Iannis Xenakis' *Musique Stochastique*: System Design and Mathematical Background

with additional mathematical and historical background to support Thomas Schmidt on *Metastasis*

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Chapter 1

Relevant Mathematical Background for *Metastasis*

1.1 Golden Ratio

The first written accounts of what we now refer to as the golden ratio were given by Euclid (325-265 BCE) in book 6 of his *The Elements*[3] and it has fascinated mathematicians and others ever since. Mathematically, the golden ratio (typically denoted with the letter Φ) can be described as an infinite sequence of equal ratios, given below. See figure 1.1 for a more visual representation.

$$\begin{aligned} \Phi &= (a : x) \\ &= (x : (a - x)) \\ &= [(a - x) : (x - (a - x))] \\ &= \ldots \end{aligned}$$

To obtain a real value for Φ we need to solve a quadratic equation using some simple algebra:

$$\begin{aligned} \Phi = \frac{a}{x} &= \frac{x}{a - x} \\ a^2 - ax &= x^2 \\ a^2 - ax - x^2 &= 0 \\ \frac{a^2}{x^2} - \frac{ax}{x^2} - \frac{x^2}{x^2} &= 0 \\ \Phi^2 - \Phi - 1 &= 0 \end{aligned}$$

We obtain the following two solutions, choosing the positive one as our value for Φ.

a		
x		$a-x$
$a-x$	$x-(a-x)$	
$\vdots$		

Figure 1.1: diagram demonstrating the proportions of the golden ratio (not to scale)

$$
\begin{aligned}
\frac{1}{2}(\sqrt{5}+1) = &\quad \Phi &\approx 1,618034\ldots \\
-\frac{1}{2}(\sqrt{5}-1) = &\quad -\Phi^{-1} &\approx -0,618034\ldots
\end{aligned}
$$

The negative solution has the interesting property of being equal to the negative of the reciprocal of Φ.

The golden ratio enjoys other curious properties such as,

$$\Phi - 1 = \Phi^{-1}$$

and

$$\Phi = \sqrt{1+\sqrt{1+\sqrt{1+\ldots}}}$$

1.2 Fibonacci Sequence

The Fibonacci sequence first appeared in 1202 in Italian mathematician Leonardo Pisano's (1170-1250, he nicknamed himself Fibonacci) *Liber Abaci*[4]. The main purpose of this book was in fact to explain the use of Arabic-Hindu numbers (as well as other material Fibonacci had picked up in his travels). The sequence can be simply defined as follows.

$$
\begin{aligned}
u_1 := u_2 &:= 1 \\
u_i &:= u_{i-2} + u_{i-1}, i = 3,4,5,6,\ldots
\end{aligned}
$$

$$1,1,2,3,5,8,21,34,55,89,144,\ldots$$

If we examine the sequence composed of the ratios of adjacent Fibonacci numbers,

$$f_n = \frac{u_{n+1}}{u_n} = \frac{1}{2}, \frac{2}{3}, \frac{3}{5}, \frac{5}{8}, \frac{8}{13}, \ldots$$

we find that,

$$\lim_{n\to\infty} f_n = \Phi$$

which is to say that this ratio approaches the golden ratio.

The golden ratio and the Fibonacci sequence crop up in an almost uncountable number of aspects of mathematics, architecture, art, and music.

1.3 The *Modulator* of Le Corbusier

Le Corbusier (born Charles-Edouard Jeanneret-Gris, 1887-1965) was a famous Swiss architect[5], who was very much interested in the golden ratio and Fibonacci sequence with respect to architecture. He was particularly interested in how these three things were interrelated along with the dimensions of the human body. Furthermore, Xenakis worked with him as an architect during the 1950's and was certainly influenced by his interest in such topics.

⋮
295,9
182,9
113,0
69,8
43,2
26,7
16,5
⋮

Figure 1.2: excerpt from the table of Le Corbusier's modulator values

The so-called "Modulator" of Le Corbusier, which appeared in his 1950 book *Mod I*, in particular plays a role in Xenakis' compositions. Baltensperger comments that the modulator "kann als ein Versuch der 'Moderne' gewertet werden, eine auf menschliches Mass bezogene, mathematische Ordnung in die Architektur wieder einzubringen." Le Corbusier was primarily interested in designing buildings, which were inherently fitting to the human form. The modulator is based upon the length 113cm, which is half the height of an average man with his arm extended straight above his head (226cm according to Le Corbusier). The actual modulator is in fact a geometric sequence, consisting of this base value with a multiplying factor of the golden ratio. See figure 1.2 for an excerpt of the values. The idea was to design buildings using these values, the intent being that the resulting structures would be better designed for human occupation and use.

The modulator, although curious and interesting, does seem to be contrived. It is not certain whether the base unit for the modulator might be adjusted for the various average heights that different cities, nation, regions, and time periods enjoy. The base unit at best is arbitrary, but none the less demonstrates

Le Corbusier's obsession with the golden ratio in architecture. The modulator appears to be more form than function. None the less, the creation of such a measure shows the importance to him of the consideration of the human form in architecture. Time Magazine, while listing him as one of the most influential people of the 20th century[6]:

> Le Corbusier was the most important architect of the 20th century. Frank Lloyd Wright was more prolific—Le Corbusier's built oeuvre comprises about 60 buildings—and many would argue he was more gifted. But Wright was a maverick; Le Corbusier dominated the architectural world, from that halcyon year of 1920, when he started publishing his magazine L'Esprit Nouveau, until his death in 1965. He inspired several generations of architects—including this author—not only in Europe but around the world. He was more than a mercurial innovator. Irascible, caustic, Calvinistic, Corbu was modern architecture's conscience.

Chapter 2

Musique Stochastique

2.1 What is *Musique Stochastique*?

Musique Stochastique (hereafter referred to as *ST*, Xenakis' own term) can be literally translated from French as "stochastic music". It is not simply one composition, but rather a system, which given arbitrary parameters produces a composition. Xenakis composed 9 works using the system, but an almost infinite number could have been created (and today still could be created). Seven of these were produced in 1962, all but one with numerically coded titles. The following table (adapted from [1] p.439) lists them, as well as, for those that have one, their common names.

code number	name / notes
ST/48 – 1,240162	
ST/10 – 1,080262	
ST/4 – 1,080262	(string quartet edition of above ST/10)
ST/4 – 1,030762	*Morsima – Amorisma*
ST/10 – 1,030762	*Amorsima – Morisma*
ST/10 – 3,060962	*Atrées*
ST/CosGauss	*Polytope de Cluny*

Additionally, parts of two others works were produced with the aid of the ST system: *Eonta* in 1964 and *Stratégie* in 1962.

2.1.1 Numeric Title System

Upon seeing the numerical names of the these works for the first time, they seem strikingly strange. However, they have a precise meaning; each numerically coded title takes the form "ST/i – v,d" where i is the number of instruments, v is the version, and d is the date of creation in the form day, month, year (each with 2 digits).

2.1.2 What does "stochastic" mean?

Baltensperger characterizes (p.563) stochastic, "als moderner Sammelbegriff, umfasst in der Mathematik und Statistik alles, was mit Wahrscheinlichkeitsrechnung zu tun hat. Sie beschäftigt sich mit der mathematischen Analyse zufälliger Ereignisse und trägt damit zur Instrumentalisierung der erkannten Gesetzmässigkeiten zum Zwecke statistischer Untersuchung bei." This description of stochastic as a concept of mathematics (more specifically statistics) can easily be seen as being consistent with the more technical nature of *ST*, for the application of probability theory deliberately plays a large role in its inner workings.

It is also interesting, however, to consider the meanings of the word outside of the statistical setting. The origin of the word 'stochastic' comes from the "Greek `stokhastikos`, from `stokhastes`, *diviner*, from `stokhazesthai`, *to guess at*, from `stokhos`, *aim, goal*."[2] Such origins are perhaps particularly poigniant due to the fact that Xenakis spent most of his early life in Greece and spoke modern Greek fluently. Stochastic from a mathematical viewpoint is generally regarded as a way to systematize random events according to some sort of overriding pattern. The origins of the word imply something more active than the mathematical usage. The focus falls on the attempt to divine meaning, to determine in some sort of supernatural way what is happening.

2.2 Aspects of Composition

2.2.1 Note Length

Note length is determined by a random generation process following an exponential distribution,

$$f(x) = \delta \cdot e^{-\delta x}$$

where δ is the 'density'. Mathematically, this density is equal to the inverse of the average note length. Qualitatively, a larger value will produce on average shorter notes and a smaller density, longer ones. The resulting note length can be *any* value greater than or equal to zero. However, as can be seen in probability density graph in figure 2.1, the longer lengths quickly become very unlikely.

Recall that a probability density function gives only the relative likelihood of all outcomes. Paradoxically each outcome has an equal probability of 0 in occurring, because there is an infinite number of outcomes, which are all possible. To obtain the probability that the outcome is between two values (say l_0 and l_1), the density function must be integrated, in this case giving,

$$P(x) = \int_{l_0}^{l_1} \delta \cdot e^{-\delta x} dx$$

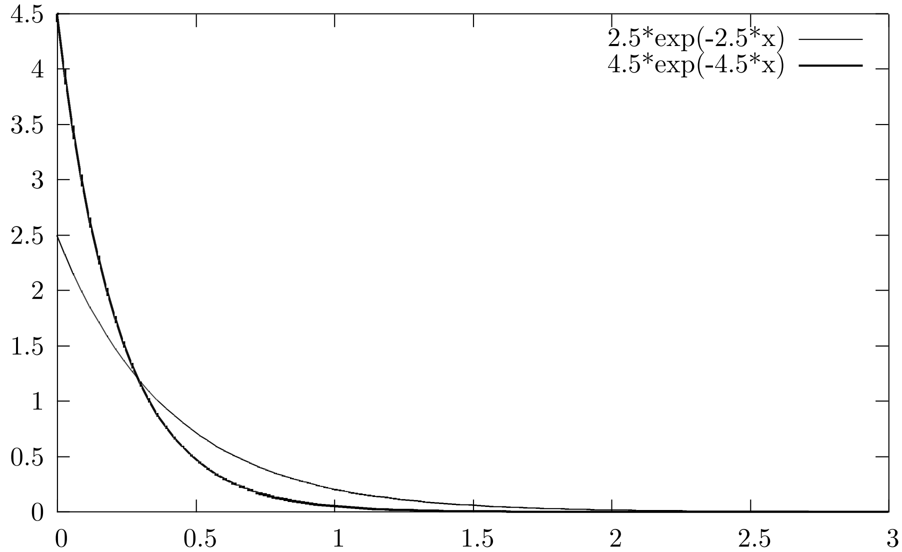

Figure 2.1: probability density of the exponential distribution for $\delta = 2.5$ and $\delta = 4.5$

2.2.2 "Note Clouds"

So-called "note clouds" consist of several randomly determined notes (Tonpunkten). Two aspects of the notes contained in each cloud are determined separately: density and pitch.

Cloud Density

Within the composition, there can be an arbitrary number of clouds, each with its own density (μ). For an entire composition, one average density parameter (μ_0) is chosen. All of these values are measured in number of notes per second. The density of individual clouds follows the Poisson distribution (figure 2.2), which is essentially a discrete version of the exponential distribution discussed in the previous section. A Poisson distribution is given by,

$$P(k) = \frac{\mu_0^k}{k!} e^{-\mu_0}$$

where k is the number of events to occur in a predetermined time period.

The density of is in fact not directly determined, but could be determined if necessary from k. When composing individual clouds, we are interested in discrete note counts, not continuous rates of production. For example, suppose it were stipulated that a 5 second cloud must perform at a rate determined from an exponential distribution (say with an average of 2.1 notes per second). Suppose that distribution produced a density of 2.3 notes per second for our

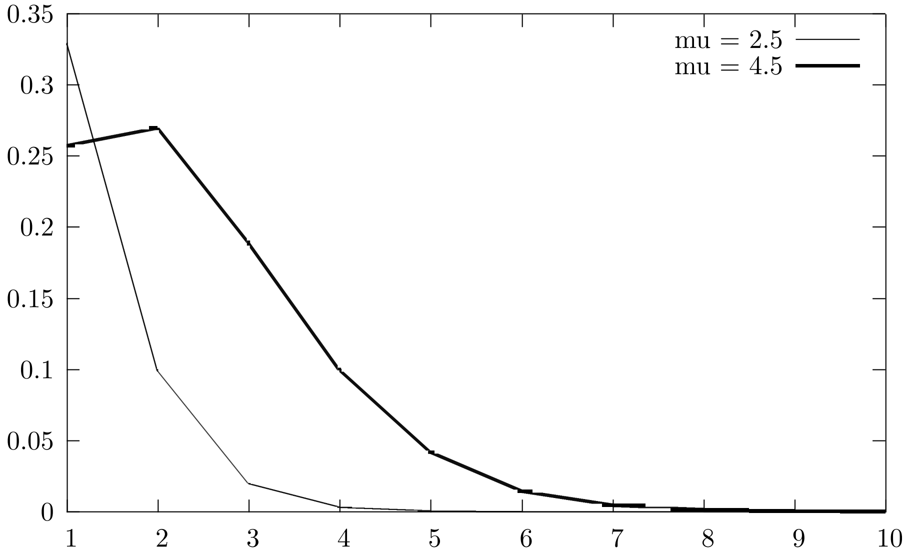

Figure 2.2: the Poisson distribution with $\mu_0 = 2.5$ and $\mu_0 = 4.5$

cloud. That would require the placement of precisely 11.5 notes. In order for this to actually work, the numbers would have to be rounded or otherwise adjusted to whole numbers. It is considerably simpler to just use a discrete Poisson distribution with a u_0 density parameter of 2.1 to begin with.

Cloud Pitch

The second aspect required for constructing note clouds is their pitch (γ). The pitches of the notes in the cloud are determined by a starting pitch along with the intervals between each pair. Baltensperger does not describe how the starting pitch is chosen, but it is presumably chosen in some random manner befitting the instrument to be playing the cloud. The intervals are determined according to a linear probability distribution (figure 2.3),

$$\Theta(\gamma)d\gamma = \frac{2}{a}(1 - \frac{\gamma}{a})d\gamma$$

where a is maximum interval value specified by the composer. Additionally, a simple random variable equivalent to the flipping of a coin is used to determine whether the interval is rising or falling.

Such a linear distribution is used so that the intervals tend towards smaller values, but not so strongly as with an exponential distribution. Furthermore, the maximum interval limit helps reduce the production of sequences that are unnatural sounding or difficult to play.

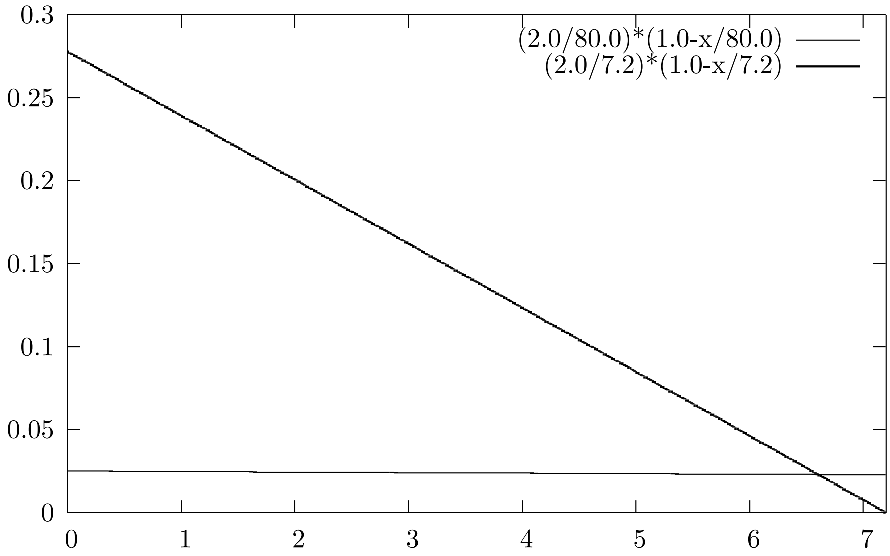

Figure 2.3: linear distribution with $a = 80$ and $a = 7.2$

2.2.3 Glissando Speed

ST (and other works of Xenakis) often make use of glissandos. In *ST* the parameter for their speeds is determined according to a normal distribution (figure 2.4),

$$f(v) = \frac{1}{a\sqrt{2\pi}} \cdot e^{-\frac{v^2}{2a^2}}$$

where a is the so-called "aggregate temperature" parameter. The inspiration for the use of the distribution as well as the name for the parameter is kinetic gas theory, something Xenakis would have come across in his studies in engineering. Statistically speaking, this number is also known as the standard deviation or square root of the variance.

2.2.4 Dynamics

Dynamics are divided into four distinct zones: *ppp*, *p*, *f*, and *ff*. From these four zones, there are 64 sequences of size 3, but only 44 are musically distinguishable. Each possible combination is given an equal probability ($\frac{1}{44}$) of being used for a given portion of the composition, resulting in a uniform distribution.

2.2.5 Instrument Choice

First, the composer must separate all timbres into similar classes, which are then given a unique number. Next, according to a linear distribution, a percentage is

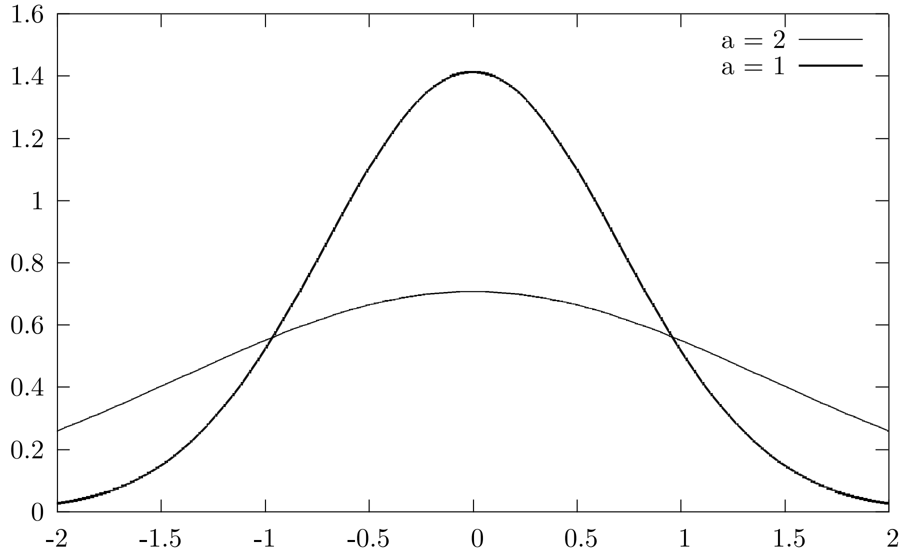

Figure 2.4: normal distribution with $a = 1$ and $a = 2$

determined for each class. This percentage dictates the proportion of the total notes in the composition that group will play. Which particular notes are played by each instrument are then randomly determined.

A metaphor concerning of a bag of marbles best illustrates this process. The bag starts out containing n marbles, one for each note that needs to be played throughout the composition. Marbles are then randomly distributed according to the proportions determined for each group, taking into account the physical constraints (pitch, length, dynamics) of the instruments.

2.3 Composition Process

The composition process can be briefly described in 8 phases divided between two parts:

First Part: Preparation

1. Original ideas (Conceptions initiales)
2. Definition of the planned sound elements (Définition d'êtres sonores)
3. Macro-composition: Definition of the transformations (Définition des transformations)
4. Micro-composition: Definition of the complete mathematical model
5. Sequential programming of the model (Programmation séquentielle)

Second Part: Production

6. Execution of the calculations (Effectuation des calculs)
7. Transcription of the numerical results onto a score (Résultat final symbolique)
8. Sonic Realization (Incarnation sonore)

2.4 Implementation Software

The logical flow of the software is described below.

- Preparation (compute constants and tables)
- Sequences (repeat until there are enough sequences)
  - Length (exponentially distributed)
  - Average Density (Poisson distributed)
  - Timbre Distribution / Orchestration
  - Notes[1] (repeat until there are enough notes)
    - Placement (exponentially distributed)
    - Instruments ("marbles in bag"')
    - Pitch (linearly distributed)
    - Glissando Speed (normally distributed)
    - Length (exponentially distributed)
    - Dynamics (uniform distribution)
- Transcription to Score

The *ST* system was written using the FORTRAN programming language and not in function paradigm, but instead used a plethora goto statements (so-called "spaghetti code"). Perhaps counterintuitive to a musician (but not to a computer scientist), the system computed compositions in a sequential manner (one instrument at a time). This presumably reduced the complexity of programming such a system, although similarly limited the system's ability to compose one segment or instrument with knowledge of another. Furthermore, the computation of a piece was not (and could not) be performed in real time. The system printed out sequences of numbers as it computed the composition. Afterwards, human labor had to be used to convert and transcribe these pages and pages of numbers into a music score, at which point it was only then able to be played and heard.

[1]To prevent the case where a note is generated that cannot be played on the given instrument, all produced notes are checked against a table of acceptable length and dynamics. Should a note be problematic, it is simply discarded, and a new one regenerated.

2.5 Implementation Hardware

Most of the various compositions were produced in 1962, using an IBM 7090 mainframe computer. At the time it was quite cutting edge and used primarily in large corporations and government agencies. It came equiped with a 460kHz processor, 32kB of RAM, and occupied the better part of a large room[7]. The price tag for one at that time was approximately $3,000,000, although the vast majority were leased monthly for $70,000 (roughly $450,000 using today's dollars). It was no doubt an expensive and unusual undertaking at the time.

2.6 Closing Comments

Every composer who has ever lived and written one or more works at some point died or will someday. Their admirers as well as their critics certainly will always wonder, what else could have been written? What would it have been like? Good? Bad? Revolutionary? And this is the same for Xenakis. However, assuming the hardware and software were available (or some sort of porting possible), a new *ST* piece could be created today. Even though nobody would have ever heard it before, it would be immediately recognizable as something by Xenakis. It would certainly be his composition, because it was his ideas and effort that gave rise to this composition. Xenakis is one of the forefathers of the electronic meta-composition—something truly revolutionary.

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[7] `http://www.frobenius.com/7090.htm`