A personal note on Hochschild and Cyclic homology

by Jinhyun Park

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1. Definition of Hochschild homology and its properties

Let k be a commutative ring and A an associative k-algebra, not necessarily commutative, not necessarily unitary.

Definition 1.1 (Hochschild (standard) complex, or Cyclic Bar complex)**.** *Let M be an A-bimodule. Let $C_n(A,M) := M \otimes_k A^{\otimes n}$. Define operators $d_i : C_n(A,M) \to C_{n-1}(A,M)$ by*

$$d_0(m \otimes a_1 \otimes \cdots \otimes a_n) = ma_1 \otimes a_2 \otimes \cdots \otimes a_n$$

$$d_i(m \otimes a_1 \otimes \cdots \otimes a_n) = m \otimes a_1 \otimes \cdots \otimes a_i a_{i+1} \otimes \cdots \otimes a_n, \quad i \leq i \leq n$$

$$d_n(m \otimes a_1 \otimes \cdots \otimes a_n) = a_n m \otimes a_1 \otimes \cdots \otimes a_{n-1}.$$

Note that for $i < j$, $d_i d_j = d_{j-1} d_i$ so that $(C_n(A,M), d_i)$ is a presimplicial module, and for $b = \sum_{i=0}^{n} (-1)^i d_i$, $(C_n(A,M), b)$ is a complex called the Hochschild (standard) complex. *When $M = A$, we call it the* Cyclic Bar complex.

Definition 1.2 (Hochschild Homology)**.** *The n-th homology $H_n(A,M) := H_n(C_*(A,M), b)$. When $M = A$, we denote it by $HH_n(A)$.*

Remark. (1) Though $H_n(A,M)$ doesn't mention k, k plays an important role. For example, if $k = \mathbb{C}$, $HH_1(\mathbb{C}) = 0$ but, for $k = \mathbb{Q}$, $HH_1(\mathbb{C}) \neq 0$.

(2) (Functoriality) Hochschild homology $H_n(A,M)$ is covariant on both places. For $f : M \to M'$, an A-bimodule homomorphism, then, $f_* : H_n(A,M) \to H_n(A,M')$, $f_*(m \otimes a_1 \otimes \cdots \otimes a_n) = f(m) \otimes a_1 \otimes \cdots \otimes a_n$ is well-defined.

For $g : A \to A'$, a k-algebra homomorphism, M', an A'-bimodule, then, $g_* : H_n(A,M') \to H_n(A',M')$, $g_*(m \otimes a_1 \otimes \cdots \otimes a_n) = m \otimes g(a_1) \otimes \cdots \otimes g(a_n)$ is well-defined.

(3) (Respect products) $HH_n(A \times A') \simeq HH_n(A) \oplus HH_n(A')$, for A, A': k-algebras. [Hint: if A, A' are flat k-algebras, then, use the Tor definition which will be given soon, and apply the Künneth formula. Otherwise, construct a homotopy. Later we will give a geeneralized result.

(4) Let $Z(A) = \{z \in A | za = az, a \in A\}$ be the center of A. Then, naturally $C_*(A,M)$ is a $Z(A)$-module via $z \cdot (m \otimes a_1 \otimes \cdots \otimes a_n) = (zm) \otimes a_1 \otimes \cdots \otimes a_n$. By the commutativity of z, it commutes with b, so that $H_n(A,M)$ is also a $Z(A)$-module.

Example 1.3. From the Hochchild complex $C_1(A,M) = M \otimes A \xrightarrow{b} C_0(A,M) = M \to 0$, obviously we have

$$H_0(A,M) = M_A = M/\mathrm{im} b = M/\{am - ma | a \in A, m \in M\}.$$

In particular, $HH_0(A) = A/[A,A]$. When $A = M = k$, then the Hochschild complex is

$$\to k \xrightarrow{1} k \xrightarrow{0} k \xrightarrow{1} \cdots \xrightarrow{1} k \xrightarrow{0} k$$

so that $HH_0(k) = k$ and $HH_n(k) = 0$ for $n > 0$.

Lemma 1.4. *Let R be any unitary ring. Then, the abelianized trace map $Tr : M_r(R) \to R \to R/[R,R]$ induces an isomorphism*

$$Tr : M_r(R)/[M_r(R), M_r(R)] \xrightarrow{\simeq} R/[R,R].$$

Proof. Use elementary matrices to show that $\ker Tr = [M_r(R), M_r(R)]$.

□

Corollary 1.5. $HH_0(M_r(A)) = A/[A,A]$.
This is a special case of Morita invariance.

Proposition 1.6. *If A is commutative, then $HH_1 \xrightarrow{\simeq} \Omega^1_{A/k}$ canonically. Further if M is a symmetric bimodule, then, $H_1(A,M) \simeq M \otimes_k \Omega^1_{A/k}$.*

Proof. A being commutative, $C_1(A,A) = A \otimes A \xrightarrow{b} C_0(A,A) = A$ is 0. Hence, $HH_1(A) = A \otimes A/ab \otimes c - a \otimes bc + ca \otimes b$. Define $\phi : A \otimes A \to \Omega^1_{A/k}$ by $a \otimes b \mapsto adb$. Then, $\phi(ab \otimes c - a \otimes bc + ca \otimes b) = abdc - adbc + cadb = abdc - abdc - acdb + cadb = 0$ so that we have $\bar{\phi} : HH_1(A) \to \Omega^1_{A/k}$. Conversely, we can give $\phi : \Omega^1_{A/k} \to HH_1(A)$ by $adb \mapsto a \otimes b$. It is trivial to check that they are inverse to each other.

□

2. The 2nd description of Hochchild Homology

Now we assume that A is a unitary k-algebra.

Definition 2.1 (Standard complex, or Bar complex). *Let $C'_n(A) = C^{bar}_n(A) = A \otimes A^{\otimes n} \otimes A = A^{\otimes n+2}$ with $d_i : C'_n(A) \to C'_{n-1}(A)$ defined only for $0 \le i \le n-1$, which is a presimplicial module. Let $b' = \sum_{i=0}^{n-1}(-1)^i d_i$ and $b' : C'_0(A) = A \otimes A \to A$ be the multiplication which is an augmentation for the complex.*

Let $A^e = A \otimes A^{op}$, then, above complex is a complex of left A^e-modules with $a \otimes a' \in A \otimes A^{op}$ acts via

$$(a \otimes a')(a_0 \otimes \cdots \otimes a_{n+1}) = (aa_0) \otimes a_1 \otimes \cdots \otimes a_n \otimes (a_{n+1}a').$$

Proposition 2.2 (and Definition). *The augmented bar complex is a resolution of the A^e-module A called the bar resolution of A.*

Remark. An n-chain of the bar resolution is often denoted by $a_0[a_1|\cdots|a_n]a_{n+1}$.

Proof. The cokernel of the last map is clearly $\mu : A \otimes A \to A$. Define a homotopy $s : A^{\otimes n+1} \to A^{\otimes n+2}$, $s(a_0 \otimes \cdots \otimes a_n) = 1 \otimes a_0 \otimes \cdots \otimes a_n$. Then, $d_i s = s d_{i-1}$ for $i \ge 1$ and $d_0 s = id$. Hence $b's + sb' = id$ i.e. b'-complex is acyclic.

□

Remark. Note that $C_n(A, A \otimes A^{op}) = (A \otimes A^{op}) \otimes A^{\otimes n} \simeq A \otimes A^{\otimes n} \otimes A = C'_n(A)$.

In case A is a flat k-algebra, we have the following 2nd description of the Hochschild homology.

Proposition 2.3 (2nd description of Hochschild Homology). *Let A be a flat k-algebra. Then, for any A-bimodule M,*

$$H_n(A,M) = \mathrm{Tor}^{A^e}_n(M,A).$$

Proof. A being k-flat. $A^{\otimes n}$ is k-flat, so that $C'_n(A) = A \otimes A^{\otimes n} \otimes A = A \otimes A^e \otimes A^{\otimes n}$ is A^e-flat.

Now, by tensoring with M over A^e, we have $1_M \otimes_{A^e} b' = b$ via

$$M \otimes_{A^e} C'_n(A) = M \otimes_{A^e} (A \otimes A^{\otimes n} \otimes A) \simeq M \otimes_{A^e} (A \otimes A^{op} \otimes A^{\otimes n}) \simeq M \otimes_k A^{\otimes n} = C_n(A,M)$$

whose homology is by definition $H_n(A,M)$ and $\mathrm{Tor}^{A^e}_n(M,A)$.

□

Definition 2.4 (Relative Hochschild Homology)**.** *Let $I \subset A$ be a two-sided ideal. Consider the canonical morphism $C_*(A) \to C_*(A/I)$ of complexes. Let $K_*(A, I) = \ker(C_*(A) \to C_*(A/I))$. Then, define $HH_n(A, I) := H_n(K_*(A, I))$. Obviously, by definition, it has a long exact sequence:*

$$\cdots \to HH_n(A, I) \to HH_n(A) \to HH_n(A/I) \to HH_{n-1}(A, I) \to \cdots .$$

Definition 2.5 (Birelative Hochschild Homology)**.** *Let I, J be two two-sided ideals of A. Consider*

$$\begin{array}{ccccccccc} 0 & \longrightarrow & K_*(A,I) & \longrightarrow & C_*(A)C_*(A/I) & \longrightarrow & 0 & & \\ & & \downarrow & & \downarrow & & & & \\ 0 & \longrightarrow & K_*(A/J, I+J/J) & \longrightarrow & C_*(A/J) & \longrightarrow & C_*(A/I+J) & \longrightarrow & 0 \end{array} .$$

Here, vertical arrows are surjective. Hence Let $K_*(A; I, J)$ be $\ker(K_*(A, I) \to K_*(A, J, I + J/J))$, which gives a short exact sequence of complexes:

$$0 \to K_*(A; I, J) \to K_*(A, I) \to K_*(A/J, I + J/J) \to 0.$$

Define $HH_n(A; I, J) = H_n(K*(A; I, J))$ then, we have the following long exact sequence:

$$\cdots \to HH_n(A; I, J) \to HH_n(A, I) \to HH_n(A/J, I + J/J) \to HH_{n-1}(A; I, J) \to \cdots .$$

Proposition 2.6 (Localization)**.** *Let $S \subset Z(A)$ be a multiplicative subset of the center and $1 \in S, 0 \notin S$. Define $M_S := Z(A)_S \otimes_A M$. When A is a flat k-algebra, we have*

$$H_n(A, M)_S \simeq H_n(A, M_S) \simeq H_n(A_S, M_S).$$

Proof. A-being k-flat, Hochschild homologies are derived functors, so that it is enough to check it for $n = 0$, in which case, it is obvious.

□

Proposition 2.7 (localization of the ground ring)**.** *Let $S \subset k$ be a multiplicative subset. When A is flat over k, the natural map $HH_*(A/k) \otimes_k k_S \to HH_*(A_S/k_S)$ is an isomorphism. In particular, if A is a $\mathbb{Q}$-algebra,*

$$HH_*(A/\mathbb{Z}) \otimes \mathbb{Q} \simeq HH_*(A/\mathbb{Q}).$$

3. The Dennis Trace map and Morita invariance

Let M be an A-bimodule, A is a k-algebra, k is a commutative ring. For the obvious map $M_r(M) \hookrightarrow M_{r+1}(M)$, we form $\varinjlim_r M_r(M) = M(M)$ which we also denote by $gl(M)$.

Definition 3.1 (The Dennis Trace map)**.** *$Tr : M_r(M) \otimes M_r(A^{\otimes n}) \to M \otimes A^{\otimes n}$ is defined by*

$$Tr(m(0) \otimes a(1) \otimes \cdots \otimes a(n)) = \sum_{1 \le i_0, \cdots, i_n \le r} m(0)_{i_0,i_1} \otimes a(1)_{i_1,i_2} \otimes \cdots \otimes a(n)_{i_n,i_0}.$$

Remark. *Any $M_r(M) = M_r(k) \otimes_k M$.*

Lemma 3.2. *Let $u_i \in M_r(k)$, $a_0 \in M$, $a_i \in A$ for $i \ge 1$. Then, $Tr(u_0 a_0 \otimes u_1 a_1 \otimes \cdots \otimes u_n a_n) = Tr(u_0 \cdots u_n) a_0 \otimes a_1 \otimes \cdots \otimes a_n$, where the 2nd trace map is the ordinary trace map.*

The proof is obvious.

Corollary 3.3. *The Dennis Trace map gives a morphism of complexes*

$$C_*(M_r(A), M_r(M)) \to C_*(A, M).$$

Proof. Enough to show that $Tr \circ d_i = d_i \circ Tr$ on $u_0 a_0 \otimes \cdots \otimes u_n a_n$ which is obvious. □

Theorem 3.4 (Morita invariance for matrices)**.** *Let $i : M \hookrightarrow M_r(M)$. Then, for all $r \geq 1$ (including $r = \infty$),*

$$Tr_* : H_*(M_r(A), M_r(M)) \to H_*(A, M)$$
$$i_* : H_*(A, M) \to H_*(M_r(A), M_r(M))$$

are isomorphisms and inverse to each other.

Proof. $Tr \circ i = id$ is obvious. So, ETS, $i \circ Tr \sim id$. We can construct an explicit homotopy. See Loday. □

Definition 3.5 (Morita equivalence)**.** *R, S, k-algebras, are said to be* Morita equivalent *if there are (R, S)-bimodule P, (S, R)-bimodule Q and isomorphisms $u : P \otimes SQ \simeq R$, $v : Q \otimes_R P \simeq S$.*

Example 3.6. $R = A$, $S = M_r(A)$ are Morita equivalent: take $P = A^r$(row vectors) $Q = A^r$ (column vectors).

Theorem 3.7. *If R, S are Morita equivalent and M is an R-bimodule, there is a natural isomorphism*

$$H_*(R, M) \simeq H_*(S, Q \otimes_R M \otimes_R P).$$

Proof. See Loday. □

We record the following useful result. See Loday.

Theorem 3.8. *Let A, A' be k-algebras, and M is an (A, A')-bimodule. Let $T = \begin{pmatrix} A & M \\ 0 & A' \end{pmatrix}$. Then, projections $T \to A, A'$ induce an isomorphism $HH_*(T) \simeq HH_*(A) \oplus HH_*(A')$.*

Remark. The Dennis trace map is transitive:

$$M_{rs}(A) \xrightarrow{Tr} M_r(M_s(A)) \xrightarrow{Tr} A$$

is the same as the trace map for $rs \times rs$ matrices.

4. Introduction to Kähler differentials and Hochschild Homology

Recall that for $a \in A$ and M an A-bimodule, we have an inner derivation

$$ad(a) : M \to M, \quad ad(a)(m) = [a, m] = am - ma.$$

Remark. $ad(a)$ extends to $C_n(A, M)$:

$$ad(a)(a_0 \otimes \cdots \otimes a_n) = \sum_{i=0}^{n} (a_0 \otimes \cdots \otimes a_{i-1} \otimes [a, a_i] \otimes a_{i+1} \otimes \cdots \otimes a_n).$$

Proposition 4.1. *Define $h(a) : C_n(A, M) \to C_{n+1}(A, M)$ by*

$$h(a)(a_0 \otimes \cdots \otimes a_n) = \sum_{i=0}^{n} (-1)^i (a_0 \otimes \cdots \otimes a_i \otimes a \otimes a_{i+1} \otimes \cdots \otimes a_n).$$

Then, $bh(a) + h(a)b = -ad(a)$ so that $ad(a)_* : H_n(A, M) \to H_n(A, M)$ is 0.

It is just a direct computation.

Definition 4.2 (Antisymmetrization map)**.** *For $\sigma \in S_n$ and $a_0 \otimes \cdots \otimes a_n \in C_n(A, M)$, define*

$$\sigma \cdot (a_0 \otimes \cdots \otimes a_n) = a_0 \otimes a_{\sigma^{-1}(1)} \otimes a_{\sigma^{-1}(2)} \otimes \cdots \otimes a_{\sigma^{-1}(n)}$$

which extends to an action of $k[S_n]$ on $C_n(A, M)$. Let $\epsilon_n := \sum_{\sigma \in S_n} (sgn\sigma) \cdot \sigma \in k[S_n]$ which is called the antisymmetrization map.

Remark. *ϵ_n defines $\epsilon_n : M \otimes \Lambda^n A \to C_n(A, M)$, $a_0 \otimes a_1 \wedge \cdots \wedge a_n = \epsilon_n(a_0 \otimes \cdots \otimes a_n)$.*

Definition 4.3 (Chevalley-Eilenberg map)**.** *$\delta : M \otimes \Lambda^n A \to M \otimes \Lambda^{n-1} A$ defined by*

$$\delta(a_0 \otimes a_1 \wedge \cdots \wedge a_n) = \sum_{i=1}^{n} (-1)^i [a_0, a_i] \otimes a_1 \wedge \cdots \wedge \hat{a_i} \wedge \cdots \wedge a_n$$

$$+ \sum_{1 \le i < j \le n} (-1)^{i+j-1} a_0 \otimes [a_i, a_j] \wedge a_1 \wedge \cdots \wedge \hat{a_i} \wedge \cdots \wedge \hat{a_j} \wedge \cdots \wedge a_n.$$

Proposition 4.4. *The following diagram is commutative:*

$$\begin{array}{ccc} M \otimes \Lambda^n A & \xrightarrow{\epsilon_n} & C_n(A, M) \\ \downarrow \delta & & \downarrow b \\ M \otimes \Lambda^{n-1} A & \xrightarrow{\epsilon_{n-1}} & C_{n-1}(A, M) \end{array} .$$

Note that when A is commutative and M is symmetric, $b \circ \epsilon_n = 0$.

Proof. Proof is done by induction on n using $h(a)$ defined before. □

Let A be commutative. Then,

Proposition 4.5. *There is a canonical map*

$$\epsilon_n : M \otimes_A \Omega^n_{A/k} \to H_n(A, M), \quad a_0 \otimes da_1 \wedge \cdots \wedge da_n \mapsto a_0 \otimes a_1 \otimes \cdots \otimes a_n.$$

In particular, if $M = A$, then, we have $\epsilon_n : \Omega^n_{A/k} \to HH_n(A)$.

Proposition 4.6. *Let A be commutative. Then there is a canonical map*

$$\pi : H_n(A, M) \to M \otimes_A \Omega^n_{A/k} \quad a_0 \otimes \cdots \otimes a_n \mapsto a_0 \otimes da_1 \wedge \cdots \wedge da_n.$$

If $M = A$, $\pi_n : HH_n(A) \to \Omega^n_{A/k}$.

Proposition 4.7. *$\pi_n \circ \epsilon_n : M \otimes_A \Omega^n_{A/k} \to M \otimes_A \Omega^n_{A/k}$ is multiplication by $n!$ so that if $k \supset \mathbb{Q}$, $M \otimes A\Omega^n_{A/k}$ is a direct summand of $H_n(A, M)$.*

Remark. If A is smooth, they are in fact isomorphic, which is a theorem of Hochschild-Kostant-Rosenberg.

5. Definition of Cyclic Homology and its properties

5.1. **Cyclic homology; 1st description (general case).** Define the cyclic action of $\mathbb{Z}/(n+1)\mathbb{Z}$ on $C_n(A) = A \otimes A^{\otimes n} = A^{\otimes n+1}$ via the action of its generator $t = t_n$:

$$t_n(a_0 \otimes a_1 \otimes \cdots \otimes a_n) = (-1)^n a_n \otimes a_0 \otimes \cdots \otimes a_{n-1}$$

called the cyclic operator. Let $N = 1 + t + \cdots + t^n$ called the norm operator.

Consider b and b' introduced in the preveious sections.

Lemma 5.1. *$(1 - t)b' = b(1 - t)$ and $b'N = Nb$.*

Proof. Let $J = d_0 t$ and observe that $t_i J t^{-i-1} = (-1)^i d_i$ for $0 < i < n$ and $t^n J t^{-n-1} = J$. Then, it is just a direct straightforward computation. See Loday or Husemöller for detail. $\square$

So that we have the following *cyclic bicomplex* CC_{**}:

$$\begin{array}{ccccccccc}
\downarrow b & & \downarrow -b' & & \downarrow b & & \downarrow -b' & & \\
A^{\otimes 3} & \xleftarrow{1-t} & A^{\otimes 3} & \xleftarrow{N} & A^{\otimes 3} & \xleftarrow{1-t} & A^{\otimes 3} & \xleftarrow{N} & \\
\downarrow b & & \downarrow -b' & & \downarrow b & & \downarrow -b' & & \\
A^{\otimes 2} & \xleftarrow{1-t} & A^{\otimes 2} & \xleftarrow{N} & A^{\otimes 2} & \xleftarrow{1-t} & A^{\otimes 2} & \xleftarrow{N} & \\
\downarrow & & \downarrow -b' & & \downarrow b & & \downarrow -b' & & \\
A & \xleftarrow{1-t} & A & \xleftarrow{N} & A & \xleftarrow{1-t} & A & \xleftarrow{N} &
\end{array}$$

Note that odd columns are exact, if A is unitary. Here $CC_{pq} = C_q(A) = A^{\otimes q+1}$.

Definition 5.2 (Cyclic homology)**.** *The cyclic homology* $HC_n(A/k) := HC_n(A) := H_n(totCC_{**}(A))$. *Here we did not assume that A is unitary.*

Remark. (1) (Functoriality) Let $f : A \to A'$ be a morphism of k-algebras. It induces $f_* : CC_{**}(A) \to CC_{**}(A')$ in an obvious way, so that we have $f_* : HC_n(A) \to HC_n(A')$.

(2) (Ground ring) If $k \to K \to A$ is a sequence of ring homomorphisms, then, we have $HC_n(A/k) \to HC_n(A/K)$.

5.2. **Cyclic homology; 2nd description ($k \supset \mathbb{Q}$).**

Definition 5.3 (The Connes complex)**.** *Let* $C_n^\lambda(A) := C_n(A)_{1-t} = \mathrm{coker}(1-t) = A^{\otimes n+1}/\mathrm{im}(1-t)$*, which is the coinvariant space of* $A^{\otimes n+1}$ *for the action of* $\mathbb{Z}/(n+1)\mathbb{Z}$*.*

This is called the Connes complex*. Let* $H_n^\lambda(A) = H_n(C_*^\lambda(A))$*.*

Consider the natural surjection $p : tot(CC_{**}(A)) \to C_*^\lambda(A)$ which is the quotient map $A^{\otimes n+1} \to A^{\otimes n+1}/1-t$ on the first column and 0 on other columns.

Theorem 5.4 (The 2nd description of Cyclic homology)**.** *Assume that* $k \supset \mathbb{Q}$*. Then,* $p_* : HC_*(A) \to H_*^\lambda(A)$ *is an isomorphism.*

Proof. Let $\theta = -(t + 2t^2 + \cdots + t^n)$. Then, by a simple computation, we can check that $n+1 = N + \theta(1-t)$, i.e. $id = \frac{1}{n+1}N + \frac{\theta}{n+1}(1-t) = N\frac{1}{n+1} + (1-t)\frac{\theta}{n+1}$. Hence, $\frac{1}{n+1}$ and $\frac{\theta}{n+1}$ define the following homotopy:

$$\begin{array}{ccccccc}
A^{\otimes n+1} & \xleftarrow{1-t} & A^{\otimes n+1} & \xleftarrow{N} & A^{\otimes n+1} & \xleftarrow{1-t} & A^{\otimes n+1} \\
\downarrow id & \searrow \frac{\theta}{n+1} & \downarrow id & \searrow \frac{1}{n+1} & \downarrow id & \searrow \frac{\theta}{n+1} & \downarrow id \\
A^{\otimes n+1} & \xleftarrow{1-t} & A^{\otimes n+1} & \xleftarrow{N} & A^{\otimes n+1} & \xleftarrow{1-t} & A^{\otimes n+1}
\end{array}$$

and id is homotopic to 0, i.e. it is contractible, hence acyclic. Hence, the row is an acyclic augmented complex with $H_0 = C_n^\lambda(A)$.

Consider the standard vertical increasing filtrations on $CC_{**}(A)$. Since each row is an acyclic augmented complex with $H_0 = C_n^\lambda(A)$, we have

$$E^1_{p,q} = \begin{cases} 0 & p > 0 \\ C_q^\lambda(A) & p = 0 \end{cases}$$

$$E^2_{p,q} = \begin{cases} 0 & p > 0 \\ H^\lambda_q(A) & p = 0 \end{cases}$$

and it generates at $r = 2$. Hence, having $E^r_{p,q} \Rightarrow HC_n(A)$, we must have $HC_q(A) = H^\lambda_q(A)$. □

5.3. **Cyclic homology; 3rd description (A unitary).** Now we go to the third description. We assume that A is a unitary k-algebra, i.e. it has a unity. Then, the odd degree columns with b' of the cyclic bicomplex are contractible (having s; extra degeneracy as a homotopy) hence acyclic. We try to simplify $CC_{**}(A)$ to obtain another simpler complex $\mathcal{B}(A)$.

Lemma 5.5 (Killing contractible complexes)**.** *Let*

$$\cdots \to A_n \oplus A'_n \xrightarrow{d=\begin{pmatrix}\alpha & \beta \\ \gamma & \delta\end{pmatrix}} A_{n-1} \oplus A'_{n-1} \to \cdots$$

be a complex of k-modules such that (A'_*, δ) is a contractible complex with a contraction homotopy $h : An' \to A'_{n+1}$. Then, the following inclusion of complexes is a quasi-isomorphism:

$$(id - h\gamma) : (A_*, \alpha - \beta h\gamma) \hookrightarrow (A_* \oplus A'_*, d).$$

Proof. We need to see

$$\begin{pmatrix}\alpha & \beta \\ \gamma & \delta\end{pmatrix} \cdot \begin{pmatrix} id \\ -h\gamma \end{pmatrix} = \begin{pmatrix} id \\ -h\gamma \end{pmatrix} \cdot (\alpha - \beta h\gamma).$$

Note that $LHS = \begin{pmatrix}\alpha - \beta h\gamma \\ \gamma - \delta h\gamma\end{pmatrix}$ and $RHS = \begin{pmatrix}\alpha - \beta h\gamma \\ -h\gamma\alpha + h\gamma\beta h\gamma\end{pmatrix}$.

Also note that $d^2 = \begin{pmatrix}\alpha & \beta \\ \gamma & \delta\end{pmatrix}^2 = \begin{pmatrix}\alpha^2 + \beta\gamma & \alpha\beta + \beta\delta \\ \gamma\alpha + \delta\gamma & \gamma\beta + \delta^2\end{pmatrix} = 0$ and $\delta^2 = 0$ implies that $\gamma\alpha + \delta\gamma = 0$ and $\gamma\beta = 0$.

Hence, we need to see $\gamma - \delta h\gamma = -h\gamma\alpha$ i.e. $\gamma = \delta h\gamma - h\gamma\alpha$. Since $\gamma\alpha = -\delta\gamma$, we have $\delta h\gamma - h\gamma\alpha = \delta h\gamma + h\delta\gamma = (\delta h + h\delta)\gamma = \gamma$. Hence $(id, -h\gamma) : A_n \to A_n \oplus A'_n$ is a morphism of complexes.

Since $\ker(id, -h\gamma) = 0$ and $\operatorname{coker}(id, -h\gamma) \simeq (A'_*, \delta)$ which is acyclic, $(id, -h\gamma)$ is a quasi-isomorphism.

□

Using above lemma with $d = \begin{pmatrix}\alpha & \beta \\ \gamma & \delta\end{pmatrix} = \begin{pmatrix} b & 1-t \\ N & -b'\end{pmatrix}$ and $h = -s$ (extra degeneracy), we can define the *Connes operator* (or the *Connes boundary map*) $B = (1-t)sN : C_q(A) \to C_{q+1}(A)$.

Proposition 5.6. *For above B, $b^2 = 0$, $B^2 = 0$ and $Bb + bB = 0$.*

Proof. $b^2 = 0$ was already done.

$B^2 = (1-t)sN(1-t)sN = (1-t)s(N(1-t))sN = 0$ because $N(1-t) = 0$.

$Bb + bB = (1-t)sNb + b(1-t)sN = (1-t)sb'N + (1-t)b'sN = (1-t)(sb' + b's)N = (1-t)N = 0$.

□

Hence we have a double complex $\mathcal{B}(A)$ $\mathcal{B}_{pq} = \left\{ \begin{array}{ll} A^{\otimes q-+1} & q \geq p \\ 0 & \text{otherwise} \end{array} \right.$ with

$$\begin{array}{ccccc}
A^{\otimes 3} & & A^{\otimes 3} & & A^{\otimes 3} \\
{\scriptstyle b}\downarrow & \nwarrow^{B} & {\scriptstyle b}\downarrow & \nwarrow^{B} & {\scriptstyle b}\downarrow \\
A^{\otimes 2} & & A^{\otimes 2} & & A^{\otimes 2} \\
{\scriptstyle b}\downarrow & \nwarrow^{B} & {\scriptstyle b}\downarrow & \nwarrow^{B} & {\scriptstyle b}\downarrow \\
A & & A & & A
\end{array}$$

i.e.

$$\begin{array}{ccccc}
A^{\otimes 3} & \xleftarrow{B} & A^{\otimes 2} & \xleftarrow{B} & A\ . \\
{\scriptstyle b}\downarrow & & {\scriptstyle b}\downarrow & & \\
A^{\otimes 2} & \xleftarrow{B} & A & & \\
{\scriptstyle b}\downarrow & & & & \\
A & & & &
\end{array}$$

Remark. $Bb + bB = 0$ implies that B induces a map on Hochschild homology $B_* : HH_n(A) \to HH_{n+1}(A)$.

Note that we have a natural injection

$$tot(\mathcal{B}(A)) \hookrightarrow tot(CC_{**}(A))$$

sending $x \in \mathcal{B}(A)_{pq} = C_{q-p}(A)$ to $x \oplus sN(x) \in C_{q-p}(A) \oplus C_{q-p+1}(A) = CC_{2p,q-p} \oplus CC_{2p-1,q-p+1} \subset tot(CC(A))_{p+q}$. The killing lemma implies that it is an isomorphism so that we have

Theorem 5.7 (3rd description of Cyclic homology)**.** *Let A be a* unitary *k-algebra. Then, $tot(\mathcal{B}(A)) \hookrightarrow tot(CC(A))$ is a quasi-isomorphism so that $H_n(tot(\mathcal{B}(A))) = HC_n(A)$.*

In general, we have the following concept which generalizes above idea:

Definition 5.8 (Mixed complex)**.** *Let $\mathcal{A}$ be an abelian category. A* mixed complex *X is a triple (X_*, b, B) where X_* is a $\mathbb{Z}$-graded object in $\mathcal{A}$, $b : X_* \to X_*$ is a morphism of degree -1, $B : X_* \to X_*$ is a morphism of degree $+1$, satisfying $b^2 = B^2 = Bb + bB = 0$.*

A morphism $f : X \to Y$ of mixed complexes is a morphism of $\mathbb{Z}$-graded objects such that $bf = fb$ and $Bf = fB$. A mixed complex is called positive *if $X_q = 0$ for $q < 0$.*

Remark. B in the cyclic homology can be written explicitly as follows: $B : A^{\otimes n+1} \to A^{\otimes n+2}$ is given by

$$B(a_0 \otimes \cdots a_n) = \sum_{i=0}^{n-1} (-1)^{ni} 1 \otimes a_i \otimes \cdots \otimes a_n \otimes a_0 \otimes \cdots \otimes a_{i-1} - (-1)^{n(i-1)} a_{i1} \otimes 1 \otimes a_i \otimes \cdots \otimes a_n \otimes a_0 \otimes \cdots \otimes a_{i-2}.$$

In low degrees, we have $B(a_0) = 1 \otimes a_0 - a_0 \otimes 1$ and $B(a_0 \otimes a_1) = (1 \otimes a_0 \otimes a_1 - 1 \otimes a_1 \otimes a_0) + (a_0 \otimes 1 \otimes a_1 - a_1 \otimes 1 \otimes a_0)$.

6. Some elementary properties of cyclic homology

Proposition 6.1. *The followings are true:*

1. $HC_0(A) = HH_0(A) = A/[A, A]$.
2. *Though $HH_n(A)$ are $Z(A)$-modules, for $HC_n(A)$, it is not true. Try for $HC_1(A)$.*

(3) *If A is commutative and unitary, then $HC_1(A) \simeq \Omega^1_{A/k}/dA$.*

(4) *(Relative cyclic homology) Let $I \subset A$ be a twosided ideal. Let $CC(A, I) = \ker(CC(A) \to CC(A/I))$ and let $HC_n(A, I) = H_n(tot(CC(A, I)))$. Then, we have a long exact sequence*

$$\cdots \to HC_n(A, I) \to HC_n(A) \to HC_n(A/I) \to HC_{n-1}(A, I) \to \cdots .$$

(5) *Without any condition on the characteristic, still we always have*

$$HC_1(A) = H_1^\lambda(A)$$

.

(6) *Let A^{op} be the opposite ring of A. Then, there are canonical isomorphisms*

$$HH_n(A) \simeq HH_n(A^{op}), \quad HC_n(A) \simeq HC_n(A^{op}).$$

[Use $\omega_n(a-0, \cdots a_n) = (a_0, a_n, a_{n-1}, \cdots, a_2, a_1)$.]

Proof. All of the above are very easy. Let's try (3) for example. By the 3rd description of cyclic homology, we can look at $\mathcal{B}(A)$:

$$\begin{array}{ccc} A^{\otimes 3} & & \\ \downarrow b & & \\ A^{\otimes 2} & \xleftarrow{B} & A \\ \downarrow b & & \\ A & & \end{array} \quad .$$

A being commutative, $b : A^{\otimes 2} \to A$, $a \otimes b \mapsto ab - ba = 0$ so that $HC_1(A) = A \otimes A/\mathrm{im} b, \mathrm{im} B$. Note that $\mathrm{im} b$ is generated by $ab \otimes c - a \otimes bc + ca \otimes b$ and $\mathrm{im} B$ is generated by $1 \otimes a - a \otimes 1$. Define $\phi : A \otimes A \to \frac{\Omega^1_{A/k}}{dA}$ by $a \otimes b \mapsto adb$. Obviously, the 1st one dies in $\Omega^1_{A/k}$ and the 2nd one dies in the quotient. Hence we have $HC_1(A) \to \frac{\Omega^1_{A/k}}{dA}$. The obvious inverse is given by $adb \mapsto a \otimes b$.

□

7. Connes exact couple

Theorem 7.1 (Connes exact couple)**.** *Let A be a k-algebra. Then, we have an exact couple, called* the Connes exact couple

$$\begin{array}{ccc} HC_*(A) & \xrightarrow{S} & HC_*(A) \\ & {\scriptstyle I}\nwarrow \quad \swarrow{\scriptstyle B} & \\ & HH_*(A) & \end{array}$$

with $\deg S = -2$, $\deg B = 1$ *and* $\deg I = 0$.

Proof. Consider the cyclic bicomplex $CC_{**}(A)$ and let $CC_{**}(A)^{\{2\}}$ be the bicomplex consisting of the first two columns of $CC_{**}(A)$. Then, we have an exact sequence of bicomplexes:

$$0 \to CC_{**}(A)^{\{2\}} \to CC_{**}(A) \to CC_{**}(A)[2, 0] \to 0$$

where $(CC_{**}(A)[2, 0])_{p,q} = CC_{p-2,q}(A)$. It gives a rise to an exact sequence of the corresponding total complexes, and obviously $tot(CC_{**}(A)^{\{2\}})$ is quasiisomorphic to $(C_*(A), b)$ so that we have the exact couple, as required.

□

Corollary 7.2. *Let $f : A \to A'$ be a k-algebra homomorphism. Ten, $f_* : HH_*(A) \to HH_*(A')$ is an isomorphism if and only if $f_* : HC_*(A) \to HC_*(A')$ is an isomorphicm.*

Proof. In low dimensions, from the Connes couple, we have an exact sequence

$$HC_1 \to HC_{-1} = 0 \to HH_0 \to HC_0 \to HC_{-2} = 0$$

i.e. $HH_0 \simeq HC_0$. Hence it is true for $n = 0$. In general, $f : A \to A'$ induces $f_* : CC(A) \to CC(A')$ so that we have a commutative diagram with exact rows:

$$\begin{array}{ccccccccc} \cdots \longrightarrow & HH_n(A) & \xrightarrow{I} & HC_n(A) & \xrightarrow{S} & HC_{n-2}(A) & \xrightarrow{B} & HH_{n-1}(A) & \xrightarrow{I} \cdots. \\ & \downarrow & & \downarrow & & \downarrow & & \downarrow & \\ \cdots \longrightarrow & HH_n(A') & \xrightarrow{I} & HC_n(A') & \xrightarrow{S} & HC_{n-2}(A') & \xrightarrow{B} & HH_{n-1}(A') & \xrightarrow{I} \cdots \end{array}$$

If $HH_n(A) \to HH_n(A')$ is an isomorphism, then, by induction and by the five lemma, $HC_{n-1}(A) \simeq HC_{n-1}(A')$, $HC_{n-2}(A) \simeq HC_{n-2}(A')$ implies $HC_n(A) \simeq HC_n(A')$.

Conversely, if $HC_*(A) \to HC_*(A')$ is an isomorphism, just a simple application of the five lemma gives the result. □

Corollary 7.3. *If $k \supset \mathbb{Q}$, then,*

$$\cdots \to HH_n(A) \xrightarrow{I} H_n^\lambda(A) \xrightarrow{S} H_{n-2}^\lambda(A) \xrightarrow{B} HH_{n-2}(A) \xrightarrow{I} \cdots$$

is exact.

Proof. If $k \supset \mathbb{Q}$, $HC_*(A) \simeq H_*^\lambda(A)$. □

Theorem 7.4 (Morita invariance for cyclic homology)**.** *If A, A' are Morita equivalent, then there is a canonical isomorphism $HC_*(A) \to HC_*(A')$.*

8. Differential forms and Cyclic homology

Let A be a commutative unitary k-algebra. Recall that we have the exterior derivative

$$d : \Omega^n_{A/k} \to \Omega^{n+1}_{A/k}, \quad d(a_0 da_1 \wedge \cdots \wedge da_n) := da_0 \wedge da_1 \wedge \cdots \wedge da_n.$$

Since $d1 = 0$, we have $d^2 = 0$ so that we hace a complex, called the de Rham complex

$$A = \Omega^0_{A/k} \xrightarrow{d} \Omega^1_{A/k} \xrightarrow{d} \Omega^2_{A/k} \xrightarrow{d} \cdots$$

and its cohomology is denoted by $H^n_{DR}(A)$. Note that $(\Omega^*_{A/k}, d)$ is a DG-algebra with $(a_0 da_1 \wedge \cdots \wedge da_n) \wedge (a'_0 da'_1 \wedge \cdots \wedge da'_m) = a_0 a'_0 da_1 \wedge \cdots \wedge da_n \wedge da'_1 \wedge \cdots \wedge da'_m$.

We state the following propositions, whose proofs are straightforward:

Proposition 8.1. *Let A be a commutative unitary k-algebra. Then, the following diagrams are commutative:*

$$\begin{array}{ccc} \Omega^n_{A/k} & \xrightarrow{d} & \Omega^{n+1}_{A/k} \\ \epsilon_n \downarrow & & \downarrow \epsilon_{n+1} \\ HH_n(A) & \xrightarrow{B_*} & HH_{n+1}(A) \end{array}$$

$$\begin{array}{ccc} HH_n(A) & \xrightarrow{B_*} & HH_{n+1}(A) \\ {\scriptstyle \pi_n}\downarrow & & \downarrow{\scriptstyle \pi_{n+1}} \\ \Omega^n_{A/k} & \xrightarrow{(n+1)d} & \Omega^{n+1}_{A/k} \end{array}.$$

Corollary 8.2. *Let A be a commutative unitary k-algebra. Then, there is a functorial map*

$$\epsilon_n : \Omega^n_{A/k}/d\Omega^{n-1}_{A/k} \to HC_n(A).$$

Proof. If we look at $\mathcal{B}(A)$, $B_* : HH_{n-1}(A) \to HH_n(A)$ factors through HC_{n-1} so that

$$\begin{array}{ccccccc} \Omega^{n-1} & & \xrightarrow{\quad d \quad} & & \Omega^n & & \\ {\scriptstyle \epsilon_{n-1}}\downarrow & & & & \downarrow{\scriptstyle \epsilon_n} & & \\ HH_{n-1} & \xrightarrow{I} & HC_{n-1} & \xrightarrow{B} & & HH_n & \xrightarrow{I} HC_n \end{array}$$

is commutative. Now, $I \circ B : HC_{n-1} \to HC_n$ is 0, being a part of the Connes exact sequence, we have $I\epsilon_n(d\Omega^{n-1}) = 0$ as well. Hence, ϵ_n induces a map $\Omega^n/d\Omega^{n-1} \to HC_n$ as desired. □

Proposition 8.3. *When $k \supset \mathbb{Q}$ and A is commutative and unitary, we have a natural map*

$$\pi_n : HC_n(A) \to \Omega^n_{A/k}/d\Omega^{n-1}_{A/k} \oplus H^{n-2}_{DR}(A) \oplus H^{n-4}_{DR}(A) \oplus \cdots .$$

Proof. Since $k \supset \mathbb{Q}$, $\frac{1}{n!}\pi_n$ induces a morphism of mixed complexes $\mathcal{B}(A) = (C(A), b, B) \to \mathcal{D}(A) = (\Omega^*_{A/k}, 0, d)$ where $\mathcal{D}(A)$ is called the reduced Delign complex with $\mathcal{D}(A)_{p,q} = \Omega^{q-p}$ if $q \geq p$ and 0 otherwise. Give the vertical increasing filtration and look at the map of spectral sequences from $\mathcal{B}(A) \to \mathcal{D}(A)$. For $\mathcal{D}(A)$, note that

$$E^1_{p,q} = H^{hor}_{p,q}(\mathcal{D}(A)) = \begin{cases} H^{q-p}_{DR}(A) = H^{n-2p}_{DR}(A) & p > 0 \\ \Omega^q/d\Omega^{q-1} & p = 0 \end{cases}.$$

Since the vertical maps are all 0, it generates at E^1 so that we have a natural map

$$HC_n(A) = H_n(tot(\mathcal{B}(A))) \to \coprod_{p+q=n} E^1_{p,q} = \Omega^n/d\Omega^{n-1} \oplus H^{n-2}_{DR}(A) \oplus H^{n-4}_{DR}(A) \oplus \cdots .$$

□

9. Cohomology

Recall that $(C_*(A, M), b) = (M \otimes_{A^e} C'_*(A), b')$, so that $H_n(A, M) = H_n(M \otimes_{A^e} C'_*(A), b')$. In the same fasion, we define the Hochschild cohomology as follows:

Definition 9.1 (Hochschild cohomology)**.** *The Hochschild cohomology of A with coefficients in M is $H^n(A, M) = H_n(\mathrm{Hom}_{A^e}(C'_*(A), M))$ and $\beta'(\phi) = -(-1)^n \phi \circ b'$.*

Explicitly, if for a cochain ϕ, $f : A^{\otimes n} \to M$ satisfies

$$\phi(a_0[a_1|\cdots|a_n]a_{n+1}) = a_0 f(a_1 \otimes \cdots \otimes a_n)a_{n+1}$$

then

$$\beta(f)(a_1 \otimes \cdots a_{n+1}) = a_1 f(a_2 \otimes \cdots \otimes a_{n+1}) + \sum_{0<i<n+1} (-1)^i f(a_1 \otimes \cdots \otimes a_i a_{i+1} \otimes \cdots \otimes a_{n+1}) + (-1)^{n+1} f(a_1 \otimes \cdots a_n)a_{n+1}$$

Hence we have, in fact,

$$H^n(A, M) = H_n(C^*(A, M), b)$$

where $C^n(A, M) = \mathrm{Hom}_k(A^{\otimes n}, M)$.

Remark. (1) If $n = 0$, $H^0(A, M) = M^A = \{m \in M | am = ma, \forall a \in A\}$.
(2) If $H^1(A, M) = Der(A, M)/\{\text{ inner derivations }\}$.
(3) If $M = A^* = \mathrm{Hom}_k(A, k)$, then we define $HH^n(A) := H^n(A, A^*)$.
(4) We also ahve a cotrace map and Morita invariance.
(5) If A is unitary and k-flat, then,

$$H^n(A, M) = \mathrm{Ext}^n_{A^e}(A, M).$$

(6) If $g : A \to A'$ is a k-morphism, then, we have $g^* : HH^n(A') \to HH^n(A)$.

9.1. **Duality.** Let M, M' be two A-bimodules. Then we have

$$C^n(A, M) \times C_n(A, M') \to M \otimes_{A^e} M'$$
$$(f, m' \otimes a_1 \otimes \cdots \otimes a_n) \mapsto f(a_1 \otimes \cdots \otimes a_n) \otimes m'.$$

This is obviously satisfying

$$< \beta(f), x >=< f, b(x) >$$

for $f \in C^n(A, M)$ and $x \in C_{n+1}(A, M')$. Hence we have

$$< \, , \, >: H^n(A, M) \otimes H_n(A, M') \to M \otimes_{A^e} M'.$$

Here, the left hand side $\otimes$ can be taken over $Z(A)$.

Remark. (1) If $n = 0$, above pairing is the surjection $M \otimes_{Z(A)} M' \to M \otimes_{A^e} M'$.
(2) If $n = 1$, let $D \in Der(A, M)$, $(D) \in H^1(A, M)$, then, for $M' = A$,

$$< \, , \, >: H^1(A, M) \otimes_{Z(A)} \Omega^1_{A/k} \to M_A = H_0(A, M)$$
$$(D) \otimes adb \mapsto aDb.$$

Similar jobs can be done for cyclic cohomologies as well.

10. Normalized Complexes

10.1. **Normalized Hochschild complex.** Let A be a unitary k-algebra. Then, $C_*(A, M)$ has a large subcomplex D_* which is acyclic; $D_n \subset M \otimes A^{\otimes n}$ is generated by elements $m \otimes a_1 \otimes \cdots \otimes a_n$ with one of $a_i = 0$. $M \otimes A^{\otimes n}/D_n$ is called the normalized Hochschild complex. Let $\bar{A} = A/k$. Then, we have $\bar{C}_n(A, M) := M \otimes A^n/D_n = M \otimes \bar{A}^{\otimes n}$. Obviously, D_* being acyclic, the quotient map $C_*(A, M) \to \bar{C}_*(A, M)$ is a quasi-isomorphism.

10.2. **Normalized (b, B) complex.** Let A be a unitary k-algebra. The (b, B)-complex $\mathcal{B}(A)$ for $HC_n(A)$ can be further simplied as well. Define $\bar{\mathcal{B}}(A)$ as follows:

$$\begin{array}{ccccc}
\downarrow{\scriptstyle b} & & \downarrow{\scriptstyle b} & & \downarrow{\scriptstyle b}\\
A\otimes\bar{A}^{\otimes 2} & \xleftarrow{\bar{B}} & A\otimes\bar{A} & \xleftarrow{\bar{B}} & A\\
\downarrow{\scriptstyle b} & & \downarrow{\scriptstyle b} & & \\
A\otimes\bar{A} & \xleftarrow{\bar{B}} & A & & \\
\downarrow{\scriptstyle b} & & & & \\
A & & & &
\end{array}$$

where $\bar{B} = sN : A \otimes \bar{A}^{\otimes n} \to A \otimes \bar{A}^{\otimes n+1}$. Explicitly,

$$\bar{B}(a_0 \otimes \cdots \otimes a_n) = \sum_{i=0}^{n} (-1)^{ni} 1 \otimes a_i \otimes \cdots \otimes a_n \otimes a_0 \otimes \cdots \otimes a_{i-1}.$$

For lower degrees, we have

$$\bar{B}(a) = 1 \otimes a, \quad \bar{B}(a \otimes a') = 1 \otimes a \otimes a' - 1 \otimes a' \otimes a.$$

Columns are normalized Hochschild complexes whose homologies are still the Hochschild homologies. Hence, by a standard spectral sequence argument, the surjection $\mathcal{B}(A) \to \bar{\mathcal{B}}(A)$ is a quasi-isomorphis, so that $H_*(tot(\bar{\mathcal{B}}(A))) = HC_*(A)$.

Remark (Summary)**.** Let A be a k-algebra. There are three canonical morphisms of complexes

$$tot(\bar{\mathcal{B}}(A)) \stackrel{1}{\leftarrow} tot(\mathcal{B}(A)) \stackrel{2}{\to} tot(CC(A)) \stackrel{3}{\to} C^{\lambda}(A).$$

Note that 2 is always a quasi-isomorphism. 1 is a quasi-isomorphism if A is unitary. 3 is a quasi-isomorphism if $k \supset \mathbb{Q}$.

Example 10.1. When $A = k$, then, $A \otimes \bar{A}^{\otimes n} \simeq \begin{cases} k & \text{if } n = 0 \\ 0 & \text{if } n > 0 \end{cases}$, so that $\bar{\mathcal{B}}(k)$ is in fact, the following one:

$$\begin{array}{ccccc} 0 & \longleftarrow & 0 & \longleftarrow & k \ . \\ \downarrow & & \downarrow & & \\ 0 & \longleftarrow & k & & \\ \downarrow & & & & \\ k & & & & \end{array}$$

Then, $(tot\bar{\mathcal{B}}(k))_{2n} = k$ and $(tot\bar{\mathcal{B}}(k))_{2n+1} = 0$ and obviously, $HC_{2n}(k) = k$ and $HC_{2n+1}(k) = 0$.

11. Reduced Hochschild and Cyclic homology

11.1. **Reduced Hoshchild homology.** Assume that $k \hookrightarrow A$. Let $k[0]$ be the complex consisting in k in degree 0. The reduced Hochshild complex is defined by the following short exact sequence:

$$0 \to k[0] \to (\bar{C}_n(A), b) \to (\bar{C}_n(A), b)_{red} \to 0$$

and its homology is called the reduced Hochschild homology and denoted by $\overline{HH}_n(A)$. From the definition, we just see that

$$0 \to HH_1(A) \to \overline{HH}_1(A) \to k \to HH_0(A) \to \overline{HH}_0(A) \to 0$$

and $\overline{HH}_n(A) \to HH_n(A)$ for $n \geq 2$.

11.2. **Reduced Cyclic homology.** Assume that $k \hookrightarrow A$. Then the reduced Cyclic homology $\overline{HC}_n(A)$ is the homology of the total complex of the bicomplex $\mathcal{B}(A)_{red}$ which is defined by

$$0 \to \bar{\mathcal{B}}(k) \to \bar{\mathcal{B}}(A) \to \mathcal{B}(A)_{red} \to 0.$$

From the homology long exact sequence, we have

$$\cdots \to HC_n(k) \to HC_n(A) \to \overline{HC}_n(A) \to HC_{n-1}(k) \to \cdots .$$

Remark. We have the following reduced Connes exact couple:

$$\begin{array}{ccc}
\overline{HC}_*(A) & \xrightarrow{S;-2} & \overline{HC}_*(A) \\
{\scriptstyle I;0}\nwarrow & & \swarrow{\scriptstyle B;+1} \\
& \overline{HH}_*(A) &
\end{array}$$

where the numbers are the degrees of the maps.