The Two Envelopes Paradox
in a Short Story

Jan Poland

Last Sunday, I went to see my rich uncle. He lives in a large villa in Zürich, just on the lakefront. He somehow gives the impression that he does not know where to spent his money. He has never been married, and his only hobby is chess playing.

That day, he seemed particularly bored to me. He did not even agree to play a chess game with me (he always wins, since I am not a good chess player). So we sat silently on his large sofa and had some tea. But suddenly, he took two closed envelopes out of his pocket. Slightly surprised, I listened as he suggested me a game.

"You see these two envelopes? There is money in both of them. Actually, one contains twice the amount as the other. Now you may take one envelope, open it, and then decide. Either you keep the money. Or you not keep it, in which case you open the other envelope and take the money in there."

Almost all visits at my uncle's ended by receiving more or less money. Last month for example he had given me 10'000 Swiss francs. The unusual thing was that game. I carefully felt the envelopes – no clue. Most probably there was a cheque in both of them. "Very well," I agreed. I took one envelope and opened it. It contained a cheque of

$$8'000 \text{ Swiss francs.}$$

I was about to take the second envelope, but my uncle stopped me. "Hang on. You may only take the other envelope if you decide *not* to keep the 8'000 francs." "Right." So I had to decide. Should I keep the first envelope or not?

"Nonsense," I grumbled. "It is just as likely to hit the higher amount in the first trial as in the second trial. It does not matter if I change the envelope or not." But then another idea occurred me. The probability that the second envelope will contain twice the amount of money should be one half, just as the probability that it contains half the amount. Given that 8'000 francs are in the first envelope, with probability $\frac{1}{2}$ I will *gain* 8'000 francs by changing and with probability $\frac{1}{2}$ I will *loose* 4'000 francs by changing my choice. Or, more mathematically, the expected gain when taking the other envelope is

$$\mathbf{E}(\text{gain if I change}) = \tfrac{1}{2} \cdot 8'000 + \tfrac{1}{2} \cdot (-4'000) = 4'000.$$

So I may expect to gain 4'000 francs if I change? This would suggest me to do so.

I was confused. Still the game seemed to be perfectly symmetric, I could as well have picked the other envelope at first by chance. I cautiously asked my uncle. “You see, that’s like the stock market is. You have to make up your mind by yourself,” was his response. He always gained at the stock market, while I had never tried.

My uncle relaxed and closed his eyes. He seemed to have no hurry, so I decided to think over the matter carefully. If it had been not 8’000 but 8’000’000 francs, I would be pretty sure: keep the money. I know that my uncle’s bank balance must be about 10’000’000, and he is a banker. So he would never sign a cheque of 16’000’000 francs. On the other hand, if the amount had been only 500 francs, I would choose the other envelope. My uncle has never given me less than 1’000 francs. But he had given me 4’000 francs a couple of times, and he had given me even as much as 20’000 francs. So this did not help very much. And anyway, if I knew nothing about my uncle’s habits of giving money to his nephew, there were still the paradox:

$$\mathbf{E}(\text{gain if I change}) = 4'000 \quad \textit{or} \quad \text{“changing does not matter”?}$$

In other words: Assume that I observed the amount x in the first envelope. Can it possibly be that the probability that the second envelope contains $2x$ is always $\frac{1}{2}$? Certainly not for all x, as I had already seen: if x is large, then the uncle does not have $2x$ francs. On the other hand, if x is some odd amount like 1.23 francs, then it is not divisible by 2, and therefore the other envelope must contain $2x$. But the paradox remains even ignoring these subtleties. So I decided to put the problem in a simple mathematical form.

Assumption. Let $f : [0, \infty) \to [0, \infty)$ be the probability density of the *larger* amount of both envelopes.

This means in particular that each non-negative amount can occur, and the uncle has potentially infinite money. (For convenience, we do not admit concentrations of probability mass to particular amounts.) So under this assumption, the probability density of the *smaller* amount is $\tilde{f}(x) = 2f(2x)$. Then I may express the probability density of the amount in the first envelop I chose:

$$g(x) = \frac{1}{2}\left(f(x) + \tilde{f}(x)\right) = \frac{f(x)}{2} + f(2x). \tag{1}$$

So if I observe x in the first envelope, the expected gain by changing is

$$\mathbf{E}(\text{gain if I change}|x) = x \cdot \frac{f(2x)}{g(x)} - \frac{x}{2} \cdot \frac{f(x)}{2g(x)}. \tag{2}$$

Integrating with respect to g gives

$$\int_0^\infty \mathbf{E}(\text{gain if I change}|x) g(x)dx = \int_0^\infty xf(2x)dx - \frac{1}{4}\int_0^\infty xf(x)dx = 0,$$

provided that the expectations of f and therefore also of g exist.

I relaxed. This answers the paradox: switching the envelope implies no gain and no loss in the expectation. This also clarifies the other problem: The probability of gain x cannot be $\frac{1}{2}$ for all x observed in the first envelope. Namely, this would imply that $g(x) = g(2x) = g(4x) = g(8x) = \ldots$, since the probability density of the amount in the second envelope is g, too. For some integer $k \in \mathbb{Z}$, the interval $[2^k, 2^{k+1})$ has non-zero mass under g:

$$\int_{2^k}^{2^{k+1}} g(x)dx = a.$$

Then the mass of $[2^{k+1}, 2^{k+2})$ would be $2a$, etc. Hence the total mass of $[0, \infty)$ would be ∞. This corresponds to the fact that there is no uniform distribution on the positive reals or natural numbers.

In the meantime, my uncle had started snoring. I reconsidered the envelope and decided to keep the 8'000 francs: he had very rarely given me more than 10'000. It was still only intuition, but I did not feel like recalling all past gifts and estimating a probability density. Yet, a slight uneasiness concerning the paradox remained. Was it *really* completely arbitrary to switch the envelope? The answer was yes if I looked at the expected gain. But if I looked at something other?

Assume that I know the probability density of the larger amount, f. Then I also know the probability density of the first observed amount, g. Also, given the first observation x (which is distributed according to g) I can decide if I should change or not. Precisely, I stay at my choice if $\mathbf{E}(\text{gain if I change})|x) < 0$. According to (2), this happens for

$$x \in A = \big\{x : f(x) > 2f(2x)\big\}. \tag{3}$$

Suppose for example that $f = \frac{1}{U}\mathbf{1}_{[0,U]}$, i.e. that the larger amount is uniformly drawn from $[0, U]$ for some $U > 0$. Then condition (3) holds for $x \in [\frac{U}{2}, U]$. Then the mass of A under g is

$$\int_{\frac{U}{2}}^{U} g(x)dx = \int_{\frac{U}{2}}^{U} \frac{f(x)}{2}dx = \frac{1}{4}$$

according to (1). So *only for a quarter of the observations, staying is favorable.* Thus in three quarter of all cases I should change.

Has the paradox returned? No, not really. I have only discovered the fact that the there is only symmetry regarding the expected gain. For the above example, in the cases where changing is favorable (of mass $\frac{3}{4}$) I might win or loose, where in the other cases I loose for sure.

Can the mass of observations where staying is preferable ever exceed $\frac{1}{2}$? No, because I can prove the following statement.

Theorem. *Let f be an arbitrary probability density, $g(x) = \frac{f(x)}{2} + f(2x)$, and $A = \big\{x : f(x) > 2f(2x)\big\}$. Then $\int_A g(x)dx \leq \frac{1}{2}$ holds. So the probability of first observations where changing is preferable is always at least $\frac{1}{2}$.*

Proof. Assume first that f is a step function, i.e. there are finitely many intervals on which f is constant. Then f has compact support contained in $[0, c]$ for suitable $c > 0$. Let $f_1 := f$ and $A_1 := A$ and assume that

$$b = \int_{\frac{c}{2}}^{c} f_1(x)dx < 1.$$

(If $b = 1$, then there is nothing to show, since then $\int_{A_1} \big[f_1(x)+2f_1(2x)\big]dx = 1$.) We construct a step function f_2 with support $[0, \frac{c}{2}]$ by $f_2(x) = \frac{f_1(x)}{1-b}$ on $[0, c/2]$ and $f_2(x) = 0$ otherwise. Then $\int f_2(x)dx = \frac{1-b}{1-b} = 1$. Let $A_2 = \{x : f_2(x) > 2f_2(2x)\}$. Then the following implication holds:

$$\int_{A_2} \big[f_2(x) + 2f_2(2x)\big]dx <= 1 \implies \int_{A_1} \big[f_1(x) + 2f_1(2x)\big]dx <= 1, \tag{4}$$

since by construction, $A_1 \subset A_2 \cup (\frac{c}{2}, c]$, and therefore

$$\begin{aligned} &\int_{A_1} \big[f_1(x) + 2f_1(2x)\big]dx \\ &\leq \int_{A_2} \big[f_1(x) + 2f_1(2x)\big]dx + \int_{\frac{c}{2}}^{c} \big[f_1(x) + 2f_1(2x)\big]dx \\ &= (1-b)\int_{A_2} \big[f_2(x) + 2f_2(2x)\big]dx + \int_{\frac{c}{2}}^{c} f_1(x)dx \leq (1-b) + b = 1. \end{aligned}$$

In this way one can proceed to f_3 and A_3, then f_4 and A_4, etc. After finitely many steps, f_n is a constant on some interval, for which $\int_{A_n} \big[f_n(x) + 2f_n(2x)\big]dx <= 1$ is obvious. Then backtracking via (4) proves the assertion for step functions. Finally, any density f can be approximated from below by step functions. Therefore the general assertion follows. □

Is this bound sharp? Yes, if for example $f = \mathbf{1}_{[1,2]}$, then $\int_A g(x)dx = \frac{1}{2}$. Can I also prove a lower bound on $\int_A g(x)dx$ besides the trivial one, zero? The answer is no. For the following example, we admit concentration of probability mass. Let n be a positive integer and f be the discrete uniform distribution on $\{1, 2, 4, \ldots, 2^n\}$. Then $A = \{2^n\}$ and $\int_A g = \frac{1}{2n}$. So $\int_A g$ may be arbitrarily small.

With a loud snore, my uncle woke up. “Did you make up your mind?” “Yes, I keep this one,” I answered. Smiling, he opened the other envelope and showed me a cheque of 16’000 francs.