DIMACS Series in Discrete Mathematics
and Theoretical Computer Science

Stationary sets, Chang's Conjecture and Partition theory

Matthew Foreman

1. Introduction and notation

This paper is intended as a contribution to some areas of interest to Andras Hajnal. Much of the paper is outgrowth of seminal work done by Hajnal in combinatorial set theory.

The principal subject matter of the paper is the study of stationary sets and their reflection properties. This subject matter has a very large "cast of characters" ranging from Shelah's PCF theory, the study of weak square properties to partition theory, Chang's Conjecture and generic elementary embeddings. Much of this has been well exposited elsewhere so the focus of this paper is on material that is perhaps less "standard" and well known.

A rather surprising consequence of recent work in the theory of stationary sets has been the discovery of refined versions of classical theorems in logic, such as the Löwenheim-Skolem theorem. These are discussed in section 2 of the paper.

Section 2 also introduces and discusses stationary set reflection as well as "localized" versions of stationary set reflection. These local versions of stationary set reflection have been used to explore classical questions in set theory such as the Singular Cardinals Hypothesis. The problem of relating the localized versions of reflection to standard reflection is also discussed and some partial results relating the two are given.

An interesting development in this study has been a nascent theory of stationary subsets of singular cardinals, even of countable cofinality. This theory avoids the obvious (and trivial) counterexamples to Fodor's Lemma and other facts, by using a more sophisticated definition of "stationary" that is an outgrowth of some recent work. Many open problems have arisen in this study and an attempt to motivate these problems and explain their background is made in this paper. This is surveyed in section 3.

In section 4 we consider Chang's Conjecture. It is remarked that this study has many relations with the theory of mutually stationary sets. Moreover, forms of Chang's Conjecture are equivalent to various "square bracket" partition relations. Several results are presented in this section in the "Chang's Conjecture" language

---

1991 *Mathematics Subject Classification.* Primary:03e02, Secondary:Hajnalalia.
Partially supported by NSF grant DMS-9803126 and Equip d'Analyse, Univ. de Paris VI.

©0000 (copyright holder)

and a guide to translation to partition language is given. Included are the proofs of several unpublished results of the author.

We have tried to follow standard notations and terminology as much as possible. We review some of this here.

We will use the terminology that a structure $\mathfrak{A}$ has type (κ, λ) iff $\mathfrak{A}$ has a domain of cardinality κ and a distinguished unary predicate of cardinality λ. In taking substructures we will be interested in their cardinality and the cardinality of the intersection of the domain of the elementary substructre with the interpretation of the unary predicate.

For the purposes of discussing stationary sets it suffices to consider structures whose type only consists of functions, and are fully Skolemized. We will call such structures *algebras*. When the particular Skolemization doesn't matter we will often ignore this requirement. As is customary, we will write $N \prec \mathfrak{A}$ to mean that N is an elementary substructure of $\mathfrak{A}$. A systematic abuse will be to not distinguish between the universe of N and the *structure* N.

We investigate stationary sets using two extremal tools, $\square$ and stationary set reflection. We will say that $\square_\kappa$ holds if there is a $\square_\kappa$ sequence where:

DEFINITION 1. *A $\square_\kappa$ sequence is a sequence of sets $\langle C_\alpha : \alpha \in \kappa^+ \rangle$ with the following properties:*

1. *C_α is closed and unbounded in α.*
2. *The order type of C_α is less than or equal to κ.*
3. *If γ is a common limit point of C_α and C_β then $C_\alpha \cap \gamma = C_\beta \cap \gamma (= C_\gamma)$.*

This formulation is slightly different (but equivalent) to the usual formulation, and makes generalizations easier to state.

We will occasionally be touching on Shelah's PCF theory. Nothing in the paper requires a deep understanding of the subject, but a knowledge of some of the definitions is important.

A typical situation in the PCF theory is that one wants to study a singular cardinal κ. To do this one considers an increasing cofinal sequence of regular cardinals $\langle \kappa_i : i \in cf(\kappa) \rangle$, and an ideal I on $cf(\kappa)$. Then properties of the reduced product $\prod_{i \in cf(\kappa)} \kappa_i / I$ are used to bootstrap properties of the κ_i to κ^+.

One of the main tools is the notion of a *scale* in $\prod_{i \in cf(\kappa)} \kappa_i / I$. The reduced product $\prod_{i \in cf(\kappa)} \kappa_i / I$ has a natural partial ordering given by setting $f <_I g$ iff $\{i : f(i) \geq g(i)\} \in I$. A *scale* in $\prod_{i \in cf(\kappa)} \kappa_i / I$ is a sequence $\{f_\alpha : \alpha < \gamma\}$ such that:

1. $\alpha < \beta < \gamma$ implies $f_\alpha <_I f_\beta$.
2. For all $f \in \prod_{i \in cf(\kappa)} \kappa_i / I$ there is an $\alpha < \gamma, f <_I f_\alpha$.

If $\{f_\alpha : \alpha < \gamma\}$ is a scale, then any cofinal subsequence is a scale and hence we can assume that γ is always a regular cardinal. In this case we will say that the *true cofinality* of $\prod_{i \in cf(\kappa)} \kappa_i / I$ is γ. (We note that this is well defined.)

Shelah has shown that there is always a sequence of cardinals κ_i such that $\prod_{i \in cf(\kappa)} \kappa_i / I$ has a scale of length κ^+, where I is the ideal of bounded subsets of $cf(\kappa)$.

We will assume some familiarity with forcing axioms, at least by name. The only one we really use is *Martin's Maximum*.

DEFINITION 2. *Suppose that $\mathbb{P}$ is a partial ordering. Then $\mathbb{P}$ is* stationary set preserving *iff for all stationary subsets $S \subset \omega_1$*

$$1 \Vdash S \text{ is stationary.}$$

Martin's Maximum is the following statement:

> If $\mathbb{P}$ is a stationary set preserving partial ordering and $\mathcal{D}$ is a collection of dense sets that has cardinality ω_1 then there is a filter $G \subset \mathcal{P}$ such that for all $D \in \mathcal{D}, G \cap D \neq \emptyset$.

We will refer to Martin's Maximum as "MM". It has a host of consequences ([**6**]) such as:

- $2_0^{\aleph} = \aleph_2$ (more generally for all regular cardinals $\kappa \geq \omega_2, \kappa^{\omega_1} = \kappa$)
- Singular cardinals hypothesis
- Chang's Conjecture
- The non-stationary ideal on ω_1 is ω_2-saturated.

Some of these are discussed in the text.

2. Stationary sets

For purposes of illustration we now give the "modern" definition of stationary sets. While used by Shelah since the early 80's it has been utilized extensively by Woodin and others.

DEFINITION 3. *Let A be a set. Let $S \subset P(A)$. Then S is* stationary *iff for all algebras $\mathfrak{A} = \langle A, f_i \rangle_{i \in \omega}$ there is an $N \prec \mathfrak{A}$ such that $N \in S$.*

Thus to say that S is stationary is to say that S satisfies a version of the Löwenheim–Skolem theorem.

This definition of stationarity is extremely convenient in that it makes it quite clear that the property of being stationary is just as canonical as the property of being a cardinal. To whit we have the following simple proposition:

PROPOSITION 4. *Let A and B be sets and $f : A \to B$ be a bijection. Suppose that $S \subset P(A)$ and $T = \{x : \text{for some } y \in S, x = f[y]\}$. Then S is stationary iff T is stationary.*

More generally we can easily show that the stationary sets form a "tower".

PROPOSITION 5. *Let A and B be sets and $f : A \to B$ be a surjection. Suppose that $S \subset P(A)$ is stationary, then $\{f[x] : x \in S\} \subset B$ is stationary. Moreover, if $T \subset P(B)$ is stationary, then $\{x \subset A : f[x] \in T\}$ is stationary.*

It is also readily seen that the collection of *non-stationary* subsets of $P(A)$ form a normal ideal (NS_A) and the completeness of this ideal restricted to various sets is easily computed. Proposition 5 then states that if $f : A \to B$ is a surjection then the f-projection of NS_A is NS_B.

For the purposes of the following examples we will refer to Definition 3 as the "new" definition of stationary and the classical definitions as the "old" definitions. See e.g. Jech's book [**13**] for the original definitions of stationary.

EXAMPLE 6. *We examine how the "new definition" compares with the usual notions found in standard textbooks:*

(1) *By the downwards Löwenheim–Skolem theorem, for all infinite cardinals* $\kappa \leq |A|$ *the set* $[A]^\kappa$ *is stationary in the new sense. In particular the collection of countable subsets of* A *is stationary. Thus the new notion is not equivalent to the old notion.*
(2) *Suppose that* $A = \lambda$ *and* λ *is a regular cardinal. For* $S \subset P(\lambda) \cap \{ordinals\}$ *(i.e.* $S \subset \lambda$*) the old and new definitions of stationary coincide.*
(3) *Suppose that* $A = \lambda$ *is a cardinal and* $S \subset \{N \in [\lambda]^{<\kappa} : N \cap \kappa \in \kappa\}$*. Then* S *is stationary in the new sense iff is stationary in the classical sense (as originally defined by Jech). This was shown by Kueker (*[**15**]*).*
(4) *Suppose that* $(\kappa, \lambda) \twoheadrightarrow (\kappa', \lambda')$*. Then the collection of non-stationary subsets (in the new sense) of* $\{X \subset \kappa : o.t.X = \kappa' \text{ and } o.t.(X \cap \lambda) = \lambda'\}$ *forms a* λ'*-complete normal ideal (the "Chang Ideal"). This new phenomenon is part of the advantage of the new definition.*

2.1. Strengthening the Löwenheim–Skolem Theorem. As noted above, the downward Löwenheim–Skolem theorem implies that for every set A, the collection of countable subsets of A is a stationary set. In this section we discuss some recent refinements of the downward Löwenheim–Skolem theorem.

One way of refining the Löwenheim–Skolem theorem is to demand more information about the elementary substructure. In particular, one can ask that the supremum of the elementary substructure lie in a given stationary set. For one cardinal (or even a finite set of cardinals, subject to mild restrictions, see Theorem 42) this is not difficult. However for infinitely many cardinals the situation is more involved, as we shall see in the section "towards a theory of stationary subsets of a singular cardinal". For countable elementary substructures we content ourselves with stating the following two results.

THEOREM 7. *(Foreman, Magidor,* [**9**]*) Suppose that* K *is a set of regular cardinals, and that* $\{S_\kappa : \kappa \in K\}$ *is a collection of stationary sets with* $S_\kappa \subset \kappa \cap cof(\omega)$*.*

Then the collection S *of countable subsets of* $sup(K)$ *with the property that for all* $N \in S$ *and all* $\kappa \in N \cap K$ *we have*

$$\sup(N \cap \kappa) \in S_\kappa.$$

is stationary.

In other words, given a sequence of stationary sets of points of cofinality ω and a structure $\mathfrak{A}$ we can always find an elementary substructure N such that the suprema of the intersection of N with the various regular cardinals belonging to N lie in the given stationary sets.

An even more refined version of this asks for control, not only of the supremum of the structure N below κ for $\kappa \in K$, but also control over a cofinal sequence of intervals in $N \cap \kappa$. To state the next theorem precisely we need some more definitions:

DEFINITION 8. *(Shelah) Let* κ *be a regular cardinal and* $\vec{S} = \{S_\alpha : \alpha \in T \subset \kappa\}$ *be a sequence of sets such that* S_α *is unbounded in* α*. Then* $\vec{S}$ *is a* club guessing sequence *iff for all closed unbounded sets* $D \subset \kappa$ *there is a stationary collection of* $\alpha \in T$ *such that* $S_\alpha \subset D$*.*

It is clear that a $\diamondsuit$ sequence is a club guessing sequence. However the following remarkable theorem is also true:

THEOREM 9. *(Shelah) Let κ and λ be infinite regular cardinals and suppose that $\kappa^+ < \lambda$. Then there is a club guessing sequence $\vec{S} = \langle S_\alpha : \alpha \in \lambda \cap cof(\kappa)\rangle$.*

Note that if we build a new sequence by taking an unbounded subset of each S_α we still have a club guessing sequence and thus without loss of generality we can assume that the order type of each S_α is κ.

Let N be a countable set and κ a regular cardinal. Suppose that $\vec{S}_\kappa = \langle S_\alpha : \alpha \in T_\kappa\rangle$ is a club guessing sequence on a set of ordinals of countable cofinality. We make the following definition:

DEFINITION 10. *Let $\alpha = \sup(N \cap \kappa)$ and suppose that $\alpha \in T_\kappa$. Enumerate S_α as $\{\alpha_n : n \in \omega\}$ in increasing order. The* pattern *of N at κ is:*

$$pat(N, \kappa) = \{n : N \cap [\alpha_n, \alpha_{n+1}) \neq \emptyset\}$$

The following theorem then says that we can control not only the supremum of N below κ but also the sequence of intervals below its supremum that N intersects.

THEOREM 11. *(Foreman, Todorcevic [**10**]) Fix a set of successor cardinals K and a collection of club guessing sequences $\{\vec{S}_\kappa : \kappa \in K\}$ that are defined on ordinals of countable cofinality. Let $f : K \to P(\omega)$ be a function that choses an infinite subset of ω for each cardinal in K. Then*

$$S_f = \{N : \text{for all } \kappa \in N \cap K, pat(N, \kappa) = f(\kappa)\}$$

is a stationary set.

2.2. Stationary Set Reflection. In this section we discuss some new and old results about stationary set reflection.

DEFINITION 12. *Suppose that $B \subset A$. Then a set $S \subset P(A)$ is said to reflect to B iff $S \cap P(B)$ is stationary.*

Notice that "reflecting to B" is a strong version of the Löwenheim–Skolem theorem. It says that every structure on A has an elementary substructure in S that is a subset of B. Again this definition coincides with the usual notion of stationary set reflection where the two overlap.

Many statements of the form:

> Every stationary set with property P reflects to a set with property Q.

follow directly from large cardinals, yet hold consistently on small cardinals. They often carry much of the combinatorial strength of the large cardinal axioms and hence provide surrogates for large cardinals at accessible cardinals.

For such a statement to hold it must be the case that the collection of sets with property Q are stationary.

The following results are well known. ([**12**],[**20**] and others):

THEOREM 13. *The following are equiconsistent:*

1. *There is a Mahlo cardinal.*
2. *Every stationary subset of ω_2 consisting of ordinals of cofinality ω reflects to an ordinal.*

We now want to investigate reflection for subsets of $P_\kappa(\lambda)$. A typical statement here is reflection to a set of cardinality μ for some μ having some structural property.

The simplest case of this is where we ask for reflection to sets of cardinality ω_1. The first structural distinction one makes for sets X of size ω_1 is whether or not $X \supset \omega_1$. One easily checks (for example)

REMARK 14. *$\{X \subset \omega_2 : |X| = \omega_1$ and $X \cap \omega_1$ is countable$\}$ is stationary iff Chang's Conjecture holds.*

Chang's conjecture is a very interesting topic that will be discussed in later sections, however we want to separate (to the extent possible) the study of Chang's Conjecture from the study of reflection, so we make the following definition:

DEFINITION 15. *(The "Weak Reflection Principle.") Let $S \subset P(A)$. Then S reflects to a set of size ω_1 iff there is a set X with $|X| = \omega_1 \subset X$ such that S reflects to X.*

The following is a simple consistency result of [**6**]:

THEOREM 16. *Let κ be supercompact, and let $G \subset Col(\omega_1, < \kappa)$ be generic. Then :*

> *$V[G] \models$ for every set A, every stationary subset of $[A]^{<\omega_1}$ reflects to a set of size ω_1.*

This property given in definition was studied in [**6**], where various results were shown. (Later elaborations of it have been studied by many authors, e.g. [**27**].)

Prominent among these results is the following theorem of Todorcevic:

THEOREM 17. *Suppose the weak reflection principle holds. Then:*

$$2^{\omega} \leq \omega_2$$

The weak reflection principle has many other consequences for example the following strong Chang's Conjecture holds:

> For every structure $\mathfrak{A} = \langle \omega_2; \omega_1, f_i \rangle_{i \in \omega}$ of type (ω_2, ω_1) there is a closed unbounded set $C \subset \omega_1$ such that $\alpha \in C$ implies that there is a $\mathfrak{B} \prec \mathfrak{A}$ of type $(\aleph_1, \aleph_0)$ such that $\mathfrak{B} \cap \omega_1 = \alpha$.

In [**6**] this is deduced by a circuitous appeal to a result of Shelah on Namba Forcing. We point out a more direct proof.

To begin with one can check ([**22**]) that the strong Chang's Conjecture follows from the following statement:

> For all large λ and all structures $\mathfrak{A}$ expanding $\langle H(\lambda), \in, \Delta \rangle$ there is a closed unbounded set $C \subset [H(\lambda)]^{\aleph_0}$ such that for all $N \in C$ there are unboundedly many $\alpha \in \omega_2$ with the property that:
>
> $$N \cap \omega_1 = Sk^{\mathfrak{A}}(N \cup \{\alpha\}) \cap \omega_1$$

This can be seen from the weak reflection property as follows:

Suppose that there is a stationary set S of counterexamples to the preceding statement. Let $X \prec \mathfrak{A}$ have cardinality ω_1 reflecting S. Let $f : \omega_1 \to X$ be a bijection. There there is a stationary set $A \subset \omega_1$ such that for all $\delta \in A$, f"$\delta \cap \omega_1 = \delta$ and f"$\delta \in S$.

We claim that there is a $\delta \in A$ such that for an unbounded set of $\alpha \in \omega_2, Sk^{\mathfrak{A}}(f\text{"}\delta \cup \{\alpha\}) \cap \omega_1 = \delta$. This clearly suffices.

If it failed there would be an α_0 such that for all $\delta \in A, \alpha > \alpha_0, Sk^{\mathfrak{A}}(f\text{"}\delta \cup \{\alpha\}) \cap \omega_1 > \delta$.

Let $M \prec \mathfrak{A}$ be a countable structure such that $\alpha_0 \in M$, M is closed under f and $\delta =_{def} M \cap \omega_1 \in A$. Then $M \cap \omega_1 = f``\delta \cap \omega_1, \alpha_0 \in M$ and $f``\delta \in S$, a contradiction.

We note other properties of the weak reflection principle shown in [**6**]:

- The nonstationary ideal on ω_1 is presaturated.
- A partial ordering is semi-proper iff it preserves stationary subsets of ω_1.

2.3. Localized Reflection to Sets of Size ω_1. We can ask more refined questions about stationary set reflection. The next definition gives a list of various uniformly definable structural properties of sets.

Definition 18. *Let $N \prec H(\lambda)$ be a set of cardinality κ with $\kappa \subset N$ (for some "large" regular λ) and μ be a regular cardinal. Then:*

(1) *N is* internally approachable *of length μ (in symbols $N \in IA(\mu)$) iff there is a sequence $\langle N_\alpha : \alpha < \mu \rangle$ that is increasing with α (under $\subset$), continuous at limit stages, has union N and is such that $\langle N_\alpha : \alpha < \beta \rangle \in N$ for all $\beta < \mu$.*

[If $\mu = \kappa$, we often ask that each N_α have cardinality less than μ and be an elementary substructure of N. For regular uncountable κ this gives an equivalent definition.]

(2) *N is* internally club *iff $N \cap P_\kappa(N)$ is closed and unbounded.*

(3) *N is* internally stationary *iff $N \cap P_\kappa(N)$ is stationary.*

(4) *N is* internally unbounded *iff $N \cap P_\kappa(N)$ is unbounded.*

(5) *N has* uniform cofinality *μ iff for all regular cardinals η, either $\eta \subset N$ or the cofinality of $sup(N \cap \eta)$ is μ. (This is equivalent to $N \cap \lambda$ having cofinality μ and for all regular $\eta \in N$ either $\eta \subset N$ or the cofinality of $N \cap \eta$ is μ.)*

(6) *If δ is a singular cardinal less than λ and $\{\kappa_i : i \in cf(\delta)\}$ is a cofinal sequence then N is* tight *for $\{\kappa_i : i \in cf(\delta)\}$ iff $cf(\delta) \subset N$ and $N \cap \prod_{i \in cf(\delta)} \kappa_i$ is cofinal in $\prod_{i \in cf(\delta)} N \cap \kappa_i$*

For $\lambda' < \lambda$, we say that N has a property "below λ'" we will mean that $N \cap H(\lambda')$ has that property.

If $\mu = \kappa$, it is easy to check that 1-4 are decreasing in strength and imply that N has uniform cofinality κ. If $N \prec H(\lambda)$ for some regular $\lambda > \kappa, \kappa \in N$ and $\kappa^{<\kappa} = \kappa$ then 1-4 are equivalent.

Being internally approachable of length μ implies that N has uniform cofinality μ, and if $cf(\delta) < \mu$ then internal approachability of length μ implies tightness for all sequences $\{\kappa_i : i \in cf(\delta)\} \in N$. However, it is not difficult to construct stationarily many N that have uniform cofinality but are not tight. (See [**9**].)

There are many situations where having uniform cofinality together with tightness implies internal approachability. Moreover for uniform tight structures there is a definable way of assigning a finite number of ordinal "invariants" to the structure N in a way that completely captures N. These results will appear in [**4**]. We will content ourselves with stating the following special case of a more general proposition:

Proposition 19. *Suppose that $N \prec H(\lambda^+)$ has the property that for each regular cardinal $\kappa \leq \lambda^+$, the supremum of $N \cap \kappa$ has uncountable cofinality. Then $N \cap \lambda^+$ is countably closed (as a set of ordinals).*

The converse clearly holds as well. In particular, we see that for N having cardinality $\aleph_1$, the $N \cap OR$ is countably closed iff N has uniform cofinality ω_1.

See [**8**] for more information about internally approachable structures.

Setting $\kappa = \omega_1$ in Definition 18, these stationary sets give targets for reflection. Indeed these are considered in the literature, a couple of sample results include:

THEOREM 20. *(Velickovic)*[**26**] *Suppose every stationary subset of* $[H(\lambda)]^{<\omega_1}$ *reflects to an internally approachable set, then the singular cardinal hypothesis holds.*

We state another example from the PCF theory of Shelah. We state the result for $\aleph_\omega$ and $\{\aleph_n : n \in \omega\}$ for simplicity. Fix a scale $\{f_\alpha : \alpha \in \aleph_{\omega+1}\} \subset \prod_{n\in a} \aleph_n$ for some set a. (Shelah has shown that such a set always exists. [**23**])

In the collection of regularity properties of scales (useful, e.g., for uniformization) one of the strongest is the property of being a *Very Good Scale* (see [**3**]). This means that for every α of uncountable cofinality, there is a closed unbounded subset C of α and a $k \in \omega$ such that for all $n \in a\backslash k, \{f_\beta(n) : \beta \in C\}$ is a strictly increasing sequence.

It is proved in [**3**] that

THEOREM 21. *Suppose* $\square_{\aleph_\omega,\aleph_n}$ *for some* n *and* $\prod_{n\in a} \aleph_n$ *has true cofinality* $\aleph_{\omega+1}$ *then there is a very good scale for* $\prod_{n\in a} \aleph_n$.

Relating this with reflection:

THEOREM 22. [**3**] *Suppose that every stationary subset of* $[H(\aleph_{\omega+1})]^{<\omega_1}$ *reflects to a stationary set of size* ω_1 *that has uniform cofinality* ω_1. *Then there are no very good scales.*

In particular, this shows that reflection to sets of uniform cofinality ω_1 implies the failure of square in a very strong way.

On examination, the proof of Theorem 16 shows that if you collapse a supercompact cardinal to be ω_2 then every stationary subset of $[H(\lambda)]^{<\omega_1}$ reflects to an internally approachable set of length and cardinality ω_1.

REMARK 23. *It is not known if the following are equivalent:*

- *Every stationary subset of every* $[H(\lambda)]^{<\omega_1}$ *reflects to a set of size* ω_1.
- *Every stationary subset of every* $[H(\lambda)]^{<\omega_1}$ *reflects to a set of size* ω_1 *that is internally approachable.*

In each of the known models for the first statement, the second statement also holds. In particular, it is easy to see that the methods of [**6**] imply:

THEOREM 24. *Assume MM. Then every stationary subset of* $[H(\lambda)]^{\aleph_0}$ *reflects to an internally approachable set of cardinality* ω_1.

The proof of this is the same as the proof of Theorem 13 in [**6**] except that one uses a larger collection of dense sets.

2.4. Simultaneous Reflection. We do have some partial progress on the problem mentioned in Remark 23.

DEFINITION 25. *We say that stationary sets* $\{S_i : i \in I\}$ *simultaneously reflect iff there is a* B *such that for all* $i \in B \cap I, S_i \cap P(B)$ *is stationary.*

Even when $|I| = 2$ simultaneous reflection is not equivalent to reflection. For example, while Theorem 13 says that reflection for all stationary subsets of $\omega_2 \cap cof(\omega)$ is equiconsistent with a Mahlo cardinal the following result shows that simultaneous reflection of every pair of sets is stronger:

THEOREM 26. *(Magidor* [**20**]*)The following are equiconsistent:*

(1) *There is a weakly compact cardinal*
(2) *For all stationary $S, T \subset \omega_2 \cap cof(\omega)$ there is an α such that both $S \cap \alpha$ and $T \cap \alpha$ are stationary.*

(It is interesting to note also that weakly compact cardinals are equiconsistent with the reflection of every stationary subset of $[\omega_2]^{<\omega_1}$ (singly).)

For successors of singular cardinals, the consistency strength of stationary set reflection has not been determined, however we know that there is no implication between reflection between reflection for one set and for many.

THEOREM 27. [**3**] *Suppose that it is consistent that there is a supercompact cardinal. Then there is a model of ZFC where the following two facts hold:*

- *Every stationary subset of $[H(\lambda)]^{<\aleph_1}$ reflects to an internally approachable set of size ω_1. (In particular, every stationary set of ordinals of countable cofinality contained in a regular cardinal λ reflects to an ordinal of cofinality ω_1.)*
- *There are two stationary subsets of $\aleph_{\omega+1}$ consisting of ordinals of countable cofinality that don't simultaneously reflect.*

We note that very similar results were proved independently by Paul Larson [**16**].

With countably many supercompact cardinals one can make a significantly more refined claim:

THEOREM 28. [**3**]

- *Assuming the consistency of ZFC with the existence of infinitely many supercompact cardinals it is consistent that every finite collection of stationary subsets of $\aleph_{\omega+1}$ simultaneously reflect to an ordinal and there is a very good scale.*

  But
- *If there is a very good scale on $\prod_{n\in a} \aleph_n$ (for some infinite $a \subset \omega$) then for every stationary set $T \subset \aleph_{\omega+1}$ there is a countable collection of stationary sets that don't simultaneously reflect.*

Similar proofs give that simultaneous reflection of any two stationary sets does not imply the simultaneous reflection of any three stationary sets etc.

The following result shows that MM implies a maximal version of simultaneous reflection:

Let $\kappa \geq \omega_2$ be a regular cardinal. Let $\langle A_\alpha : \alpha \in \kappa \rangle$ be a partition of $\kappa \cap cof(\omega)$ into stationary sets.

THEOREM 29. *(Foreman) There is a stationary collection S of $N \in [\kappa]^{\aleph_1}$ such that:*

$$\text{For all } \alpha < \kappa, A_\alpha \text{ is stationary in } \sup(N \cap \kappa) \text{ iff } \alpha \in N.$$

In particular, the supremum function is one to one on S etc.

This result was shown by the author immediately after the publication of [**6**] and will appear in a forthcoming paper. Its significance is that the set described is a typical set of measure one in any normal supercompact measure that is canonically well ordered.

We now return to the context of subsection 2.1. Fix club guessing sequences as in that section. Let X be a set of size ω_1. Suppose that $\kappa < \lambda$ is a successor cardinal and that $X \cap \kappa$ has supremum α of countable cofinality. Let $\{\alpha_n : n \in \omega\}$ be the club guessing sequence at α.

Define a subset S of ω by setting $S = \{n : X \cap [\alpha_n, \alpha_{n+1}) \neq \emptyset\}$. Then there is a closed unbounded subset C of $[X]^{<\aleph_1}$ such that for all $N \in C, pat(N, \kappa) = S$.

Now let $f, g : \lambda \cap \{regular\ cardinals\} \to P(\omega)$ be two functions taking infinite values that differ at all regular cardinals. (For example, $f(\kappa) = \{even\ numbers\}$ and $g(\kappa) = \{odd\ numbers\}$ for all κ.) By Theorem 11, S_f and S_g are stationary. By the remark in the previous paragraph, if X has cardinality ω_1 and reflects both S_f, S_g then for all successor $\kappa \in X \cap \lambda, cf(X \cap \kappa) = \omega_1$. Thus X has uniform cofinality ω_1.

Thus we have the following consequence of Theorem 11:

THEOREM 30. *(Foreman, Todorcevic) Suppose that λ is a regular cardinal and that every collection of three stationary subsets of $[H(\lambda)]^{<\aleph_1}$ reflects simultaneously to a set of cardinality ω_1. Then every stationary subset of $[H(\lambda)]^{<\aleph_1}$ reflects to a set of size ω_1 that has uniform cofinality ω_1.*

We have the following variant of Velickovic's theorem:

THEOREM 31. *(Foreman, Todorcevic) Suppose that the singular cardinal hypothesis fails. Then there is a cardinal κ of countable cofinality and a stationary subset of $[\kappa^+]^{<\aleph_0}$ that does not reflect to a set of uniform cofinality ω_1.*

We now turn to partial results relating reflection to reflection to an internally approachable set. We need to review Shelah's notion of partial squares.

DEFINITION 32. *Suppose that $S \subset \lambda^+$ is a stationary set of ordinals of having cofinality less than or equal a fixed κ. A* partial square sequence on S *is a sequence $\langle C_\alpha : \alpha \in S\rangle$ with the following properties:*

1. *C_α is closed and unbounded in α.*
2. *The order type of C_α is less than or equal to κ.*
3. *If γ is a common limit point of C_α and C_β then $C_\alpha \cap \gamma = C_\beta \cap \gamma$.*

Shelah has shown the following remarkable theorem:

THEOREM 33. *Suppose that $\kappa < \lambda$ are regular. Then $\lambda^+ \cap cof(\kappa) = \bigcup_{\delta<\lambda} S_\delta$ such that each S_δ carries a partial square sequence.*

It is apparently an open problem whether there is even one stationary subset of a $\aleph_{\omega+1}$ consisting of ordinals of uncountable cofinality that carries a partial square sequence (even assuming the SCH). [**24**]

We state the next results for $\aleph_{\omega+1}$ for simplicity.

Suppose that $\aleph_\omega^\omega = \aleph_{\omega+1}$ and $\langle f_\alpha : \alpha < \aleph_{\omega+1}\rangle \subset \prod \aleph_n$ is a scale, and $S \subset \aleph_\omega \cap cof(\omega_1)$ is a stationary sequence with a partial square sequence. We begin by

polishing the scale: without loss of generality we can assume that for all $\alpha \in S$ all γ a limit point of C_α and all $2 \leq n \in \omega, f_\gamma(n) = \sup_{\beta \in C_\alpha \cap \gamma} f_\beta(n)$.

The point of the partial square sequence is that it gives a well defined notion of tightness for countable sets:

DEFINITION 34. *Suppose that $\langle f_\alpha : \alpha < \aleph_{\omega+1} \rangle$ is a polished scale. Let $N \prec H(\lambda)$ be countable and suppose that $\gamma = \sup N \cap \aleph_{\omega+1}$ is a limit point of some C_α. Then N is* tight *iff $C_\alpha \cap \gamma \subset N$ and for all but finitely many $n, \sup(N \cap \omega_n) = f_\gamma(n)$.*

The following result states that tight elementary substructures exist:

LEMMA 35. *Suppose that $\langle f_\alpha \rangle, S, \langle C_\alpha \rangle$ are as above. (i.e. $\langle f_\alpha \rangle$ is polished etc.) Then there is a stationary set of countable tight structures.*

Tight structures are recoverable from a finite set of ordinals and the scale $\langle f_\alpha \rangle$. In particular we can completely recover $N \cap \aleph_{\omega+1}$ from the ordinals γ and $\{\sup(N \cap \omega_n) : \sup(N \cap \omega_n) \neq f_\gamma(n)\}$.

Suppose now that N is a tight structure and $N \prec X$ with $X \prec H(\lambda)$ of uniform cofinality ω_1. Then by Proposition 19, the finite set of ordinals needed to define N lie inside X.

We have outlined the proof of the following theorem:

THEOREM 36. *(Foreman, Todorcevic) Suppose that there is a partial square sequence on a stationary set of points of cofinality ω_1 in $\aleph_{\omega+1}$ and that every collection of 4 stationary subsets of $[H(\lambda)]^{<\omega_1}$ simultaneously to a stationary set of cardinality ω_1. Then every stationary subset of $[\aleph_{\omega+1}]^{<\omega_1}$ reflects to an internally stationary set.*

COROLLARY 37. *Under the same hypothesis, if the CH holds then every stationary set reflects to an internally approachable set.*

2.5. Reflection at Cardinalities Above ω_1. The situation for reflection at cardinalities above ω_1 is more complicated. This is partially due to the fact that every countable $N \prec H(\lambda)$ is internally approachable (of length ω), but this fails for uncountable N.

In [**8**] the following results are shown:

THEOREM 38. *Suppose that κ is a supercompact cardinal. Then for every cardinal $\mu < \kappa, V^{Col(\mu^+, <\kappa)}$ satisfies:*

> *Every stationary subset of $IA(\mu)$ reflects to a stationary subset of some element of $IA(\mu^+)$.*

However this appears to be near the best possible positive result:

THEOREM 39. *Suppose that κ is a regular cardinal. Let $\lambda \geq \omega_3$ be a regular cardinal. For $i, j \in \{0, 1\}$ $S_{i,j} = \{N \prec H(\lambda) : |N| = \omega_1$ and $cf(N \cap \omega_2) = \omega_i, cf(N \cap \omega_3) = \omega_j\}$. Then $S_{0,0} \cup S_{0,1}$ and $S_{1,0}$ contain non-reflecting stationary sets.*

Since the CH implies that if $N \in S_{1,1}$ then $N \cap H(\omega_3) \in IA(\omega_1)$, we see that Theorem 38 implies that it is consistent that every stationary subset of $S_{1,1}$ reflects.

Shelah proved an even better result than Theorem 39:

THEOREM 40. [**21**] *Suppose that κ is a singular cardinal. Then there is a non-reflecting stationary subset of $[\kappa^+]^{<\aleph_2}$ consisting of sets with uniform cofinality ω_1. (In other words, sets whose intersection with the ordinals is countably closed.)*

3. Towards a Theory of Stationary Subsets of Singular Cardinals

In this section we define the notion of *mutual stationarity* which gives a weak notion of stationary set for singular cardinals (even of countable cofinality).

DEFINITION 41. *Let S be a stationary subset of $P(H(\lambda))$ for some λ. Let $K \subset \lambda$ be a set of regular cardinals and $\{S_\kappa : \kappa \in K\}$ be such that $S_\kappa \subset \kappa$.*

The sequence $\{S_\kappa : \kappa \in K\}$ is S-stationary iff for all structures $\mathfrak{A}$ expanding $H(\lambda)$, there is an $N \in S$ such that $N \prec \mathfrak{A}$ and for all $\kappa \in K, \sup(N \cap \kappa) \in S_\kappa$.

If $S = P(H(\lambda))$ we say that the sets $\{S_\kappa : \kappa \in K\}$ are *mutually stationary*. If S is the collection of tight subsets of $H(\lambda)$, then we say that S is *tightly stationary*, etc.

We begin our discussion with some easy remarks:

- The choice of $H(\lambda)$ is irrelevant. A simple Skolemization argument shows that the sequence $\{S_\kappa : \kappa \in K\}$ is mutually stationary (or not) no matter which $H(\lambda)$ we choose as long as $\lambda \geq \sup(K)$.
- If $\{S_\kappa : \kappa \in K\}$ is mutually stationary then each S_κ is stationary.
- If λ is a singular cardinal and $\langle \kappa_i : i \in cf(\lambda)\rangle$ is a discrete sequence of regular cardinals increasing to λ, then every subset A of λ gives rise to a sequence of subsets $S_i = A \cap \kappa_i$. Hence we seek to study subsets of λ by studying the mutual stationarity properties of the sequence $\langle S_i : i \in cf(\lambda)\rangle$.

To justify the claim that this is a theory of subsets of a singular cardinal we must show that the notion of mutual stationarity is not a "local" notion (at least for sequences of bounded cofinality). The following is a slight extension of a theorem of Baumgartner [**1**]:

THEOREM 42. [**9**] *Let ν be a regular cardinal less than the least element of K. If $\{S_\kappa : \kappa \in K\}$ is mutually stationary, and for all κ, $S_\kappa \subset cof(\leq \nu)$ then for all $\lambda_1, \ldots, \lambda_n$ greater than ν and not in K, and all sequences of stationary sets $S_{\lambda_i} \subset \lambda_i \cap cf(\leq \nu)$ the sequence $\{S_\kappa : \kappa \in K\} \cup \{S_{\lambda_1}, \ldots, S_{\lambda_n}\}$ is mutually stationary.*

In other words any finite modification of a mutually stationary sequence is mutually stationary (subject to a mild cofinality restriction.)

3.1. When Mutual Stationarity is Equivalent to Stationarity.

For stationary sets of ordinals of countable cofinality mutual stationarity is easy to understand. The following is a restatement of Theorem 7.

THEOREM 43. *Suppose that $\{S_\kappa : \kappa \in K\}$ is a sequence of stationary sets of ordinals of cofinality ω, then $\{S_\kappa : \kappa \in K\}$ is mutually stationary.*

This has corollaries in the theory of saturated ideals, for example:

COROLLARY 44. *Let $\lambda = \aleph_\alpha$ for some singular α. Then the non-stationary ideal on $[\lambda]^{<\aleph_1}$ is not $\lambda^{|\alpha|}$-saturated.*

A version of Theorem 43 is true for subsets of a small collection of measurable cardinals (or a former measurable cardinal after a Prikry sequence has been shot through it).

THEOREM 45. [**4**] *Let K be a collection of measurable cardinals and suppose that $\min K > |K|$. Then for every sequence of stationary sets $\vec{S} = \{S_\kappa : \kappa \in K\}$, $\vec{S}$ is mutually stationary.*

Moreover, if κ is a single measurable cardinal and $\mathbb{P}$ is the partial ordering for shooting a Prikry sequence $\langle \kappa_n : n \in \omega \rangle$ through κ, then there is a condition $p \in \mathbb{P}$ which forces

> *"If $S_n \subset \kappa_n$ is stationary then the sequence $\langle S_n : n \in \omega \rangle$ is mutually stationary."*

3.2. Sets Containing Ordinals of a Fixed Cofinality. We now focus our discussion of mutual stationarity to sequences of sets of ordinals of a fixed cofinality. As in the case of stationary set reflection, the mutual stationarity of sets concentrating on cofinalities different from those covered in Theorem 42 implies instances of Chang's Conjecture and we consider those in the next section.

We begin by showing that the notion of mutual stationarity is not trivial:

THEOREM 46. [**9**] *Suppose that $V = L$. Let $1 \leq k \in \omega$. Then there is a sequence of stationary sets $\vec{S} = \langle S_n : k < n \in \omega \rangle$ such that $S_n \subset \omega_n \cap cof(\omega_k)$ such that $\vec{S}$ is not mutually stationary.*

Welch has improved this theorem in two ways, to produce more examples of stationary but not mutually stationary sequences in L and to extend this result to larger fine structural models.

We now explore ways that this theory is analogous to stationary subsets of regular cardinals.

We begin by remarking that the non-mutually stationary sequences of sets do not seem to form an ideal, although many of combinatorial properties that follow from being an ideal still hold.

EXAMPLE 47. *Suppose that $\langle T_i \subset \omega_i : k \leq i \in \omega, i \text{ odd} \rangle$ and $\langle U_i \subset \omega_i : k \leq i \in \omega, i \text{ even} \rangle$ are non-mutually stationary sequences of sets of ordinals of cofinality ω_k. (For example, each T_i and U_i empty.) Consider the sequences:*

$$S^0_n = \begin{cases} T_n & \text{if } k \leq n \text{ and } n \text{ is odd} \\ \omega_n \cap cof(\omega_k) & \text{if } n \text{ is even} \end{cases}$$

and

$$S^1_n = \begin{cases} U_n & \text{if } k \leq n \text{ and } n \text{ is even} \\ \omega_n \cap cof(\omega_k) & \text{if } n \text{ is odd} \end{cases}$$

Then neither $\langle S^0_n \rangle$ nor $\langle S^1_n \rangle$ are mutually stationary, but the sequence $\langle S^0_n \cup S^1_n \rangle$ is clearly mutally stationary.

One of the basic theorems of the theory of stationary subsets of a regular cardinal is *Fodor's Lemma* that a regressive function on a stationary set is constant on a substationary set. A consequence of this is *Solovay's Theorem* that every stationary subset of a regular cardinal κ can be split into κ disjoint stationary sets.

In our current state of knowledge we cannot prove analogous results for mutually stationary sets. However for tightly stationary sequences there are versions of these theorems and we have partial results for mutually stationary sequences that have surprising connections.

Most of the next few results assume that $2^\omega < \aleph_\omega$. (Or, in the more general situations we do not discuss, $2^{cf(\lambda)} < \lambda$.) We do not know if this is necessary.

The next result is a weak substitute for the property of being an ideal:

THEOREM 48. *Assume that $2^{\aleph_0} = \aleph_k < \aleph_\omega$. Suppose that $S^n \subset \aleph_n$ $(n > k)$ is a mutually stationary sequence of sets. If we split each S^n into two sets S^n_0 and S^n_1, then there is a function $f : \omega \to 2$ such that the sequence $S^n_{f(n)}$ is mutually stationary.*

(Note that since we can make finite variations on mutually stationary sets the restriction that S^n is only defined for $n > k$ is not onerous.)

We now define a version of stationary set splitting:

DEFINITION 49. *Let $S^n \subset \aleph_n$ be a mutually stationary sequence of sets. We say that the sequence* weakly splits *if there is a partition of each S^n into S^n_0 and S^n_1 and functions $f, g : \omega \to 2$ such that for infinitely many m, $f(m) \neq g(m)$, and both $\{S^n_{f(n)} : n \in \omega\}$ and $\{S^n_{g(n)} : n \in \omega\}$ are mutually stationary.*

The sequence splits *iff there is a partition of each S^n into S^n_0 and S^n_1 and function $f : \omega \to 2$ such that both $\{S^n_{f(n)} : n \in \omega\}$ and $\{S^n_{1-f(n)} : n \in \omega\}$ are stationary. Of course, in this case, we can always find S^n_i so that f is constantly 0.*

An equivalent property to weak splitting is that there is an infinite set $A \subset \omega$ and a sequence of partitions $S^n = S^n_0 \cup S^n_1$ such that for $i = 0$ or 1, the sequence T^n defined by taking $T^n = S^n_i$ for $n \in A$ and $T^n = S^n$ for $n \notin A$ is mutually stationary.

Neither of these is really the optimal version of splitting for mutually stationary sequences that one could hope for; we do not know how to prove:

Conjecture: *Suppose that $\langle S_n : k \leq n \in \omega\rangle$ is a mutually stationary sequence. Then there are partitions $S_n = \bigcup_{\alpha<\omega_n} S^\alpha_n$ such that for a function $f \in \prod_{k\leq n\in\omega} \omega_n$ the sequence $\langle S^{f(n)}_n : k \leq n \in \omega\rangle$ is mutually stationary.*

We do not even know if every mutually stationary sequence weakly splits. We have partial results that show that every mutually stationary weakly splits in L.

DEFINITION 50. *Suppose that $A \subset \omega$ and $\vec{U} = \langle U^n : n \in A\rangle$ and $\vec{T} = \langle T^n : n \in A\rangle$ are sequences of mutually stationary sets. We say that $\vec{U} \subset \vec{T}$ iff for all $n \in A, U^n \subset T^n$. If $\mathcal{F}$ is a collection of mutually stationary sequences of sets $\vec{T} = \langle T^n : n > k\rangle$, then we say that $\mathcal{F}$ is a* filter *provided that $\mathcal{F}$ is closed upwards under $\subset$ and whenever $\vec{T}, \vec{U}$ are elements of $\mathcal{F}$, there is a $\vec{V} \in \mathcal{F}$ such that for all large enough $n \in \omega, V^n \subset U^n \cap T^n$.*

Let $\langle S^n : n > k\rangle$ be a mutually stationary sequence that does not weakly split. Suppose that the sequence concentrates on ordinals of cofinality $\aleph_k$. Then the next theorem says that for sequences of subsets of the S^n, being mutually stationary acts similarly to being a closed unbounded subset of a regular cardinal.

THEOREM 51. [**9**] *Assume that $2^{\aleph_0} < \aleph_\omega$ and suppose that $\vec{S} = \langle S^n : n > k\rangle$ is a mutually stationary sequence of sets that doesn't weakly split with $S^n \subset \omega_n$. Let $\mathcal{F}$ be the collection of sequences $\vec{T} \subset \vec{S}$ that are mutually stationary. Then $\mathcal{F}$ is a filter that is closed under $\subset$-descending sequences of uncountable cofinality.*

Let $\mathcal{U}$ be an ultrafilter on ω. For each set $X \subset \aleph_\omega$, associate the sequence $X_n = (X \cap S^n) \backslash \omega_{n-1}$. Define an ultrafilter $\mathcal{V}$ on $\aleph_\omega$, by putting $X \in \mathcal{V}$ iff there is a set $A \in \mathcal{U}$ such that the sequence:

$$Y^n = \begin{cases} X^n & \text{if } n \in A \\ S^n & \text{otherwise} \end{cases}$$

is mutually stationary.

THEOREM 52. [**9**] *If $\vec{S}$ does not weakly split and $2^{\aleph_0} < \aleph_\omega$ then $\mathcal{V}$ is an ultrafilter on $\aleph_\omega$ that is κ-indecomposible for all κ between $2^{\aleph_0}$ and $\aleph_\omega$.*

In fact $\mathcal{V}$ is descendingly κ complete for all κ between $2^{\aleph_0}$ and $\aleph_\omega$.

COROLLARY 53. *Suppose that $2^{\aleph_0} < \aleph_\omega$. If there is a mutually stationary sequence that does not weakly split, then there is an inner model with a measurable cardinal. (See* [**5**]*) In particular, if $V = L$, every sequence of mutually stationary sets weakly splits.*

3.3. A Fodor's Lemma for Singular Cardinals. For tightly stationary sequences we have a version of Fodor's Lemma. To formulate this we need first to examine what constitutes a reasonable Fodor's Lemma. Again we consider $\aleph_\omega$ for convenience.

Suppose that $A \subset \aleph_\omega$. Define $A_n = A \cap \omega_n$, and let $f(\beta) = max\{n : \omega_n \leq \beta\}$. Then f cannot be constant on any unbounded subset of A.

A more reasonable request of f is that f is constant (or bounded) on a relatively large subset of every A_n, in the sense that the sequence of subsets be mutually stationary. In other words that there is a function $g \in \prod \omega_n$ that bounds f on a mutually stationary sequence of sets.

This is the content of the next theorem, however we need to work with *tightly stationary* sets.

THEOREM 54. [**9**] *Suppose that $\langle S^n : k < n < \omega \rangle$ is a tightly stationary sequence of sets. If $f : \aleph_\omega \to \aleph_\omega$ is regressive (i.e. for all $\alpha, f(\alpha) < \alpha$), then there is a function $g \in \prod_{n \in \omega} \aleph_n$ such that if $T^n_g = \{\alpha \in S^n : f(\alpha) < g(n)\}$, then the sequence $\langle T^n_g : k < n < \omega \rangle$ is tightly stationary.*

We note that this theorem is deficient in at least two respects. First of all it only works for tightly stationary sets, rather than mutually stationary sets. Secondly the function g simply bounds f rather than predicts f exactly. We do not know if the following is true:

Question: Suppose that $\langle S^n : k < n < \omega \rangle$ is a sequence of mutually stationary sets. If $f : \aleph_\omega \to \aleph_\omega$ is regressive, is there a function $g \in \prod_{n \in \omega} \aleph_n$ such that if $T^n_g = \{\alpha \in S^n : f(\alpha) = g(n)\}$, then the sequence $\langle T^n_g : k < n < \omega \rangle$ is mutually stationary?

4. Chang's Conjectures, Mutual Stationarity and Square Bracket Partition Relations

We now return to Chang's Conjecture.

4.1. Chang's Conjecture, Jonsson Cardinals and Mutual Stationarity. We first remark that any sequence of mutually stationary sets that concentrate on ordinals of cofinality above the minimal cardinal involved is a form of Chang's Conjecture. For example we can show:

PROPOSITION 55. *Let $m > k \geq n > l$. Then:*

- *The two stationary sets $\{\omega_m \cap cof(\omega_k), \omega_n \cap cof(\omega_l)\}$ are mutually stationary*
  implies
- *For some $p \leq n, (\omega_p, \omega_{p-1}) \twoheadrightarrow (\omega_k, \omega_{k-1})$*

The reader will easily be able to refine this proposition using Theorem 62. In another direction, $\{\omega_{n+2} \cap cof(\omega_{n+1}), \omega_n \cap cof(\omega_{n-1})\}$ are mutually stationary iff $(\omega_{n+2}, \omega_n) \twoheadrightarrow (\omega_{n+1}, \omega_{n-1})$.

However in its current form Proposition 55 is strong enough to show (assuming $l \geq 1$) that any instance of mutual stationarity of finite sets not covered by Theorem 42 implies the existence of all core models constructed at this moment in time. (see e.g.[**25**].)

For this reason the cofinality restriction in Theorem 42 is necessary. It is tempting to try to reformulate Theorem 42 to allow for finite changes by mutually stationary sequences, but the following example shows that this is not possible:

EXAMPLE 56. *Suppose that $(\omega_2, \omega_1) \twoheadrightarrow (\omega_1, \omega)$ but $(\omega_3, \omega_2) \not\twoheadrightarrow (\omega_2, \omega_1)$. Then $S_2 = \omega_2 \cap cof(\omega_1)$ and $S_1 = \omega_1$ are mutually stationary. For all $n \geq 3$ let $S_n = \omega_n \cap cof(\omega_2)$. Then $\langle S_n : n \geq 3\rangle$ is mutually stationary. However $\langle S_n : 1 \leq n \in \omega\rangle$ is not mutually stationary.*

The example of a sequence of stationary sets that is not mutually stationary given by Theorem 46 is a sequence such that for any finite number $m, \langle S_n : k < n < m\rangle$ is mutually stationary, but the whole sequence is not.

It is an open problem whether there is a mutually stationary sequence of sets $\langle S_n \subset \omega_n : k < n \in \omega\rangle$ and $\langle T_m : j < m \leq k\rangle$ such that for any finite $k', \langle T_m : j < m \leq k\rangle^\frown\langle S_n : k < n < k'\rangle$ is mutually stationary but $\langle T_m : j < m \leq k\rangle^\frown\langle S_n : k < n < \omega\rangle$ is not.

For infinitely many cardinals the result is even more amusing:

THEOREM 57. *Suppose that $2^\omega < \aleph_\omega$. Then the following are equivalent:*

- *$\aleph_\omega$ is Jonsson.*
- *There is a mutually stationary sequence of sets $S_n \subset \omega_n \cap cof(\omega_{k_n})$ such that the sequence k_n takes infinitely many values.*
- *There is a subsequence $\langle \kappa_n : n \in \omega\rangle$ of $\langle \omega_n : n \in \omega\rangle$ such that $\langle \kappa_{n+1} \cap cof(\kappa_n) : n \in \omega\rangle$ is mutually stationary.*

The proof of this is essentially Silver's proof of that if $\aleph_\omega$ is Jonsson, then there is an inner model of a measurable cardinal. (The proof follows the outline given in [**14**].)

For infinite alternations of cofinalities the situation is unclear. For example Shelah has shown:

THEOREM 58. *Suppose that $\langle \kappa_n : n \in \omega\rangle$ is a sequence of measurable cardinals. Then in the forcing extension collapsing these cardinals to be the $\aleph_n$'s for all functions $f : \omega\backslash k \to \{0, k\}$ the sequence $\langle \omega_n \cap cof(\omega_{f(n)}) : n > k\rangle$ is mutually stationary.*

(For a simple proof of this consult the upcoming [**4**].)

For mutually stationary subsets consisting only of a non-eventually constant sequence of uncountable cofinalities or more than two cofinalities there are consistency results proved by Liu [**18**] and Liu and Shelah [**19**]:

THEOREM 59. *(Liu) Suppose it is consistent to have an elementary embedding $j : V \to M$ with M a transitive class such that that $V_\lambda \subset M$ where λ is an ordinal fixed point above the first ordinal moved by j. Then for all $k \in \omega$ it there is a forcing extension in which the sets defined for $n > k$ as:*

$$S_n = \begin{cases} \omega_n \cap cof(\omega_k) & \text{if } n \text{ is divisible by three} \\ \omega_n \cap cof(\omega_{k-1}) & \text{otherwise.} \end{cases}$$

are mutually stationary.

The result in [**19**] gets more cofinalities, but loses control over where they sit:

THEOREM 60. *Suppose that $\{A_0, \ldots, A_n\}$ are disjoint subsets of ω such that the reduced product $\prod_{n \in A_i} \omega_n / \mathcal{F}r$ has true cofinality $\aleph_{\omega+i}$ (where $\mathcal{F}r$ is the Frechet filter on ω). Then for any $\{k_0, \ldots, k_n\}$ a tail of the sets:*

$$S_n = \begin{cases} \omega_n \cap cof(\omega_{k_i}) & \text{if } n \in A_i \\ \omega_n & \text{if } n \notin \bigcup_{i \in n+1} A_i \end{cases}$$

is mutually stationary.

By Theorem 42 it is clear that we can take the tail of the S_n's to be all S_n for n that are greater than k_i all i.

It is an interesting open problem if there are any restrictions on the cofinalities of mutually stationary sequences.

4.2. Chang's Conjecture and Square Bracket Partition Relations. The following definition comes from [**11**]:

DEFINITION 61. *Let κ, λ, μ be cardinals and γ be an ordinal. We write:*

$$\kappa \to [\mu]^\lambda_\gamma$$

for the assertion that for all functions $f : [\kappa]^\lambda \to \gamma$ there is a set $Y \subset \kappa$ having cardinality μ and an $\alpha \in \gamma$ such that for no $z \in [Y]^\lambda, f(z) = \alpha$.

In other words if it is possible to "omit" a color from γ. There is a clear generalization for the statement $\kappa \to [\mu]^{<\lambda}_\gamma$. With this notation κ is Jonsson just in case $\kappa \to [\kappa]^{<\omega}_\kappa$. So square bracket arrow relations with exponent less than ω are strengthenings of the statement that Jonsson Algebras exist.

The square bracket relations can also state some instances of Chang's Conjectures directly: for a regular cardinal ρ, $\kappa \to [\mu]^{<\omega}_{\rho^+}$ is equivalent to the statement that $(\kappa, \rho^+) \twoheadrightarrow (\mu, \rho)$.

We now show a stronger result. The following first appeared in [**8**](Theorem 2.15).

THEOREM 62. *(Foreman, Magidor) (ZFC) Let $\kappa < \lambda < \mu$ be three consecutive cardinals. Then there is a structure $\mathfrak{A} = (\mu, f_i)_{i \in \omega}$ such that if $\mathfrak{B} \prec \mathfrak{A}$ with $|\mathfrak{B}| = \lambda$, and $|\mathfrak{B} \cap \lambda| = \kappa$, then $cf(\sup \mathfrak{B} \cap \lambda) = cf(\kappa)$. In particular if $\kappa \subset \mathfrak{B}$, then $\mathfrak{B} \cap \mu$ is $< \kappa$-closed in $\sup(\mathfrak{B} \cap \mu)$ (as a set of ordinals).*

To illustrate some consequences of this suppose that $\kappa = \omega_1, \lambda = \omega_2$ and $\mu = \omega_3$ and the CH holds. Let $N \prec H(\lambda)$ be a structure such that $N \cap \omega_3 \prec \mathfrak{A}$, $|N| = \omega_2$ and $|N \cap \omega_2| = \omega_1$. Lemma 66 (below) shows that we can assume that $\omega_1 \subset N$. We claim that every countable subset of $N \cap \omega_3$ belongs to N. We show this by induction on ω_i for $i = 1, 2, 3$. For $i = 1$, note that since N contains all of the countable ordinals, the CH implies that N must include all of the real numbers. For $i = 2$, we note that by Theorem 62 any countable subset of $\omega_2 \cap N$ must be bounded by some $\delta \in \omega_2 \cap N$. Since N contains a bijection f_δ between δ and $|\delta|$ we can apply the induction to see that every countable subset of $N \cap |\delta|$ belongs to N and hence every countable subset of $\delta \cap N$ belongs to N. The case of $i = 3$ follows similarly since the order type of $N \cap \omega_3$ must be ω_2.

In particular we can immediately see:

COROLLARY 63. *Suppose that* $(\omega_3, \omega_2) \twoheadrightarrow (\omega_2, \omega_1)$ *and the CH. Then*

$$\omega_3 \to [\omega_2]^\omega_{\omega_2}$$

In particular, since Laver, using a model essentially due to Kunen, showed that $(\omega_3, \omega_2) \twoheadrightarrow (\omega_2, \omega_1)$ is consistent, we see that the "infinite exponent" partition relation is consistent.

$\vdash$ To see the corollary from the theorem. Let $f : [\omega_3]^\omega \to \omega_2$. Then by $(\omega_3, \omega_2) \twoheadrightarrow (\omega_2, \omega_1)$ we see that there is an elementary substructure N of $\langle H(\lambda), f\rangle$ such that $(N \cap \omega_3)^\omega \subset N$, $|N \cap \omega_3| = \omega_2$ and $N \cap \omega_2$ is a proper subset of ω_2. Since N is closed under f, we see that $N \cap \omega_3$ witnesses the square bracket relation. $\dashv$

More generally, one sees under the GCH that if κ is a regular cardinal and $(\kappa^{+2}, \kappa^+) \twoheadrightarrow (\kappa^+, \kappa)$ then $\kappa^{+2} \to [\kappa^+]^{<\kappa}_{\kappa^+}$. Such Chang's Conjectures are consistent, even simultaneously (See [**7**]).

We now explore some relations of $\square$ and Chang's Conjecture. (Thus indirectly illuminating square bracket partition relations.)

Some aspects of Chang's Conjecture are *local* and thus to contradict them one requires a non-local square condition (e.g. many local square conditions simultaneously).

To illustrate this phenomenon in one particular case, suppose one has $(\omega_2, \omega_1) \to \to (\omega_1, \omega)$. Then for all cardinals κ above ω_2 one sees easily that $(\kappa, \omega_1) \twoheadrightarrow (\omega_1, \omega)$. It is easy to force $\square_{\omega_2}$ by ω_2 strategically closed forcing. Since ω_2 strategically closed forcing preserves the property $(\omega_2, \omega_1) \twoheadrightarrow (\omega_1, \omega)$, we get:

REMARK 64. *Suppose that* $(\omega_2, \omega_1) \twoheadrightarrow (\omega_1, \omega)$. *Then there is a forcing extension in which* $(\omega_2, \omega_1) \twoheadrightarrow (\omega_1, \omega)$ *and* $\square_{\omega_2}$ *both hold. In particular we have the consistency of* $\square_{\omega_2}$ *and* $(\omega_3, \omega_1) \twoheadrightarrow (\omega_1, \omega)$

The reader will easily be able to create multiple other examples of the consistency of Chang's Conjectures with local square properties. In particular small variants of this example show that it is consistent to have $\square_{\omega_3}$ and

$$\omega_4 \to [\omega_2]^\omega_{\omega_2, \omega_1}$$

hold in the same model of ZFC.

We now show that with many local square properties Chang's Conjecture must fail. (The earliest version of this theorem was first proved by Todorcevic in the case of $\square_{\omega_1}$ and $(\omega_2, \omega_1) \twoheadrightarrow (\omega_1, \omega)$.)

THEOREM 65. *Let κ and μ be cardinals and assume (for simplicity of statement) that $\kappa < \aleph_{\mu^+}$. Suppose that $\square_\rho$ holds for all ρ with $\mu^+ \leq \rho \leq \kappa$. Then for all ρ between μ and κ, the Chang's Conjecture: $(\kappa^+, \rho) \twoheadrightarrow (\mu^+, \mu)$ fails.*

To prove this theorem we will use the following standard fact (see e.g. [**17**]):

Fact: *Let $\mathfrak{B} \prec H(\lambda)$ for some large regular λ. Suppose that $\eta \in \mathfrak{B}$ and let $\mathfrak{B}_\eta$ be the Skolem hull in $H(\lambda)$ of $\mathfrak{B} \cup \eta$. Then for all regular $\xi \in (\mathfrak{B} \cup \{\lambda\})$ with $\xi > \eta$, the supremum of $\mathfrak{B}_\eta \cap \xi$ is the same as the supremum of $\mathfrak{B} \cap \xi$.*

We begin the proof with a lemma:

LEMMA 66. *Suppose that $(\kappa^+, \rho) \twoheadrightarrow (\mu^+, \mu)$, for some $\mu < \kappa$. Then there is some $\lambda \leq \kappa$ such that $(\lambda^+, \lambda) \twoheadrightarrow (\mu^+, \mu)$ in the strong form that for all structures $\mathfrak{A}$ on λ^+ in a countable language there is a $\mathfrak{B} \prec \mathfrak{A}$ such that $|\mathfrak{B}| = \mu^+, |\mathfrak{B} \cap \lambda| = \mu$ and $\mu \subset \mathfrak{B}$.*

$\vdash$ We go by contradiction. Let $\mathfrak{A}$ be a sufficiently rich algebra on κ^+ (e.g. coding an elementary substructure of $H(\kappa^{+20})$) and let $\mathfrak{B} \prec \mathfrak{A}$ be a Chang elementary substructure witnessing $(\kappa^+, \rho) \twoheadrightarrow (\mu^+, \mu)$. Let $\mathfrak{B}_\mu$ be the Skolem hull of $\mathfrak{B} \cup \mu$ in $\mathfrak{A}$. Let $\xi \leq \kappa^+$ be the least cardinal in $\mathfrak{B}_\mu$ such that $|\mathfrak{B}_\mu \cap \xi| = \mu^+$. Then:

- Since $\kappa < \aleph_{\mu^+}$, we must have that ξ is a successor cardinal. Let λ be its immediate cardinal predecessor. (So $\xi = \lambda^+$.)
- $\xi > \mu^+$ since $\sup(\mathfrak{B}_\mu \cap \mu^+) = \sup(\mathfrak{B} \cap \mu^+)$.
- The order type of $\mathfrak{B} \cap \xi$ is μ^+, since $|\mathfrak{B} \cap \lambda| = \mu$.

Now suppose $\kappa = \aleph_\eta$ and $\mathfrak{A}$ is a structure of cardinality κ^+ that codes an elementary substructure of some $\langle H(\theta), \in, \Delta, \langle \mathfrak{C}_\gamma : \gamma < \eta \rangle \rangle$, where $\mathfrak{C}_\gamma$ is an algebra on $\aleph_{\gamma+1}$ witnessing that $(\aleph_{\gamma+1}, \aleph_\gamma) \not\twoheadrightarrow (\mu^+, \mu)$. Let $\mathfrak{B}$ be the Chang elementary substructure witnessing $(\kappa^+, \rho) \twoheadrightarrow (\mu^+, \mu)$. Consider $\mathfrak{B}_\mu$ and suppose that $\xi = \aleph_{\gamma+1}$. Then:

- $\gamma \in \mathfrak{B}_\mu$, so a code for $\mathfrak{C}_\gamma$ is in $\mathfrak{B}_\mu$.
- Thus $\mathfrak{B}_\mu \cap \xi \prec \mathfrak{C}_\gamma$.

However, $\mathfrak{C}_\gamma$ is a witness to $(\xi, \lambda) \not\twoheadrightarrow (\mu^+, \mu)$, and thus we have reached a contradiction. $\dashv$

To prove the theorem we need to recall an important special case of Definition 32:

DEFINITION 67. *We will say that $\square_\kappa(cof(\leq \mu))$ holds provided that a partial $\square_\kappa$ sequence holds on the stationary set of ordinals in κ having cofinality less than or equal to μ.*

Results of Jensen and Cummings, Foreman and Magidor ([**3**]) show that if ordinary square holds "everywhere" between μ and $\kappa < \aleph_{\mu^+}$ then $\square_\kappa(cof(\leq \mu))$ holds. If in addition Jensen's "Global Square" holds then for all $\kappa \geq \mu$, we know that $\square_\kappa(cof(\leq \mu))$ holds.

We now finish the proof of Theorem 65:

$\vdash$ If the theorem fails, by the lemma we can assume that κ is a cardinal such that $(\kappa^+, \kappa) \twoheadrightarrow (\mu^+, \mu)$ in the strong sense that for all structures $\mathfrak{A}$ there is a Chang elemenary substructure $\mathfrak{B}$ such that $\mu \subset \mathfrak{B}$.

Let λ be a regular cardinal much bigger than κ. If $(\kappa^+, \kappa) \twoheadrightarrow (\mu^+, \mu)$ then we can find an elementary substructure N of $H(\lambda)$ such that $|N| = |N \cap \kappa^+| = \mu^+$

and $|N \cap \kappa| = \mu$ and $\mu + 1 \subset N$. Using the fact that $\kappa < \aleph_{\mu^+}$ we can assume all cardinals below κ belong to N. By the minimality of κ we can now assume that $|N \cap \rho| = \mu$ for all cardinals $\mu \leq \rho \leq \kappa$.

Let $\gamma^* = \sup(N \cap \kappa^+)$. For each limit point ξ of N, let ξ^* be the least element of N greater than or equal to ξ. It is easy to check that if $\xi \notin N$, then $cf(\xi^*) > \mu$.

Let η be the least regular cardinal such that for stationarily limit points ξ of $N \cap \gamma^*$, the cofinality of ξ^* is η. Since κ is between μ and $\aleph_{\mu^+}$, we see that some such η exists and that if $\eta > \mu$ then $\eta = \nu^+$ for some ν. Fix this ν. (If $\eta \leq \mu$, let $\nu = \mu$.)

Let $\langle C_\gamma : \gamma \in cof(\leq (\eta \cup \mu^+)) \cap \kappa^+ \rangle$ be a $\square(cof(\leq (\eta \cup \mu^+)))$ sequence on κ^+.

We start by considering any ξ with $cf(\xi^*) = \eta$ where ξ is a common limit point of N and C_{γ^*}. Since $\xi^* \in N$, we know that $C_{\xi^*} \in N$. Both C_{ξ^*} and C_{γ^*} are unbounded in ξ and hence $C_\xi = C_{\xi^*} \cap \xi = C_{\gamma^*} \cap \xi$. In particular we have that $C_{\gamma^*} \cap \xi = C_{\xi^*} \cap \xi$.

Let N_1 be the Skolem hull of $N \cup \nu$. Let $\delta = N_1 \cap \eta$. Note that while $|N_1| = \max\{\nu, \mu\}$, if $\beta \in N$ is an ordinal of cofinality at least η, we know by the "Fact" that $\sup(N_1 \cap \beta) = \sup(N \cap \beta)$. In particular, we see that $\sup(N_1 \cap \xi^*) = \xi$ and $\sup(N_1 \cap \kappa^+) = \gamma^*$.

We now split into two cases:

Case 1: The first case is when $\eta \geq \mu^+$

Since $\nu \subset N_1$, for all ξ with $cf(\xi^*) = \eta$, $C_{\xi^*} \cap \xi \subset N_1$ and $C_{\xi^*} \cap N_1$ has order type $N_1 \cap \eta = \delta$. Thus, the order type of $C_{\gamma^*} > \delta$.

Now choose a ξ a common limit point of N and C_{γ^*} such that ξ^* has cofinality η and $C_{\gamma^*} \cap \xi$ is unbounded in ξ and has order type greater than δ. Then $C_{\gamma^*} \cap \xi = C_{\xi^*} \cap \xi$, so $C_{\xi^*} \cap \xi$ has order type greater than δ, a contradiction.

Case 2: The second case is when $\eta \leq \mu$.

In this case $\nu = \mu$, $N_1 = N$ and $N \cap \mu^+$ is an ordinal. We have that $\xi^* = \xi$ and the order type of $C_{\xi^*} \leq N \cap \mu^+$. On the other hand, C_{γ^*} has order type μ^+, and hence we can find a limit point ξ of $C_{\gamma^*} \cap N$ where $cf(\xi^*) = \eta$ and the order type of $C_{\gamma^*} \cap \xi > N \cap \mu^+$. This implies that the order type of C_{ξ^*} is greater than $N \cap \mu^+$, again a contradiction. $\dashv$

We now examine another case where Chang's Conjecture fails at a singular cardinal under an assumpion must weaker than square.

Definition 68. [**3**] ADS_κ holds *iff there exists* $\langle A_\alpha : \alpha < \kappa^+ \rangle$ *such that*

1. A_α *is unbounded in* κ
2. *o.t.* $A_\alpha = cf(\kappa)$.
3. *For all* $\beta < \kappa^+$ *there exists* $g : \beta \to \kappa$ *such that the sequence* $\langle A_\alpha \setminus g(\alpha) : \alpha < \beta \rangle$ *consists of pairwise disjoint sets.*

The following result indicates some of the utility of ADS_κ.

Fact 69 (Shelah [**22**]). *If* $V \subseteq W$ *are two models of set theory such that*

$$V \models \text{"}\kappa \text{ is a cardinal and } \mathrm{ADS}_\kappa \text{ holds"}$$

and κ^+_V *is a cardinal in* W*, then* $W \models cf(\kappa) = cf(|\kappa|)$.

See [**2**] for a generalisation of this result.

It is a theorem in [**3**] that $\square^*_\kappa$ implies ADS_κ as does the existence of a *very good scale*. In the same paper it was shown that ADS_κ contradicts the statement that every stationary set in $[\kappa^+]^{<\aleph_1}$ reflects to as set of uniform cofinality.

THEOREM 70. *Suppose that* ADS_κ *holds. Then* $(\kappa^+,\kappa) \not\twoheadrightarrow (cf(\kappa)^+, cf(\kappa))$.

In particular if $\mathrm{ADS}_{\aleph_\omega}$ holds then $(\aleph_{\omega+1},\aleph_\omega) \not\twoheadrightarrow (\omega_1,\omega)$.

$\vdash$ Let N be an elementary substructure of $H(\lambda)$ containing the sequence $\langle A_\alpha : \alpha < \kappa^+\rangle$ such that $|N \cap \kappa^+| = cf(\kappa)^+$ and $|N \cap \kappa| = cf(\kappa)$. Again without loss of generality we can assume that $cf(\kappa) \subset N$. By ADS_κ, we can choose disjoint tails of the $\langle A_\alpha : \alpha \in N \cap \kappa^+\rangle$. Since each $A_\alpha \subset N \cap \kappa$, this implies that $|N \cap \kappa| = \kappa^+$, a contradiction. $\dashv$

In fact the proof just uses the fact that one can find disjoint tails for any collection of $cf(\kappa)^+$ many A_α's, a weaker property than ADS_κ.

5. Open Problems

In this section the author gives a short list of problems (many more occur in [**3, 4, 9**] etc.). While some of these problems have intimate links to classical problems of set theory and thus are likely to be difficult, most of the problems simply haven't been investigated closely, and are thus likely to succumb to a determined attack. The problems are listed in their simplest interesting case. The reader is left to extend the problem list herself.

We begin with some questions involving the existence or non-existence of various sequences:

(1) Let $\langle S_n : k < n \in \omega\rangle$ be a mutually stationary sequence with $S_n \subset \omega_n$. What can you say about the cofinalities of ordinals in the various S_n's? Can they have any finite pattern? An infinite pattern? Are there any patterns (non-trivially) forbidden by ZFC?
(2) Can you prove in ZFC that there is a sequence of stationary sets $S_n \subset \omega_n \cap cof(\omega_1)$ that is not mutually stationary?
(3) Can you always force such a sequence?
(4) Can you show in ZFC that there is always a mutually stationary sequence that is not tightly stationary? (Such sequences are forced in [**4**].)

Questions involving Fodor's Lemma and splitting mutually stationary sequences:

(5) Suppose that $\langle S^n : k < n < \omega\rangle$ is a sequence of mutually stationary sets. If $f : \aleph_\omega \to \aleph_\omega$ is regressive, is there a function $g \in \prod_{n\in\omega} \aleph_n$ such that if $T^n_g = \{\alpha \in S^n : f(\alpha) = g(n)\}$, then the sequence $\langle T^n_g : k < n < \omega\rangle$ is mutually stationary?
(6) Suppose that $\langle S_n : k \le n \in \omega\rangle$ is a mutually stationary sequence. Are there partitions $S_n = \bigcup_{\alpha<\omega_n} S^\alpha_n$ such that for a function $f \in \prod_{k\le n\in\omega} \omega_n$ the sequence $\langle S^{f(n)}_n : k \le n \in \omega\rangle$ is mutually stationary?

    Are there partitions $S_n = S^0_n \cup S^1_n$ such that for $i = 0, 1$ the sequences $\langle S^i_n : n \in \omega\rangle$ are mutually stationary?

Questions exploring the theory of mutually stationary sets:

(7) Several places in the theory we use $2^{\aleph_0} < \aleph_\omega$ (e.g. Theorems 48,51), can this be eliminated? Similarly, the places in the theory where we use that a mutually stationary sequence concentrates on ordinals of a fixed cofinality?

(8) Is there a mutually stationary sequence of sets $\langle S_n \subset \omega_n : k < n \in \omega\rangle$ and $\langle T_m : j < m \leq k\rangle$ such that for any finite k', $\langle T_m : j < m \leq k\rangle^\frown \langle S_n : k < n < k'\rangle$ is mutually stationary but $\langle T_m : j < m \leq k\rangle^\frown \langle S_n : k < n < \omega\rangle$ is not?

(9) Is there is a sequence $\langle S_\alpha : \alpha < \omega_1\rangle$ such that $S_\alpha \subset \kappa_\alpha$ and for every $a \in [\omega_1]^{\aleph_0}$, $\langle S_\alpha : \alpha \in a\rangle$ is mutually stationary but $\langle S_\alpha : \alpha < \omega_1\rangle$ is not?

Finally some very elementary questions for non-CH models:

(10) Suppose that $N \prec H(\lambda)$ has cardinality ω_1 and is internally stationary. Is it internally club? Internally approachable? If it is internally club is it necessarily internally approachable? What is the relation between these ideas? What about for large N's?

References

[1] James E. Baumgartner. On the size of closed unbounded sets. *Ann. Pure Appl. Logic*, 54(3):195–227, 1991.

[2] James Cummings. Collapsing successors of singulars. *Proc. Amer. Math. Soc.*, 125(9):2703–2709, 1997.

[3] James Cummings, Matthew Foreman, and Menachem Magidor. Squares, scales and stationary reflection. *Journal of Mathematical Logic*, 1:1–69, To appear.

[4] James Cummings, Matthew Foreman, and Menachem Magidor. Stationary, tight and approachable. *Cummings' Computer*, (Growing Daily), To appear.

[5] Hans-Dieter Donder. Regularity of ultrafilters and the core model. *Israel J. Math.*, 63(3):289–322, 1988.

[6] M. Foreman, M. Magidor, and S. Shelah. Martin's maximum, saturated ideals, and nonregular ultrafilters. I. *Ann. of Math. (2)*, 127(1):1–47, 1988.

[7] Matthew Foreman. More saturated ideals. In *Cabal seminar 79–81*, pages 1–27. Springer, Berlin, 1983.

[8] Matthew Foreman and Menachem Magidor. Large cardinals and definable counterexamples to the continuum hypothesis. *Ann. Pure Appl. Logic*, 76(1):47–97, 1995.

[9] Matthew Foreman and Menachem Magidor. Mutually stationary sequences of sets and the non-saturation of the non-stationary ideal on $p_\kappa(\lambda)$). *Acta Mathematica*, 37 pages, To appear.

[10] Matthew Foreman and Stevo Todorcevic. Strong Löwenheim–Skolem theorems and Simultaneous Reflection. pp 1–∞, To appear.

[11] András Hajnal and Peter Hamburger. *Set theory*. Cambridge University Press, Cambridge, 1999. Translated from the 1983 Hungarian original by Attila Máté.

[12] Leo Harrington and Saharon Shelah. Some exact equiconsistency results in set theory. *Notre Dame J. Formal Logic*, 26(2):178–188, 1985.

[13] Thomas Jech. *Set theory*. Academic Press [Harcourt Brace Jovanovich Publishers], New York, 1978. Pure and Applied Mathematics.

[14] A. Kanamori and M. Magidor. The evolution of large cardinal axioms in set theory. In *Higher set theory (Proc. Conf., Math. Forschungsinst., Oberwolfach, 1977)*, pages 99–275. Springer, Berlin, 1978.

[15] David W. Kueker. Countable approximations and Löwenheim-Skolem theorems. *Ann. Math. Logic*, 11(1):57–103, 1977.

[16] Paul Larson. Separating stationary reflection principles. *J. Symbolic Logic*, 65(1):247–258, 2000.

[17] Jean-Pierre Levinski, Menachem Magidor, and Saharon Shelah. Chang's conjecture for $\aleph_\omega$. *Israel J. Math.*, 69(2):161–172, 1990.

[18] Kecheng Liu. Stationary subsets of $[\aleph_\omega]^{<\omega_n}$. *J. Symbolic Logic*, 58(4):1201–1218, 1993.

[19] Kecheng Liu and Saharon Shelah. Cofinalities of elementary substructures of structures on $\aleph_\omega$. *Israel J. Math.*, 99:189–205, 1997.
[20] Menachem Magidor. Reflecting stationary sets. *J. Symbolic Logic*, 47(4):755–771 (1983), 1982.
[21] Menachem Magidor and Saharon Shelah. When does almost free imply free? (For groups, transversals, etc.). *J. Amer. Math. Soc.*, 7(4):769–830, 1994.
[22] Saharon Shelah. *Proper forcing*. Springer-Verlag, Berlin, 1982.
[23] Saharon Shelah. *Cardinal arithmetic*. The Clarendon Press Oxford University Press, New York, 1994. Oxford Science Publications.
[24] Saharon Shelah. On what i do not understand (and have something to say). *Fundamenta Mathematicae*, 166:1–82, 2000.
[25] John R. Steel. *The core model iterability problem*. Springer-Verlag, Berlin, 1996.
[26] Boban Veličković. Forcing axioms and stationary sets. *Adv. Math.*, 94(2):256–284, 1992.
[27] W. Hugh Woodin. *The axiom of determinacy, forcing axioms, and the nonstationary ideal*. Walter de Gruyter & Co., Berlin, 1999.

Mathematics Department, University of California, Irvine, USA