Weak Categories

Slides prepared by

Timothy Porter and Peter May

for a talk given by

André Joyal, Timothy Porter, and
Peter May

Survey: definitions and comparisons
of various notions of 'weak category'
and of enriched and internal variants.

- Quasi-categories
- Segal cats and complete Segal cats
- A_∞-categories
- Top-Cats and $\mathcal{S}$-cats ($\mathcal{S}$=sSets)
- Philosophy: weak $(\infty, 1)$-categories
- An analogue: weak complicial sets

Two related problems:

(i) ('small' version) need 'adequate' kinds of (small) weak $(-)$ categories, for instance to model homotopy types, stacks, higher dimensional automata, and so forth.

$(-)$ might be n or ∞, or $\cdots$

(ii) ('large' version) need an adequate weak $(-)$ category theory, including the essence of homotopy coherence to handle/study the objects of (i), and to implement potential applications.

Weakening categories

Understand $\mathcal{C}at$ in $\mathcal{S}$ (simplicial sets)

then weaken structure.

Quasi-categories

If $\mathbb{C}$ is a (small) category, then $Ner(\mathbb{C})$

is a weak or restricted Kan complex.

(Fillers for inner horns.)

If $\mathbb{G}$ is a groupoid, then $Ner(\mathbb{G})$ is a

Kan complex.

Definition

A *quasi-category* is a restricted Kan

complex.

Simplicial characterization of categories

Segal Maps

Let $p > 0$. We have increasing

maps $e_i : [1] \to [p]$ given by

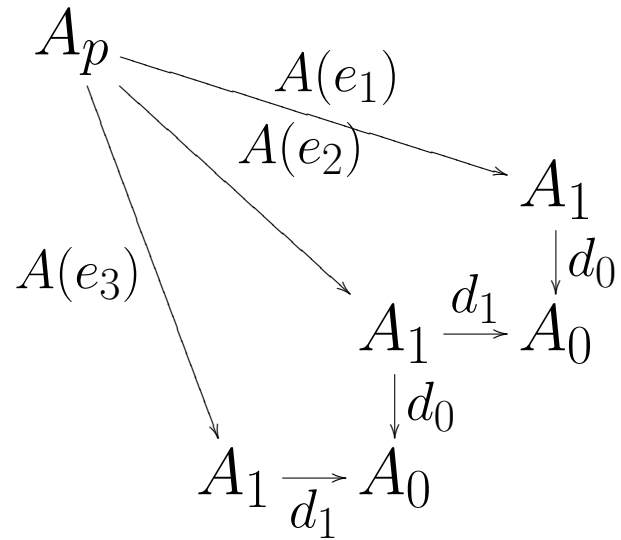$e_i(0) = i$ and $e_i(1) = i + 1$.

Note that $e_i(1) = e_{i+1}(0)$.

For a simplicial set $A : \mathbf{\Delta}^{op} \to Sets$,

evaluate A on the e_i to get functions

$A_p \to A_1$.

These yield a cone diagram,

for instance, when $p = 3$:

$$A_p \xrightarrow{A(e_1)} A_1$$

$$A(e_2)$$

$$A(e_3)$$

$$A_1 \xrightarrow{d_1} A_0 \quad \downarrow d_0$$

$$\downarrow d_0$$

$$A_1 \xrightarrow[d_1]{} A_0$$

We get "Segal maps"

$$\delta[p] : A_p \to A_1 \times_{A_0} A_1 \times_{A_0} \ldots \times_{A_0} A_1.$$

If $A = Ner(\mathbb{C})$ for a small category $\mathbb{C}$, then the Segal maps are bijections.

The converse holds:

If A is a simplicial set such that the Segal maps are bijections, then there is a category structure on the directed graph

$$A_1 \rightrightarrows A_0$$

making it into a category whose nerve is isomorphic to A.

Complete Segal categories

Definition

Let $\mathcal{C}$ be a Cartesian monoidal category and $\mathcal{W}$ be a subcategory of "weak equivalences".

A *complete Segal category* in $(\mathcal{C}, \mathcal{W})$ is a simplicial object A in $\mathcal{C}$ such that the Segal maps of A are in $\mathcal{W}$.

$(\mathcal{C}, \mathcal{W}) = (\text{Sets, Bijections})$: Get $\mathcal{C}at$.

Object sets play a distinguished role.

Segal categories

Assume that $\mathcal{C}$ has coproducts.

Define $\delta\colon Sets \to \mathcal{C}$ via coproducts

of copies of the terminal object $*$.

This defines "discrete objects" in $\mathcal{C}$.

Definition

A *Segal category* in $(\mathcal{C}, \mathcal{W})$ is a

complete Segal category A such

that A_0 is a discrete object of $\mathcal{C}$.

The original notion of Segal space is

$(\mathcal{C}, \mathcal{W}) = (\mathcal{T}op,$ weak equivalence$)$.

Can replace $\mathcal{T}op$ by $\mathcal{S}$ or $\mathcal{C}at$.

The starting point, $n = 2$, of the

Tamsamani-Simpson definition of

weak n-category is

$(\mathcal{C}, \mathcal{W}) = (\mathcal{C}$at, equivalence$)$.

A_∞ spaces and categories

Segal categories in $(\mathcal{T}op, \text{weak equiv})$

with object set $*$ model loop spaces.

Stasheff associahedra operad $\mathcal{K} = \{\mathcal{K}(q)\}$.

A_∞ spaces,

$$\mathcal{K}(q) \times X^j \to X,$$

also model loop spaces.

Many object version:

A_∞-category in $\mathcal{T}op$.

DG A_∞ algebras and categories

$C_*(\mathcal{K})$: operad of chain complexes.

DG A_∞ algebra:
$$C_*(\mathcal{K}(q)) \otimes A^q \to A.$$

Many object version:

DG A_∞-category.

(Fukaya, Kontsevich,
mirror symmetry.)

A generalization gives a crude initial pattern for the operadic definitions of weak n-categories of Batanin, Leinster, Trimble, and May.

A_∞-category

Let $\mathcal{C}$ be a monoidal category with
unit object $*$ and a subcategory $\mathcal{W}$.
An A_∞-operad in $(\mathcal{C}, \mathcal{W})$ is an
operad $\mathcal{O}$ in $\mathcal{C}$ with an augmentation
$\mathcal{O}(q) \to Ass$ such that $\mathcal{O}(q) \to *$
is in $\mathcal{W}$ for each q.

Definition

An A_∞-category $\mathbb{C}$ in $(\mathcal{C}, \mathcal{W})$ is an
$\mathcal{O}$-category for an A_∞ operad $\mathcal{O}$.

$$\mathcal{O}(q) \otimes \mathbb{C}(S_{q-1}, S_q) \otimes \ldots \otimes \mathbb{C}(S_0, S_1)$$
$$\longrightarrow \mathbb{C}(S_0, S_q)$$

Enriched $\mathcal{V}$-categories

Let $\mathcal{V}$ be a closed symmetric monoidal category with unit object I.

Change point of view, allow classes of objects in our structured categories $\mathcal{C}$.

Enrichment over $\mathcal{V}$:

Objects $\mathcal{C}(X, Y)$ of $\mathcal{V}$ for objects

X and Y of $\mathcal{C}$

$$\mathcal{C}(S, T) \otimes \mathcal{C}(R, S) \to \mathcal{C}(R, T)$$

$$\mathsf{Id} \colon I \to \mathcal{C}(S, S)$$

$\mathcal{V}$-Cat $= \mathcal{V}$-enriched small categories.

Internal categories in $\mathcal{V}$

Let $\mathcal{V}$ be bicomplete.

An internal category $\mathbb{C}$ in $\mathscr{V}$ consists

of objects $\mathbb{O}(\mathbb{C})$ and $\mathbb{M}(\mathbb{C})$ of $\mathcal{V}$ with

maps S, T, Id, and C in $\mathcal{V}$ which

satisfy the category axioms.

$\mathcal{C}\mathrm{at}(\mathcal{V}) =$ internal categories in $\mathscr{V}$.

Functor $\mathcal{V}\text{-cat} \rightarrow \mathsf{Cat}(\mathcal{V})$

δ on objects gives $\mathbb{O}(\mathbb{C})$, coproduct

of the $\mathbb{C}(S,T)$ gives $\mathbb{M}(\mathbb{C})$.

Get "object-discrete internal categories".

Why enrich and why $\mathcal{S}$?

Problems (i) and (ii) suggest that a 'good' category of (small) weak $(-)$ categories might be expected to itself be a (large) weak $(-)$ category and also a category 'weakly enriched' over weak $(-)$ categories.

Categories are intrinsically intertwined with simplicial sets, and simplicial sets are good for doing homotopy theory.

$\mathcal{S}$-enriched categories may be a good starting point.

Digression on alternatives

Joyal: Enrichment over quasi-categories may be an even better choice.

Porter: I do not favour Top-enriched categories as I have found them awkward to use in the applications that I have been studying.

May: I favor Top-enriched categories as they are indispensible to many applications I have been studying.

Example: topologized fundamental groupoid

Internal categories in Top are good.

ΠX has object space X, morphism space of Moore paths (varying length) in X, and $X \simeq B\Pi X$. Get quotient topological groupoid $\pi^{Top}X$ (identify homotopic paths) such that

$$B\pi^{Top}X \simeq B\pi X.$$

(πX is the discrete fundamental groupoid).

$$X \simeq B\Pi X \to B\pi^{Top}X \simeq B\pi X.$$

How to enrich over $\mathcal{S}$

Let $\mathcal{C}$ be symmetric monoidal and

$F \colon \Delta \to \mathcal{C}$ be a cosimplicial object.

$$[n] \to \mathcal{C}(X \otimes F([n]), Y)$$

defines enriched Hom.

Top, $\mathcal{S}$ Ch_k (chain complexes).

$\mathcal{S}m/k$ (Smooth schemes over k).

$\mathcal{C}$at: two interesting choices of F.

$[n] \mapsto$ Total order with $n+1$ objects,

$[n] \mapsto$ Groupoid with $n+1$ uniquely isomorphic objects.

Analogous construction works when $\mathcal{C}$ is tensored over $\mathcal{S}$.

In general:

For any category $\mathcal{C}$, $Simp(\mathcal{C}) = \mathcal{C}^{\Delta^{op}}$ is an $\mathcal{S}$-category.

Let K be a simplicial set. The comma category Δ/K has objects $([n], x)$ with $x \in K_n$. The morphisms

$$\mu : ([n], x) \to ([m], y)$$

are those $\mu\colon [n] \to [m]$ in Δ such that $K(\mu)(y) = x$.

There is a forgetful functor

$$\delta_K : \Delta/K \to \Delta, \quad \delta_K([n], x) = [n].$$

Given $X, Y \in Simp(\mathcal{A})$, define

$$Simp(\mathcal{A})(X, Y)_n = Nat(X\delta^{op}_{\Delta[n]}, Y\delta^{op}_{\Delta[n]}).$$

$\mathcal{S}$-categories versus $Simp(\mathcal{C}at)$

Let $\mathbb{B} : \boldsymbol{\Delta}^{op} \to Cat$ be a simplicial

object in Cat. It has a simplicial set

B of objects. Say that $\mathbb{B}$ is object

discrete if B is discrete. Then,

for $(x, y) \in B$, let

$$\mathbb{B}(x, y)_n = \{\sigma \in \mathbb{B}_n | S(\sigma) = x, T(\sigma) = y\}.$$

With faces and degeneracies induced

from those of $\mathbb{B}$, $\{\mathbb{B}(x, y)_n | n \in \mathbb{N}\}$

is a simplicial set $\mathbb{C}(x, y)$.

Levelwise composition in $\mathbb{B}$ induces

$$\mathbb{C}(x, y) \times \mathbb{C}(y, z) \to \mathbb{C}(x, z).$$

This gives an $\mathbb{S}$-enriched category $\mathbb{C}$.

Conversely, a small $\mathcal{S}$-category $\mathbb{C}$ gives an object discrete simplicial category $\mathbb{B}$: $\mathcal{S}$-categories are object discrete simplicial categories.

Recall: Segal categories are object discrete complete Segal categories. When regarding simplicial categories and complete Segal categories as bisimplicial sets, the two notions of "object discrete" differ by which 0th simplicial coordinate is required to be discrete.

Weak categories

Slogan:

"Weak category $= (\infty, 1)$-category".

"An (∞, n)-category is an ∞-category
with invertible i-morphisms, $i > n$".

An $(\infty, 0)$-category is a space.

That is, we can take any model for

the homotopy theory of spaces,

any $(\mathcal{C}, \mathcal{W})$ with correct $\mathcal{C}[\mathcal{W}^{-1}]$.

Examples: $\mathcal{T}\mathrm{op}$, $\mathcal{S}$, $\mathcal{C}\mathrm{at}$ (Thomason).

Baby Comparison Program

Compare notions of weak category.

Examples ($\mathcal{W} =$ 'weak equivalence'):

Quasi-category

Complete Segal category

Segal category

A_∞-category

$\mathcal{S}$-category

$\mathcal{T}op$-category

Weakening of strict ∞-Categories

"Complicial set": a simplicial set A with a stratrification by "thin" simplices $t_n A_n \subset A_n$ for all n (including all degenerate simplices) satisfying certain axioms.

Equivalent to strict ∞-Categories.

Weak complicial sets: weaken axioms.

Street's weak ∞ categories:

Intrinsic stratification on simplicial sets.

Require: this is a weak complicial set.

$\mathcal{S}$-categories and homotopy coherence

Part of large version of the general problem. Given a small category $\mathbb{D}$, use free-forget monad to get a free simplicial resolution

$$S(\mathbb{D}) \in Simp(Cat),$$

similar to free group simplicial resolution or bar resolution. If the monad 'remembers identities', then

$$S(\mathbb{D}) \in \mathcal{S}\text{-}Cat.$$

Vogt 1973, Cordier 1982:

A homotopy coherent diagram in Top of shape $\mathbb{D}$ is the same as an $\mathcal{S}$-functor

$$S(\mathbb{D}) \to Top.$$

Definition (Cordier (1980), based on earlier ideas of Vogt, Boardman-Vogt.)

Let $\mathcal{B}$ be simplicially enriched.

The *homotopy coherent nerve* of $\mathcal{B}$, denoted $Ner_{h.c.}(\mathcal{B})$, is the simplicial 'set' with

$$Ner_{h.c.}(\mathcal{B})_n = \mathcal{S}\text{-}Cat(S[n], \mathcal{B}).$$

If $\mathcal{B}$ is a locally Kan $\mathcal{S}$-category, then $Ner_{h.c.}(\mathcal{B})$ is a quasi-category.

(Cordier–Porter, 1986)

Question: If $\mathcal{B}$ is locally complicial, is $Ner_{h.c.}(\mathcal{B})$ a weak complicial set?

How else can one construct $\mathcal{S}$-cats?
Hammocks!

For a category $\mathcal{C}$ and subcategory $\mathcal{W}$,
construct an $\mathcal{S}$-category $L^H(\mathcal{C}, \mathcal{W})$,
or $L^H\mathcal{C}$, the *hammock localisation*
of $(\mathcal{C}, \mathcal{W})$, as follows:

The objects of $L^H\mathcal{C}$ are those of $\mathcal{C}$.

For objects X and Y, the k-simplices
of $L^H\mathcal{C}(X, Y)$ are the "reduced
hammocks of width k and any length"
between X and Y. Such a thing is a
commutative diagram of the form

$$
\begin{array}{ccccc}
C_{0,1} & \!\!\!\!\rule[0.5ex]{2em}{0.4pt}\!\!\!\! & C_{0,2} & \!\!\!\rule[0.5ex]{1em}{0.4pt}\cdots\rule[0.5ex]{1em}{0.4pt}\!\!\! & C_{0,n-1} \\
\downarrow & & \downarrow & & \downarrow \\
C_{1,1} & \!\!\!\!\rule[0.5ex]{2em}{0.4pt}\!\!\!\! & C_{1,2} & \!\!\!\rule[0.5ex]{1em}{0.4pt}\cdots\rule[0.5ex]{1em}{0.4pt}\!\!\! & C_{1,n-1} \\
\downarrow & & \downarrow & & \downarrow \\
\vdots & & \vdots & & \vdots \\
\downarrow & & \downarrow & & \downarrow \\
C_{k-1,1} & \!\!\!\!\rule[0.5ex]{2em}{0.4pt}\!\!\!\! & C_{k-1,2} & \!\!\!\rule[0.5ex]{1em}{0.4pt}\cdots\rule[0.5ex]{1em}{0.4pt}\!\!\! & C_{k-1,n-1} \\
\downarrow & & \downarrow & & \downarrow \\
C_{k,1} & \!\!\!\!\rule[0.5ex]{2em}{0.4pt}\!\!\!\! & C_{k,2} & \!\!\!\rule[0.5ex]{1em}{0.4pt}\cdots\rule[0.5ex]{1em}{0.4pt}\!\!\! & C_{k,n-1}
\end{array}
$$

with X on the left connecting to $C_{0,1}$, $C_{1,1}$, $C_{k-1,1}$, $C_{k,1}$, and Y on the right connecting to $C_{0,n-1}$, $C_{1,n-1}$, $C_{k-1,n-1}$, $C_{k,n-1}$
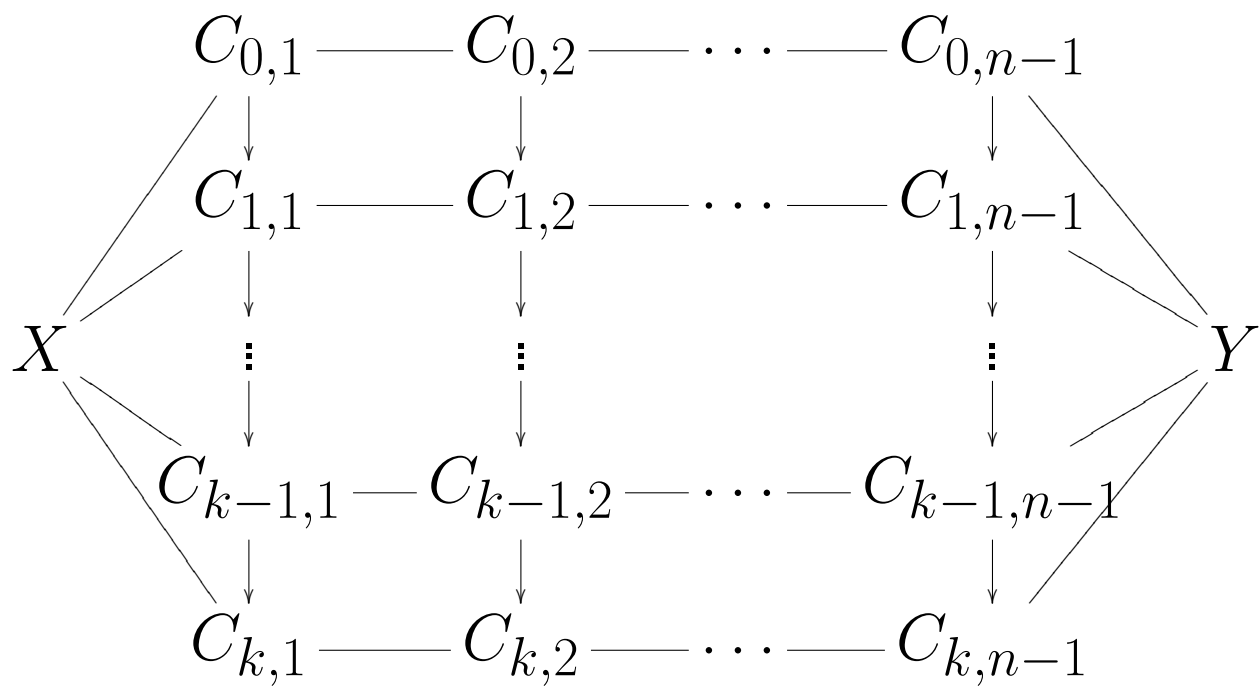
in which

(i) the length n of the hammock is an integer ≥ 0,

(ii) all the vertical maps are in W,

(iii) in each column of horizontal maps, all maps go in the same direction; if they go left, they must be in W.

The hammocks are subject to two irredundancy conditions.

(iv) the maps in adjacent columns go in different directions, and

(v) no column contains only identity maps.

Why the hammock localization?

Its category of components is $\mathcal{C}[\mathcal{W}^{-1}]$.

Version of inverting $\mathcal{W}$ that retains

higher homotopical information.

(Massey products or Toda brackets).

Baby camparison should give that the

hammock localizations of all models

for weak categories have equivalent

hammock localizations.

Model category theory shows how.