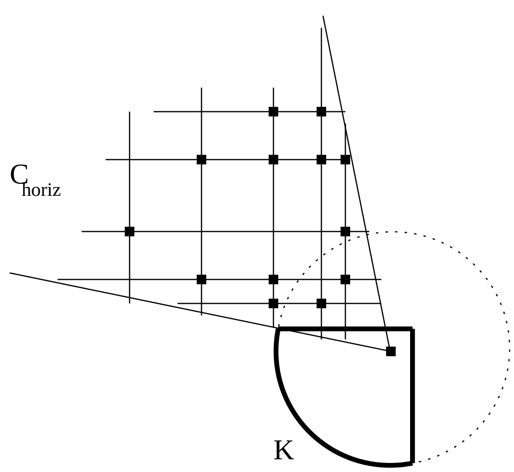

Figure 3.3: Lattice structure in C_{horiz}

Using counting arguments and two applications of Cauchy-Schwarz we find that there are are $\Omega(n^4/t^4)$ rectangles in this $t \times t$ lattice. See [40] for arguments of this nature. Given any rectangle, it is clear that whenever we are measuring from the northeast corner to the southwest corner we are using Euclidean distance. Therefore, $|\Delta_K(E \bigcap C_{horiz})| \gtrsim n^4/t^4$.

We aim to bound the number of incidences, $I(n, sn)$, between the $n/4$ points in C_{horiz} and the at most s circles about each point that capture all the other points. Instead of using the whole circle, we in fact only draw in the quarter circle in the third quadrant (as it is part of K). By the above remarks we have that $I(n, sn) \gtrsim n^4/t^4$. And by hypothesis $I(n, sn) \lesssim n^\alpha (ns)^\beta$. Therefore $s \gtrsim \left(\frac{n^{4-\alpha-\beta}}{t^4}\right)^{1/\beta}$. Then the number of distinct distances is $\gtrsim \max\{t, \left(\frac{n^{4-\alpha-\beta}}{t^4}\right)^{1/\beta}\}$, which is minimized when $t = n^{\frac{4-\alpha-\beta}{4+\beta}}$. So, in all cases we are done. □

Remarks 3.2.3.2. The best known bound to date for this problem is $\alpha = 6/11$ and $\beta = 9/11$. As a consequence we get $f_K(n) = \Omega(n^{29/53})$ for all baseball metrics K. This is not a significant improvement from $\Omega(n^{1/2})$, which we get from Theorem 2.1.2.6, but it does show that $n^{1/2}$ is not tight for such metrics.

Notice that it is much more difficult to extend this result to more general metrics, than it was in Theorem 1.4.1.1. We cannot arbitrarily change the sliced regions, as we depend on the fact that they force points of E in C_{horiz} into a lattice structure. Furthermore, we cannot change the Euclidean portion, for the reasons described in remark 3.1.3.4. We should be able to generalize the theorem to metrics formed by taking two big slices, that are not necessarily perpendicular, off the Euclidean cirlce. This is because we will still get a lattice like structure.

3.3 From $n^{2/3}$ to $n^{4/5}$

The next result on the EDP after Moser was published over 30 years later in 1984 by Fan Chung [12], where she showed that $f_2(n) \gtrsim n^{5/7}$. The methods used were similar to those of Moser. The argument again begins by choosing two points of minimal distance and then looking at a sector with cn points in it and subdividing that sector into many small annular regions. As before, we choose a and consider the cases where either half of the points are in annuli with greater than a points or are in annuli with fewer than a points. In the first case we get $\gtrsim n/\sqrt{a}$ points using Lemma 3.1.3.1. In the second case Chung observes that one can break the points up further into boxes B_i, such that $a < b < |B_i| < 2b$, and again look at the points with minimal distance within B_i and take another sector. This also breaks into two cases. In one we get $\gtrsim \sqrt{ab}$ distances and in the other we get $\gtrsim an/b$ distinct distances. Letting $a = n^{4/7}$ and $b = n^{6/7}$ we have $f_2(n) \gtrsim n^{5/7}$ in all cases.

At the end of Chung's $n^{5/7}$ paper, she notes that J. Beck proved that $f_2(n) \gtrsim n^{58/81-\epsilon}$ and that she could further improve this to $n^{8/11}$. Neither of these results were ever published, though there are many references in the literature to a preprint by Beck. These results were all improved upon soon thereafter using new tools.

In a lengthy paper by Clarkson et al. [15] the authors use incidence theory to show that $f_K(n) \gtrsim n^{3/4}$ for all strictly convex asymmetric metrics K. They do so by showing that for n points and m K-circles (not necessarily of the same radius) $I(n,m) \lesssim n^{3/5}m^{4/5} + m + n$. Suppose that t is the maximum number of distances that occurs from any of the n points. Now draw in the at most nt (K)-circles about each point that capture all the other points. Then we get that $n^2 = I(n, nt) \lesssim n^{3/5}(nt)^{4/5} = n^{7/5}t^{4/5}$, or $t \gtrsim n^{3/4}$. And so we see, in fact, that this statement is even stronger than $f_K(n) \gtrsim n^{3/4}$, in that there exists a single point from which there are $n^{3/4}$ distinct distances. This is the strongest known result that works for all strictly convex asymmetric metrics. In fact, with the exception of the next result we mention, all the remaining results are of this stronger form.

The final paper on the EDP before Székely's $n^{4/5}$ argument was published by Chung, Szemerédi and Trotter [12] in 1992. They showed that $f_2(n) \gtrsim n^{4/5}/(\log n)^c$ for an absolute constant $c > 0$, where c was quite large in the proof (10^{200}) but the authors noted that they could probably do much better. The proof uses Theorem 4.0.4.3, which we encounter in the next chapter and in Székely's argument, but it still relies heavily on geometric properties and builds on previous constructions.

CHAPTER 4

Crossing numbers

The *crossing number* of a graph G, denoted $\mathrm{cr}(G)$, is the minimum number of crossings over all possible drawings of G. In fact, we will use *topological multigraphs*, which are multigraphs with its vertices represented by distinct points in the plane and its edges by continuous arcs between these pairs. Furthermore, we allow edges to pass through vertices without containing them (i.e. without creating new edges). Topological multigraphs with this property were first used by Pach and Sharir in [43]. In this situation, two edges are said to *cross* if they have some point in common that is not an endpoints of both curves.

Here we state two results on crossing numbers that we will use later. Both theorems say that if there are sufficiently many edges in comparison to the number of vertices, then we are guaranteed a certain number of crossings.

Theorem 4.0.4.1 (Leighton [35]; Ajtai, Chvatal, Newborn and Szemerédi [2]). *For any simple graph G with n vertices and $e \geq 4n$ edges, cr(G) $\gtrsim e^3/n^2$.*

Theorem 4.0.4.2 (Székely [50]). *Suppose G is a (topological) multigraph with n nodes, e_G edges and maximum edge multiplicity m. Then either $e_G < 5nm$ or cr(G) $\gtrsim e_G^3/(n^2m)$.*

In [50] Székely states this theorem only for multigraphs, but Solymosi and Tóth assert in [45] that the proof works verbatim for topological multigraphs.

The following theorems follow easily from Theorem 4.0.4.1 and Theorem 4.0.4.2, respectively. We only sketch the proof of the second, but the first follows in exactly the same manner, see [50].

Theorem 4.0.4.3 (Szemerédi and Trotter [52]). *Given n points in the plane and l lines, let L_k be the number of lines incident to at least k points and let $I(n,l)$ denote the number of incidences between points and lines. Then we have the following*

1. $L_k \lesssim n^2/k^3$ *if* $2 \leq k \leq \sqrt{n}$,

2. $L_k \lesssim n/k$ *if* $k \geq \sqrt{n}$, *and*

3. $I(n,l) \lesssim (nl)^{2/3} + n + l$

where the constants are independent of n, l and k.

For Theorem 5.2.5.1 we will need a modified version of the previous theorem to account for the fact that bisectors [1] of two points will not be lines in general for non-Euclidean metrics. We will discuss bisectors further in the next chapter.

Theorem 4.0.4.4. *Given n points and b bisectors in the plane, let m be the maximum number of bisectors passing through any two of the points. Let B_k be the number of bisectors incident to at least k points and $I(n,b)$ denote the number of incidences between points and bisectors. Suppose also that any two bisectors intersect in at most $c_o < \infty$ points. Then*

1. $B_k \lesssim mn^2/k^3$ *if* $2 \leq k \leq \sqrt{n}$,

2. $B_k \lesssim mn/k$ *if* $k \geq \sqrt{n}$, *and*

[1] We define the *bisector* of two points x and y to be the set $\{z \in \mathbb{R}^2 : ||x-z||_K = ||y-z||_K\}$, and denote it $\mathcal{B}(x,y)$; in other words, the set of points equidistant from x and y.

3. $I(n,b) \lesssim m^{1/3}(nb)^{2/3} + nm + b$,

where the constants depend on c_o*, but are independent of* n*,* b *and* k*.*

Sketch of Proof. For parts (1) and (2), we construct a graph G out of those bisectors which pass through at least k points. Let the arcs connecting the points on these bisectors (forgetting about the infinite arcs) be the edges of G. If $e_G \geq 5nm$, apply Theorem 4.0.4.2 and note that $\mathrm{cr}(G) \leq c_o B_k^2$ and then solve for B_k. On the other hand, if $e_G < 5nm$ then one can bound B_k by observing that $e_G \geq (k-1)B_k$. Forcing k to be either larger or smaller than $\sqrt{n}$ gives us the desired bounds. For part (3) we construct the same graph, but we use all b bisectors. Call this graph G', and note that $e_{G'} \geq I(n,b) - b$. As in the proof of (1) and (2), we apply Theorem 4.0.4.2 and get the desired result. □

CHAPTER 5

The $n^{4/5}$ theory

First we will run through Székely's argument showing $f_2(n) = \Omega(n^{4/5})$ from [50]. We will follow Székely's method closely in order to prove Theorem 5.2.5.1 which shows that Theorem 5.1.5.1 can be modified to work for a class of strictly convex asymmetric metrics. Specifically, we will require that for any two pairs of points, the bisectors of each pair can intersect in at most finitely many points. Later in this chapter we will see that we can broaden this restriction to include the L_p metrics for $1 < p < \infty$.

5.1 Székely's argument

Theorem 5.1.5.1 (Székely [50]). *Given a set E of n points in the plane, one of them determines $\gtrsim n^{4/5}$ distinct Euclidean distances from the others.*

Sketch of Proof. Let $t = \max_{p\in E} |\{||p-q||_2 : q \in E\}|$. We may assume that $t = o(n)$, otherwise we are done. Draw in all the circles centered about each point that go through all the other points. From this we construct a multigraph, whose vertices are the set E and whose edges are those arcs that connect consecutive points on the circles. For technical reasons, we delete edges that lie on circles that contain only 1 or 2 points. Since $t = o(n)$ there are still $\gtrsim n^2$ edges remaining in this multigraph G after removing these edges.

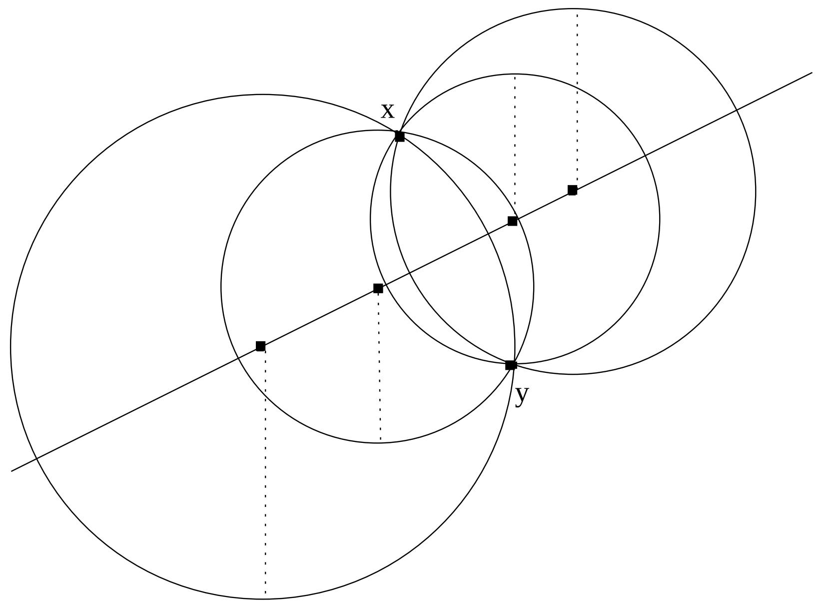

Figure 5.1: Multiplicity of k in G

We can see that $\mathrm{cr}(G) \leq 2n^2t^2$ by counting. And thus we wish to apply Theorem 4.0.4.2 to get a lower bound on $\mathrm{cr}(G)$ in terms of n and t. Unfortunately we could have very high multiplicity in G. The rest of the proof consists of finding out how much multiplicity we can get rid of while keeping $\gtrsim n^2$ edges of G. We observe that whenever we have multiplicity of at least k between two points of G, it means that the bisector of those two points must have at least k points on it. See Figure 5.1. The following proposition (over)counts the number of edges of G that join pairs with multiplicity at least k.

Proposition 5.1.5.2. *The number of pairs (f, a) such that f is a line with at least k points on it, and a is an edge in G with endpoints that have f as their bisector, is $\lesssim tn^2/k^2 + tn \log n$.*

Proposition 5.1.5.2 is proved by considering the cases $k \leq \sqrt{n}$ and $\sqrt{n} < k < n$ and using Theorem 4.0.4.3 and an easy inclusion-exclusion argument, respectively (see [50] for the complete proof). If we set $k = K\sqrt{t}$ for an appropriate constant K, then deleting all edges contributing to multiplicity of least k will yield a multigraph with cn^2 edges. Applying Theorem 4.0.4.2 we get that $\mathrm{cr}(G) \gtrsim e^3/n^2\sqrt{t} \gtrsim n^4/\sqrt{t}$. Combining this with the upper bound of $2n^2t^2$ on the crossing number we have the desired result. □

5.2 Potato metrics

Theorem 5.2.5.1. *Given a set E of n points in the plane, one of them determines $\Omega(n^{4/5})$ distinct (K)-distances from the others if our metric K is strictly convex, and all pairs of bisectors intersect in at most c_o points, for some constant c_o.*

Note that this condition on the bisectors excludes the Euclidean metric for instance, because many pairs of points have the same line as their bisector and thus they intersect infinitely often. Furthermore, this condition forces our norm to have no axes of symmetry for the same reason. We note that bisectors in metrics which are convex but not strictly convex, like the L_∞ metric, can have two dimensional regions. But for strictly convex metrics, bisectors are homeomorphic to a line, see [36]. Recall that all metrics that satisfy the hypotheses of Theorem 5.2.5.1 are called *potato metrics*. See Figure 5.2.

Proof. As in the proof of Theorem 5.1.5.1, we draw at most t (K)-circles about each point which capture all the other points of E. For technical reasons that will become apparent later we delete those circles that contain fewer than 3 points. Again, we consider the multigraph G whose vertices are the points of E and whose edges are the arcs connecting consecutive points on the circles. Since $t = o(n)$ we have that $e_G \gtrsim n^2$, where e_G is the number of edges in G.

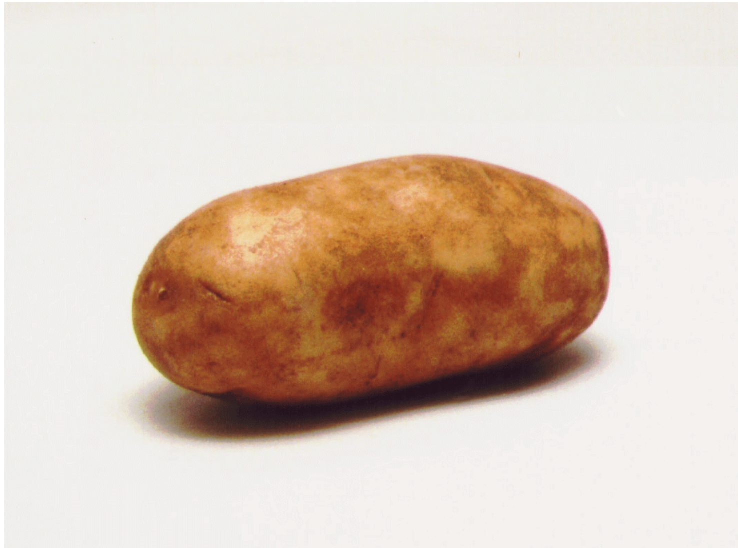

Figure 5.2: Potato

As before, we aim to bound the crossing number of G from above and below by expressions that depend only on n and t. We easily get such an upper bound of $2n^2t^2$, by Lemma 2.1.2.2. Again, we want to apply Theorem 4.0.4.2 to obtain a lower bound for the crossing number of G, but at this point all we can say is that the multiplicity of G is at most n. This bound is too weak, as $e_G < (n-1)n < 5n^2$, so we are not even in a position to apply Theorem 4.0.4.2. Since the multiplicity can indeed be as high as n we will do as in Székely's proof and eliminate edges that contribute to"high" multiplicity while retaining at least Cn^2 edges of G. Therefore, we need to better understand when multiplicity occurs in G.

Note that if there are k edges of G connecting two points, then there must be at least k points on the bisector of these points. Thus we can utilize Theorem 4.0.4.4 which gives us bounds on the number of bisectors that can go through k points. In order to use Theorem 4.0.4.4, however, we need to know how many bisectors can go through two points of E. Let us construct another multigraph H, whose vertices are the points in E and whose edges are arcs on the $\binom{n}{2}$ bisectors that connect points of E. We delete the two arcs that extend to infinity on each bisector. We aim to find the multiplicity, m, of H.

To start, we make an easy observation regarding m: For two points $x, y \in E$ there are at most t^2 bisectors $\mathcal{B}(x', y')$, $x', y' \in E$ that can pass through both of them. This can be seen by recalling that there are at most t circles about x and at most t about y and so there are at most t^2 pairs of points that could determine bisectors passing through them. This is a good first bound on m, but it turns out that we can do much better.

We really only need to include a point on the bisector $\mathcal{B}(x, y)$ in the multigraph H if some circle about that point creates an edge in G between x and y, otherwise it is not contributing to multiplicity in G (which is our ultimate concern). Let us call this modified multigraph H'. Then H' is a topological multigraph, as bisectors may not contain some of the points that they pass through. As noted previously, we can still apply Theorem 4.0.4.4 to such graphs.

Claim 5.2.5.2. *The maximum edge multiplicity m' of H' is at most $2t$.*

Assuming this claim for the moment, Theorem 4.0.4.4 implies that the number of bisectors in H' that contain at least k points is $\lesssim tn^2/k^3$ as long as $k \leq \sqrt{n}$ and $\lesssim tn/k$ if $k \geq \sqrt{n}$. Thus we get the same bound on the number of pairs of points in G that have multiplicity at least k. Multiplying by the number of edges connecting those pairs tells us that by deleting all the edges of G that contribute to multiplicity of at least k, the most edges we can remove is:

$$\sum_{i:k<2^i\lesssim\sqrt{n}} \underbrace{\frac{tn^2}{2^{3i}}}_{\text{bisectors}} \underbrace{2^i}_{\text{arcs}} + \sum_{i:\sqrt{n}<2^i<n} \underbrace{\frac{tn}{2^i}}_{\text{bisectors}} \underbrace{2^i}_{\text{arcs}} \lesssim \frac{tn^2}{k^2} + tn\log_2 n. \qquad (5.2.5.3)$$

If we let $k = K\sqrt{t}$ for an appropriate constant $K > 0$, then we will still have $\gtrsim n^2$ edges left in G after removing all edges where there is multiplicity at least $K\sqrt{t}$. Finally we can apply Theorem 4.0.4.2 to get that $cr(G) \gtrsim (e_G)^2/\sqrt{n}n^2 \gtrsim n^6/\sqrt{t}n^2$. This combined with the upper bound of $2n^2t^2$ on the crossing number of G gives the desired result; namely that $t \gtrsim n^{4/5}$. $\square$

Proof of Claim 5.2.5.2

All that remains in the proof of Theorem 5.2.5.1 is to show that the multigraph H' has maximum edge multiplicity $2t$. In order to prove this we need one lemma on how (K)-circles intersect.

Recall Lemma 2.1.2.2 which states that two (K)-circles can intersect at most twice if K is strictly convex. We use this to prove something slightly stronger; namely, we determine where those intersections can occur in relation to the origins of the two (K)-circles.

Lemma 5.2.5.4. *Suppose $C_K(x, r)$ and $C_K(y, s)$ intersect in two points. Let x_o and x_e be the points on $C_K(x, r)$ that are closest to and farthest from y* [1]*, respectively. Then each of the intersections will lie on different sides of the line $\overline{x_o x_e}$.*

Proof. We can assume that $x \neq y$, as in this case there are no intersections unless $r = s$. Define $h(x) := ||y - x||$ on $C_K(x, r)$, starting at $x = x_0$ and traversing clockwise around $C_K(x, r)$. It suffices to show that h is strictly increasing between x_o and x_e, and strictly decreasing otherwise.

To see this, consider the arcs A_+ and A_- on $C_K(x, r)$ from x_o and x_e travelling clockwise and counterclockwise, respectively. Clearly these two arcs are on

[1] *We are referring to the closest and farthest K-distance here.*

opposite sides of the $\overline{x_o x_e}$. Then showing that h is strictly increasing between x_o and x_e is equivalent to showing that there is exactly one intersection on A_-. Similarly, showing that h is strictly decreasing elsewhere gives that there is precisely one intersection on A_+. See Figure 5.3.

We proceed by contradiction. Consider arc A_- and suppose there exists $x_1 < x_2$ such that $h(x_1) < h(x_2)$. Take x_3 between x_1 and x_2, with $h(x_1) < h(x_3) < h(x_2)$. Then there are three points on $C_K(x,r)$ that attain the value $h(x_3)$, by applying the Intermediate Value Theorem on the intervals $[x_o, x_1]$ and $[x_2, x_e]$. But this contradicts Lemma 2.1.2.2, as it means that $|C_K(y,s) \bigcap C_K(x, ||x - x_3||)| \geq 3$. We also violate Lemma 2.1.2.2 if there is an interval for which $h'(x) \equiv 0$, because that would mean that some circle about y intersects infinitely often with $C_K(x,r)$. Similarly, we can show that h is strictly decreasing outside $[x_o, x_e]$ and we are done. □

Remarks 5.2.5.5. We note that in Lemma 5.2.5.4 the line $\overline{x_o x_e}$ is the same as the line $\overline{xy}$ when the metric is symmetric. This means that if we take the points y_o and y_e on $B_y(s)$ that are closest to and furthest from x, respectively, then the line $\overline{y_o y_e} = \overline{xy}$ as well. However, if we have an asymmetric metric the lines $\overline{x_o x_e}$, $\overline{y_o y_e}$ and $\overline{xy}$ may all be different. This will not matter in the proof of Claim 5.2.5.2, which we are now in a position to prove.

Proof of Claim 5.2.5.2. Take two points x and y in H'. We may assume that there are t (K-)circles about each point (those from the graph G) and that all of the circles about x intersect all of the circles about y in two places, as this is the worst case scenario for multiplicity. Call these circles $C_K(x, r_i)$ and $C_K(y, s_i)$, where $1 \leq i \leq t$ and $r_i < r_j$ if $i < j$ and similarly for the $s_i's$. Then there are t^2 pairs of points whose bisectors go through x and y, precisely the pairs $C_K(x, r_i) \bigcap C_K(y, s_j) := (x_{ij}, y_{ij})$. Recall that in the multigraph H', however,

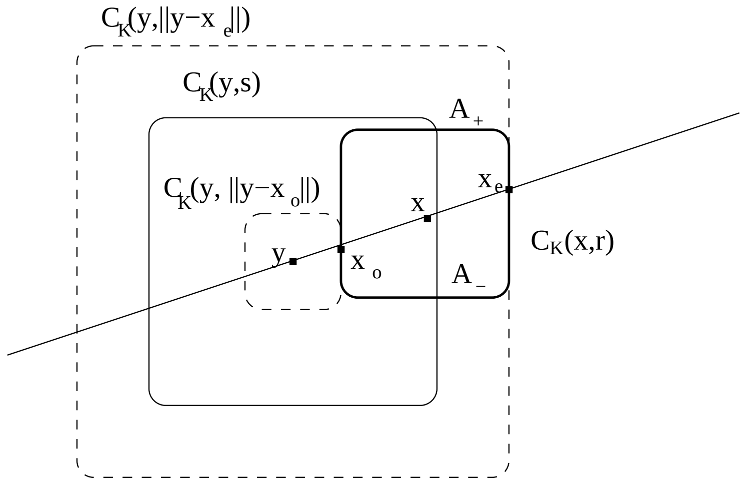

Figure 5.3: Intersections of $C_K(x,r)$ and $C_K(y,s)$

we retain points on a bisector only if the arcs from x_{ij} to y_{ij} are edges on both $C_K(x,r_i)$ and $C_K(y,s_j)$ in our graph G. We will show that this can happen at most $2t$ times by showing that for a fixed circle about x, $C_K(x,r)$, only 2 of the t possible pairs satisfy the necessary conditions. Then, summing over all the circles about x we get our result.

As in the proof of Lemma 5.2.5.4, let x_o and x_e be the points on $C_K(x,r)$ that are closest to and farthest from y, respectively. And call the two intersections of $C_K(y,s_i)$ with $C_K(x,r)$, x_i and y_i, where x_i in on A_+ and y_i is on A_-. See Figure 5.4. Moving clockwise around $C_K(x,r)$ from the point x_o we see in order: $x_o, x_1, x_2, \ldots, x_t, x_e, y_t, y_{t-1}, \ldots, y_1, x_o$. Denote the arc between two points a and b as arc(a,b). Then it suffices to show that arc(x_i, y_i) on $C_K(x,r)$ is in G for at most two $1 \leq i \leq t$.

If arc(x_{rj_1},y_{rj_1}) is in G on $C_K(x,r)$ then it necessarily contains either x_o or x_e. Assuming without loss of generality that it contains x_o we have that none of arc(x_{rj},y_{rj}) are in G for $j < j_1$. If any other arc(x_{rj_2},y_{rj_2}) is on $C_K(x,r)$ then

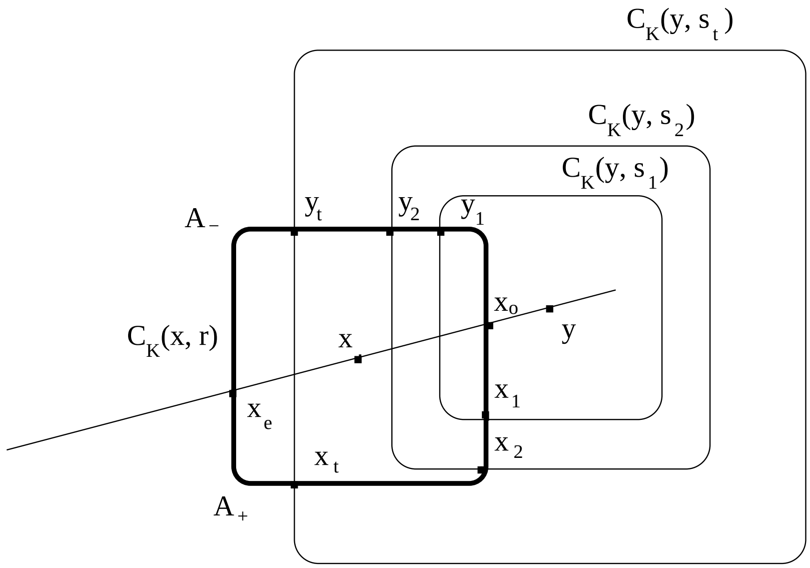

Figure 5.4: Nested Intersections

it must contain x_e and we exclude arc(x_{rj},y_{rj}) for $j > j_2$. Then there cannot be any more such arcs of G on $C_K(x, r)$, otherwise we would exclude one of the previously chosen arcs. Thus we have our result, as x and y were arbitrarily chosen. □

The reader may notice that in the proof of Claim 5.2.5.2 we did not use all of our restrictions to get the result. Namely, we only considered whether our arcs were on the circle $C_K(x, r)$ and did not discuss whether it was an edge on the circle centered about y. Thus it is reasonable to ask whether we can get lower multiplicity (and thus a better result in Theorem 5.2.5.1). The following example shows that the bound $2t$ cannot be lowered.

Example 5.2.5.6. Take t to be an even integer, and let $\text{arc}_c(a, b)$ denote the clockwise arc from a to b. Then the following edges can be in the graph G: On circles about x we can get: $\text{arc}_c(y_{ii},\ x_{ii})$ and $\text{arc}_c(x_{i,(t+1-i)},\ y_{i,(t+1-i)})$ for $1 \leq i \leq t/2$, and $\text{arc}_c(x_{ii},\ y_{ii})$, $\text{arc}_c(y_{i,(t+1-i)}, x_{i,(t+1-i)})$ for $t/2 < i \leq t$. Similarly, on the cir-

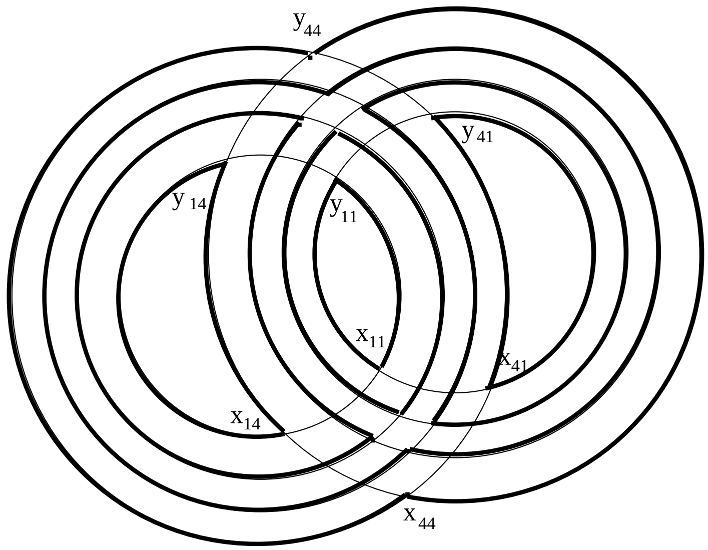

Figure 5.5: Tightness of Claim 5.2.5.2

cles about y we can get the arcs: $\text{arc}_c(x_{ii},\ y_{ii})$ and $\text{arc}_c(y_{i,(t+1-i)},\ x_{i,(t+1-i)})$ for $1 \leq i \leq t/2$, and $\text{arc}_c(y_{ii},\ x_{ii})$ and $\text{arc}_c(x_{i,(t+1-i)}, y_{i,(t+1-i)})$ for $t/2 < i \leq t$. See Figure 5.5. This example shows that $2t$ is tight in Claim 5.2.5.2.

5.3 Finitely many axes of symmetry

In Theorem 5.2.5.1 we only dealt with metrics that had bisectors intersecting in finitely many points. This property forced the metric to have no axes of symmetry, otherwise bisectors are lines whenever two points are on a line that is perpendicular to that axis of symmetry. Here we argue that Theorem 5.2.5.1 can be extended to metrics for which there are finitely many axes of symmetry.

Suppose we have M axes of symmetry in our metric K. And suppose that for the non-linear bisectors we still have the property that they intersect in only finitely many points. Using the method of proof from Theorems 5.1.5.1 and

5.2.5.1, we hope that after removing all edges of G contributing to multiplicity of $\sqrt{t}$ or higher we still have $\gtrsim n^2$ edges. We know that if all the bisectors are lines (Euclidean metric) or if all pairs of bisectors intersect in only finitely many points (potato metrics), then this is the case. We now face a hybrid situation, where we have some control over the number of linear bisectors. Note, however, that edges between two points x and y in G are only generated from points on $\mathcal{B}(x, y)$, so we do not get mixing of the two cases.

Thus, our main concern is determining how many lines of any given slope go through at least 2^i points. Clearly this quantity is bounded above by $n/2^i$. Thus, we are in the situation from Theorem 5.1.5.1 and we see that each of these lines can be associated with at most $2^i 2t$ edges that contribute to 2^i edge multiplicity in G. Summing, we get that the number of such edges is at most:

$$M \sum_{i:k<2^i<n} \underbrace{\frac{n}{2^i}}_{\text{lines}} \underbrace{2^i 2t}_{\text{arcs}} \lesssim tn \log_2 n.$$

Since $t = o(n)$ this term is dominated by the first term in equation 5.2.5.3 from the proof of Theorem 5.2.5.1. Note that M will effect the constant K in the proof, but does not affect the asymptotic behavior. Thus we can make Theorem 5.2.5.1 work for a broader class of metrics. We state this formally now.

Theorem 5.3.5.1. *Given a set E of n points in the plane, one of them determines $\gtrsim n^{4/5}$ distinct (K)-distances from the others if our metric K is strictly convex, has only finitely many axes of symmetry and all pairs of non-linear bisectors intersect in at most c_o points, for some constant c_o.*

5.4 L_p metric

In this section we aim to show that the L_p metric satisfies the conditions of Theorem 5.3.5.1. It should be clear that we must investigate the behavior of L_p bisectors in order to make this claim. It is well known that L_p bisectors are continuous curves homeomorphic to lines [36], but we need slightly more. We need to be assured that there are only a few instances where we get linear portions in the bisectors (those resulting from the axes of symmetry) and also that two pairs of points with non-linear bisectors do not create the same curve. We state these as lemmas here and prove them in the next section.

Lemma 5.4.5.1. *The L_p bisector of two points z_1 and z_2 is a line whenever the slope of the line $\overline{z_1 z_2}$ is equal to $0, 1, -1$ or ∞. Otherwise, the bisector contains no line segments.*

Lemma 5.4.5.2. *Given two non-linear L_p bisectors $\mathcal{B}_{(z_1,z_2)}$ and $\mathcal{B}_{(z_3,z_4)}$, then $\mathcal{B}_{(z_1,z_2)} = \mathcal{B}_{(z_3,z_4)}$ implies that $(z_1, z_2) = (z_3, z_4)$.*

Lemma 5.4.5.2 tells us that two non-linear L_p bisectors act differently at infinity unless they are generated by the same two points. Therefore, outside some large ball the bisectors do not intersect at all. And inside this ball the non-linear L_p bisectors intersect only finitely often by compactness since the curves are analytic. We would like, however, to have a uniform number $c_o < \infty$ that works for all p, $1 < p < \infty$. To obtain this we employ the following theorem from algebraic geometry.

Theorem 5.4.5.3 (Khovanskiĭ, [24], [25]). *Consider a system of n equations*

$$P_1 = ... = P_n = 0$$

with n real unknowns $x = x_1, \ldots, x_n$, in which the P_i are polynomials of degree m_i in $n+k$ variables $x, y_1, \ldots, y_k$ and $y_j = e^{<a_j, x>}$, $a_j = a_j^1, \ldots, a_j^n$ for $j =$

$1, \ldots, k$. *The number of non-degenerate solutions of this system is finite and at most* $m_1 \ldots m_n(\sum m_i + 1)^k 2^{k(k-1)/2}$.

This theorem, together with Lemmas 5.4.5.1 and 5.4.5.2 indeed gives us the following result. Of course, this theorem could also help to get results for metrics besides L_p.

Corollary 5.4.5.4. *Given a set E of n points in the plane, one of them determines at least $cn^{4/5}$ distinct L_p-distances from the others, where c is a positive constant independent of p and n.*

Proof. By the above remarks it suffices to show that two non-linear L_p bisectors cannot intersect more than c_o times, for some $c_o < \infty$. Take two arbitrary pairs of points from our set E that have non-linear L_p bisectors, (z_1, z_2) and (z_3, z_4), where $z_i = (x_i, y_i)$.

Then $\mathcal{B}(z_1, z_2) = |x - x_1|^p + |y - y_1|^p = |x - x_2|^p + |y - y_2|^p$ and $\mathcal{B}(z_3, z_4) = |x - x_3|^p + |y - y_3|^p = |x - x_4|^p + |y - y_4|^p$. We need to make these two equations fit the hypotheses of Theorem 5.4.5.3. First of all, we can eliminate the absolute values and simply apply Theorem 5.4.5.3 to several pairs of equations, those that describe each section of the bisectors. Next we make the following change of variables. Let $x - x_i = e^{u_i}$ and let $y - y_i = e^{v_i}$ for $i = 1, 2, 3, 4$. Now $\mathcal{B}(z_1, z_2)$ and $\mathcal{B}(z_3, z_4)$ can be written as,

$$e^{u_1 p} + e^{v_1 p} = e^{u_2 p} + e^{v_2 p}$$

and

$$e^{u_3 p} + e^{v_3 p} = e^{u_4 p} + e^{v_4 p},$$

respectively.

Written this way, we have 10 equations and 10 unknowns that fit the hypotheses of Theorem 5.4.5.3, where $m_i = 1$ and $k = 8$. Thus the number of non-degenerate solutions to this system is at most 2^{36}. Multiplying this number by 36 to accommodate for the 6 ways in which we must write each bisector without absolute values we obtain our c_o. Clearly c_o does not depend on the value of p.

Finally, we need not worry about degenerate solutions by the following reasoning. From Lemma 5.4.5.2 we know that there are only finitely many solutions, and so we only need to concern ourselves with finitely many tangential meetings. Suppose there were d tangential meetings. If we shift the bisector up or down by some very small amount ϵ we would have between 0 and $2d$ crossings in addition to the original crossings. Thus twice the number of tangential crossings plus the transversal ones cannot exceed c_o. □

Remarks 5.4.5.5. We conjecture that c_o is much smaller, perhaps as low as 5 for all p, but are unable to prove it thus far. We can improve upon the above bound slightly, as we know something about the convexity of the L_p bisectors from Lemma 5.4.5.1. Thus we can say that certain portions of the bisectors intersect at most twice. Unfortunately we cannot get c_o anywhere close to our conjectured bound.

Properties of L_p bisectors

Proof of Lemma 5.4.5.1. Let $p > 2$. It is trivial to check that the L_p bisector $\mathcal{B}(z_1, z_2)$ is a line whenever $\overline{z_1 z_2}$ has slope equal to $0, 1, -1$ or ∞. It remains to show that all other pairs yield bisectors without any line segments. Without loss of generality we may take our two points to be $z_1 = (t, -1)$ and $z_2 = (-t, 1)$, where $0 < t < 1$. Dilation does not affect the proof and so the rest of the cases follow by symmetry. Then $\mathcal{B}(z_1, z_2)$ is an increasing curve passing through $(0,0)$ and the portion in the third quadrant is simply a rotation of the first quadrant. We prove the claim by showing that there are precisely three inflection points, one in each of the first and third quadrants and one at $(0,0)$, and otherwise the bisector is either concave up or concave down.

We divide the first quadrant into three different parts, so that we can avoid the absolute value signs in the L_p metric. We show that $\mathcal{B}(z_1, z_2)$ has positive second derivative for $0 \leq x < t, y < 1$ and negative second derivative for $y > 1$. In the remaining region, $x > t$ and $y < 1$, we find that $\mathcal{B}(z_1, z_2)$ has precisely one inflection point and otherwise has non-zero second derivative. In fact, we break this last region into two subsections and show that the second derivative is strictly positive for $x < 1$ and therefore the inflection point occurs when $x > 1$.

Case 1: $0 \leq x < t, y < 1$**.** Here $\mathcal{B}(z_1, z_2)$ has the form

$$(t+x)^p - (t-x)^p = (1+y)^p - (1-y)^p.$$

Calculation yields that,

$$y' = \frac{(t+x)^{p-1} + (t-x)^{p-1}}{(1+y)^{p-1} + (1-y)^{p-1}} := \frac{a(x)}{b(y)}$$

and

$$y'' = (p-1)\frac{b(y)^2((t+x)^{p-2} - (t-x)^{p-2}) - a(x)^2((1+y)^{p-2} - (1-y)^{p-2})}{b(y)^3}.$$

Note that $y''(0,0) = 0$. Since $b(y) > 0$ and $p > 2$ we get that,

$$
\begin{aligned}
y'' > 0 \Leftrightarrow & \frac{((t+x)^{p-1}+(t-x)^{p-1})^2}{(t+x)^{p-2}-(t-x)^{p-2}} < \frac{((1+y)^{p-1}+(1-y)^{p-1})^2}{(1+y)^{p-2}-(1-y)^{p-2}} \\
\Leftrightarrow & (t+x)^p-(t-x)^p + \frac{4t^2(t^2-x^2)^{p-2}}{(t+x)^{p-2}-(t-x)^{p-2}} \\
& < (1+y)^p-(1-y)^p + \frac{4(1-y^2)^{p-2}}{(1+y)^{p-2}-(1-y)^{p-2}} \\
\Leftrightarrow & \frac{t^2(t-x)^{p-2}}{1-\frac{(t-x)^{p-2}}{(t+x)^{p-2}}} < \frac{(1-y)^{p-2}}{1-\frac{(1-y)^{p-2}}{(1+y)^{p-2}}}.
\end{aligned}
$$

By continually noting that $y < x$ and $t < 1$ we see that this last equation holds, and thus we get that our bisector has positive second derivative in this region.

Case 2: $x > t$ **and** $y < 1$**.** Our bisector is described by

$$(x+t)^p-(x-t)^p = (1+y)^p-(1-y)^p$$

in this region, and we note that the point $(1,t) \in \mathcal{B}(z_1,z_2)$ here. Again, calculation reveals that,

$$y' = \frac{(x+t)^{p-1}-(x-t)^{p-1}}{(1+y)^{p-1}+(1-y)^{p-1}} := \frac{a(x)}{b(y)}$$

and

$$y'' = (p-1)\frac{b(y)^2((x+t)^{p-2}-(x-t)^{p-2})-a(x)^2((1+y)^{p-2}-(1-y)^{p-2})}{b(y)^3}.$$

Thus $y'' > 0$ if and only if the numerator is positive, which is equivalent to showing that

$$\left(\frac{1-y}{x-t}\right)^{p-2}\frac{1-(\frac{x-t}{x+t})^{p-2}}{1-(\frac{1-y}{1+y})^{p-2}} > t^2.$$

When $x < 1$ and $y < t$ it is easy to check that $y'' > 0$ simply by comparing both sides. Therefore let us focus on the region where $x > 1$ and $t < y < 1$. Call the left hand side of the last inequality $g(x,y) = h_1(x,y)h_2(x,y)$. We see immediately that $g(1,t) = 1$ and $g(\cdot,1) = 0$. Thus, we will have our result if we can show that $g(x,y)$ is a strictly decreasing function. So, it suffices to show that $h_1(x,y)$ and $h_2(x,y)$ are strictly decreasing, since each component is non-negative.

Let us verify that $h_1(x,y)$ is decreasing in this region. We take (x_0, y_0) and (x_1, y_1) both on our bisector. Assume that $x_0 < x_1$. This implies that $y_0 < y_1$ since the bisector is increasing. We have:

$$\begin{aligned} h_1(x_0,y_0) &> h_1(x_1,y_1) \\ \Leftrightarrow \left(\frac{1-y_0}{x_0-t}\right)^{p-2} &> \left(\frac{1-y_1}{x_1-t}\right)^{p-2} \\ \Leftrightarrow (1-y_0)(x_1-t) &> (x_0-t)(1-y_1). \end{aligned}$$

This is true since both $(1-y_0) > (1-y_1)$ and $(x_1-t) > (x_0-t)$. Exactly the same argument shows that $h_2(x,y)$ is also strictly decreasing.

Case 3: $y > 1$**.** Here the bisector can be written as,

$$(x+t)^p - (x-t)^p = (y+1)^p - (y-1)^p.$$

Again, calculation gives that,

$$y' = \frac{(x+t)^{p-1} - (x-t)^{p-1}}{(y+1)^{p-1} + (y-1)^{p-1}} := \frac{a(x)}{b(y)}$$

and

$$y'' = (p-1)\frac{b(y)^2((x+t)^{p-2} - (x-t)^{p-2}) - a(x)^2((y+1)^{p-2} - (y-1)^{p-2})}{b(y)^3}.$$

Thus, as in the previous case, $y'' < 0$ if and only if:

$$\left(\frac{y^2-1}{x^2-t^2}\right)^{p-2} \frac{(x+t)^{p-2} - (x-t)^{p-2}}{(y+1)^{p-2} - (y-1)^{p-2}} < t^2.$$

Thus, it suffices to show that

$$\left(\frac{y^2}{x^2}\right)^{p-2} \frac{(x+t)^{p-2} - (x-t)^{p-2}}{(y+1)^{p-2} - (y-1)^{p-2}} < t^2,$$

which is equivalent to:

$$\left(\frac{y^{p-1}}{x^{p-1}}\right)\left(\frac{c_1 + c_2\frac{t^2}{x^2} + c_3\frac{t^4}{x^4} + \dots}{c_1 + c_2\frac{1}{y^2} + c_3\frac{1}{y^4} + \dots}\right) < t,$$

where the c_i are positive constants depending only on p. But this is clearly true since $y < xt^{\frac{1}{p-1}}$ and $yt < x$.

This completes the proof. As expected, the proof for p such that $1 < p < 2$ follows similarly. □

Some basic calculus gives us the following corollary.

Corollary 5.4.5.6. *Non-linear L_p bisectors can intersect any given line in at most 5 places.*

Before proving Lemma 5.4.5.2 we establish some definitions and look at the asymptotic behavior of L_p bisectors at infinity. Let $\mathcal{B}$ be the bisector of the two points $(x_0 + eh, y_0 - h)$ and $(x_0 - eh, y_0 + h)$, where $h > 0$ and $e \neq 0$. We will call e the *eccentricity* and h the *height* of $\mathcal{B}$. It is easy to check that $\mathcal{B}$ passes through the point (x_0, y_0). Let us call this point the *origin* of the bisector. Then for x and y large and positive $\mathcal{B}$ can be written as,

$$((x - x_0) + eh)^p - ((x - x_0) - eh)^p = ((y - y_0) + h)^p - ((y - y_0) - h)^p.$$

Expanding reveals that,

$$(y - y_0) = e^{\frac{1}{p-1}}(x - x_0)\left(1 + \binom{p}{3}\left(\frac{e^2h^2}{(x - x_0)^2} - \frac{1}{(y - y_0)^2}\right) + O\left(\frac{1}{x^2y^2}\right)\right)^{\frac{1}{p-1}}.$$

And thus, as x and y go to positive infinity, we get that $(y - y_0) \to (x - x_0)e^{\frac{1}{p-1}}$. To obtain more precise asymptotics, we let $(y - y_0) = (x - x_0)e^{\frac{1}{p-1}} + \varepsilon$ in the initial equation and solve for ε. Doing so we get:

$$\varepsilon = \frac{(p-2)h^2(e^3 - e^{\frac{p-3}{p-1}})}{6(x - x_0)e^{\frac{p-2}{p-1}}} := \frac{c(e, h)}{(x - x_0)}$$

and therefore

$$(y - y_0) = (x - x_0)e^{\frac{1}{p-1}} + \frac{c(e,h)}{(x - x_0)} + O\left(\frac{1}{(x - x_0)^2}\right)$$

or

$$y = xe^{\frac{1}{p-1}} + (y_0 - x_0 e^{\frac{1}{p-1}}) + \frac{c(e,h)}{x} + O\left(\frac{1}{x^2}\right)$$

Note that $c(e, h) < 0$ for all $p > 2$ and thus the second derivative will be negative at infinity as we saw was the case in the previous lemma.

Proof of Lemma 5.4.5.2. Take two non-linear bisectors, $\mathcal{B}_1$ and $\mathcal{B}_2$, where $\mathcal{B}_i$ has origin (x_i, y_i) and height and eccentricity e_i and h_i respectively. From above, we get that asymptotically the bisectors look like:

$$y = xe_1^{\frac{1}{p-1}} + (y_1 - x_1 e_1^{\frac{1}{p-1}}) + \frac{c(e_1, h_1)}{x} + \ldots$$

and

$$y = xe_2^{\frac{1}{p-1}} + (y_2 - x_2 e_2^{\frac{1}{p-1}}) + \frac{c(e_2, h_2)}{x} + \ldots .$$

Obviously, for the first coefficient to be the same we need that $e_1 = e_2$. And for the second coefficients to be equal we must have $(y_1 - y_2) = (x_1 - x_2)e^{\frac{1}{p-1}}$. This means that both have their origin on the line $(y - y_1) = (x - x_1)e^{\frac{1}{p-1}}$. Finally, in order for the third coefficients to be equal we get that $h_1^2 = h_2^2$ which implies that $h_1 = h_2$ since we have defined the height to be positive.

What remains to show is that if the two bisectors are the same (i.e. all their coefficients are the same), then they are generated by the same pair of points. From the above deductions, this means that we need to show that two translates of the same curve are the same only if they have the same origin (as we have already established that both pairs of points share the same eccentricity and height). Furthermore, we know that both have their origin on the line $(y - y_1) =$

$(x - x_1)e^{\frac{1}{p-1}}$. Each curve only intersects this line at their origin, however. Thus they could not be identical unless they met in this point, so we are done. $\square$

5.5 Failures

Though we do not know of any strictly convex metrics for which there exists a point set E of n points in the plane such that Theorem 5.2.5.1 fails, it is clear that our method of proof cannot easily be extended. Here we will give examples of strictly convex metrics where the bisectors intersect in ways that are problematic. Note that Székely's method works if all the bisectors are lines, or as we showed in Theorem 5.3.5.1, his method can be extended to work if relatively few bisectors are lines and otherwise pairs of bisectors intersect in fewer than c_o points. We seem unable, however, to accommodate metrics where there are "many" linear portions in "many" of the bisectors. We look at a few examples of metrics that we cannot deal with presently.

There are two obvious classes of metrics for which bisectors will have "many" linear pieces. The first is when we have a metric K that looks like the Euclidean metric in some places and looks like a potato metric in others. In this case we will have linear portions in bisectors whenever the Euclidean parts of the metric interact. See Figure 5.6. The other class of metrics which gives rise to many linear bisectors are those which have infinitely many axes of symmetry, but otherwise act like potato metrics. In each of these cases it is quite possible that we can use a counting argument to account for these linear portions, though neither seems immediately clear.

Another property we depend on is the fact that non-linear bisectors in Theorems 5.2.5.1, 5.3.5.1 and 5.4.5.4 intersect in at most $c_o < \infty$ points. In fact, the

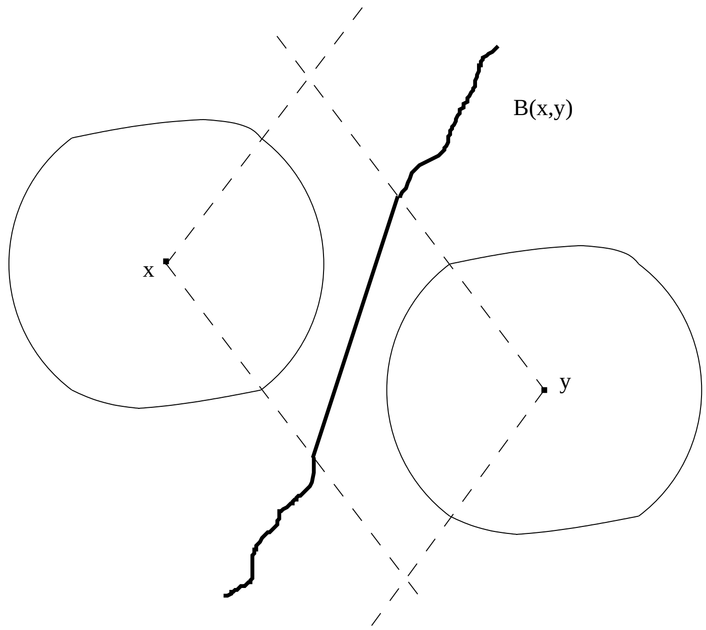

Figure 5.6: Example of bad bisector

constants in all of these theorems depend on c_o. We have not constructed metrics that intersect infinitely often that are not coincident, but expect such metrics to exist.

CHAPTER 6

The $n^{6/7}$ theory and beyond

The next big advance on the Erdős distance problem came several years later in 2001 when Solymosi and Tóth [45] proved that $f_2(n) = \Omega(n^{6/7})$. They rely heavily on Székely's method from [50], but also use Theorem 4.0.4.3 and Beck's theorem [6] on how many lines are incident to at least two points. We will begin this chapter by sketching this proof, as all advances on the EDP since have built on it. In fact, the advancements made by Tardos [53] and Katz and Tardos [20] improve upon a single lemma within this proof. We will briefly discuss this key lemma and its limitations in proving the conjecture that $f_2(n) = \Omega(n^{1-\epsilon})$ for all $\epsilon > 0$. In this chapter we will also discuss the problem of extending [45] to other convex metrics.

6.1 Solymosi and Tóth's argument

As mentioned, the proof that $f_2(n) = \Omega(n^{6/7})$ will require several new tools. In addition to using Theorem 4.0.4.3, we will also use the following theorem, which we state without proof.

Theorem 6.1.6.1 (Beck, [6]). *Given n points in the plane, at least one of the following statements must hold:*

1. *There exists a line incident to at least $n/100$ points.*

2. *There are $\Omega(n^2)$ lines incident to at least two points.*

The authors of [45] note the interconnectedness of the tools used in their proof. Theorem 6.1.6.1 can be proved using Theorem 4.0.4.3, which in turn can be proved using Székely's methods, see [46] and [50] respectively.

Theorem 6.1.6.2 (Solymosi and Tóth, [45]). *Given a set E of n points in the plane there exists a point determining $\Omega(n^{6/7})$ distinct Euclidean distances from the others.*

Sketch of Proof. As in the proof of 5.1.5.1 we let $t = \max_{p \in E} |\{||p-q||_2 : q \in E\}|$. We may assume that $t = o(n)$ and $t \gtrsim n^{3/4}$. Let $\mathcal{L}$ denote the set of lines passing through at least two points of E. Since $t = o(n)$ we can use an averaging argument to deduce from Theorem 6.1.6.1 that there exists an absolute constant c_1 such that at least $c_1 n$ points of of E are incident to at least $c_1 n$ lines of $\mathcal{L}$. Then let B be the set of such points, and take some $a \in B$.

Draw in the lines through a that go through points of E. We know there must be at least $c_1 n$ such lines. Take only one point other than a on each of these lines for the time being. Draw in the circles around a that hit those chosen points and delete those capturing fewer than 3 points. On each circle break the points in triples, possibly deleting as many as 2 from each. We still have $\gtrsim n$ points left by our hypothesis.

We call a triple "bad" if all three bisectors formed from its points go through at least k points. And we call the initial point a from B "bad" if at least half of its triples are bad. Our first step is to choose m such that at least half the points of B are bad. Then we will get upper and lower bounds on the number of incidences $I(L_k, p)$ between k-rich lines and bad points.

Clearly, the smaller k is the "easier" it is to get k-rich lines and thus more bad points. Therefore we want the largest such k. The idea is to construct a multigraph from the points in the triples and the arcs on the circles, as in the proof of Theorem 5.1.5.1, and then apply Theorem 4.0.4.2. Doing this we find that we can take $k = \frac{c_2 n^2}{t^2}$ and at least $c_1 n/2$ points of B will be bad, see [45] for the details.

Finding an upper bound on $I(L_k, p)$ is straightforward. We simply apply Theorem 4.0.4.3 to find a bound on the number of k-rich lines and use it a second time to get that $I(L_k, p) \lesssim t^4/n^2$. Getting a lower bound on the quantity $I(L_k, p)$ in terms of n and t is somewhat harder. We will need the following lemma, which is tight as stated. We omit the proof, see [45], but will discuss its importance in subsequent sections.

Lemma 6.1.6.3. *Let T be a set of N triples (a_i, b_i, c_i) of distinct real numbers such that $a_i < b_i < c_i$ for $i = 1, \dots, N$ and $c_i < a_{i+1}$ for all but at most $t-1$ of the i. Let $W = \{\frac{a_i+b_i}{2}, \frac{a_i+c_i}{2}, \frac{b_i+c_i}{2} : i = 1, \dots, N\}$. Then $|W| \gtrsim \frac{N}{t^{2/3}}$.*

For each point $p \neq a$ in a bad triple about a bad point a, map p to the orientation of the ray $\overrightarrow{ap}$. By construction this map is an injection, and W corresponds to k-rich lines. Therefore the number of k-rich line incident to a is $\gtrsim n/t^{2/3}$. And since a was an arbitrary element of B, we get that $I(L_k, p) \gtrsim n^2/t^{2/3}$. Combining the two bounds we have our result. □

6.2 Extending to other convex metrics

We would hope that the proof of Theorem 6.1.6.2 could be extended to work for other strictly convex metrics, just as the $n^{4/5}$ argument could be modified. At first glance, it appears that the use of Beck's Theorem will create difficulties, but we will not have to modify this part at all. Another thing we need to look at is Lemma 6.1.6.3. The purpose of Lemma 6.1.6.3 is to make sure that we are not counting any k-rich bisector multiple times. In the case of potato metrics, however, we know that any two different pairs of points necessarily have different bisectors. Thus we get that $I(B_k, p) \gtrsim n^2$ quite easily, where B_k is a k-rich bisector.

Having addressed the issue of using Beck's Theorem and Lemma 6.1.6.3 when extending Theorem 6.1.6.2 to potato metrics, it seems hopeful that we may get a similar result for potato metrics. Let us go through the rest of the proof of Theorem 6.1.6.2, modifying it for potato metrics in the most obvious ways. After the basic set up, the proof of Theorem 6.1.6.2 has three main parts. The first step is finding the largest value k such that at least half the points of E are bad. The other two parts are devoted to obtaining upper and lower bounds on $I(B_k, p)$. We have already found an obvious lower bound on $I(B_k, p)$, so we focus on the other two steps now.

We can simply follow Theorem 6.1.6.2 verbatim to get that if $k = \frac{c_2 n^2}{t^2}$ then at least $c_1 n/2$ points of B will be bad (using the notation from the proof of Theorem 6.1.6.2). Then we apply Theorem 4.0.4.4 twice (instead of applying Theorem 4.0.4.3) to get that

$$\begin{aligned} I(B_k, p) &\lesssim m^{1/3}(nB_k)^{2/3} + nm + B_k \\ &\lesssim m^{1/3} n^{2/3} \left(\frac{mn^2}{k^3} \right)^{2/3} \\ &\lesssim mt^4/n^2 \end{aligned}$$

where m is the number of bisectors that go through two points.

If we take pairs instead of triples in the initial stages, we are essentially back in the proof of Theorem 5.1.5.1. In that case we can say that $m \leq 2t$, and comparing the upper and lower bounds on $I(B_k, p)$, we get that $n^2 \lesssim t^5/n^2$, or $t \gtrsim n^{4/5}$, as we had before. Therefore, we must see how m changes when we take triples. Also, we look to see if we can further improve Lemma 6.1.6.3 when considering potato metrics.

6.3 Improvements on Lemma 6.1.6.3

As mentioned, the most recent progress on the Erdős distance problem has been made by improving Lemma 6.1.6.3. We will not go into detail on the methods involved in improving this lemma, but we will give a sketch of how it implies improved bounds on the EDP. In [53], Tardos gives a thorough description of the problem and we basically use his notation here.

First, we need to give some definitions. Given a real valued $n \times s$ matrix $A = (a_{ij})$, let $S(A) = \{a_{ij} + a_{ik} | 1 \leq i \leq n, 1 \leq j < k \leq s\}$. Then define $g_s(n) = \min\{|S(A)| : \text{all } sn \text{ entries of } A \text{ are distinct}\}$. It turns out that if one can get $g_s(n) = \Omega(n^\alpha)$ for some $s > 0 \in \mathbb{Z}$ and $\alpha > 0 \in \mathbb{R}$, then it implies that $f_2(n) = \Omega(n^{\frac{4}{5-\alpha}})$. The reason is that if we can get such estimates for $t \leq n$, then in Lemma 6.1.6.3 we get that $|W| \gtrsim \frac{n}{2t} g_s(t)$. Therefore the lower bound on $I(L_k, p)$ in Theorem 6.1.6.2 is $\frac{n^2}{t^{1-\alpha}}$. Keeping the previous upper bound on $I(L_k, p)$ of t^4/n^2 implies that $t \gtrsim n^{\frac{4}{5-\alpha}}$ as desired.

In [45], Solymosi and Tóth implicitly use that $g_3(n) = n^{1/3}$ and thus get that $t \gtrsim n^{6/7}$. It is also worth noting that since $g_2(n) = 1$, we can easily retrieve Székely's result of $t \gtrsim n^{4/5}$ in this language. Since $g_3(n)$ is tight (as noted in [45]) it is necessary to investigate $g_s(n)$ for $s > 3$ and try to get larger values of α in order to get improvement in the EDP. In terms of Theorem 6.1.6.2 this means that instead of looking at triples, we look at s-tuples in the initial set up of the proof. Solymosi and Tóth noted that their bound could probably be improved by doing this, but they did not formulate Lemma 6.1.6.3 in this way.

We give a list of improved bounds on $g_s(n)$, but as Tardos notes in [53], beyond $s = 4$ the correct orders of magnitude are still not known. In [53] Tardos gives an gives an elementary proof that $g_5(n) = \Omega(n^{4/11})$, and using significantly more complex methods he shows that for all $\epsilon > 0$ there exists a positive integer $s = s(\epsilon)$ such that $g_s(n) \geq n^{1/e-\epsilon}$, where e is the base of the natural logarithm. The most recent improvement was given by Katz in [20] in 2004, where he shows that $g_5 = \Omega(n^{7/19})$. This final improvement yields the following as a corollary:

Theorem 6.3.6.1 (Katz and Tardos [20]). *Given a set E of n points in the plane, one of them determines $\gtrsim n^{19/22}$ distinct Euclidean distances from the others.*

While improving the bounds on $g_s(n)$ can give improved bounds on the EDP, an (unpublished) example by Imre Ruzsa shows that $g_s(n) = O(n^{\frac{1}{2}-\frac{1}{2s-2}})$ for even values of s and so one cannot hope to get beyond $n^{8/9}$ with such methods. Furthermore, the experts in this area do not feel confident that the upper bound on $g_s(n)$ is any closer to the truth than the lower bounds that they have obtained.

In relation to the problem of changing the metric in the EDP these recent improvements to Lemma 6.1.6.3 do not seem to help us at all. Since all bisectors are different in potato metrics we do not have to worry about the fact that two different pairs could have the same bisector. Thus, as mentioned before, we get that $|W| \gtrsim n$ for potato metrics which is already much better than the best possible outcome for the Euclidean metric where one would get $|W| \gtrsim n/\sqrt{t}$.

CHAPTER 7

Related problems

In this chapter we discuss problems that are closely related to the Erdős distance problem. For each related problem we give the connection to the EDP, as well as a brief outline of the history and progress on the problem. Whenever possible we will also mention results dealing with non-Euclidean metrics. The purpose of this chapter is to introduce the reader to the broader area of distance set problems and provide other approaches to the EDP. We do not prove any new results here.

7.1 Unit distances in the plane

Here we discuss another problem of Erdős which asks: What is the maximum number of unit distances for n points in the plane? More formally, we want to find $g_2(n) = \max_{|E|=n} |\{(u,v) \mid u,v \in E, ||u-v||_2 = 1\}|$. Erdős asked this question in the same paper that he asked the EDP [17] and showed that $n^{1+c/\log\log n} < g_2(n) < n^{3/2}$, again using the lattice example to get the lower bound. He conjectured that the lower bound should be the truth.

To start, let us see why this is a stronger statement and implies results on the EDP. We will show that if $g_2(n) \lesssim n^{\alpha}$ then $f_2(n) \gtrsim n^{2-\alpha}$. By scaling, we can assume that the unit distance occurs more than any other distance, so that $g_2(n)$ is the maximum number of times any distance can be repeated. Let t be the number of distinct distances. Then it is clear that $t \times g_2(n) \geq \binom{n}{2}$ and thus

$f_2(n) \gtrsim n^2/n^\alpha$ as desired.

With this implication in hand, it is interesting to note that Erdős actually had two proofs that $f_2(n) \gtrsim n^{1/2}$ in [17] since he also proved that $g_2(n) < n^{3/2}$. He did not mention the connection between the two problems. In fact, his proof regarding $g_2(n)$ relied on the strict convexity of the L_2 metric, whereas his proof that $f_2(n) \gtrsim n^{1/2}$ did not.

We now give a brief summary of the results on the unit distance problem. The next improvement by Jozsa and Szermerdi [19] proved that $g_2(n) = o(n^{3/2})$, which Beck and Spencer [7] improved to $n^{1.44\cdots}$. Then, in 1984, J. Spencer, E. Szemerédi and W.T. Trotter showed in [49] that $g_2(n) = O(n^{4/3})$, using a proof sharing many of the same lemmas and techniques as in [14]. By the above discussion this last result implies that $f_2(n) = \Omega(n^{2/3})$, which we recall was proved directly by Moser over 30 years earlier [41]. Furthermore, the proof in [49] requires a fair amount of work to set up compared with Moser's result. Clearly, progress on the unit distance problem is considerably slower, but if the conjecture is true it would prove the EDP conjecture as well.

To date, there have not been any further improvements on the unit distance problem. However, others have showed the $n^{4/3}$ bound using various methods. For example, Székely proves this bound in an elegant way using Theorem 4.0.4.1 in [50]. Of particular interest to the work presented here is a result of Clarkson et al., in [15]. They show that the maximum number of incidences between n points and m unit circles is $\lesssim (nm)^{2/3} + m + n$ which gives that $g_2(n) \gtrsim n^{4/3}$. Simply let $m = n$ and draw in the unit circles around each point and note that any pair of points that are distance one from one other accounts for two incidences. Moreover, they proved this for any strictly convex metric, so that $g_K(n) \lesssim n^{4/3}$ (and therefore $f_K(n) \gtrsim n^{2/3}$) for any strictly convex asymmetric metric K.

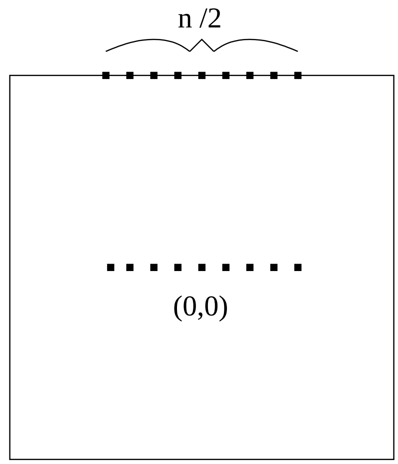

Figure 7.1: Many unit distances for L_1

While this result of Clarkson et al [15] is the best known bound for strictly convex metrics, Peter Brass obtained several results for convex metrics which are not strictly convex in [10]. Using his notation, let $\theta(||\cdot||_K)$ denote the (K)-length of the longest line-segment in the unit disk $\{x : ||x||_K = 1\}$. He first proves that $0 \leq \theta(||\cdot||_K) \leq 2$ for all convex norms K, with $\theta(||\cdot||_K) = 0$ if and only if K is strictly convex, and $\theta(||\cdot||_K) = 2$ if and only if the unit disk is a parallelogram. Therefore, we have from above that $n \log n \lesssim g_K(n) \lesssim n^{3/4}$ for $\theta(||\cdot||_K) = 0$. Brass goes on to show that $g_K(n) = \lfloor n^2/4 \rfloor$ if $0 < \theta(||\cdot||_K) \leq 1$ and $g_K(n) = \lfloor (n^2+n)/4 \rfloor$ if $1 < \theta(||\cdot||_K) \leq 2$. See Figure 7.1 to see how one gets at least $\lfloor (n^2+n)/4 \rfloor$ distances for the L_∞ metric. We note that in [10] Brass also gives precise bounds for the maximum number of occurrences of the largest distance and smallest distance for n points in the plane, depending on the value of $\theta(||\cdot||_K)$, which is another related problem in the area.

7.2 The k most frequent distance problem

Another way to arrive at results on the Erdős distance problem is by looking at the related problem of incidences between circles and points. In [46], Solymosi, Tardos, and Tóth improve the upper bound on the number of incidences between n points and n families of at most k concentric circles in the plane. They use methods very closely related to those in [45] and arrive at the current world record for the EDP. It should be clear, given the proof of the $n^{6/7}$ argument that this should be the case. See [46] for references to other such results in incidence theory.

As for the question of bounding the number of incidences between any n points and any m circles, it is conjectured that $I(n,m) = O((nm)^{2/3}\log^c(nm) + n + m)$ (see [46]), which would be asymtotically tight by the $\sqrt{n}\times\sqrt{n}$ lattice example. In [3], Aronov and Sharir show that $I(n,m) = O((nm)^{2/3} + n^{6/11+3\epsilon}m^{9/11-\epsilon} + n + m)$ for any $\epsilon > 0$. Note that this implies that $f_2(n) \gtrsim n^{7/9}$. Simply let t be the maximum number of distinct distances from any of the n points, and as usual draw in the circles around each point that capture the remaining $n-1$ points. Then $m = nt$ and combining this with the fact that $I(n,m) \gtrsim n^2$, we get the result.

In fact, this is the general method that Agarwal, Nevo, Pach, Pinchasi, Sharir and Smorodinsky used in [1], where they show that $f_K(n) \gtrsim n^{(7/9)}/\kappa_s(n)$ for a broad class of strictly convex metrics K, where $\kappa_s(n) = (\log n)^{O(\alpha^s(n))}$, where $\alpha(n)$ is the (extremely slowly growing) inverse Ackermann function and s depends on K. In this paper the authors were able to extend the result from [3] that is mentioned above to work for many strictly convex shapes in addition to circles. The proof tends to depend on smoothness properties and therefore it holds for a rather general class of asymmetric metrics, see [1] for the precise conditions on the metrics. This result gives hope that many strictly convex metrics (if not all) will behave similarly to the Euclidean metric for the EDP.

7.3 Well-distributed sets

In the Erdős distance problem we want to find the minimum number of distinct distances amongst any set of n points in the plane, but there are several variants that look at more specific point sets. For instance, several people (including Erdős) looked at points set in convex and general positions. Another popular type of sets to consider is well-distributed point sets. There is some motivation to look at such sets from certain problems in analysis, see [28] and references therein. Furthermore, as noted in [47], all the examples of sets with small distance sets relative to their size (such as the lattice) are homogeneous.

We present (roughly) the notation used in [31] and [23]. A set E in the plane is called well-distributed if there exists a constant C such that every cube side of length C contains at least one point of E. Then clearly $E_{\sqrt{n}} := E \cap [-\sqrt{n}/2, \sqrt{n}/2]^2$ has $\gtrsim n$ points. So, for the analog of the EDP in this setting (called the Asymptotic EDP in [5]) it is conjectured that $|\Delta_2(E_{\sqrt{n}})| \gtrsim n^{1-\epsilon}$. For a general convex metric K, let $E_{\sqrt{n}}$ as above and let $\Delta_K(E_{K,\sqrt{n}}) = \{||x-y||_K | x, y \in E_{K,\sqrt{n}}\}$.

Most of the results in this area can be found in [31] and [23]. In [31], Iosevich and Łaba show that if a well-distributed set $E_{\sqrt{n}}$ has $|\Delta_K(E_{K,\sqrt{n}})| \lesssim n^{3/4-\epsilon}$, then the metric K must be a polygon. Furthermore, if under the same assumptions $|\Delta_K(E_{K,\sqrt{n}})| \lesssim \sqrt{n}$ then K can only have finitely many sides. In [23], Konyagin and Łaba give necessary and sufficient conditions for the existence of a well-distributed set $E_{\sqrt{n}}$ such that $|\Delta_K(E_{K,\sqrt{n}})| \lesssim \sqrt{n}$. In particular, they show that there must be a coordinate system for which the slopes of all the sides in the finitely-sided polygon K are algebraic, otherwise $|\Delta_K(E_{K,\sqrt{n}})| = \omega(\sqrt{n}\log\sqrt{n}/\log\log\sqrt{n})$.

Contrasting these results with example 2.2.2.1, we see that the above results depend heavily on the well-distributivity property imposed on the metric. As noted in [31], if the slopes of the line segments connecting points in $E_{\sqrt{n}}$ is not dense, then the metric can be changed arbitrarily in those directions.

7.4 EDP in higher dimensions

We have focused solely on the plane thus far, but the Erdős distance problem can be formulated in higher dimensions as well. One simply considers the minimum number of distances determined by a set E of n points in $\mathbb{R}^d$; let us denote this minimum by $f_{K,d}(n)$. In Erdos's original paper on the subject [17] he showed $n^{1/d} \lesssim f_{L_2,d}(n) \lesssim n^{2/d}$. Again it is conjectured that the upper bound is closer to the truth, namely that for every $\epsilon > 0$ there exists a constant C_ϵ such that $|\Delta_2(E)| \geq C_\epsilon n^{2/d-\epsilon}$. The upper bound is attained by the $n^{1/d} \times n^{1/d} \times \ldots \times n^{1/d}$ integer lattice in $\mathbb{R}^d$.

The most obvious way to approach the case of higher dimensions is by induction. In order to do this it is necessary to have that results on $\mathbb{R}^d$ also work on S^d. The argument goes as follows. Suppose that any set E of n points in $\mathbb{R}^d$ or S^d determines at least n^s distinct distances. Now take any set of n points in $\mathbb{R}^{d+1}$, and let t be the maximum number of points that is determined from any point in E. Let u be some constant such $0 < u < 1$. If $t \geq n^u$ then we have at least n^u distances. Otherwise, there exists some sphere S^d with at least n^{1-u} points on it. And by the induction hypothesis we have at least $n^{(1-u)s}$ distinct distances. Therefore $f_{L_2,d+1} \gtrsim min\{n^u, n^{(1-u)s}\}$ distances. This is maximized when $u = s/(s+1)$. If we do this from $d = 2$ with $f_{K,2}(n) \gtrsim n^s$, and we additionally have our result on spheres, then we get that $f_{K,d}(n) \gtrsim n^{s/(1+(d-2)s)}$.

In [28] Iosevich notes that many authors have shown that their results in the plane hold on the sphere, since many of the geometric properties that hold in the plane transfer to the sphere. However, even if we achieve $s = 1$, as is conjectured in the plane, we only get $f_{L_2,d}(n) \gtrsim n^{1/(d-1)}$ in $\mathbb{R}^d$ for $d \geq 3$, which is not very close to the conjectured bound of $n^{2/d-\epsilon}$. Of course, if one gets bounds in dimension $d \geq 3$, then induction can lead to better bounds for $d_o > d$.

To see the next result for $d = 3$ we refer the reader back to the first part of this chapter, where we discussed the function $g_2(n)$. To keep our notation consistent, let us denote the maximum number of times the same distance can occur amongst n points in $\mathbb{R}^d$ by $g_{K,d}(n)$. Clarkson et al. showed in [15] that $g_{L_2,3} = O(n^{3/2+\epsilon})$, which we recall implies that $f_{L_2,3} = \Omega(n^{1/2-\epsilon})$. Aronov, Pach, Sharir, and Tardos [4] recently improved this by showing that $f_{L_2,3} = \Omega(n^{77/141-\epsilon}) = \Omega(n^{.546\cdots})$, and by showing that the same holds on the three dimensional sphere, they obtain analogous results in higher dimensions by the above induction argument. Namely, they get that $f_{L_2,d}(n) = \Omega(n^{1/(d-90/77)-\epsilon})$ for $d \geq 3$. Most recently, Solymosi and Vu show in [48] that $f_{L_2,d}(n) = \Omega(n^{\frac{2}{d}-\frac{2}{d(d+2)}})$ for $d \geq 3$, which is very close to the conjectured bound. They show that $n^{2/d}$ is near optimal, in the sense that the exponent cannot be replaced by $(2-\epsilon)/d$ for any $\epsilon > 0$ for d large enough. Moreover, for $d = 3$ they get $f_{L_2,3} = \Omega(n^{.5643})$, by using the improved values of α [20] discussed in chapter 6.

We also note that there has been work done on the case of well-distributed (or homogeneous) point sets in higher dimensions. In [28] Iosevich showed using Moser's argument from [41] that $f_{L_p,d}(n) \gtrsim n^{3/2d}$ for $1 < p < \infty$, for any n

points well-distributed in $\mathbb{R}^d$. In fact, this argument should work for all point sets, but it is very quick if they are well-distributed. Recently in [47], Solymosi and Vu proved that $f_{L_2,d}(n) = \Omega(n^{2/d-1/d^2})$ for n points well-distributed in $\mathbb{R}^d$, which improves slightly on their bound for arbitrary sets. It should be noted that their definition of well-distributed (which they call homogeneous) is slightly less restrictive than in [28].

It is probably a relatively easy exercise to show that Theorems 5.2.5.1 and 5.4.5.4 also hold on $S^2 \subset \mathbb{R}^3$, which by the above argument would give us results in higher dimensions by induction. The bounds obtained this way are not very good (i.e. $\Omega(n^{4/9})$ for $d=3$) and so it would be more productive to attack $d=3$ directly. For instance, the methods in [4] rely primarily on methods like incidence thoery that we have encountered in this work, so there is a decent chance that it could be modified to work for L_p metrics as well.

We note that there has been recent progress on the EDP on the sphere by Iosevich and Rudnev [33].

7.5 The Falconer distance problem

The well-known Falconer distance conjecture is essentially the continuous analog of the Erdős distance problem. In dimension d it says: If E is a compact subset of $\mathbb{R}^d$ with $\dim(E) \geq d/2$, then $\dim(\Delta_2(E)) = 1$, where $\Delta_2(E) := |E - E| = \{||x-y||_2 : x, y \in E\}$ and dimension refers to Hausdorff dimension. Here we will give a brief outline on the progress on this problem, as well as mention its relationship to the EDP. For more thorough descriptions of both, see [39], [5] and [55].

The problem was first explored by Falconer in [18], where he showed that if

$\dim(E) \geq (d+1)/2$ then $\dim(\Delta_2(E)) = 1$. Bourgain [8] improved this result for all d and showed $\dim(E) \geq 13/9$ implies that $\dim(\Delta_2(E)) = 1$ for $d = 2$. In 1999 Wolff proved the best known result for $d = 2$ in [54] using a Fourier analytic approach. He showed that if $\dim(E) \geq 4/3$ then $\dim(\Delta_2(E)) = 1$, and also explained that these method could do no better in the plane. In 2003 Erdoğan [16] gave another proof of Wolff's theorem and showed that if $\dim(E) \geq d(d+2)/2(d+1)$ then $\dim(\Delta_2(E)) = 1$. Furthermore, Erdoğan remarks that his result holds for any metric K whose unit ball is smooth with non-vanishing Gaussian curvature.

In addition to these results, there are several in the reverse direction for $d = 2$. In [38] Mattila showed that if $\dim(E) > 1$ then $\dim(\Delta_2(E)) \geq 1/2$. In this same paper Mattila also gave a useful reduction for solving the Falconer distance conjecture that provided the framework for all future results. The most recent result on this related problem is due to Bourgain; in [9] he showed that there exists $c > 0$ such that $\dim(E) > 1$ implies that $\dim(\Delta_2(E)) \geq 1/2 + c$.

There are several related results. Most relevant to this work is the work of Iosevich and Łaba [32], where they give various results with respect to general metrics. Also in this vein, Konyagin and Łaba [22] obtain results for polygonal norms, see [23] and [22]. Very recently, Iosevich and Rudnev [34] have shown that the Falconer distance problem is true when considering lattice like sets in higher dimensions. Additionally, their proof works for any metric that has a sufficiently smooth unit ball.

There is an obvious connection between the EDP and the Falconer distance conjecture, in that both are trying to determine the size of distance sets in terms of the size of the original set. Beyond that, there is no direct relationship between the two problems, though one might hope to borrow methods from one problem to use on the other. However, in [30] Iosevich and Hofmann show that the Falconer distance conjecture does imply an asymptotic version of the EDP, which was dicussed in section 7.3.

CHAPTER 8

Contributions and future directions

Here the primary goal was to get bounds on $f_K(n)$ for different classes of convex metrics K. While results of this nature are mentioned in the literature, this is the first comprehensive analysis of the question that we know of.

We were very happy to have proved that $f_K(n) \geq cn^{1/2}$ for all convex asymmetric metrics K, where c does not depend on K. Since $n^{1/2}$ is tight for all convex symmetric metrics that are not strictly convex, this is the best possible result we could have hoped for. The question of asymmetric metrics that are convex, but not strictly convex, is only partially answered here; for instance, the bounds we found for baseball metrics can surely be improved upon.

In terms of strictly convex metrics, we did not make a huge improvement on previously known bounds. Our methods, however, yield relatively short proofs and give the current best known bound of $n^{4/5}$ for the L_p metric $(1 < p < \infty)$. Based on other results, it seems likely that this bound can be improved upon and also extended to other strictly convex metrics. We expect to explore this further.

Throughout this work there are many open problems stated implicitly. In particular, it would be very interesting to find examples of metrics K for which $f_K(n) << n^{1-\epsilon}$. Another approachable project would be to establish what the broadest class of metrics for which our $n^{4/5}$ argument can be extended.

Finally, we mention the connection between the work done here and Voronoi diagram problems. Given sites P_i in the plane, a Voronoi diagram is the partition of the plane such that $x \in \mathbb{R}^2$ is assigned to its nearest site. Clearly, the bisectors of sites is important to this problem, so results such as ours may be useful in finding Voronoi diagrams for non-Euclidean metrics.

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