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Counting Unrooted Loopless Planar Maps

Valery A. Liskovets and Timothy R. Walsh

Abstract. We present a formula for the number of n-edge unrooted loopless planar maps considered up to orientation-preserving isomorphism. The only sum contained in this formula is over the divisors of n.
Résumé. Nous présentons une formule pour le nombre de cartes planaires sans boucles avec n arêtes, à isomorphisme près préservant l'orientation. La seule somme contenue dans cette formule est prise parmi les diviseurs de n.

1. Introduction

At the end of the 1970s the first-named author developed a general method of counting planar maps up to orientation-preserving isomorphism ("unrooted") which is based on using quotient maps [**Li81**] (cf. also [**Li85, Li98**]). It results in a formula which represents the number of unrooted planar n-edge maps of a given class in terms of the numbers of rooted maps of the same class and of their quotient maps with respect to orientation-preserving isomorphism. Based on Burnside's (orbit counting) lemma, this reductive formula contains a sum over the orders of automorphisms of the maps under consideration; as a rule, these are the divisors of n. Generally the formula may contain other summations and need not be very simple since quotient maps may form a fairly complicated class of maps.

Until now, this method was applied successfully to several natural classes of planar maps. Namely, simple formulae have been obtained for counting all maps, homogeneous maps and so called strongly self-dual maps; this last formula contains no sums [**Li81**]. (We add also that two related problems were solved in [**BoLL00, B-MS00**]. Moreover, a formula of this kind has been obtained for the first time in another way in [**Wk72**] for plane trees. It is a particular case of the formula for homogeneous maps, and in [**BnBLL00**] it was generalized to planar m-ary cacti.) Later on, we applied this method to obtain similar formulae for non-separable maps [**LiW83**] (see also [**LiW87**]) and for eulerian and unicursal planar maps [**LiW02**]. All these classes have a remarkable property in common: the number of rooted maps in them is expressed by a simple sum-free formula. This feature immediately implies a simple explicit form of the formula for counting unrooted maps in the simplest cases when the class of quotient maps coincides, or almost coincides, with the initial class. These cases include in particular the types of maps considered in [**Li81**]. However for non-separable, eulerian and unicursal maps the quotient maps are not identical, or even nearly so, to the original maps; so we cannot assume a priori that the corresponding rooted quotient maps are also enumerated by simple sum-free formulae which eliminate additional sums and auxiliary terms in the formula for unrooted

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maps and simplify it significantly. Nevertheless, quite unexpectedly, this property is shared by all the cases considered so far and we have found, for the corresponding unrooted maps, counting formulae that contain only a sum over the divisors of n and a bounded number of additional terms.

The aim of the present article is to investigate one more natural class of maps, loopless maps, which have the same property with respect to rooted enumeration and to establish the existence of a similar simple formula for the number of unrooted maps in it. Again, we do not have any direct explanation for this phenomenon.

Loopless maps have attracted much attention in enumerative combinatorics. Let $L'(n)$ be the number of rooted loopless planar maps with n edges. It was shown in [**WlL75**] and in several more recent publications (see, in particular, [**Wo80, BeW85**]) that

$$L'(n) = \frac{2(4n+1)!}{(n+1)!(3n+2)!} = \frac{2(4n+1)}{(n+1)(3n+1)(3n+2)}\binom{4n}{n}, \quad n \geq 0. \tag{1.1}$$

Let $L^+(n)$ denote the number of unrooted loopless planar maps with n edges counted up to orientation-preserving isomorphism. In this article we prove

THEOREM 1.1. *For $n \geq 1$,*

$$\begin{aligned} L^+(n) &= \frac{1}{2n}\Bigg[L'(n) + \sum_{t<n,\, t|n} \phi\Big(\frac{n}{t}\Big)\frac{(t+1)(3t+1)(3t+2)}{2(4t+1)}\,L'(t) \\ &+ \begin{cases} n^2\,L'\Big(\dfrac{n-1}{2}\Big) & \text{if } n \text{ is odd} \\ \dfrac{4n(n-1)(2n-1)}{3(3n+2)}\,L'\Big(\dfrac{n-2}{2}\Big) & \text{if } n \text{ is even} \end{cases}\Bigg], \end{aligned} \tag{1.2}$$

where $\phi(n)$ is the Euler totient function.

Substituting (1.1) into formula (1.2) we can represent it in the following explicit form:

COROLLARY 1.2.

$$\begin{aligned} L^+(n) &= \frac{1}{2n}\Bigg[\frac{2(4n+1)}{(n+1)(3n+1)(3n+2)}\binom{4n}{n} + \sum_{t<n,\, t|n} \phi\Big(\frac{n}{t}\Big)\binom{4t}{t} \\ &+ \begin{cases} \dfrac{2n}{n+1}\dbinom{2n}{\frac{n-1}{2}} & \text{if } n \text{ is odd} \\ \dbinom{2n}{\frac{n-2}{2}} & \text{if } n \text{ is even} \end{cases}\Bigg]. \end{aligned} \tag{1.3}$$

The article is organized as follows. Section 2 contains a general description of planar maps, their automorphisms and quotient maps. Section 3 contains a general "reductive" enumerative formula for loopless planar maps and a description of their quotient maps. These quotient maps are enumerated in Section 4. From these results, formula (1.2) is derived in Section 5, which also includes a table of values and some open questions.

2. Maps and quotient maps

A *map* is a 2-cell imbedding of a connected planar graph in a closed orientable surface; if the surface is a sphere, then the map is *planar*. A well-known combinatorial model of maps on an orientable surface represents a map as a pair of permutations (σ, α) acting on a finite set D of *darts* or *edge-ends* such that α is a fixed-point-free involution and the group generated by σ and α is transitive on D. The vertices, edges and faces are, respectively, the cycles of σ, α and $\sigma\alpha$; σ corresponds to counter-clockwise rotation around a vertex from one dart to the next, α corresponds to going from one end of an edge to the other, and $\sigma\alpha$

corresponds to walking clockwise from one edge to the next around the boundary of a face. A map is planar if it satisfies Euler's formula:

$$\#(vertices) + \#(faces) - \#(edges) = 2. \tag{2.1}$$

In what follows, a map is assumed to be planar. An *automorphism* of a combinatorial map is a permutation of D that commutes with σ and α; it corresponds to an orientation-preserving homeomorphism of a (topological) map. Topological and combinatorial models of maps are known to be equivalent (see [**JoS78**]); we will need them both.

A map is *rooted* by distinguishing a dart as the *root*. It was shown in [**Tu63**] (and follows easily from the combinatorial model) that only the trivial automorphism of a planar map fixes the root. Consequently, rooted maps can be counted without considering their symmetries. By *counting unrooted maps* we mean counting isomorphism classes of maps (with respect to orientation-preserving isomorphism).

The method developed in [**Li81, Li85**] (and slightly simplified and modified by form in [**Li98**]) makes it possible to count unrooted maps of classes more complex than plane trees. It relies on constructing and counting quotient maps and uses significantly the familiar property that for any non-trivial orientation-preserving automorphism ρ of a map Γ, the map can be drawn on the sphere so that ρ represents a (geometrical) *rotation* of the sphere about a well-defined axis which intersects the map in two *elements* (vertices, edges or faces) called *axial*, which, for the sake of brevity, we call the *poles* (see loc. cit. for the necessary references). Geometrically, the points of intersection of the axis with edges and faces are their midpoints. On the other hand, as follows from the combinatorial model (the transitivity property), any automorphism of a map is *regular* - all the dart-cycles are of the same length. There is a bijection between the maps fixed by an automorphism and the isomorphic submaps into which the automorphism divides the maps, and this fact provides a way for counting unrooted maps using Burnside's lemma.

Given a map Γ and a non-trivial (orientation-preserving) automorphism ρ of it which is presented geometrically as a rotation and determined by the pair of poles, the order $p \geq 2$ (the period of rotation) and the angle of rotation $2\pi k/p$ (where $k,\ 1 \leq k < p$, is prime to p), the *quotient map* Δ of Γ with respect to ρ is constructed by cutting the sphere into p identical sectors whose common edge is the axis of rotation, choosing one of those sectors, expanding it into a sphere and closing it. In fact, Δ depends only on the cyclic group generated by ρ. If a pole of Γ is an edge, then it turns into a "half-edge" in Δ, that is into an edge which contains a single dart; so an additional vertex of valency 1 (endpoint), called a *singular vertex*, is created. A singular vertex contains no darts and it is identified with the corresponding pole. If Δ contains one or two singular vertices, then $p = 2$. If Γ is rooted, then among the p sectors we choose the one that contains the root, so that Δ is also rooted.

We define a *q-map* to be a planar map with 0, 1 or 2 vertices of valency 1 distinguished as singular vertices and two elements distinguished as axial (poles) which are either vertices or faces and must include all the singular vertices. Given a q-map Δ and an integer $p \geq 2$, the map Γ and the pair of poles such that Δ is the quotient map with respect to an automorphism of order p about an axis intersecting that pair of poles can be retrieved by a process called *lifting*: a semicircular cut whose diameter intersects the two poles is made in the sphere containing Δ, the sphere is then shrunk into a sector of dihedral angle $2\pi/p$, any singular vertex (if any for $p = 2$) is deleted leaving its incident edge with a single dart, and p copies of this sector are pasted together to make a sphere containing Γ. If Δ is rooted, then the root of one of these copies is chosen to be the root of Γ.

3. The quotient map of a loopless map

A map is called *loopless* if its graph does not contain loops. Below we give a construction for a quotient map of a loopless map. Let $L'_0(n)$, $L'_1(n)$ and $L'_2(n)$ be the number of rooted n-edge q-maps with 0, 1 or 2 singular vertices, respectively, whose liftings are rooted loopless maps. The following formula is a direct

consequence of the general enumerative scheme of [**Li81, Li85**] described in the form presented in [**Li98**, Sect. 8.7]:

PROPOSITION 3.1.

$$2nL^+(n) = L'(n) + \sum_{t<n,\, t|n} \phi\Big(\frac{n}{t}\Big) L'_0(t) + \begin{cases} L'_1\big((n+1)/2\big) & \text{if } n \text{ is odd} \\ L'_2\big((n/2)+1\big) & \text{if } n \text{ is even.} \end{cases} \tag{3.1}$$

□

Each term in the sum in (3.1) is contributed by the automorphisms of order $p = n/t$ and the factor $\phi(n/t)$ is the number of such automorphisms. Below we prove (1.2) by finding expressions for $L'_i(n)$, $i = 0, 1, 2$, which are sums of one or two terms, and substituting them into (3.1).

To evaluate $L'_i(n)$, $i = 0, 1, 2$, we must consider two cases: either the quotient map has no loops or it has at least one loop. The former case is easily tractable by adding 0, 1 or 2 singular vertices to a rooted loopless map; the latter requires a characterization of a rooted q-map that has at least one loop but is lifted into a rooted map with no loops.

LEMMA 3.2. *A loop ℓ in a q-map Δ is destroyed by lifting if and only if ℓ separates the poles of Δ.*

PROOF. Suppose that the loop ℓ separates the poles. Then ℓ can be drawn as a circle that separates the poles. The cut made as the first step in lifting Δ (see Section 2) will intersect ℓ and it can be arranged not to intersect the vertex v incident to ℓ. The sector will contain v with one dart of ℓ on either side of it. When p such sectors are pasted together, each one will have a copy of v, and adjacent copies of v will be joined by a *link* (non-loop edge) consisting of one dart of ℓ from one sector and the other dart of ℓ from the adjacent sector. The loop ℓ will thus be replaced by p links.

Suppose that ℓ does not separate the poles. Then the cut can be arranged not to intersect ℓ. The sector will have the loop ℓ and the lifted map will have p loops. □

LEMMA 3.3. *Lifting a q-map Δ creates a loop if and only if one pole of Δ is a singular vertex v and the other pole is the vertex adjacent to v.*

PROOF. Suppose that one pole is a singular vertex v and the other pole is the vertex adjacent to v. The sector will contain v and a single dart d. Since v is a pole, the lifted map will have a single copy of v with two copies of d joined together into a single edge, a loop incident to v.

Suppose that Δ has no singular vertex. Then each link of Δ is lifted into p links of Γ. Now suppose that Δ has a singular vertex, which must of necessity be a pole. If the adjacent vertex v is not the other pole, then it will be lifted into two vertices joined together by the link whose darts are the two copies of the edge joining v to the singular vertex of Δ. In either case, no loop will be created. □

THEOREM 3.4. *A map Γ lifted from a q-map Δ is loopless if and only if Δ satisfies the following two conditions:*

1. *if Δ has loops, each of them separates the poles, and*
2. *if one pole is a singular vertex s, then the other pole is not the vertex adjacent to s.*

PROOF. An easy consequence of Lemmas 3.2 and 3.3. □

DEFINITION 3.5. An *ℓ-map* is a q-map which has at least one loop but whose liftings have no loops.

We present a construction of an ℓ-map Δ, an analogue of the s-map of [**LiW83**] that consists of a chain of blocks.

Suppose that Δ contains $k-1$ loops, $k > 1$. Arbitrarily call one pole the outer pole and the other one the inner pole. By Theorem 3.4, condition (1), the $k-1$ loops are all nested one inside the other in linear order $\ell_1, \ell_2, \ldots, \ell_{k-1}$, with the outer pole strictly outside the outermost loop ℓ_1, and therefore not the vertex incident to ℓ_1 and the inner pole strictly inside the innermost loop ℓ_{k-1}, and therefore not the vertex incident

to ℓ_{k-1}. The outer pole belongs to the submap M_1 of Δ that is outside of ℓ_1 (if the outside of ℓ_1 is empty, then M_1 is just the vertex-map). The inner pole belongs to the submap M_k that is inside of ℓ_{k-1}. For $i = 2, 3, ..., k-1$ we denote by M_i the submap that is inside of ℓ_{i-1} but outside of ℓ_i. All the M_i are loopless. We call the M_i the *components* of Δ; M_1 and M_k are the *extremal* components and the other components are the *internal* ones.

An example of a loopless map and its quotient map, which is an ℓ-map, is depicted in Fig. 1.

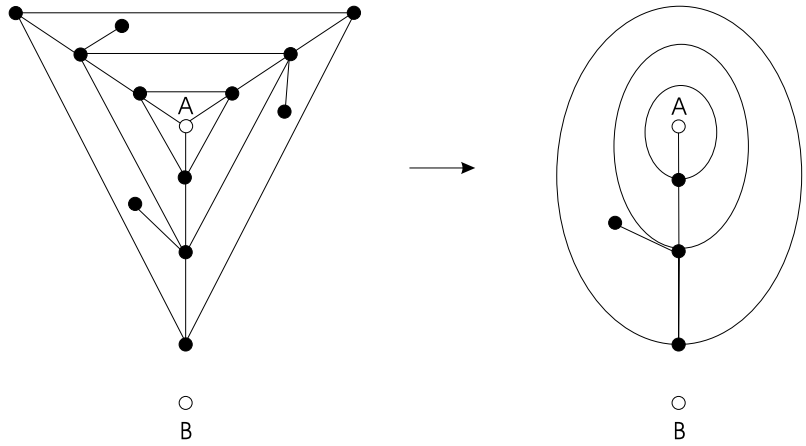

FIGURE 1. A loopless map (left) and its quotient map (right) with respect to rotations of order 3 around the axis AB (where B is the midpoint of the outer face)

We construct Δ from the outside in. We begin with a loopless map M_1. We insert an empty loop ℓ_1 into M_1, at the one vertex of M_1 if M_1 is a vertex-map or between two consecutive darts of a vertex of M_1 otherwise. After this insertion, one of the darts d of ℓ_1 will still have the property that $\sigma(d) = \alpha(d)$ (a counter-clockwise rotation about the vertex incident to ℓ_1 starting at d traverses the empty inside of ℓ_1 and then encounters the other dart of the loop) - we call d the *right* dart of ℓ_1 and $\alpha(d)$ its *left* dart. Then a rooted loopless map M_2 is inserted into ℓ_1 so that the root of M_2 (if M_2 is not a vertex-map) becomes $\sigma(d)$. If $k > 2$, then another empty loop ℓ_2 is inserted into M_2 and another rooted loopless map M_3 inserted into ℓ_2 and so on until the innermost loop ℓ_{k-1} has been inserted into M_{k-1} and the innermost rooted loopless map M_k has been inserted into ℓ_{k-1}. If Δ is not to have any singular vertices, then the outer pole is chosen to be some vertex or face of M_1 but not the vertex incident with ℓ_1 and the inner pole is chosen to be some vertex or face of M_k but not the vertex incident with ℓ_{k-1}. The modification of this construction to account for singular vertices is discussed in Sections 4.3 and 4.4.

4. Enumeration of quotient maps of rooted loopless maps

4.1. **Enumeration of rooted ℓ-maps by the sizes of the extremal components.** We now proceed to enumerate rooted ℓ-maps with n edges and no singular vertices such that M_1 has a edges, M_k has b edges, and for $2 \leq i \leq k-1$, M_i has n_i edges, so that $n_2 + \cdots + n_{k-1} = n - (a+b) - (k-1)$. For the moment we distinguish the poles as outer and inner (to distinguish between M_1 and M_k) but in this subsection we do not include the number of choices of poles in the enumeration formulae.

Suppose for the moment that the root of Δ belongs to M_1 (if M_1 is a vertex-map, then the root is the left dart of ℓ_1). If M_1 is not a vertex-map, then there are $2a$ places to insert ℓ_1; otherwise there is one place. For $2 \leq i \leq k$ there is one place to insert M_i into ℓ_{i-1}. For $2 \leq i \leq k-1$ there are $2n_i + 1$ places to insert

ℓ_i into M_i: for any dart d of M_i, ℓ_i can be inserted between d and $\sigma(d)$, or else ℓ_i can be inserted between the root of M_i and the right dart of ℓ_{i-1}. The number of ℓ-maps whose root belongs to M_1 is thus

$$L'(a)L'(b)\prod_{i=2}^{k-1}(2n_i+1)L'(n_i)\cdot\begin{cases}2a & \text{if } a>0\\ 1 & \text{if } a=0.\end{cases} \tag{4.1}$$

Before continuing with the enumeration we formally state a folkloric lemma and provide two proofs, special cases of which appear in numerous places in the literature.

LEMMA 4.1 (the Little Labeling Lemma). *Suppose that there are two sorts of labels for a combinatorial object, each with the property that only the trivial automorphism preserves the labels. If the object can be labeled in x ways with labels of the first sort and y ways with labels of the second sort, then the numbers x' and y' of equivalence classes of labelings of the two sorts, where two labelings are equivalent if the object has an automorphism taking one set of labels into the other, are in the same proportion $x : y$. This proportion extends by summation to any set of objects with the same proportion $x : y$ of ways of labeling them with labels of the two sorts.*

PROOF. *Proof 1:* Let A be the number of automorphisms of the object. Then $x' = x/A$ and $y' = y/A$.

Proof 2: We count the number of inequivalent ways to apply labels of both sorts at once. Since either sort of labeling destroys all non-trivial automorphisms, once labels of one sort have been applied, all the ways of applying labels of the other sort are inequivalent. There are thus $x'y$ inequivalent ways to apply labels of the first sort followed by labels of the second sort and $y'x$ ways to apply labels of the second sort followed by labels of the first sort; so $x'y = y'x$. □

Resuming the enumeration, suppose now that the root can be any dart of the map. Then the factor $2a$ or 1 of (4.1) is replaced by $2n$ (here we are using Lemma 4.1, where one sort of labeling is a rooting in M_1 and the other sort is a rooting anywhere in the map) provided that the poles are actually distinguished as outer and inner. If $a \neq b$, we can call the outer pole the one that belongs to the component with more edges by insisting that $a > b$. If $a = b$, then the distinction between the poles remains arbitrary; removing it is equivalent to dividing the number of rooted ℓ-maps by 2 (here we are using Lemma 4.1, where one sort of labeling includes distinguishing the poles as well as rooting the map and the other sort does not).

Let $C'(a, b; n)$ be the number of rooted n-edge ℓ-maps whose extremal components have a and b edges (we no longer distinguish the poles as inner and outer). Applying to (4.1) the discussion of the previous paragraph and then summing first over all sequences of $n_2, \ldots, n_{k-1}$ which add to $n-(a+b)-(k-1)$ and then over k from 2 to $n-2$ we obtain

$$C'(a,b;n) = nL'(a)L'(b)\sum_{k=2}^{n-2}\sum_{\substack{n_2+\cdots+n_{k-1}\\=n-(a+b)-(k-1)}}\prod_{i=2}^{k-1}(2n_i+1)L'(n_i)\cdot\begin{cases}2 & \text{if } a\neq b\\ 1 & \text{if } a=b.\end{cases} \tag{4.2}$$

Let

$$g(x) = \sum_{n=0}^{\infty} L'(n)x^n. \tag{4.3}$$

It was shown in [**WlL75**] that

$$g(x) = 1 + z - z^2 - z^3, \tag{4.4}$$

where

$$z = x(1+z)^4. \tag{4.5}$$

We will use these formulae repeatedly. By differentiating (4.3) we find that

$$\sum_{n=0}^{\infty}(2n+1)L'(n)x^n = 2xg'(x) + g(x), \tag{4.6}$$

We evaluate $g'(x)$ by differentiating (4.4) with respect to z and then dividing by dx/dz as evaluated from (4.5) and then we multiply by x, again from (4.5), and simplify to obtain

$$xg'(x) = z(1+z)^2. \tag{4.7}$$

Substituting from (4.7) and (4.4) and simplifying, we obtain

$$2xg'(x) + g(x) = (1+z)^3. \tag{4.8}$$

We denote by $[x^n]\, f$ the coefficient of x^n in the power series f. Substituting from (4.8) and (4.6) we find that the inner sum in (4.2) is $[x^{n-(a+b)-(k-1)}]\,(1+z)^{3(k-2)} = [x^{n-(a+b)-1}]\,x^{k-2}(1+z)^{3(k-2)}$, so that the outer sum is

$$[x^{n-(a+b)-1}]\,(1-x(1+z)^3)^{-1}. \tag{4.9}$$

Substituting from (4.5) for x into (4.9), simplifying and substituting into (4.2), we find that

$$C'(a,b;n) = nL'(a)L'(b)\cdot[x^{n-(a+b)-1}]\,(1+z)\cdot\begin{cases} 2 & \text{if } a\neq b \\ 1 & \text{if } a=b. \end{cases} \tag{4.10}$$

We could use Lagrange inversion [**La81**] to evaluate $C'(a,b;n)$ explicitly but we do not need that formula in what follows.

In the rest of Section 4 the choice of poles will be included in the enumeration formulae.

4.2. **No singular vertices.** A pole of an ℓ-map can be any vertex or face of an extremal component except the vertex the component shares with a loop. If the component has m edges, then by Euler's formula (2.1) there are a total of $m+1$ vertices and faces, not counting the forbidden vertex; so the number of rooted ℓ-maps with n edges and no singular vertices is

$$\sum_{\substack{a\geq b\geq 0\\ a+b\leq n}}(a+1)(b+1)C'(a,b;n), \tag{4.11}$$

which, by (4.10), is equal to

$$n\sum_{\substack{a\geq b\geq 0\\ a+b\leq n}}(a+1)L'(a)(b+1)L'(b)\cdot[x^{n-(a+b)-1}]\,(1+z). \tag{4.12}$$

In a manner similar to the derivation of (4.6) and (4.8) we find that

$$\sum_{a=0}^{\infty}(a+1)L'(a)x^a = xg'(x)+g(x) = (1+z)^2. \tag{4.13}$$

Substituting from (4.13) into (4.12) and simplifying, we obtain

$$n\cdot[x^{n-1}]\,(1+z)^5. \tag{4.14}$$

The derivative of $(1+z)^5$ is $5(1+z)^4$; so by Lagrange inversion (4.14) is equal to

$$5\frac{n}{n-1}\cdot[z^{n-2}]\,(1+z)^{4n} = \frac{5n}{n-1}\binom{4n}{n-2} = \frac{5n(4n)!}{(n-1)!(3n+2)!}. \tag{4.15}$$

Comparing the right side of (4.15) with (1.1), we express the number of rooted n-edge ℓ-maps as a multiple of $L'(n)$:

$$\frac{5nL'(n)}{4n+1}\binom{n+1}{2}. \tag{4.16}$$

The number of rooted loopless n-edge q-maps is

$$\binom{n+2}{2} L'(n) \tag{4.17}$$

because by Euler's formula (2.1) there are a total of $n+2$ faces and vertices and any pair can be chosen to be the poles.

Adding (4.16) and (4.17) we find that

$$L'_0(n) = \frac{(n+1)(3n+1)(3n+2)}{2(4n+1)} L'(n) = \binom{4n}{n} \tag{4.18}$$

(the last equality is obtained by using (1.1)).

4.3. **One singular vertex.** We construct a rooted ℓ-map with n edges and one singular vertex by taking a rooted ℓ-map with $n-1$ edges and no singular vertices, inserting a singular vertex and its incident edge into M_1 (which has a edges) making the singular vertex the outer pole and choosing one of the $b+1$ possible inner poles in M_k which has b edges. There are $2a+1$ slots into which to insert the dart opposite the singular vertex: as $\sigma(d)$, where d can be either any dart of M_1 or else the right dart of ℓ_1. This augmented ℓ-map has $2n-1$ darts that can be the root, as opposed to $2n-2$ for the original ℓ-map. To get the number of maps we substitute $n-1$ for n in (4.10), multiply by $(2n-1)/(2n-2)$ to account for the extra possible root (by Lemma 4.1), by $2a+1$ to account for the insertions and by $b+1$ to account for the inner pole, and then sum over a and b. We obtain

$$\frac{2n-1}{2} \sum_{\substack{a,b\geq 0 \\ a+b\leq n-1}} (2a+1)L'(a)(b+1)L'(b) \cdot [x^{n-(a+b)-2}]\,(1+z) \tag{4.19}$$

$$= \frac{2n-1}{2} \cdot [x^{n-2}]\,(2xg'(x)+g(x))\,(xg'(x)+g(x))\,(1+z). \tag{4.20}$$

Substituting from (4.8) and (4.13) into (4.20) and simplifying we obtain

$$\frac{2n-1}{2} \cdot [x^{n-2}]\,(1+z)^6. \tag{4.21}$$

The derivative of $(1+z)^6$ is $6(1+z)^5$; so by Lagrange inversion, (4.21) is equal to

$$\frac{3(2n-1)}{n-2} \cdot [z^{n-3}]\,(1+z)^{4n-3} = \frac{3(2n-1)}{n-2}\binom{4n-3}{n-3}. \tag{4.22}$$

Comparing the right side of (4.22) with (1.1), we find that the number of rooted ℓ-maps with n edges and one singular vertex is

$$(n-1)(2n-1)L'(n-1). \tag{4.23}$$

We construct a rooted loopless q-map with n edges and one singular vertex by taking a rooted loopless map with $n-1$ edges, inserting a vertex of valency 1 and its incident edge into one of the $2(n-1)$ possible slots, making the singular vertex one pole, choosing another pole and letting the set of possible roots include the dart opposite the singular vertex. The number $L'(n-1)$ gets multiplied by $2n-2$ for the insertions, by $(2n-1)/(2n-2)$ for the extra possible root (by Lemma 4.1), and by n for the choice of the second pole: by Euler's formula (2.1) there are a total of $n+1$ vertices and faces aside from the first pole, but by Theorem 3.4, condition (2), the vertex adjacent to it is ineligible to be a pole. The number of rooted loopless q-maps with n edges and one singular vertex is thus

$$n(2n-1)L'(n-1). \tag{4.24}$$

By adding (4.23) and (4.24), we find that

$$L'_1(n) = (2n-1)^2 L'(n-1). \tag{4.25}$$

4.4. **Two singular vertices.** We construct a rooted ℓ-map with two singular vertices by taking a rooted ℓ-map with $n-2$ edges and no singular vertices, inserting a singular vertex and its incident edge into M_1 (which has a edges) and another one into M_k (which has b edges), making each of these singular vertices a pole and allowing the set of possible roots to include the darts opposite both singular vertices. There are $2a+1$ possible insertions into M_1 and $2b+1$ possible insertions into M_k. To get the number of maps, we substitute $n-2$ for n in (4.10), multiply by $(2a+1)(2b+1)$ to account for the insertions, multiply by $(2n-2)/(2n-4)$ to account for the two extra possible roots (by Lemma 4.1) and sum over a and b. We obtain

$$(n-1)\sum_{\substack{a,b\geq 0\\ a+b\leq n-2}} (2a+1)L'(a)(2b+1)L'(b)\cdot[x^{n-(a+b)-3}]\,(1+z) \tag{4.26}$$

$$= (n-1)\cdot[x^{n-3}]\,\left(2xg'(x)+g(x)\right)^2(1+z). \tag{4.27}$$

Substituting from (4.8) into (4.27) and simplifying we obtain

$$(n-1)\cdot[x^{n-3}]\,(1+z)^7. \tag{4.28}$$

The derivative of $(1+z)^7$ is $7(1+z)^6$; so, by Lagrange inversion, (4.28) is equal to

$$\frac{7(n-1)}{n-3}\cdot[z^{n-4}]\,(1+z)^{4n-6} = \frac{7(n-1)}{n-3}\binom{4n-6}{n-4}. \tag{4.29}$$

We construct a rooted loopless q-map with n edges and two singular vertices by taking a rooted loopless map with $n-2$ edges and inserting two singular vertices and their incident edges into $2n-4$ possible slots, making both the singular vertices poles, and allowing the set of possible roots to include the darts opposite the two singular vertices. The number $L'(n-2)$ gets multiplied by $(2n-2)/(2n-4)$ to account for the two extra possible roots (by Lemma 4.1); to account for the insertions it gets multiplied by $(2n-4)(2n-3)/2$ instead of $(2n-4)(2n-5)/2$ because both opposite darts can be inserted into the same slot. The number of rooted loopless q-maps with n edges and two singular vertices is thus

$$(n-1)(2n-3)L'(n-2). \tag{4.30}$$

Adding (4.29) and (4.30) and comparing with (1.1) we get two expressions for $L_2'(n)$:

$$L_2'(n) = \frac{4(n-1)(2n-3)(4n-5)}{3(3n-2)}\,L'(n-2) = \binom{4n-4}{n-2}, \tag{4.31}$$

and we keep them both because the one that is not a multiple of $L'(n-2)$ is simpler.

5. The result. Discussion

Substituting from (4.18), (4.25) and (4.31) into (3.1) we obtain (1.2), thus proving Theorem 1.1. □

Table 1 contains the values of $L'(n)$ and $L^+(n)$ for $0\leq n\leq 20$. These latter values were verified for up to 7 edges by comparison with the number of unrooted loopless maps generated by computer [**Wl83**].

We note here that there is another way to derive formula (1.2): we express an ℓ-map as a chain of blocks, at least one of which is a loop, whose extremal components contain the poles as internal elements, with a rooted loopless map inserted between each pair of darts d, $\sigma(d)$.

An interesting open problem would be a proof of formulae (1.2) or (1.3) (and the analogous formulae for unrooted non-separable, eulerian and unicursal maps) that involves natural bijections instead of Lagrange inversion, thus possibly explaining the absence of a rational factor to be multiplied by $\binom{4\,t}{t}$ for $t<n$ in (1.3), which is a special case of a general phenomenon discussed in more detail in [**Li04**]. Another open problem is counting unrooted loopless maps (as well as eulerian and unicursal maps) by number of edges and vertices. This problem is probably easier to solve than the previous one because it has already been solved quite effectively for all maps and non-separable maps by the second-named author [**Wl03**]. In general,

TABLE 1. The number of rooted and unrooted loopless planar maps

#(edges)	#(rooted maps)	#(unrooted maps)
0	1	1
1	1	1
2	3	2
3	13	5
4	68	14
5	399	49
6	2530	240
7	16965	1259
8	118668	7570
9	857956	47996
10	6369883	319518
11	48336171	2199295
12	373537388	15571610
13	2931682810	112773478
14	23317105140	832809504
15	187606350645	6253763323
16	1524813969276	47650870538
17	12504654858828	376784975116
18	103367824774012	2871331929096
19	860593023907540	22647192990256
20	7211115497448720	180277915464664

there is no necessity to restrict oneself to classes of maps for which rooted maps are enumerated by sum-free formulae. For instance, it would be interesting to count unrooted n-edge planar maps without either loops or isthmuses; for counting such rooted maps see [**WlL75**].

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Institute of Mathematics, National Academy of Sciences, Minsk, BELARUS, 220072
E-mail address: liskov@im.bas-net.by

Département D'Informatique, Université du Québec à Montréal, Montréal (Québec), CANADA, H3C 3P8
E-mail address: walsh.timothy@uqam.ca