Crossing Number is Hard for Cubic Graphs

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Abstract. It was proved by [Garey and Johnson, 1983] that computing the crossing number of a graph is an NP-hard problem. Their reduction, however, used parallel edges and vertices of very high degrees. We prove here that it is NP-hard to determine the crossing number of a simple cubic graph. In particular, this implies that the minor-monotone version of crossing number is also NP-hard, which has been open till now.

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1 Background on Crossing Number

We assume that the reader is familiar with basic terms of graph theory. In this paper we consider finite simple graphs, unless we specifically speak about multigraphs. A graph is cubic if it has all vertices of degree 3.

In a *(proper) drawing* of a graph G in the plane the vertices of G are points and the edges are simple curves joining their endvertices. Moreover, it is required that no edge passes through a vertex (except at its ends), and that no three edges intersect in a common point which is not a vertex. An *edge crossing* is an intersection point of two edges-curves in the drawing which is not a vertex. The *crossing number* $\mathrm{cr}(G)$ of a graph G is the minimum number of edge crossings in a proper drawing of G in the plane (thus, a graph is planar if and only if its crossing number is 0). We remark that there are other possible definitions of crossing number which are supposed, but not(!) known [9], to be equivalent to each other.

Crossing number problems were introduced by Turán, whose work in a brick factory during the Second World War led him to inquire about the crossing number of the complete bipartite graphs $K_{m,n}$. Turán devised a natural drawing of $K_{m,n}$ with $\lfloor\frac{m}{2}\rfloor\lfloor\frac{m-1}{2}\rfloor\lfloor\frac{n}{2}\rfloor\lfloor\frac{n-1}{2}\rfloor$ crossings, but the conjecture of Zarankiewic

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that such a drawing is the best possible, is still wide open. (Look at an interesting story of a false "proof" of the conjecture [6].) Not surprisingly, exact crossing numbers are in general very difficult to compute. As an example of another graph family whose crossing number has been deeply studied, we mention the Cartesian products of cycles $C_m \times C_n$ — their crossing number $m(n-2)$ for $m \geq n$ was conjectured in [7]. There has been a number of particular results on this difficult problem (such as [8] as one example), and, remarkably, the problem is almost solved now [4]. That is one of only a few nontrivial exact crossing numbers known today.

The algorithmic problem <u>CrossingNumber</u> is given as follows:

Input: A multigraph G and an integer k.
Question: Is it true that $\mathrm{cr}(G) \leq k$?

Computing the crossing number has important applications in, for example, VLSI design, or in graph visualization. The problem is in NP since one could guess the optimal drawing, replace the crossings in it with new (simultaneously subdividing) vertices, and verify planarity of the resulting graph. It has been proved by Garey and Johnson [3] that CrossingNumber is an NP-complete problem for k on the input.

Since then, a new significant complexity result about graph crossing number has appeared only recently — a paper by Grohe [5] presenting a quadratic-time (FPT) algorithm for CrossingNumber(k) with constant k. There is also a long-standing open question, originally asked by Seese: What is the complexity of CrossingNumber for graphs of fixed tree-width? (Here we leave aside other results dealing with various restricted versions of the crossing number problem appearing in connection with VLSI design or with graph drawing, such as the "layered" or "rectilinear" crossing numbers etc.)

Before the above mentioned FPT algorithm of Grohe for crossing number appeared; Fellows [1] had observed that there are finitely many excluded minors for the cubic graphs of crossing number at most k, which implied a (non-constructive) algorithm for CrossingNumber(k) with constant k over cubic graphs. That observation might still suggest that CrossingNumber was easier to solve over cubic graphs than in general. However, that is not so, as we show in this paper.

2 Crossing Number and OLA

We first define another classical NP-complete combinatorial problem [2] called <u>OptimalLinearArrangement</u>, which is given as follows:

Input: An n-vertex graph G, and an integer a.
Question: Is there a bijection $\alpha : V(G) \to \{1, \ldots, n\}$ (a *linear arrangement* of vertices) such that the following holds

$$\sum\nolimits_{uv \in E(G)} |\alpha(u) - \alpha(v)| \;\leq\; a\,? \tag{1}$$

The sum on the left of (1) is called the *weight* of α.

The above mentioned paper [3] actually reduces CrossingNumber from OptimalLinearArrangement. We, however, consider that reduction "unrealistic" in the following sense: The reduction in [3] creates many large classes of parallel edges, and it uses vertices of very high degrees. (There seems to be no easy modification avoiding those.) So we consider it natural to ask what can be said about the crossing number problem on simple graphs with small vertex degrees.

It might be tempting to construct a "nicer" polynomial reduction for CrossingNumber from another NP-complete problem called Planar-SAT (a version of the satisfiability problem with a planar incidence graph). There have been, to our knowledge, a few attempts in this directions, so far unsuccessful. We consider this phenomenon remarkable since Planar-SAT seems to be much closer to crossing-number problems than the Linear Arrangement is.

Still, we have found another construction reducing CrossingNumber from OptimalLinearArrangement, which produces cubic graphs. The basic idea of our construction is similar to [3], but the restriction to degree-3 vertices brings many more difficulties to the proofs. The construction establishes our main result which reads:

Theorem 2.1. *The problem* CrossingNumber *is NP-complete for* 3*-connected (simple) cubic graphs.*

Let us, moreover, define the *minor-monotone crossing number* $\mathrm{mcr}(G)$: A *minor* F of a graph G is a graph obtained from a subgraph of G by contractions of edges. Then $\mathrm{mcr}(G)$ as the smallest crossing number $\mathrm{cr}(H)$ over all graphs H having G as a minor. The traditional versions of crossing number do not behave well with respect to taking minors; one may find graphs G such that $\mathrm{cr}(G) = 1$ but $\mathrm{cr}(G')$ is arbitrarily large for a minor G' of G. On the other hand, $\mathrm{mcr}(G') \leq \mathrm{mcr}(G)$ for a minor G' of G by definition. The algorithmic problem MM-CrossingNumber (from "Minor-Monotone") is defined as follows:

Input: A multigraph G and an integer k.
Question: Is it true that $\mathrm{mcr}(G) \leq k$?

Our main result immediately extends to a proof that also $\mathrm{mcr}(G)$ is NP-hard to compute, which has been an open question till now.

Corollary 2.2. *The problem* MM-CrossingNumber *is NP-complete.*

Observation. Let a cubic graph G be a minor of a multigraph H. Then some subdivision of G is contained as a subgraph in H. Hence $\mathrm{cr}(G) \leq \mathrm{cr}(H)$.

Thus $\mathrm{cr}(G) = \mathrm{mcr}(G)$ for cubic graphs, and the corollary follows directly from Theorem 2.1.

3 The Cubic Reduction

Let us call a *cubic grid* the graph illustrated in Figure 1 (looking like a "brick wall"). We say that the cubic-grid *height* equals the number of the "horizontal" paths, and the *length* equals the number of edges on the "top-most" horizontal path. (The positions are referred to as in Figure 1.) Formally, the cubic grid of even height h and length ℓ, denoted by $\mathcal{C}'_{h,\ell}$, is defined

$$\begin{aligned} V(\mathcal{C}'_{h,\ell}) &= \{v_{i,j} : i = 1, 2, \ldots, h;\ j = 0, 1, \ldots, \ell\} \cup \\ &\cup \{w_{i,j} : i = 2, 3, \ldots, h-1;\ j = 1, 2, \ldots, \ell\}\,, \\ E(\mathcal{C}'_{h,\ell}) &= \{v_{2i-1,j} v_{2i,j} : i = 1, 2, \ldots, h/2;\ j = 0, 1, \ldots, \ell\} \cup \\ &\cup \{w_{2i,j} w_{2i+1,j} : i = 1, 2, \ldots, h/2 - 1;\ j = 1, 2, \ldots, \ell\} \cup \\ &\cup \{v_{i,j-1} w_{i,j}, w_{i,j} v_{i,j} : i = 2, 3, \ldots, h-1;\ j = 1, 2, \ldots, \ell\} \cup \\ &\cup \{v_{i,j-1} v_{i,j} : i = 1, h;\ j = 1, 2, \ldots, \ell\}\,. \end{aligned}$$

Suppose we now identify the "left-most" vertices in the grid $\mathcal{C}'_{h,\ell}$ with the "right-most" ones, formally $v_{i,0} = v_{i,\ell}$ for $i = 1, 2, \ldots, h$, and simplify the resulting graph. Then we obtain the *cyclic cubic grid* $\mathcal{C}_{h,\ell}$ (which is, indeed, a cubic graph).

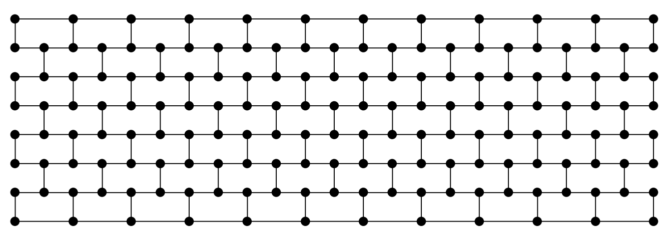

Fig. 1. An illustration of a cubic grid (a fragment of length 11 and height 8)

Let us have a cubic grid $\mathcal{C}'_{h,\ell}$ or $\mathcal{C}_{h,\ell}$ as above. We say that an edge f is *attached to the grid at low position* j if the edge $v_{1,j-1}v_{1,j}$ is subdivided with a vertex x_f, where x_f is an endvertex of f as well. We say that f is *attached at high position* j if an analogous construction is done for the edge $v_{h,j-1}v_{h,j}$. Notice that the new vertex x_f introduced when attaching an edge f has degree 3, and that the degrees of other vertices are unchanged. Similarly, a vertex x is *attached to the grid at position* j if two new edges f, f' with a common endvertex x are attached via their other endvertices at low and high positions j, respectively, to our cubic grid. This is illustrated on a detailed picture in Figure 2.

In a cyclic cubic grid $\mathcal{C}_{h,\ell}$, the cycles M^i on vertices $v_{i,0}w_{i,1}v_{i,1}w_{i,2}$-$\ldots v_{i,\ell-1}w_{i,\ell}$ for $i = 2, 3, \ldots, h-1$, and on vertices $v_{i,0}v_{i,1}\ldots v_{i,\ell-1}$ for $i = 1, h$, are called the *main cycles* of the grid $\mathcal{C}_{h,\ell}$. M^1 and M^h are also referred to as the *outer* main cycles. We use the same names, main cycles, for the subdivisions of the cycles M^i in graphs created from the grid $\mathcal{C}_{h,\ell}$ by attaching edges.

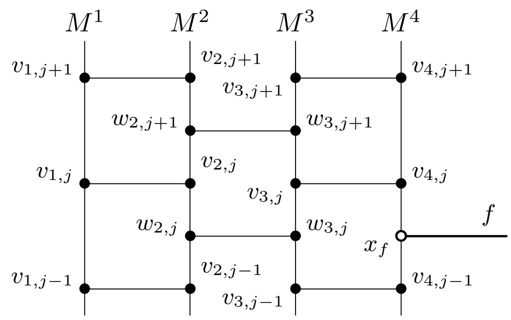

Fig. 2. A detail of the cyclic cubic grid $\mathcal{C}_{4,\ell}$, with an edge f attached at high position j

Assume now that we are given a graph G on n vertices. In order to prove Theorem 2.1, we are going to construct a cubic graph H_G depending on G. (Although our graph H_G is huge, it has polynomial size in G.) We show then how one can compute the weight of an optimal linear arrangement for G from the crossing number $\mathrm{cr}(H_G)$, and vice versa. Our construction uses several size parameters defined next:

$$\begin{aligned} n &= |V(G)|,\;\; m = |E(G)|, \\ t &= 2mn, \qquad (2) \\ r &= t^2 = 4m^2n^2, \\ s &= m^3r = 4m^5n^2, \\ q &= (m^3+n+1)r = 4m^5n^2 + 4m^2n^3 + 4m^2n^2, \\ z &= 2((s+rn)nt + r) = 16m^6n^4 + 16m^3n^5 + 8m^2n^2. \end{aligned}$$

Without loss of generality we may assume that the graph G is sufficiently large, say

$$m > n > 100\,. \qquad (3)$$

We start with two copies B_1, B_2 of the cyclic cubic grid $\mathcal{C}_{z,q}$, called here the *boulders* (for their huge size that keeps the rest of our graph "in place"). Then we make n disjoint copies $R_1, \ldots, R_n$ of the cyclic cubic grid $\mathcal{C}_{t,q}$, called here the *rings*. An intermediate step in the construction – our graph $H_{m,n}$, is obtained by the following operations:

- Start with the disjoint union $B_1 \cup B_2 \cup R_1 \cup \ldots \cup R_n$ of the two boulders and the n rings.
- For every pair of integers $0 \le i < m^3$ and $0 \le j < r$, take a new edge κ_{i+jm^3}, and attach κ_{i+jm^3} at low positions $i + j(m^3+n+1) < q$ to the boulder B_1 via one end, and to B_2 via the other end. These s new edges $\kappa_0, \ldots, \kappa_{s-1}$ are called the *free spokes* in $H_{m,n}$.

– For every pair of integers $1 \le i \le n$ and $0 \le j < r$, set $p = i - 1 + m^3 + j(m^3 + n + 1) < q$, and take two new vertices $\nu_{i,j}$ and $\nu'_{i,j}$ connected by an edge $\mu^3_{i,j}$. Then attach a new edge $\mu^1_{i,j}$ with one end $\nu_{i,j}$ (new edge $\mu^5_{i,j}$ with one end $\nu'_{i,j}$) to the boulder B_1 (boulder B_2) at low position p via the other end. Finally, attach a new edge $\mu^2_{i,j}$ with one end $\nu_{i,j}$ (new edge $\mu^4_{i,j}$ with one end $\nu'_{i,j}$) to the ring R_i at low (high) position p via the other end. The path formed by three edges $\mu^1_{i,j}, \mu^3_{i,j}, \mu^5_{i,j}$ is called the *j-th ring spoke* of R_i in $H_{m,n}$.

We remark that the above construction attaches only one edge at the same position of each of the boulders and rings, and so the operations are well-defined. (Figure 3.) This remark applies also to further constructions on the graph H_G.

To simplify our notation, the above names of the boulders B_1, B_2 and the rings R_i are inherited to the subdivisions of those boulders and rings created in the construction of $H_{m,n}$. The same simplified notation is used further for the graph H_G, too.

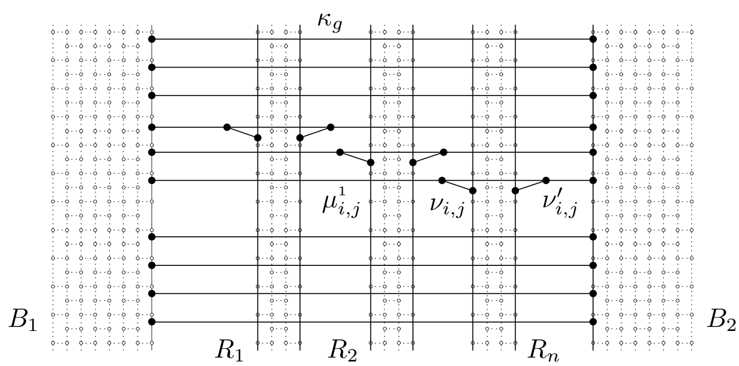

Fig. 3. How to attach free and ring spokes in the graph $H_{m,n}$

So far, the constructed graph $H_{m,n}$ does not depend on a particular structure of G, but only on its size and our choice of the parameters (2). One may say that $H_{m,n}$ acts as a skeleton in the forthcomming construction, in which the rings of $H_{m,n}$ shall model the vertices of G, and the order the rings are drawn in shall correspond to a linear arrangement of vertices of G. The following simple lemma shows necessary "flexibility" of drawings of $H_{m,n}$ with any order of the rings. (Actually, the number of crossings in the lemma is optimal, as we implicitly show in Section 4.)

Lemma 3.1. *For any permutation π of the set $\{1, 2, \ldots, n\}$, there is a drawing of the graph $H_{m,n}$ with $(s + rn)nt$ crossings conforming to the following: The subdrawings of all the rings are pairwise disjoint, each ring separates the two boulders in $H_{m,n}$ from each other, and any free spoke in the drawing intersects all the rings in order $R_{\pi(1)}, \ldots, R_{\pi(n)}$ from B_1 to B_2.*

Finally, the particular graph H_G needed for our polynomial reduction from G is constructed as follows:

– Start with the graph $H_{m,n}$, for $n = |V(G)|$ and $m = |E(G)|$. Number the vertices $V(G) = \{1, 2, \dots, n\}$.
– For every ordered pair $0 < i, j \leq n$ such that $\{i, j\} \in E(G)$, set $p = (i - 1 + jn - n)4m^2(m^3 + n + 1) + m^3 + n < q$. In the graph $H_{m,n}$, attach new vertices χ_{ij}, χ'_{ij} to the rings R_i, R_j, respectively, at positions p, and add a new edge $\chi_{ij}\chi'_{ij}$. The subgraph $X_{i,j}$ induced on the five new edges incident with χ_{ij}, χ'_{ij} is called a *handle* of the edge ij in H_G. (Figure 4.)

That is, the rings in H_G model the vertices of G, and the handles model the edges of G. As we show later, an optimal drawing of H_G uniquely determines an ordering of the rings of $H_{m,n}$, and hence the weight of an optimal linear arrangement of G corresponds to the number of crossings between the rings and the handles in an optimal drawing of the graph H_G.

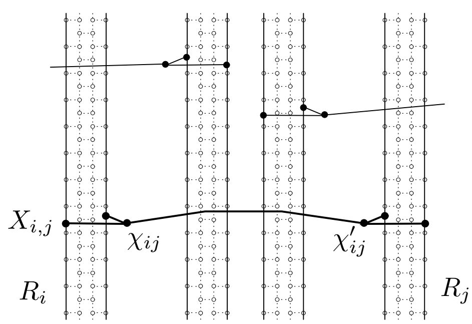

Fig. 4. How to attach handles of the edges of G in the graph H_G

We conclude this section with an upper bound on the crossing number of our constructed graph, which naturally follows from the drawings introduced in Lemma 3.1.

Lemma 3.2. *Let us, for a given graph G, construct the graph H_G as described above. If G has a linear arrangement of weight A, then the crossing number of H_G is*

$$\mathrm{cr}(H_G) \leq (s + rn)nt + 2(A + m)t - 4m\,,$$

where the weight of a linear arrangement is defined by (1) *on page 2, and m, n, r, s, t are given by* (2) *on page 5.*

Corollary 3.3. *For any G conforming to* (3), $\mathrm{cr}(H_G) < z/2 = (s + rn)nt + r$.

4 Sketch of the Proof

To prove correctness of our reduction, we now have to show a lower bound on the crossing number of our graph H_G, depending on weight of the optimal linear arrangement of G. We achieve this goal by showing that an optimal drawing of H_G has to look (almost) like the drawing described in the proof of Lemma 3.2. Since the whole proof is quite long and technical, and it uses more topological rather than combinatorial arguments, we give here only a brief outline of the main steps.

We argue that the boulders of H_G have to be drawn without crossings at all, and that each ring has to separate the two boulders from each other (and hence the rings are "nested" in each other). Such a configuration already forces the number of crossings of $H_{m,n}$ as in Lemma 3.1. Then we identify a linear ordering of the rings, and show that every edge handle in H_G generates at least as many additional crossings as expected from the ordering of rings. A special attention has to be paid to proving that no edge crossing is counted twice in our arguments.

The following particular claim will be useful during the proof.

Lemma 4.1. *Let k, ℓ, t be integers, and let $(p_1, p_2, \ldots, p_k)$ be an increasing sequence of integers such that $p_1 > 4kt$, $p_k < \ell$, and $p_{j+1} - p_i \geq 4kt$ for $j = 1, 2, \ldots, k$. Assume that the graph F is constructed from the cyclic cubic grid $\mathcal{C}_{t,\ell}$ by attaching a vertex z_j at position p_j for each $j = 1, 2, \ldots, k$. Then* $\mathrm{cr}(F) = k(t-2)$.

We need to be a bit more formal in this section. A *curve* γ is a continuous function mapping the interval $[0, 1]$ to a topological space. A curve γ is a *closed* curve if $\gamma(0) = \gamma(1)$. A closed curve γ is *contractible* in a topological space if γ can be continuously deformed to a single point there. We call a *cylinder* the topological space obtained from the unit square by identifying one pair of opposite edges in the same direction. (A cylinder has two disjoint closed curves as the boundary.)

Recall the notation from Section 3, and assume that G is a graph on the vertex set $\{1, 2, \ldots, n\}$. Let H_G denote the graph constructed along the description on page 7. The following statement, together with Lemma 3.2, validates our reduction.

Proposition 4.2. *If an optimal linear arrangement of a graph G has weight A, then the crossing number of the graph H_G is at least*

$$\mathrm{cr}(H_G) \geq (s + rn)nt + 2(A + m)t - 8m\,.$$

(See (2) *and Lemma 3.2 for details on the notation.)*

We proceed the proof of Proposition 4.2 along the following sequence of claims. Assume that we have an optimal drawing of the graph H_G at hand.

Lemma 4.3. *In the optimal drawing of H_G, the boulders B_1, B_2 are drawn with no edge crossings.*

Hence, in particular, the first main cycles N_j of the boulder B_j, $j = 1, 2$ are drawn with no crossings. Then there is a uniquely defined cylinder Π with the boundary curves N_1 and N_2 in the plane. Realize that the whole subgraph $H_G - V(B_1) - V(B_2)$ is drawn on Π.

Lemma 4.4. *In the optimal drawing of H_G, each main cycle M of every ring R_i, $i \in \{1, 2, \dots, n\}$ is drawn as a closed curve separating the subdrawing of the boulder B_1 from the subdrawing of B_2, i.e. noncontractible on Π.*

This claim is the first key step in the proof of Proposition 4.2. The idea behind is that a main cycle drawn as a contractible curve on Π would have to have too many crossings with the free spokes of H_G.

Corollary 4.5. *In the optimal drawing of H_G, there are at least $(s + rn)nt$ crossings between edges of the main cycles of the rings and edges of the free and ring spokes in H_G.*

Lemma 4.6. *There is a selection of main cycles $M_i \subset R_i$, $i = 1, 2, \dots, n$ of the rings in H_G, such that the cycles $M_1, \dots, M_n$ are drawn as pairwise disjoint closed curves in the above optimal drawing of H_G. Hence, there is a permutation π of $\{1, \dots, n\}$ such that, for each $j = 1, \dots, n$, the closed curve $M_{\pi(j)}$ separates the subdrawing $B_1 \cup M_{\pi(1)} \cup \ldots \cup M_{\pi(j-1)}$ from the subdrawing $B_2 \cup M_{\pi(j+1)} \cup \ldots \cup M_{\pi(n)}$.*

Lemma 4.7. *For every $k = 0, 1, \dots, 4n^2 - 1$, there is an index $c_k \in C_k = \{km^5 - 2m^4, \dots, km^5 + 2m^4\}$ such that the edge of the c_k-th free spoke κ_{c_k} is crossed exactly once by each of the main cycles of all the rings, and that κ_{c_k} has no more crossings than those in the optimal drawing of H_G.*

Recall that the vertices of G are numbered as $\{1, 2, \dots, n\}$, and that $X_{i,j}$ denotes the subgraph of the handle in the constructed graph H_G corresponding to an edge $ij \in E(G)$ (page 7).

Lemma 4.8. *Let π be the permutation from Lemma 4.6, let Π be the cylinder defined after Lemma 4.3 for the optimal drawing of H_G, and let $\{i, j\} \in E(G)$ be an edge. For $\ell = i + n(j - 1)$, consider the indices $c_{4\ell-2}$ and $c_{4\ell+2}$ given by Lemma 4.7, and denote by Σ_ℓ the region on Π bounded by the drawings of the $c_{4\ell-2}$, $c_{4\ell+2}$-th free spokes and containing the subdrawing of the handle $X_{i,j}$. Then Σ_ℓ contains at least*

$$t \left(|\pi^{-1}(i) - \pi^{-1}(j)| - 1 \right)$$

crossings between edges of the subgraph $X_{i,j} \cup R_i \cup R_j$ and edges of the main cycles of other rings R_k in H_G for $k \neq i, j$.

The last claim presents the second key step of our proof of Proposition 4.2. It shows that the handles $X_{i,j}$ (of edges of G) really have to cross all the rings which are between R_i, R_j in the order given by π. Moreover, the crossings can

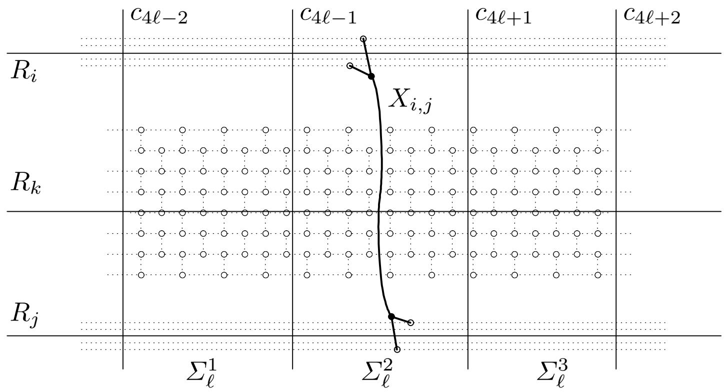

Fig. 5. An illustration to Lemma 4.8

be separated in distinct regions of the cylinder Π, and so it is clear that they will not be counted twice. See an illustration in Figure 5. Now we are ready to finish the proof.

We are going to count three collections of edge crossings in the optimal drawing of H_G. Firstly, there are (at least) $(s + rn)nt$ crossings described in Corollary 4.5. Secondly, denote by d_i the degree of the vertex i in G. Let us consider the subgraph F_i of H_G formed by the ring R_i and by $2d_i$ pairs of incident edges from all handles which are attached to R_i in H_G. Then, by Lemma 4.1, the subgraph F_i itself has at least $2d_i(t - 2)$ edge crossings in any drawing of H_G. Thirdly, the permutation π from Lemma 4.6 defines a linear arrangement $\alpha = \pi^{-1}$ of the vertices of G. An edge $\{i, j\}$ of G then contributes (via its two handles in H_G) with at least $2t\big(|\alpha(i)-\alpha(j)|-1\big)$ crossings in H_G by Lemma 4.8.

Altogether, we have found at least this many distinct edge crossings in the optimal drawing of H_G:

$$(s + rn)nt + \sum_{i\in V(G)} 2d_i(t - 2) + \sum_{\{i,j\}\in E(G)} 2t\big(|\alpha(i) - \alpha(j)| - 1\big) =$$

$$= (s + rn)nt + 2t \sum_{\{i,j\}\in E(G)} |\alpha(i) - \alpha(j)| - 2tm + 4tm - 8m =$$

$$= (s + rn)nt + 2tA + 2tm - 8m$$

Proof of Theorem 2.1. Assume that G, a is an input instance of the OPTIMALLINEARARRANGEMENT problem, and that G is sufficiently large (3). The above described graph H_G is clearly cubic, it has polynomial size in $n = |V(G)|$, and H_G has been constructed efficiently. We now ask the problem CROSSINGNUMBER on the input $\langle\, H_G,\, (s+rn)nt + 2t(a+m)\,\rangle$, and give the same answer to OPTIMALLINEARARRANGEMENT on $\langle\, G, a\,\rangle$.

If there is a linear arrangement of G of weight at most a, then our correct answer is YES according to Lemma 3.2. Conversely, if the optimal linear arrangement of G has weight greater than a, then the crossing number of H_G is by Proposition 4.2

$$\mathrm{cr}(H_G) \geq (s+rn)nt + 2t(a+1+m) - 8m >$$

$$> (s+rn)nt + 2t(a+m)\,,$$

and so the correct answer is NO. Since the OptimalLinearArrangement problem is known to be NP-complete [2], the statement of Theorem 2.1 follows.
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