CHAPTER 11

IN THIS CHAPTER WE COVER . . .

Weighted Voting Systems

Voting is often used to decide yes or no questions. Legislatures vote on bills, stockholders vote on resolutions presented by the board of directors of a corporation, and juries vote to acquit or convict a defendant. In this chapter, we shall concentrate on situations where there are just two alternatives, such as "yes" or "no." The theorem of Kenneth May quoted in the previous chapter says that majority rule is the only system with the following properties:

1. All voters are treated equally.
2. Both alternatives are treated equally.
3. If you vote "no," and "yes" wins, then "yes" would still win if you switched your vote to "yes," provided that no other voters switched their votes.
4. A tie cannot occur unless there is an even number of voters.

There are many situations in which one or more of these properties are not valid. For example, in a criminal trial the jury is required to reach a unanimous decision on a motion to convict (or on a motion to acquit); thus, if there is one "no" vote, the motion is not adopted. In this case, the alternatives are not treated equally. Here's another familiar example. Stockholders are allowed one vote per share that they own. If I own 10,000 shares and you own 100, then this voting system does not treat us equally.

Some systems where the voters appear to be unequal in power actually have all of the properties required by May's theorem. Any student of politics will attest that not all legislators are equally powerful (think of the speaker of the U.S.

House of Representatives versus a freshman member, or the prime minister versus a backbencher in Parliament). Nevertheless, the voting system actually treats the legislators equally: Each has one vote. Our interest is in the voting system itself and not in the influence that some voters might acquire as a result of experience or accomplishment.

Voting systems that treat participants unequally are often used when the participants are indeed unequal. For example, the Council of Ministers of the European Union accords more power to states such as France, which have large populations, than it does to smaller states, such as Austria. Rather than giving the larger states more representatives, as in the U.S. House of Representatives, the Council of Ministers gives the ministers from the larger states more votes.

We shall find two measures of voting power that apply when voters are not treated equally or alternatives are not treated equally, or both: the Shapley–Shubik power index, and the Banzhaf power index. The Banzhaf power index is an accurate measure of power when there is no spectrum of opinion. For example, if each voter decides which way to vote by tossing a coin, the Banzhaf power index will indicate each voter's share of power. The Shapley–Shubik index is appropriate in a process where measures are crafted so as to attract enough votes to win.

One type of voting system in which the voters or the alternatives may be treated unequally is a **weighted voting system.** Each participant has a specified number of votes, called his or her **weight.** If my voting weight is more than yours, then I might have more power than you to influence the outcome, and certainly I won't have less. (We will see that voters with different numbers of votes may actually have equal power.) In any voting system, there must be a criterion for deciding whether "yes" or "no" has won. In a weighted voting system, this is done by specifying a number called the **quota.** If the sum of the weights of all the voters who favor a motion is equal to the quota, or exceeds it, then "yes" wins. Otherwise, "no" wins. The quota must be greater than half of the total weight of all the voters, to avoid situations where contradictory motions can pass, and it cannot be greater than the total weight, or no motion would ever pass.

The European Union's Council of Ministers uses weighted voting, but in the United States, it is unusual for a legislative body to use a weighted voting system (see Spotlight 11.3). It cannot be said that there is no weighted voting in the United States, because the Electoral College, which elects the president, functions as a weighted voting system in which the voters are the states. See Spotlight 11.1.

Notation for Weighted Voting Systems

To describe a weighted voting system, you must specify the voting weights $w_1, w_2, \ldots, w_n$ of the participants, and the quota, q. The following notation is a shorthand way of making these specifications:

$$[q : w_1, w_2, \ldots, w_n]$$

SPOTLIGHT 11.1 The Electoral College

In a U.S. presidential election, the voters in each state don't actually cast their votes for the candidates. They vote for electors to represent them in the Electoral College. The number of electors allotted to a state is equal to the size of its congressional delegation, so a state with one congressional district gets 3 electors: one for its representative, and one for each of its two senators. A state with 25 representatives would get 27 electors. The District of Columbia, while not a state, is entitled by the 23rd Amendment to the U.S. Constitution to send 3 electors to the College.

All states except two select their electors in a statewide contest. Thus, all of a state's electors are committed to vote for the presidential candidate favored by a plurality of the voters in the state. For example, the candidate who gets a plurality in California receives all 55 of the state's electoral votes. In Maine and Nebraska, there is a different procedure. Two electors (corresponding to the senators) are chosen statewide, and the electors corresponding to the representatives are chosen by congressional district. Nebraska has three congressional districts. It is possible that one or two of the districts might favor one ticket, while the state as a whole might favor another.

Effectively, the Electoral College functions as a weighted voting systems in which there are 56 participants: the 50 states, the District of Columbia, three Nebraska congressional districts, and two Maine congressional districts. The weights range from 1 for individual congressional districts to 55, and the quota, 270, is a simple majority of the 538 electors.

Electoral votes cast in Ohio, election day 2004. *(Thomas Dworzak/Magnum Photos.)*

The weighted voting system [51 : 40, 60] describes a voting system in which there are two voters, with voting weights 40 and 60, and the quota is 51.

A Dictator

EXAMPLE 1

Suppose there is one voter, *D*, who has all of the power. A motion will pass if and only if *D* is in favor, and it doesn't matter how the other participants vote. Most weighted voting systems that we will consider do not have a **dictator,** but if there is one, his or her voting weight must be equal to or more than the quota. The system [51 : 40, 60] has a dictator because the weight-60 voter can pass any motion that she wants. ■

Dummy Voters

EXAMPLE 2

A voting system may include some participants—called **dummy** voters—whose votes don't count. For example, the U.S. Congress has a nonvoting delegate who represents the District of Columbia. If a voting system has a dictator, all of the participants except the dictator are dummy voters. In the voting system [8 : 5, 3, 1], the weight-1 voter is a dummy, because a motion will pass only if it has the support of the weight-5 and weight-3 voters, and then the additional 1 vote is not needed. For another example, consider [51 : 26, 26, 26, 22]. The voter with weight 22 is not needed when two of the other voters combine to support a motion; they have enough weight to pass the motion without her. If she joins forces with just one of the other voters, their total weight, 48, is not enough to win. Thus, the weight-22 voter is a dummy. ■

Three More Three-Voter Systems

EXAMPLE 3

By adjusting the quota, the distribution of power in a weighted voting system can be altered. We have seen that the weight-1 voter in [8 : 5, 3, 1] is a dummy, but by increasing the quota to 9 we obtain a system in which the power is equally distributed—unanimous support is required to pass a motion in [9 : 5, 3, 1]. The weight-1 voter is also not a dummy in [6 : 5, 3, 1] because he can join the weight-5 voter to pass a motion, even if the weight-3 voter opposes. Finally, consider [51 : 49, 48, 3]. Although it looks as if the weight-3 voter will have relatively little power, and may even be a dummy, in fact she has the same voting power as the other two voters. Any two of the three voters in this system can pass a measure. ■

Veto Power

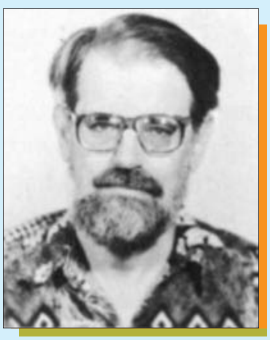
EXAMPLE 4

A voter whose vote is necessary to pass any motion is said to have **veto power.** For example, in the system [6 : 5, 3, 1], the weight-5 voter has veto power because the other two voters do not have enough combined weight to pass a motion. A dictator always has veto power, and it is possible for more than one voter to have veto power as well. In a criminal trial, each juror has veto power. In the system [8 : 5, 3, 1] the voters with weights 5 and 3 each have veto power. ■

SPOTLIGHT 11.2 Power Indices

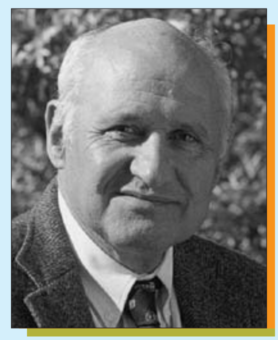
The first widely accepted numerical index for assessing power in voting systems was the **Shapley–Shubik power index**, developed in 1954 by a mathematician, Lloyd S. Shapley, and an economist, Martin Shubik. A particular voter's power as measured by this index is proportional to the number of different permutations (or orderings) of the voters in which he or she has the potential to cast the pivotal vote–the vote that first turns from losing to winning.

The **Banzhaf power index** was introduced in 1965 by John F. Banzhaf III, a law professor who is also well-known as the founder of the antismoking organization ASH (Action on Smoking and Health). The Banzhaf index is the one most often cited in court rulings, perhaps because Banzhaf brought several cases to court and continues to file *amicus curiae* briefs when courts evaluate weighted voting systems. A voter's Banzhaf index is the number of different possible voting **combinations** in which he or she casts a swing vote–a vote in favor of a motion that is necessary for the motion to pass, or a vote against a motion that is essential for its defeat.



Lloyd S. Shapley

John F. Banzhaf III



Martin Shubik

The voters in the system [6 : 5, 3, 1] are not equally powerful–the weight-5 voter has veto power and the other two don't–and yet none of the voters are dummies. We can't compare power by comparing the voting weights because the weight-3 voter has the same voting power as the weight-1 voter. Together, they can stop the weight-5 voter from passing a motion, and either one can combine with the weight-5 voter to pass a motion. A **power index** gives a way to measure the share of power that each participant in a voting system (weighted or otherwise) has. Spotlight 11.2 is a brief history of power indices.

11.1 The Shapley–Shubik Power Index

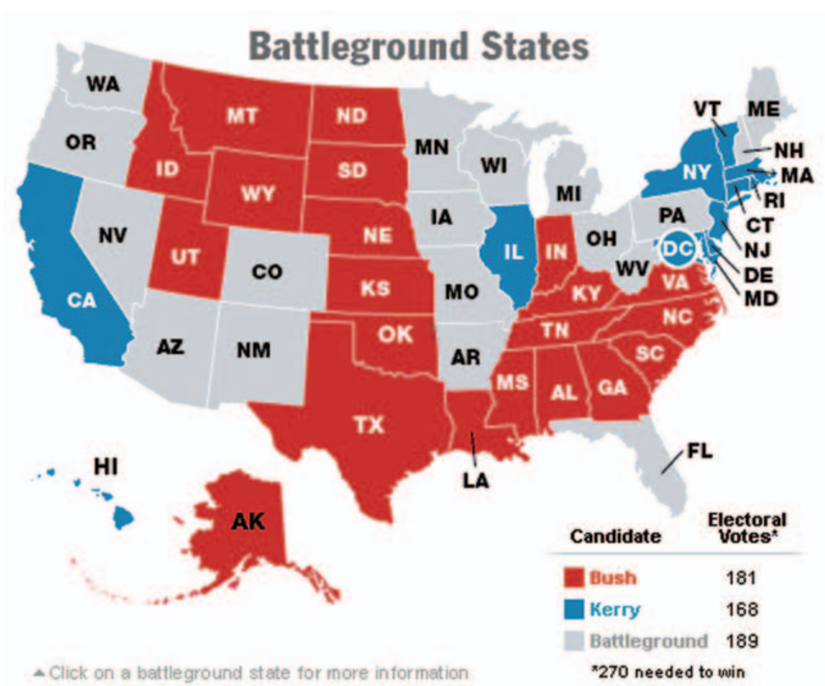
When an election looms, politicians focus on "moderate voters." These are people who could be convinced to favor one side or the other. Moderate voters can make elected officials pay attention while voters who have an extreme commitment to one side or the other are ignored. However, moderate voters achieve their influence as a result of their political position, and we are primarily interested in the power that voters acquire as a result of the system itself. For example, France has 29 votes in the European Union (EU) Council, and Austria has 10. A motion before the Council, in which both countries were moderate voters, would be more likely to be written so as to acquire France's vote rather than Austria's, because France has so many more votes.

In United States presidential elections, politicians color the states that are likely to vote Republican in the Electoral College red, and the Democratic ones blue. The ones that could go either way are the "battleground states," and the campaigns put most of their investment in these states.

In 1954, Lloyd Shapley and Martin Shubik devised a way to gauge the share of decision-making power of each participant in a voting system. A voter's share of power is called his or her *Shapley–Shubik power index*. The index is defined in terms of *permutations*.

Permutation

A **permutation** of voters is an ordering of all of the voters in a voting system.



Politicians color the states that are strongly Republican red, and the states that are strongly Democratic blue. We have colored the battleground states–which could go either way–gray. *(© 2004 Time Inc. All rights reserved.)*

Voters are ordered in accordance with their commitment to an issue, starting with those who are most favorably inclined and ending with those who are most determined to oppose. For example, suppose that the issue is animal rights. Here the spectrum might range from a voter who would outlaw the sale of cow's milk to one who would legalize cockfighting. If an animal rights bill is being drafted, it must be written so as to receive enough votes to meet the quota.

Pivotal Voter

The first voter in a permutation who, when joined by those coming before him or her, would have enough voting weight to win is the **pivotal voter** in the permutation. Each permutation has exactly one pivotal voter.

If the issue is taxation instead of animal rights, the spectrum of opinion will probably be completely different. Voters who have moderate positions on animal rights may or may not be at the extremes when the subject is taxes. Each issue being debated corresponds to some permutation–and the pivotal voter on one issue may well not be pivotal on another issue.

A successful tax cut bill must be drafted so as to secure the support of the pivotal voter of the taxation permutation, an animal welfare bill must be drafted so that the the pivotal voter of the animal rights permutation will support it, and so on.

The Shapley–Shubik Power Index

The **Shapley–Shubik power index** of each voter is computed by counting the number of permutations in which he or she is pivotal, then dividing it by the total number of permutations. Thus, if we consider each permutation to belong to the voter who is pivotal, each voter's Shapley–Shubik index is his or her share of the permutations.

If there are n voters, the number of permutations is called the **factorial** of n and is denoted $n!$. There is a simple formula for $n!$:

Factorial

For a positive whole number n,

$$n! = n \times (n-1) \times (n-2) \times \cdots \times 2 \times 1$$

and $0! = 1$.

For example,

$$1! = 1$$
$$2! = 2 \times 1 = 2$$
$$3! = 3 \times 2 \times 1 = 6$$
$$4! = 4 \times 3 \times 2 \times 1 = 24$$

To continue this list, observe that $n! = n \times (n-1)!$ for $n \geq 1$. Thus

$$5! = 5 \times 4! = 5 \times 24 = 120$$
$$6! = 6 \times 5! = 6 \times 120 = 720$$
$$7! = 7 \times 6! = 7 \times 720 = 5040$$
$$8! = 8 \times 7! = 8 \times 5040 = 40{,}320$$

and so on. You can imagine that $n!$ increases dramatically as n increases–an instance of the combinatorial explosion. You probably don't want to calculate 100!. It is a 158-digit number.

To justify the formula, suppose that we are listing all of the permutations. There are n voters who could be first; when the first voter is selected, there are $n - 1$ remaining voters who could be in second position, then $n - 2$ who could be third, and so on. When it is time to select for the last position, there is one voter left. By the fundamental principle of counting (see Chapter 1), the number of permutations is the product of the numbers of choices that we have had at each stage.

The Shapley–Shubik Power Index of a Three-Voter System

EXAMPLE 5

Let us calculate the Shapley–Shubik power index of the voting system [6: 5, 3, 1]. We will name the participants A, B, and C, and consider their $3! = 6$ permutations. Table 11.1 displays all six permutations. Next to each permutation, the total weights of the first voter, the first two voters, and all three voters are shown in sequence. The first number in the sequence that equals or exceeds the quota is underlined, and the corresponding pivotal voter's symbol is circled. We see that A is pivotal in four permutations, while B and C are each pivotal in one. Hence the Shapley–Shubik index of A is $\frac{4}{6}$, and B and C each have Shapley–Shubik indices of $\frac{1}{6}$. ■

The Corporation with Four Shareholders

EXAMPLE 6

A corporation has four shareholders, A, B, C, and D, with 40, 30, 20, and 10 shares, respectively. The corporation uses the weighted voting system

$$[51 : 40, 30, 20, 10]$$

The $4! = 24$ permutations of the shareholders are shown in Table 11.2. In 10 of the permutations, A is the pivotal voter; B and C are each pivotal voters in 6;

TABLE 11.1 Permutations and Pivotal Voters for the Three-Person Committee

Permutations			Weights		
A	B	C	5	8	9
A	C	B	5	6	9
B	A	C	3	8	9
B	C	A	3	4	9
C	A	B	1	6	9
C	B	A	1	4	9

TABLE 11.2 Permutations and Pivotal Voters for the Four-Person Corporation

Permutations				Weights				Pivot
A	B	C	D	40	70	90	100	B
A	B	D	C	40	70	80	100	B
A	C	B	D	40	60	90	100	C
A	C	D	B	40	60	70	100	C
A	D	B	C	40	50	80	100	B
A	D	C	B	40	50	70	100	C
B	A	C	D	30	70	90	100	A
B	A	D	C	30	70	80	100	A
B	C	A	D	30	50	90	100	A
B	C	D	A	30	50	60	100	D
B	D	A	C	30	40	80	100	A
B	D	C	A	30	40	60	100	C
C	A	B	D	20	60	90	100	A
C	A	D	B	20	60	70	100	A
C	B	A	D	20	50	90	100	A
C	B	D	A	20	50	60	100	D
C	D	A	B	20	30	70	100	A
C	D	B	A	20	50	60	100	B
D	A	B	C	10	50	80	100	B
D	A	C	B	10	50	70	100	C
D	B	A	C	10	40	80	100	A
D	B	C	A	10	40	60	100	C
D	C	A	B	10	30	70	100	A
D	C	B	A	10	30	60	100	B

and D is the pivotal voter in 2 permutations. Therefore, the Shapley–Shubik power index for this weighted voting system is

$$\left(\frac{10}{24}, \frac{6}{24}, \frac{6}{24}, \frac{2}{24}\right) \blacksquare$$

How to Compute the Shapley–Shubik Power Index

For voting systems with no more than four voters, the Shapley–Shubik power index may be calculated by making a list of all the voting permutations and identifying the pivotal voter in each.

Listing all of the permutations is the brute force way of calculating the Shapley–Shubik power index. As we saw in connection with the traveling salesperson problem in Chapter 2, brute force methods can be impossible to carry out, due to the combinatorial explosion. If we were to study a 10-voter system by the brute force method, we would have to list 10! permutations—that's more than $3\frac{1}{2}$ million. If we made the list using the same size page and same typeface as this text, we could probably get about 350 permutations on each page. It would take 10,000 of these pages to list all of the permutations. If we used a computer, each permutation would occupy 10 bytes, and $3\frac{1}{2}$ million of them would take 35 megabytes, a manageable size. But even a computer would not be able to handle a 100-voter system because the number of permutations is a 158-digit number. Even by using all the memory devices—chips, disk drives, magnetic tape, flash memory, and so on—in the world, you would not be able to save a file containing all of the permutations.

If all the voters have the same voting weight, you don't need to make a list of all the permutations, because each has the same share of power. If there are 100 voters, each with 1 vote, the Shapley–Shubik index of each is $\frac{1}{100}$. If all but one or two of the voters have equal power, we can still calculate the Shapley–Shubik power index of each without making a list of permutations. We use two principles to do this:

- Voters with the same voting weight have the same Shapley–Shubik power index.
- The sum of the Shapley–Shubik power indices of all the voters is 1.

A Seven-Person Committee

EXAMPLE 7

The chairperson of a committee has 3 votes, and there are six other members with 1 vote. The quota for passing a measure is a simple majority, 5 of the 9 votes. In our notation, this voting system is [5 : 3, 1, 1, 1, 1, 1, 1].

Each ordinary member has the same power index. Our strategy is to compute the index of the chair, and then divide the share of power that the chair does *not* have equally among the ordinary members.

There are 7! = 5040 permutations to consider. We will group them by the position occupied by the chairperson. Thus, *CMMMMMM*, in which the chairperson is first, is the first group. Counting from the left, we see that the votes are accumulated in the sequence 3, 4, $\underline{5}$, 6, 7, 8, 9. In these permutations (there are 6! of them), the chairperson is not the pivot; an ordinary member is. In the second group, *MCMMMMM*, the votes accumulate in the sequence 1, 4, $\underline{5}$, 6, 7, 8, 9, and again, the chairperson is not the pivot.

The chairperson is the pivot in the next three groups of permutations, *MMCMMMM*, *MMMCMMM*, and *MMMMCMM*, with vote accumulations 1, 2, $\underline{5}$, 6, 7, 8, 9; 1, 2, 3, $\underline{6}$, 7, 8, 9; and 1, 2, 3, 4, $\underline{7}$, 8, 9, respectively. In the final two groups, *MMMMMCM* and *MMMMMMC*, with vote accumulations 1, 2, 3, 4, $\underline{5}$, 8, 9 and 1, 2, 3, 4, $\underline{5}$, 6, 9, an ordinary member, not the chairperson, is the pivot again.

Each of the 7 groups of permutations is of the same size, 6!, because the 6 ordinary members can appear in any order in each. The chairperson is the pivot in three groups, for a total of $\frac{3}{7}$ of the total number of permutations. His Shapley–Shubik power index is therefore $\frac{3}{7}$. The remaining $\frac{4}{7}$ of the voting power is shared equally by the 6 ordinary members. Therefore each has $\frac{4}{7} \div 6 = \frac{2}{21}$ of the power.

The Shapley–Shubik power index of this weighted voting system is therefore

$$\left(\frac{3}{7}, \frac{2}{21}, \frac{2}{21}, \frac{2}{21}, \frac{2}{21}, \frac{2}{21}, \frac{2}{21}\right)$$

Because $\frac{3}{7} \div \frac{2}{21} = 4\frac{1}{2}$, the Shapley–Shubik model indicates that the chairperson is $4\frac{1}{2}$ times as powerful as an ordinary member, although his voting power is only 3 times as much. ■

EXAMPLE 8 A Committee with Two Co-Chairs

A committee has 7 members: two co-chairs who each have 3 votes, and five other members with 1 vote each. The quota is 7, and thus the weighted voting system is [7 : 3, 3, 1, 1, 1, 1, 1]. Our objective is to determine the Shapley–Shubik power index of each member of the committee. Our strategy this time is to determine the voting power of *A*, a weight-1 voter. *A* is pivotal when the voters coming before him in the permutation have a combined weight of exactly 6. There are two ways to meet this condition:

- $C_1C_2AX_1X_2X_3X_4$, where C_1 and C_2 represent co-chairs, and $X_1, \ldots, X_4$ represent members who are not co-chairs and are not *A*.
- $Y_1Y_2Y_3Y_4AY_5Y_6$, where one of $Y_1, \ldots Y_4$ is a co-chair, one of Y_5, Y_6 is a co-chair, and the remaining *Y*'s are weight-1 voters.

For the first type of permutation, there are 2 ways to order the co-chairs (if the co-chairs are *P* and *Q*, then C_1 could be *P* or *Q* and C_2 would be the remaining co-chair), and there are 4! ways of ordering the other 4 members. There are thus $2 \times 4! = 48$ permutations of the first type.

To count the permutations of the second type, let us start with Y_5 and Y_6. There are 2 ways to choose the co-chair and 4 ways to choose the ordinary member for these positions in the permutation. Once these are chosen, there are 2 ways to put them in order and 4! ways to put the remaining co-chair and three ordinary members in order as Y_1, Y_2, Y_3, and Y_4. Thus, the number of permutations of the second type is $2 \times 4 \times 2 \times 4! = 384$.

In all, there are $48 + 384 = 432$ permutations in which A is pivotal. The Shapley–Shubik index of A is therefore $\frac{432}{7!} = \frac{3}{35}$. The other weight-1 voters have the same Shapley–Shubik index, so the combined share of power of the five weight-1 voters is $5 \times \frac{3}{35} = \frac{3}{7}$. The two co-chairs have the remainder of the power, so each has $(1 - \frac{3}{7}) \div 2 = \frac{2}{7}$ of the power. The Shapley–Shubik index of the system is therefore

$$\left(\frac{2}{7}, \frac{2}{7}, \frac{3}{35}, \frac{3}{35}, \frac{3}{35}, \frac{3}{35}, \frac{3}{35}\right) \blacksquare$$

The Permutation in the 2004 Election

EXAMPLE 9

In the 2004 election, the Electoral College reelected the Bush–Cheney ticket. Spotlight 11.1 explains how the Electoral College operates. Although there are 538 electors in the college, all states except Maine and Nebraska select their electors in a statewide general election, so even if the popular vote in a state was close, all of the electors from that state will vote for the same ticket. There are actually 56 independent votes in the Electoral College.[1]

Each of the 56 voters in the Electoral College is selected by and represents an electorate. Some, such as Nebraska's third district, were heavily in favor of the Bush–Cheney ticket (by more than 3 to 1); others, such as Iowa, New Mexico, and Wisconsin, were almost equally split between the Bush–Cheney and Kerry–Edwards tickets; and still others, such as the District of Columbia, were strongly in the Kerry–Edwards camp (almost 10 to 1). Table 11.3 lists the 56 voters in the Electoral College, ordered by their margin in favor of the Bush–Cheney ticket. The voting weight of each is shown, and a running total of electoral votes gives the total weight of each voter and all who came before it in the table. In listing the states and other voters in this order, we have recorded a permutation of the Electoral College participants. The pivotal voter is the one that brings the running total over the quota (270). If you recall the news about the 2004 election, you will not be surprised to see which voter is pivotal—Ohio. ■

[1]Strictly speaking, the votes in Maine and Nebraska are not really independent—see Exercise 39 at the end of this chapter.

TABLE 11.3 The Permutation Resulting from the General Election for President of the United States in 2004

Voter*	Weight	Bush's Margin	Running Total	Voter*	Weight	Bush's Margin	Running Total
NE Dist 3	1	3.158	1	CO	9	1.112	222
UT	5	2.695	6	FL	27	1.107	249
WY	3	2.370	9	NV	5	1.055	254
ID	4	2.257	13	OH	20	1.051	274
NE	2	2.023	15	IA	7	1.018	281
OK	7	1.904	22	NM	5	1.016	286
ND	3	1.771	25	WI	10	0.992	296
NE Dist 1	1	1.752	26	NH	4	0.973	300
AK	3	1.721	29	PA	21	0.955	321
KS	6	1.705	35	MI	17	0.934	338
AL	9	1.697	44	MN	10	0.932	348
TX	34	1.599	78	OR	7	0.924	355
NE Dist 2	1	1.574	79	ME Dist 2	1	0.888	356
SD	3	1.558	82	NJ	15	0.879	371
MT	3	1.531	85	WA	11	0.864	382
IN	11	1.528	96	DE	3	0.858	385
MS	6	1.506	102	HI	4	0.839	389
KE	8	1.500	110	ME	2	0.836	391
SC	8	1.421	118	CA	55	0.818	446
GA	15	1.405	133	IL	21	0.815	467
LA	9	1.345	142	CT	7	0.810	474
TN	11	1.338	153	ME Dist 1	1	0.783	475
WV	5	1.296	158	MD	10	0.778	485
NC	15	1.288	173	VT	3	0.701	488
AZ	10	1.235	183	NY	31	0.700	519
AR	6	1.221	189	RI	4	0.653	523
VA	13	1.191	202	MA	12	0.595	535
MO	11	1.159	213	DC	3	0.103	538

*The voters are ordered by decreasing margin for the Bush–Cheney ticket. This margin is the number of popular votes cast for Bush–Cheney divided by the votes cast for Kerry–Edwards.

11.2 The Banzhaf Power Index

While the Shapley–Shubik power index is based on a count of permutations in which a voter is pivotal, the Banzhaf index is based on a count of coalitions in which a voter is *critical*. A **coalition** is a set of voters who are prepared to vote for, or to oppose, a motion. A **winning coalition** favors a motion, and has enough votes to pass it. A **blocking coalition** opposes a measure, and has the votes to defeat it. For example, in a dictatorship, a coalition in favor of a motion is a winning coalition if and only if the dictator belongs to it. Similarly, a coalition opposing a measure is a blocking coalition if and only if it includes the dictator.

In a winning or blocking coalition, there may be some voters whose votes are necessary to win. If any one of these voters should switch to the other side, the coalition would not have the votes it needs to have its way: It would become a **losing coalition.** These voters are called **critical voters** in the coalition.

Let's consider the presidential election of 2004 from this viewpoint. The Bush–Cheney ticket won this election with 286 electoral votes–16 more than the quota. The critical voters would be those belonging to the coalition that favored the Bush–Cheney ticket and had a voting weight of more than 16: Florida, Ohio, and Texas. If a voter with voting weight less than 16 were to leave the coalition, the Bush–Cheney ticket would still win, but with a smaller margin. Such a voter would not be critical.

By contrast, the election of 2000 was much closer, with only 271 votes for the Bush–Cheney ticket. With only one vote more than the quota, every voter in the winning coalition, except the congressional districts in Nebraska, was critical. (The Maine congressional districts voted for the Gore–Lieberman ticket and thus did not belong to the winning coalition.)

Critical Voters

EXAMPLE 10

Consider a committee of three members, *A, B,* and *C*. The chairperson of the committee, *A,* has two votes, while *B* and *C* each have one. The quota is three, and this voting system is

$$[3: 2, 1, 1]$$

The coalition {*A, B, C*} is a winning coalition because it has all four votes. Suppose that *A* decides to leave the coalition. We can indicate this situation schematically as follows:

A	*B*	*C*	Votes	Outcome
2	1	1	4	Pass
↓				
0	1	1	2	Fail

By changing her vote, *A* has changed the outcome. In this coalition, *A* is a critical voter.

Now let's go back to the original coalition and see what happens if *B* changes his vote.

A	*B*	*C*	Votes	Outcome
2	1	1	4	Pass
	↓			
2	0	1	3	Pass

This time, the outcome doesn't change, so *B* is not a critical voter in this coalition. Because *C* has the same power as *B*, he is also not a critical voter in the coalition. ■

EXAMPLE 11 Winning and Blocking

In the committee with members *A*, *B*, and *C*, and voting system [3: 2, 1, 1], *A* and *B* have formed a coalition to vote in favor of measure *X* and to oppose another measure, *Y*. Member *C* is voting against *X* and for *Y*. Because the coalition {*A*, *B*} has 3 votes (2 for *A*, and 1 for *B*), it is a winning coalition for *X* and a blocking coalition for *Y*.

Measure *X*					Measure *Y*				
A	*B*	*C*	Votes	Outcome	*A*	*B*	*C*	Votes	Outcome
2	1	0	3	Pass	0	0	1	1	Fail

■

When voting for *X*, both *A* and *B* are critical voters. However, it takes only 2 votes to block a measure. If *A* leaves the coalition and joins *C* in voting for *Y*, then *Y* will be approved. Therefore *A* is a critical voter in the coalition to block motion *Y*. Voter *B* is not critical in the blocking coalition, because if he decides to switch his vote and support *Y*, it will not change the outcome. *A* can block motion *Y* by herself. ■

The Banzhaf power index was developed in 1965 by an attorney, John F. Banzhaf III, in an analysis of weighted voting that appeared in the *Rutgers Law Review*. Mr. Banzhaf's article was entitled "Weighted Voting Doesn't Work." See Spotlight 11.3.

The Banzhaf Power Index

A voter's **Banzhaf power index** is the number of distinct winning or blocking coalitions in which his or her vote is critical.

Spotlight 11.3 A Mathematical Quagmire

A county legislature in the United States is usually called a Board of Supervisors. Unlike state legislators, who represent districts that are carefully drawn to be equal in population, supervisors in some counties represent towns within the county. Because the towns differ in population, weighted voting is used to compensate for the resulting inequity.

If each supervisor's voting weight is proportional to the population of the town he or she represents, there will be situations in which one or more supervisors on a board are dummy voters, even if no supervisor is dictator. In a 1965 law review article, John F. Banzhaf III found that three of the six supervisors of Nassau County, New York, were dummies. The article inspired legal action against several elected bodies that employ weighted voting systems.

The first legal challenge to weighted voting was to invalidate the voting system of the Board of Supervisors of Washington County, New York. In its decision, the New York State Court of Appeals provided a way to fix a weighted voting system: Each supervisor's Banzhaf power index, rather than his or her voting weight, should be proportional to the population of the district that he or she represents. The court predicted that its remedy would lead to a "mathematical quagmire."

Five lawsuits, filed over a period of 25 years, challenged weighted voting in the Nassau County Board of Supervisors. These cases proved to be the mathematical quagmire that the appeals court had feared. The courts attempted to force Nassau County to comply with the Washington County decision. Although the county made a sincere attempt to do so, every voting system that it devised faced a new legal challenge. With conflicting expert testimony, the U.S. District Court finally ruled in 1993 that weighted voting was inherently unfair.

Banzhaf's law review article, which initially drew attention to weighted voting in Nassau County, was aptly titled "Weighted Voting Doesn't Work."

Nevertheless, tradition is hard to change. Many boards of supervisors of counties, particularly in the State of New York, still use weighted voting, and legal challenges to the practice, even after the Nassau County decision, have not always been successful.

To determine the Banzhaf power index of a voter A, we must count all possible winning and blocking coalitions of which A is a member and casts a critical vote. The weight of a winning coalition must be q or more, where q is the quota. A blocking coalition must be large enough to deny the "yes" voters the q votes they need to win. If the total weight of all the voters is n, then the weight of the blocking coalition has to be more than $n - q$. Assuming that all weights are integers, this means that the weight of a blocking coalition must be at least $n - q + 1$.

To determine which voters are critical in a given winning or blocking coalition, the following principle is useful.

Extra Votes Principle

A winning coalition with total weight w has $w - q$ **extra votes**. A blocking coalition with total weight w has $w - (n - q + 1)$ extra votes. The critical voters are those whose weight is more than the coalition's extra votes. These are the voters that the coalition can't afford to lose.

We can readily identify the critical voters in any coalition by comparing each voter's weight with the number of extra votes that the coalition has.

Calculating the Banzhaf Power Index

To calculate the Banzhaf power index of a given voting system:

1. Make a list of the winning and blocking coalitions.
2. Use the **extra-votes principle** to identify the critical voters in each coalition.

A voter's Banzhaf power index is then the number of coalitions in which he or she appears as a critical voter.

EXAMPLE 12 Calculating the Banzhaf Index

We will calculate the Banzhaf index for the committee with voting system [3: 2, 1, 1].

The winning coalitions are all those whose weights sum to 3 or 4, and we will start by making a list of them:

Winning Coalition	Weight	Extra Votes	Critical Votes A	Critical Votes B	Critical Votes C
$\{A, B\}$	3	0	1	1	0
$\{A, C\}$	3	0	1	0	1
$\{A, B, C\}$	4	1	1	0	0
		Totals	3	1	1

All members of the coalitions with 0 extra votes are critical voters. Because A is the only voter with more than 1 vote, she is the only critical voter in the coalition that has 1 extra vote. We have thus found that A is a critical voter in three winning coalitions, while B and C are each critical voters in one winning coalition.

Blocking coalitions have total weights of 2, 3, or 4. Here is a list of the blocking coalitions:

Blocking Coalition	Weight	Extra Votes	Critical Votes A	B	C
$\{A\}$	2	0	1	0	0
$\{B, C\}$	2	0	0	1	1
$\{A, B\}$	3	1	1	0	0
$\{A, C\}$	3	1	1	0	0
$\{A, B, C\}$	4	2	0	0	0
		Totals	3	1	1

Again, all voters in the coalitions with 0 extra votes are critical. In the blocking coalitions with 1 extra vote, only A is critical. The 4-vote blocking coalition $\{A, B, C\}$ has 2 extra votes. Because no voter has more than 2 votes, there are no critical voters in $\{A, B, C\}$, considered as a blocking coalition. Voter A is critical in three blocking coalitions, while B and C are each critical in one. Adding up winning and blocking critical votes, we find that the Banzhaf index of A is 6, while B and C each have a Banzhaf index of 2. We will say that the Banzhaf index of this system is (6, 2, 2).

The Banzhaf index provides a comparison of the voting power of the participants in a voting system. Thus, A, with a Banzhaf index of 6, is three times as powerful as B or C. To determine the way voting power is distributed, we can add the numbers of critical voters for all three voters together to get $6 + 2 + 2 = 10$ critical votes in all. Thus, A has $\frac{6}{10} = 60\%$ of the voting power, while B and C each have 20%. By comparison, the Shapley–Shubik model gives $\frac{2}{3}$ of the power to A, while B and C each have $\frac{1}{6}$. ■

Consider the following three voting systems.

$$\text{System I: } [2 : 1, 1, 1]$$
$$\text{System II: } [3 : 2, 1, 1]$$
$$\text{System III: } [3 : 1, 1, 1]$$

We have studied system II and found that its Banzhaf power index is $(6, 2, 2)$. Although power is distributed equally in systems I and III, and both have the same Shapley–Shubik power index, these systems have different Banzhaf power indices. System III requires a unanimous vote to pass a measure. There is only one winning coalition, in which each of the three voters is critical. Each voter is also critical in exactly one blocking coalition, in which he or she stands alone against the other two voters. The Banzhaf power index for system I is therefore $(2, 2, 2)$. In system I, coalitions with total weight 2 or 3 can either block or win. The coalition with weight 3 has one extra vote, and no voter has more than one vote, so it has no critical voters. All voters are critical in the coalitions of weight 2. Each coalition is counted twice—once as a winning coalition, when both members vote "yes," and once as a blocking coalition, when both voters vote "no." Thus, the two weight-2 coalitions that include voter A, $\{A, B\}$ and $\{A, C\}$, give A a total of 4 critical votes. The Banzhaf index of system I is thus $(4, 4, 4)$.

The voters in system I have greater Banzhaf power indices than in system III. From a practical point of view, this means that in system I, an individual voter has more chances to influence the outcome if system I is used rather than system III.

In the examples that we have discussed so far, each participant has been a critical voter in exactly as many winning coalitions as blocking coalitions. This is not a coincidence, as we will now see.

Winning/Blocking Duality

The number of winning coalitions in which a given voter is critical is equal to the number of blocking coalitions in which the same voter is critical.

To understand why winning/blocking duality works, consider a voter, A, who is critical in a winning coalition, C. The voters who are not in the coalition C are voting "no" but do not have enough votes to block. However, if A changes her vote to "no," then C will be a losing coalition because A was a critical voter. When joined by A, the "no" voters form a blocking coalition, with A as a critical voter. Thus, there is a one-to-one correspondence between winning coalitions in which A is a critical voter and blocking coalitions with A as a critical voter.

By the winning/blocking duality principle, we can determine a voter's Banzhaf power index by doubling the number of winning coalitions in which he or she is a critical voter—this will account for the blocking coalitions.

EXAMPLE 13 The Corporation with Four Shareholders

The corporation with four shareholders (see Example 6) uses the weighted voting system

$$[51 : 40, 30, 20, 10]$$

Table 11.4 displays a list of all the winning coalitions of shareholders and the number of extra votes that each has. The four columns at the right are marked to indicate the critical voters in each coalition. By doubling the critical votes shown in the table, we arrive at the Banzhaf index of the corporation: $(10, 6, 6, 2)$. In this model, A has

$$\frac{10}{10 + 6 + 6 + 2} \qquad \text{or approximately } 42\%$$

TABLE 11.4 Winning Coalitions in the Four-Stockholder Corporation

Coalition	Weight	Extra Votes	Critical Voters A	B	C	D
$\{A, B, C, D\}$	100	49				
$\{A, B, C\}$	90	39	1			
$\{A, B, D\}$	80	29	1	1		
$\{A, C, D\}$	70	19	1		1	
$\{A, B\}$	70	19	1	1		
$\{B, C, D\}$	60	9		1	1	1
$\{A, C\}$	60	9	1		1	
Critical votes			5	3	3	1

of the voting power, while B and C each have 25% (even though B has more shares than C). Shareholder D has the remaining 8% of the voting power, according to the Banzhaf model. In this case, power is distributed exactly as it was by the Shapley–Shubik model. ■

How to Count Combinations

A **voting combination** is a record of how the voters cast their votes for or against a given proposition. For example, if there are three voters, A, B, and C, and A and C voted "yes" while B voted "no," we might record the voting combination as "Yes, No, Yes." A briefer notation is to visualize voting combinations as **binary numbers.**

A whole number N is represented in binary form as a sequence of binary digits, or **bits,** which can be 0 or 1. This sequence expresses the way that N can be expressed as a sum of powers of 2. Thus, if

$$N = 2^k + 2^p + 2^q + \cdots + 2^z$$

where k is the largest exponent, then the binary representation of N is

$$(N)_2 = b_k b_{k-1} \cdots b_1 b_0$$

where the bits b_k, b_p, b_q, . . . , b_z corresponding to the exponents in the sum are equal to 1, and all other bits are 0. For this and many other purposes, I recommend memorizing the first few powers of 2, as in the following table.

n	0	1	2	3	4	5	6	7	8	9	10
2^n	1	2	4	8	16	32	64	128	256	512	1024

EXAMPLE 14 Expressing the Number 49 in Binary Notation

The largest power of 2 less than 49 is $2^5 = 32$. Subtract 32 from 49 to get 17. The largest power of 2 less than 17 is $16 = 2^4$. Subtract 16 from 17 to get 1, which is a power of 2 ($1 = 2^0$). Thus $49 = 2^5 + 2^4 + 2^0$, and hence the nonzero bits of $(49)_2$ are b_5, b_4, and b_0, while $b_3 = b_2 = b_1 = 0$. Listing the bits in order, $(49)_2 =$ 110001. ■

Now suppose that we have n voters. A sequence of n bits can represent a voting combination, where each voter is associated to a particular bit, which is 1 if the voter approves and 0 if the voter disapproves. A sequence of n bits also gives the binary representation of a number between 0 (the sequence with n 0's) and $2^n - 1$, which, as a binary number, is a sequence of n 1's. It follows that a set of n voters can have 2^n different voting combinations.

The number of voting combinations with n voters and exactly k "yes" votes is denoted C_k^n. For example, there is only one combination, $000\cdots0$ where no one votes "yes," so $C_0^n = 1$. You can show that by the same reasoning, $C_n^n = 1$. There are n combinations with exactly one "yes" vote:

$$100\cdots0,\ 010\cdots0,\ \ldots,\ 000\cdots1,$$

and hence $C_1^n = n$.

If each voter in a combination with k "yes" votes and $n - k$ "no" votes were to switch his or her vote to the opposite side, there would be $n - k$ "yes" votes and k "no" votes. Thus, the number of combinations of n voters with k "yes" votes is equal to the number of combinations of n voters with $n - k$ "yes" votes. In symbols, we can state this fact as follows:

Duality Formula for Combinations

$$C_k^n = C_{n-k}^n$$

Another formula that is useful is the **addition formula.** Suppose that there are $n + 1$ voters, one of whom is called Z. There are C_k^n combinations in which Z votes "no" and k other voters vote "yes," and C_{k-1}^n combinations in which Z and $k - 1$ of the other voters all vote "yes." This encompasses all C_k^{n+1} voting combinations of the $n + 1$ voters in which there are exactly k "yes" votes. Thus we have the following:

Addition Formula

$$C_{k-1}^n + C_k^n = C_k^{n+1}$$

The addition formula enables us to calculate the numbers C^n_k. Starting with $C^0_0 = C^1_0 = C^1_1 = 1$, we obtain $C^2_1 = C^1_0 + C^1_1 = 1 + 1 = 2$. Continuing, it is convenient to display the results in triangular form:

$$
\begin{array}{c}
1\\
1\quad 1\\
1\quad 2\quad 1\\
1\quad 3\quad 3\quad 1\\
1\quad 4\quad 6\quad 4\quad 1\\
1\quad 5\quad 10\quad 10\quad 5\quad 1\\
1\quad 6\quad 15\quad 20\quad 15\quad 6\quad 1
\end{array}
$$

The entries on the left and right edges of the triangle are all equal to 1, and each entry in the interior of the triangle is obtained by adding the two entries just above it. This triangle is called **Pascal's triangle** in honor of Blaise Pascal (1623–1662), the French mathematician and philosopher credited with discovering it. The number C^n_k can be found by counting down to the nth row, remembering that the 1 on top is the 0th row, and then finding the kth entry, counting from the left, where the count starts with 0.

Pascal's triangle is an intriguing pattern, but it is only useful to calculate C^n_k when n is relatively small. The following expression gives a way to calculate C^n_k in more general situations:

To use the combination formula, cancel before multiplying.

Combination Formula

$$C^n_k = \frac{n!}{k!(n-k)!}$$

Calculate C^{40}_4 EXAMPLE 15

From the combination formula, $C^{40}_4 = \dfrac{40!}{4!\,36!}$. Notice that $40! = 40 \times 39 \times 38 \times 37 \times 36!$. Thus we can cancel 36! and obtain

$$C^{40}_4 = \frac{40 \times 39 \times 38 \times 37}{4 \times 3 \times 2 \times 1} = 91{,}390$$

To verify the combination formula, let $D^n_k = \dfrac{n!}{k!(n-k)!}$. It's our job to show that $D^n_k = C^n_k$. Recalling that $0! = 1$, we have $D^n_0 = \frac{n!}{0!n!} = 1$ and $D^n_n = \frac{n!}{n!0!} = 1$. Also, the numbers D^n_k obey the addition formula:

$$D^n_{k-1} + D^n_k = D^{n+1}_k$$

To see this, we have to add the fractions $D^n_{k-1} = \frac{n!}{(k-1)!(n-k+1)!}$ and $D^n_k = \frac{n!}{k!(n-k)!}$. Because $k! = k \times (k-1)!$ and $(n-k+1)! = (n-k+1) \times (n-k)!$, the least common denominator is $k!(n-k+1)!$. Therefore

$$D^n_{k-1} + D^n_k = \frac{k \times n!}{k!(n-k+1)!} + \frac{(n-k+1) \times n!}{k!(n-k+1)!}$$
$$= \frac{k \times n! + (n-k+1) \times n!}{k!(n-k+1)!} = \frac{(n+1) \times n!}{k!(n+1-k)!} = D^{n+1}_k$$

It follows that if we arrange the numbers D^n_k in a triangle, as we did C^n_k, we will again get Pascal's triangle, because the left and right edges are filled with 1's, and each interior entry is equal to the sum of the two entries above it. We thus conclude that $C^n_k = D^n_k$, and hence the combination formula holds.

Efficient counting methods make it possible to compute the Banzhaf power index of large weighted voting systems. The method of counting combinations applies to systems in which most of the voters have the same weight, as in the seven-person committee that we considered in Example 7.

EXAMPLE 16 The Banzhaf Index of the Seven-Person Committee

The chairperson of this committee has 3 votes. Each of the six other members has 1 vote. The quota is 5, so we are considering the voting system

$$[5 : 3, 1, 1, 1, 1, 1, 1]$$

The chairperson, whom we will call C, is a critical voter in any winning coalition with no more than 2 extra votes. To achieve the quota, C's coalition must include at least two weight-1 voters. If there are five or more weight-1 voters in the coalition, then C's vote will not be needed: C will not be a critical voter. The number of coalitions with two weight-1 voters is C^6_2, because we are counting the voting combinations of the six weight-1 voters with 2 "yes" votes. Similarly, there are C^6_3 coalitions consisting of C and three weight-1 voters, and C^6_4 coalitions with C and four weight-1 voters. Referring to Pascal's triangle, displayed on the previous page, there are

$$C^6_2 + C^6_3 + C^6_4 = 15 + 20 + 15 = 50$$

winning coalitions. Counting an equal number of blocking coalitions, the Banzhaf power index of C is 100.

When we calculated the Shapley–Shubik power index, we only had to consider the chairperson. The other members' indices could then be determined because the Shapley–Shubik indices of all the members add up to 1. Because there is no fixed sum of the Banzhaf power indices of all the participants, we have to calculate the indices of the weight-1 voters separately. These voters do have the same voting power, so we only have to consider one of them, whom we will call

M. By the extra votes principle, M is a critical voter in a winning coalition only if this coalition has exactly 5 votes. There are two ways to assemble such a winning coalition:

- A three-member coalition consisting of M, C, and one of the other five weight-1 members. There are $C_1^5 = 5$ of these coalitions, because we are considering voting combinations of the 5 weight-1 voters other than M in which there is 1 "yes" and 4 "no" votes.
- A five-member coalition consisting of M and 4 other weight-1 voters. There are $C_4^5 = 5$ of these coalitions–here we are counting the voting combinations of the 5 weight-1 voters other than M in which there are 4 "yes" votes and 1 "no."

Adding, we find that M is a critical voter in 10 winning coalitions. Doubling this to account for the blocking coalitions, the Banzhaf power index of each weight-1 voter is 20.

To summarize, the Banzhaf power index of this voting system is

$$(100, 20, 20, 20, 20, 20, 20).$$

The total number of critical votes is $100 + 6 \times 20 = 220$. Thus, according to the Banzhaf model, C has $\frac{100}{220}$, or about 45%, of the power in the committee, and each weight-1 voter has $\frac{1}{11}$, or about 9.1%, of the power. This is in pretty close agreement with the Shapley–Shubik model, where we found that C had $\frac{3}{7}$, or about 43%, of the power, while each weight-1 voter had $\frac{2}{21}$, or approximately 9.5%, of the power. ■

In Example 8, we determined the Shapley–Shubik power index of the voting system [7 : 3, 3, 1, 1, 1, 1, 1] (the committee with two co-chairs). In the following example, we will determine the Banzhaf power index of that committee.

The Committee with Two Co-Chairs

EXAMPLE 17

To determine the voting power of each voter in the system [7 : 3, 3, 1, 1, 1, 1, 1] by the Banzhaf model, let's start with a weight-1 voter, M. He will be a critical voter in a winning coalition if and only if the votes of the other members in the coalition add up to exactly 6. There are two ways to achieve this total:

- The two co-chairs, and no other weight-1 voters, could join with M. There is exactly one such coalition.
- One of the two co-chairs, and 3 of the other 4 weight-1 voters, could join with M. There are $C_1^2 \times C_3^4 = 8$ such coalitions.

Thus, M is a critical voter in 9 winning coalitions; doubling this, we find that his Banzhaf power index is 18.

Now we must determine the Banzhaf power index of a weight-3 voter, A. She will be a critical voter in a winning coalition in which the other members have a combined total of 4, 5, or 6 votes.

- If she is joined by the other co-chair, the coalition would need 1, 2, or 3 of the 5 weight-1 members. The number of coalitions of this sort is

$$C_1^5 + C_2^5 + C_3^5 = 5 + 10 + 10 = 25$$

- If the other co-chair is opposed, she could be joined by 4 or all 5 of the weight-1 members. The number of such coalitions is $C_4^5 + C_5^5 = 6$.

It follows that A is a critical voter in 31 winning coalitions; her Banzhaf power index is 62. The Banzhaf power index of this committee is $(62, 62, 18, 18, 18, 18, 18)$. ■

The total number of critical votes in the committee with two co-chairs is $(2 \times 62) + (5 \times 18) = 214$. Thus, each co-chair has $\frac{62}{214}$, or approximately 29.0%, of the power, and each weight-1 member has $\frac{18}{214}$, or about 8.4%, of the power, by the Banzhaf model. Recall that, according to the Shapley–Shubik model, the co-chairs each had $\frac{2}{7}$ (about 28.6%) of the power and the weight-1 members had $\frac{3}{35}$ (about 8.6%). The agreement between the two models is, as in the other examples that we have considered, pretty close.

There are situations in which the difference in the distribution of power given by the models is significant. In Spotlight 11.4, the Banzhaf and Shapley–Shubik power indices of the United States Electoral College are compared. While the differences may seem small, by the Shapley–Shubik model, California has about 11.0% of the voting power in the college, while by the Banzhaf model, California has 11.4% of the power. When the stakes are high, this difference is significant. The following example presents a situation in which the models give dramatically different results.

EXAMPLE 18 The Big Shareholder

B holds 100,000 shares of stock in a corporation. There is a total of one million shares of stock, and the remaining stock is held by 9000 shareholders, each of whom has 100 shares. A weighted voting system, in which each shareholder's voting weight is equal to the number of shares that he or she owns, is used.

The Shapley–Shubik index of this system is determined by the same strategy that we used in Example 7 (the seven-person committee). This time, the permutations of the stockholders are divided into 9001 groups, depending upon the location of B. Each group has the same number of permutations (9000! of them, to be precise), and B is pivotal when she appears in the 4002nd through the 5001st position. If she is 4002nd, then there are $4001 \times 100 = 400{,}100$ shares preceding her, and her 100,000 shares bring the total to a bare majority of 500,100 shares. If there are more than 5000 shares ahead of B, the 5001st, a small shareholder,

SPOTLIGHT 11.4 The Electoral College: Presidential Elections of 2004 and 2008

The following table displays the Shapley–Shubik (SSPI) and Banzhaf (BPI) power indices of the voters in the Electoral College, as compared with the voter's weight as a percent of 538 (PCT), the total weight of all of the voters. It shows that for the most part, both measures of power agree closely with the actual share of power that a participant in the college has by virtue of its voting weight. There is an exception, though. California, whose voting weight is slightly more than 20% of the quota, has more than its share of power by either measure.

Voter	Weight	PCT (%)	SSPI (%)	BPI (%)
CA	55	10.22	11.04	11.41
TX	34	6.32	6.50	6.39
NY	31	5.76	5.89	5.79
FL	27	5.02	5.09	5.01
IL, PA	21	3.90	3.91	3.87
OH	20	3.72	3.72	3.68
MI	17	3.16	3.14	3.12
GA, NC, NJ	15	2.79	2.76	2.74
VA	13	2.42	2.38	2.37
MA	12	2.23	2.20	2.19
IN, MO, TN, WA	11	2.04	2.01	2.01
AZ, MD, MN, WI	10	1.86	1.82	1.82
AL, CO, LA	9	1.67	1.64	1.64
KY, SC	8	1.49	1.45	1.46
CT, IA, OK, OR	7	1.30	1.27	1.27
AR, KS, MS	6	1.12	1.09	1.09
NM, NV, UT, WV	5	0.93	0.90	0.91
HI, ID, NH, RI	4	0.74	0.72	0.73
AK, DE, DC, MT, ND, SD, VT, WY	3	0.56	0.54	0.55
ME, NE	2	0.37	0.36	0.36
Congressional districts (5 in all)	1	0.19	0.18	0.18

would be the pivot. Thus, B is pivotal in 1000 of the 9001 groups, and her Shapley–Shubik power index is $\frac{1000}{9001}$, or about 11.1%. The 9000 small shareholders have equal shares of the remaining power: The index of each is

$$\left(1 - \frac{1000}{9001}\right) \div 9000$$

which works out to be 0.0099%.

The Banzhaf power index is much harder to calculate, and it involves numbers of critical votes that are unimaginably large. However, we can assess *B*'s share of power according to the Banzhaf model by the following consideration. By this model, each voter's share of the power is proportional to the probability that he or she would be a critical voter in a winning or blocking coalition if every voter tossed a coin and voted "yes" for heads, "no" for tails.

The shareholder *B* will be critical if between 4001 and 5000 small shareholders vote "yes." (If she votes "yes," she will be a critical voter in a winning coalition with weight between 500,100 and 600,000; if she votes "no," she will be a critical voter in a blocking coalition with between 500,000 and 599,900 shares.) Imagine that you are betting on 9000 people tossing coins. On average, there will be 4500 heads; you win if the number of heads is between 4001 and 5000. You could play this game for the rest of your life and never lose! If you remember about normal distributions (see Chapter 5), the number of heads in this experiment would have mean 4500, and the standard deviation would be approximately 50. Thus, 68% of the time there would be between 4450 and 4550 heads; 95% of the time there would be between 4400 and 4600 heads; 99.7% of the time there would be between 4350 and 4650 heads; and so on. It follows that *B* is a critical voter in almost all situations.

Now consider a shareholder *S* who owns 100 shares. He will be critical if the owners of exactly 500,000 other shares vote "yes" (this could be *B* and 4000 small shareholders, or 5000 small shareholders without *B*). Imagine again that you are betting on 9000 coin tosses, with *B* tossing first. You win if either the first toss is heads, and *exactly* 4000 of the following tosses are heads, or if the first toss is tails, and *exactly* 5000 of the following tosses are heads. Otherwise you lose. Because 4000 and 5000 are both far from the average number of heads, 4500, this is a game *S* could play for the rest of his life and *never* win! It follows that by the Banzhaf model, the small shareholders have virtually no power; the big shareholder possesses almost all of the power. ■

11.3 Comparing Voting Systems

In many cases, different weighted voting systems turn out to have the same winning coalitions. For example, a dictatorship is no different if the dictator's weight is exactly equal to the quota or if it is much more. The dictator will have the same Banzhaf power index (2^n if there are n voters, including the dictator), and the same Shapley–Shubik power index (1) by virtue of being a dictator. To compare voting systems—which may be specified with weights or in some other way—we need to know what the winning coalitions are.

If there are just two voters, *A* and *B*, the empty coalition, { }, is surely a losing coalition, and {*A, B*} is a winning coalition. There are only three distinct voting systems in this case: In the first, unanimous consent is required to pass a measure, so the only winning coalition is {*A, B*}. In the second, *A* is a dictator, and {*A*} is also a winning coalition. In the third, *B* is a dictator, and the winning coalitions are {*B*} and {*A, B*}. Although there is an unlimited number of ways to assign weights and a quota to a two-voter system, there are only three ways that the power can be distributed: *A* as dictator, *B* as dictator, or consensus rule.

Equivalent Voting Systems

Two voting systems are **equivalent** if there is a way for all of the voters of the first system to exchange places with the voters of the second system and preserve all winning coalitions.

The weighted voting systems [50 : 49, 1] and [4 : 3, 3], involving pairs of voters *A*, *B*, and *C*, *D*, respectively, are equivalent because in each system, unanimous support is required to pass a measure. We could have *A* exchange places with *C*, and *B* exchange places with *D*.

Now consider two voting systems [2 : 2, 1] and [5 : 3, 6] involving the same pair of voters, *A* and *B*. In the first, [2 : 2, 1], *A* is a dictator, while in the second, [5 : 3, 6], *B* dictates. By having *A* and *B* exchange places with each other, we see that the two systems are equivalent. "Equivalent" does not mean "the same." Voter *A* would tell you that the system where he is the dictator is not the same as the system where *B* is the dictator. The systems are equivalent because each has a dictator.

Every two-voter system is equivalent either to a system with a dictator or to one that requires consensus. As the number of voters increases, the number of different types of voting systems increases.

Minimal Winning Coalitions

A **minimal winning coalition** is a winning coalition in which each voter is a critical voter.

In a dictatorship, every coalition that includes the dictator is a winning coalition, but the only *minimal* winning coalition is the one that includes the dictator and no other voters.

Minimal Winning Coalitions: A Three-Voter System — EXAMPLE 19

The three-member committee from Example 5 uses the voting system [6 : 5, 3, 1]. Let's refer to its members as *A, B,* and *C* in order of decreasing weight. There are three winning coalitions. One, {*A, B*}, has weight 8, more than the quota, but it is minimal because both voters are critical. Another, {*A, C*}, with weight 6, is also minimal. The third winning coalition, {*A, B, C*}, is not minimal because only *A* is a critical voter. ■

EXAMPLE 20 The Four-Shareholder Corporation

Table 11.4 lists the five winning coalitions in the corporation with the voting system [51 : 40, 30, 20, 10]. In each coalition, the critical voters have been identified. The minimal ones are those in which each voter is marked as critical: {*A*, *B*}, {*A*, *C*}, and {*B*, *C*, *D*}. These minimal winning coalitions are displayed in Figure 11.1. ■

A voting system can be described completely by specifying its minimal winning coalitions. If you want to make up a new voting system, instead of specifying weights and a quota, you could make a list of the minimal winning coalitions. You would have to be careful that your list satisfies the following three requirements:

1. Your list can't be empty. You have to name at least one coalition—otherwise, there would be no way to approve a motion.
2. You can't have one minimal winning coalition that contains another one—otherwise, the larger coalition wouldn't be minimal.
3. Every pair of coalitions in the list has to overlap—otherwise, two opposing motions could pass.

In the four-shareholder corporation (see Figure 11.1), you can see that these requirements are satisfied. Now let's construct some voting systems.

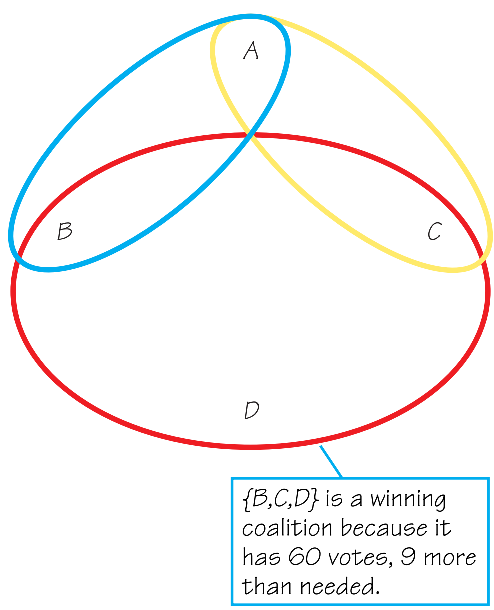


FIGURE 11.1 Each oval surrounds a minimal winning coalition for the four-shareholder corporation.

Three-Voter Systems

EXAMPLE 21

We would like to make a list of all voting systems that have three participants, *A, B,* and *C*. To keep the size of the list manageable, we will insist that no two voting systems on the list be equivalent.

To start, suppose that {*A*} is a minimal winning coalition. Requirement 3 tells us that every other minimal winning coalition must overlap with {*A*}, but the only way that could happen would be if *A* also belonged to the other coalition. In this case, requirement 2 would be violated. Thus, {*A*} can be the only minimal winning coalition. This is the voting system where *A* is dictator. Systems where *B* or *C* is dictator are not listed because they are equivalent to this one.

Now suppose that there is no dictator. Every minimal winning coalition must contain either two or all three voters. Let's consider the case in which {*A, B, C*} is a minimal winning coalition. It is the only winning coalition, because any other winning coalition would have to be entirely contained in this coalition, which requirement 2 doesn't allow. In this voting system, a unanimous vote is required to pass a measure. We will call this system *consensus rule*.

Finally, let's suppose that there is a two-voter minimal winning coalition, say {*A, B*}. If it is the only minimal winning coalition, then a measure will pass if *A* and *B* both vote "yes" and the vote of *C* does not matter: In other words, *C* is a dummy, and *A* and *B* make all the arrangements. We will call this system the *clique*. Of course, the clique could be {*A, C*} or {*B, C*}, but these systems are equivalent to the one where {*A, B*} is the clique.

There could be two two-voter minimal winning coalitions, say {*A, B*} and {*A, C*}. Neither coalition contains the other, and there is an overlap, so all of the requirements are satisfied. In this system, *A* has veto power. We have encountered this system in Example 10, and we will call it the *chair veto*. There are two other voting systems equivalent to this one, where *B* or *C* is chair.

It is possible that all three two-member coalitions are minimal winning coalitions. Because there are only three voters, any two distinct two-member coalitions will overlap, so the requirements are still satisfied. This system is called *majority rule*. ■

Table 11.5 lists all five of these three-voter systems. Each system can be presented as a weighted voting system, and suitable weights are given in the table. If we want to make a similar list of all types of four-voter systems, we can start by making each three-voter system into a four-voter system. This is done by putting a fourth voter, *D*, into the system, without including him in any of the minimal winning coalitions. This makes *D* a dummy. You may be interested to know that there are an additional nine 4-voter systems that don't have any dummies. Try to list as many of these systems as you can.

The Scholarship Committee

EXAMPLE 22

A university offers scholarships on the basis of either academic excellence or financial need. Each application for a scholarship is reviewed by two professors,

TABLE 11.5 Voting Systems with Three Participants

System	Minimal Winning Coalitions	Weights	Banzhaf Index
Dictator	$\{A\}$	[3: 3, 1, 1]	(8, 0, 0)
Clique	$\{A, B\}$	[4: 2, 2, 1]	(4, 4, 0)
Majority	$\{A, B\}, \{A, C\}, \{B, C\}$	[2: 1, 1, 1]	(4, 4, 4)
Chair veto	$\{A, B\}, \{A, C\}$	[3: 2, 1, 1]	(6, 2, 2)
Consensus	$\{A, B, C\}$	[3: 1, 1, 1]	(2, 2, 2)

who rate the student academically, and two financial aid officers, who rate the applicant's need. If both professors or both financial aid officers recommend the applicant for a scholarship, the Dean of Admissions decides whether or not to award a scholarship. Is it possible to assign weights to the professors, the financial aid officers, and the dean to reflect this decision-making system?

To answer this question, let's focus on the minimal winning coalitions. The participants are the two professors, A and B; the financial aid officers E and F; and the dean D. A scholarship will be offered if approved by the professors and the dean, or by the financial aid officers and the dean. Thus, the minimal winning coalitions are

$$\{A, B, D\} \qquad \text{and} \qquad \{D, E, F\}$$

(see Figure 11.2).

Consider the following two winning coalitions: In the first, all except the financial aid officer F favors an award; while in the second, only the professor B dissents.

$$C_1 = \{A, B, D, E\} \qquad \text{and} \qquad C_2 = \{A, D, E, F\}$$

In C_1, we notice that A is a critical voter and E isn't, while in C_2 the tables are turned because E is critical while A is not. If this were a weighted voting system, then in any winning coalition, the critical voters would all have greater weight than those who are not critical. Thus A would have to have both more weight than E (because of the situation in C_1) and less weight than E (because of C_2), which is impossible. ■

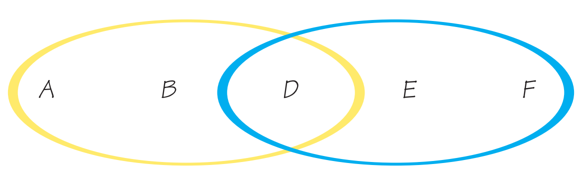


FIGURE 11.2 The Scholarship Committee: Minimal winning coalitions.

Although the scholarship committee is not equivalent to any weighted voting system, it is possible to determine the Shapley–Shubik and Banzhaf power indices of each participant.

Power Indices of the Scholarship Committee

EXAMPLE 23

The dean has veto power. Therefore, she will be the pivot in any permutation where she appears last. If she is second-to-last in a permutation, she will still be the pivot, because either both professors or both financial aid officers must come before her. In the middle position, she will be pivotal if and only if both professors or both aid officers come first. Adding this up, we have $2 \times 4! = 48$ permutations in which the dean is in fourth or fifth position. There are four permutations of the form Prof, Prof, Dean, Aid, Aid because the professors and the aid officers can be in either order, and another four of the form Aid, Aid, Dean, Prof, Prof. The dean is not the pivot when she is first or second, because at least three people have to approve a scholarship. We conclude that the dean is pivotal in $48 + 4 + 4 = 56$ permutations in all. Her Shapley–Shubik power index is therefore $\frac{56}{5!} = \frac{7}{15}$. Each of the other participants is equally powerful, and they share the remaining $\frac{8}{15}$ of the power. Thus each professor and each aid officer has a Shapley–Shubik power index of $\frac{2}{15}$.

To compute the Banzhaf index, let's list all the winning coalitions: There are 7 of them.

$$\{A, B, D\}, \{E, F, D\}, \{B, E, F, D\}, \{A, E, F, D\},$$
$$\{A, B, F, D\}, \{A, B, E, D\}, \text{ and } \{A, B, E, F, D\}$$

The dean, who has veto power, is a critical voter in each winning coalition and in 7 blocking coalitions, so her Banzhaf power index is 14. Professor A is critical in each winning coalition that includes B but not both aid officers: There are 3 of these. He is also critical in 3 blocking coalitions, so his Banzhaf power index is 6. The remaining participants, B, E, and F, have the same power, so the Banzhaf power index of the scholarship committee is

$$(14, 6, 6, 6, 6)$$

REVIEW VOCABULARY

Addition formula

$$C^n_{k-1} + C^n_k = C^{n+1}_k$$

Banzhaf power index A count of the winning or blocking coalitions in which a voter is a critical member. This is a measure of the actual voting power of that voter.

Bit A binary digit: 0 or 1.

Binary number The expression of a number in base-2 notation. If b_n denotes the n bit to the left of the binary point (n is 0 or negative for bits to the right of the binary point; this only occurs in representing numbers that are not whole), then the number being represented is the sum of the terms $b_n \times 2^{n-1}$. The binary number 11001101 has $b_1 = 1$, $b_2 = 0$, $b_3 = b_4 = 1$, $b_5 = b_6 = 0$, and $b_7 = b_8 = 1$. Hence it can be converted to decimal form as

$$2^0 + 2^2 + 2^3 + 2^6 + 2^7 = 205$$

Blocking coalition A coalition in opposition to a measure that can prevent the measure from passing.

$\boldsymbol{C^n_k}$ The number of voting combinations in a voting system with n voters, in which k voters say "yes" and $n - k$ voters say "no." This number, referred to as "n-choose-k," is given by the formula

$$C^n_k = \frac{n!}{k! \times (n-k)!}$$

Coalition The set of participants in a voting system who favor, or who oppose, a given motion. A coalition may be empty (if, for example, the voting body unanimously favors a motion, the opposing coalition is empty), it may contain some but not all voters, or it may consist of all the voters.

Critical voter A member of a winning coalition whose vote is essential for the coalition to win, or a member of a blocking coalition whose vote is essential for the coalition to block.

Dictator A participant in a voting system who can pass any issue even if all other voters oppose it and block any issue even if all other voters approve it.

Duality formula $C^n_k = C^n_{n-k}$

Dummy A participant who has no power in a voting system. A dummy is never a critical voter in any winning or blocking coaliton and is never the pivotal voter in any permutation.

Equivalent voting systems Two voting systems are equivalent if there is a way for all the voters of the first system to exchange places with the voters of the second system and preserve all winning coalitions.

Extra votes The number of votes that a winning coalition has in excess of the quota.

Extra-votes principle The critical voters in the coalition are those whose weights are more than the extra votes of the coalition. For example, if a coalition has 12 votes and the quota is 9, there are 3 extra votes. The critical voters in the coalition are those with more than 3 votes.

Factorial The number of permutations of n voters (or n distinct objects) is called *n-factorial,* or in symbols, $n!$. Because the empty coalition can be ordered in only one way, $0! = 1$. When n is a positive whole number, $n!$ is equal to the product of all the integers from 1 up to n. If n has more than one digit, then $n!$ is a pretty big number: $10!$ is more than three million, and $1000!$ has 2568 digits.

Losing coalition A coalition that does not have the voting power to get its way.

Minimal winning coalition A winning coalition that will become losing if any member defects. Each member is a critical voter.

Pascal's triangle A triangular pattern of integers, in which each entry on the left and right edges is 1, and each interior entry is equal to the sum of the two entries above it. The entry that is located k units from the left edge, on the row n units below the vertex, is C^n_k.

Permutation A specific ordering from first to last of the elements of a set; for example, an ordering of the participants in a voting system.

Pivotal voter The first voter in a permutation who, with his or her predecessors in the permutation, will form a winning coalition. Each permutation has one and only one pivotal voter.

Power index A numerical measure of an individual voter's ability to influence a decision, the individual's voting power.

Quota The minimum number of votes necessary to pass a measure in a weighted voting system.

Shapley–Shubik power index A numerical measure of power for participants in a voting system. A participant's Shapley–Shubik index is the number of permutations of the voters in which he or she is the pivotal voter, divided by the number of permutations ($n!$ if there are n participants).

Veto power A voter has veto power if no issue can pass without his or her vote. A voter with veto power is a one-person blocking coalition.

Voting combination A list of voters indicating the vote of each on an issue. There is a total of 2^n

combinations in an n-element set, and C_k^n combinations with k "yes" votes and $n - k$ "no" votes.

Weight The number of votes assigned to a voter in a weighted voting system, or the total number of votes of all voters in a coalition.

Weighted voting system A voting system in which each participant is assigned a voting weight (different participants may have different voting weights). A quota is specified, and if the sum of the voting weights of the voters supporting a motion is at least equal to that quota, the motion is approved. The notation

$$[q : w_1, w_2, \ldots w_n]$$

is used to denote a system in which there are n voters, with voting weights $w_1, w_2, \ldots w_n$; and the quota is q.

Winning coalition A set of participants in a voting system who can pass a measure by voting for it.

SKILLS CHECK

1. What would be the quota for a voting system that has a total of 20 voters and uses a simple majority quota?

(a) 10
(b) 11
(c) 20

2. For the weighted voting system [65 : 60, 30, 10],

(a) the weight-60 voter is a dictator.
(b) the weight-30 voter has veto power.
(c) the weight-10 voter is not a dummy.

3. Two daughters each hold six votes, and a son has the remaining two votes for a trust fund. The quota for passing a measure is 8. Which statement is true?

(a) The son is a dummy voter.
(b) The son is not a dummy voter but has less power than a daughter.
(c) The three children have equal power.

4. Which participants in the weighted voting system [10 : 4, 4, 3, 2] have veto power?

(a) None of the participants
(b) Only the weight-4 voters
(c) Each participant

5. What is the value of C_2^6?

(a) 12
(b) 15
(c) 32

6. For the weighted voting system [6: 4, 3, 2, 1], find the Banzhaf power index for the voter with three votes.

(a) 3
(b) 6
(c) 14

7. Calculate the Shapley–Shubik power index for the three-vote voter in the weighted voting system [6; 4, 3, 2, 1].

(a) $\frac{1}{4}$
(b) $\frac{5}{24}$
(c) $\frac{1}{12}$

8. A blocking coalition can always

(a) defeat a motion.
(b) pass a motion.
(c) force a reevaluation of a motion.

9. If a winning coalition is minimal, then the number of extra votes

(a) is zero.
(b) is less than the number of votes held by any member of the winning coalition.
(c) is less than the number of votes held by the losing coalition.

10. The best description for a voter who always wins is

(a) a voter with veto power
(b) a dictator.
(c) a pivotal voter.

11. A critical voter in a coalition is a voter

(a) who has the most votes.
(b) who has fewer votes than the number of extra votes of the coalition.
(c) whose defection changes the coalition from a winning coalition to a losing coalition.

12. The weight of a voter is

(a) the number of votes assigned to the voter.
(b) the number of times the voter is pivotal.
(c) the number of times the voter is part of a winning coalition.

13. How large is the number 8! (eight factorial)?
(a) More than a million
(b) Less than a million but more than 10,000
(c) Less than 10,000

14. In how many different ways can six voters be ordered from first to last?

(a) 30
(b) 64
(c) More than 500

15. In how many ways can six voters respond to a "yes–no" question?

(a) 12
(b) 36
(c) 64

16. What number is missing from this row of Pascal's triangle?

1 6 15 ___ 15 6 1

(a) 20
(b) 25
(c) 30

17. A pivotal voter

(a) casts the deciding vote in a permutation if each voter votes in turn.
(b) is a dictator.
(c) can defeat any motion by voting "no."

18. A voter with veto power

(a) always wins.
(b) has the greatest voting weight.
(c) is pivotal in every permutation where he or she is the last voter.

19. $C_9^{12} =$

(a) C_3^{12}
(b) 220
(c) Both of the above

20. In the weighted voting system [7 : 3, 2, 2, 2] the number of voting combinations with exactly the quota to win is

(a) 2
(b) 3
(c) 4

EXERCISES

■ **Challenge** ◆ **Discussion**

How Weighted Voting Works

◆ **1.** In the United States Senate, each of the 100 senators has one vote, and the vice president of the United States can vote also if it is necessary to break a tie.

(a) A simple majority is needed to pass a bill. What constitutes a winning coalition, and what constitutes a blocking coalition?
(b) One seat in the Senate is vacant, so that only 99 senators can vote. What constitutes a winning coalition, and what constitutes a blocking coalition for passing a bill?
(c) To ratify a treaty, a two-thirds majority is required. What constitutes a winning coalition, and what constitutes a blocking coalition?

◆ **2.** Is it possible to have a weighted voting system in which more votes are required to block a measure than to pass a measure?

3. In the weighted voting system [9 : 5, 4, 3]:

(a) Is there a dictator?

(b) Which voters, if any, have veto power?
(c) Is any voter a dummy?

4. If there are four voters, with voting weights of 30, 29, 28, and 13, and if one, and only one, of the voters has veto power,

(a) which voter has veto power?
(b) find the quota.
(c) is any voter a dummy?

Shapley–Shubik Power Index

5. Can a dummy be pivotal in any permutation? Explain why or why not.

6. A jury requires a unanimous vote to convict or to acquit. Give a quick way to determine the pivotal voter in any permutation of the jury's members.

7. For the weighted voting system [51 : 30, 25, 24, 21]:

(a) List all permutations in which the weight-30 voter is pivotal.
(b) List all permutations in which the weight-25 voter is pivotal.
(c) Calculate the Shapley–Shubik index.

8. In the voting system [7 : 3, 2, 2, 2, 2, 2]:

(a) Describe the set of permutations in which the weight-3 voter is pivotal.
(b) How many of these permutations are there?
(c) Use the answer that you have given in part (b) to determine the Shapley–Shubik index of the system.

9. How would the Shapley–Shubik index in Exercise 7 change if the quota were increased to

(a) 52?
(b) 55?
(c) 58?

10. Refer to the permutation of the 2004 presidential election (see Table 11.3). The Republican Bush–Cheney ticket carried Nevada (NV), 414,939 to 393,372. Which state would be the pivot if at the last minute their opponents, Kerry–Edwards, had broadcast an ad that convinced 1000 voters to switch from Bush–Cheney to Kerry–Edwards? Assume that no votes are changed outside of Nevada.

Banzhaf Power Index

11. (a) List the 16 possible combinations of how four voters, *A, B, C,* and *D,* can vote either "yes" (Y) or "no" (N) on an issue.
(b) List the 16 subsets of the set {*A, B, C, D*}.
(c) How do the lists in parts (a) and (b) correspond to each other?
(d) In how many of the combinations in part (a) is the vote

(i) 4 Y to 0 N?
(ii) 3 Y to 1 N?
(iii) 2 Y to 2 N?

12. List all winning coalitions in the weighted voting system, [51 : 30, 25, 24, 21].

(a) Identify the critical voters in each coalition.
(b) Determine the Banzhaf power index of each participant.
(c) List all of the blocking coalitions in which the weight-30 voter is critical, and match each with its dual winning coalition in which the same voter is critical.

13. The system in Exercise 12 is modified by increasing the quota to

(a) 52.
(b) 55.
(c) 58.
(d) 73.
(e) 76.
(f) 79.
(g) 82.

Calculate the Banzhaf index in each case. (*Hint:* Increasing the quota will reduce the number of extra votes in each of the original coalitions. When the number of extra votes becomes negative, the coalition is losing and you can cross it off the list. As the number of extra votes decreases, a member of a coalition who was originally not a critical voter will become a critical voter.)

14. Determine the number of extra votes for each winning coalition, and calculate the Banzhaf index for each of the following weighted voting systems.

(a) [51: 52, 48]
(b) [3: 2, 2, 1]
(c) [8: 5, 4, 3]
(d) [51: 45, 43, 8, 4]
(e) [51: 45, 43, 6, 6]

15. Express the following decimal numbers as binary:

(a) 585
(b) 1365
(c) 2005

16. Calculate the following:

(a) C_3^7
(b) C_{100}^{50}
(c) C_2^{15}
(d) C_{13}^{15}

17. Calculate the following:

(a) C_3^6
(b) C_2^{100}
(c) C_{98}^{100}
(d) C_5^9

18. The various weighted voting systems used by the Board of Supervisors of Nassau County, New York turned out to be the mathematical quagmire described in Spotlight 11.3. Before the county's weighted voting was declared unconstitutional by a federal district court in 1993, it was changed several times. The weights in use since 1958 were as follows:

		Weights					
Year	**Quota**	H_1	H_2	N	B	G	L
1958	16	9	9	7	3	1	1
1964	58	31	31	21	28	2	2
1970	63	31	31	21	28	2	2
1976	71	35	35	23	32	2	3
1982	65	30	28	15	22	6	7

Here H_1 is the presiding supervisor, always from the community of Hempstead; H_2 is the second supervisor from Hempstead; and *N*, *B*, *G*, and *L* are the supervisors from the remaining districts: North Hempstead, Oyster Bay, Glen Cove, and Long Beach.

(a) ◆ From 1970 on, more than a simple majority was required to pass any measure. Give an argument in favor of this policy from the viewpoint of a supervisor who would benefit from it, and an argument against the policy from the viewpoint of a supervisor who would lose some power.
(b) In which years were some supervisors dummy voters?
(c) Suppose that the two Hempstead supervisors always vote together. In which years are some of the supervisors dummy voters?
(d) Assume that the two Hempstead supervisors always agree, so that the board is in effect a five-voter system. Determine the Banzhaf index of this system in each year.
(e) In 1982, a special supermajority of 72 votes was needed to pass measures that ordinarily require a two-thirds majority. If the two Hempstead supervisors vote together, what is the Banzhaf index of the resulting five-voter system?
(f) ◆ Table 11.6 gives the 1980 census for each municipality, the number of votes assigned to each supervisor, and the Banzhaf index for each supervisor in 1982. Do you think the voting scheme was fair?

Comparing Voting Systems

19. Consider a four-person voting system with voters *A*, *B*, *C*, and *D*. The winning coalitions are

$$\{A, B, C, D\}, \{A, B, C\}, \{A, B, D\},$$
$$\{A, C, D\}, \text{ and } \{A, B\}$$

(a) List the minimal winning coalitions.
◆ **(b)** Show that $\{A\}$ is a minimal blocking coalition. Are there other minimal blocking coalitions?
(c) Determine the Banzhaf power index for this voting system.
(d) Find an equivalent weighted voting system.
(e) Calculate the Shapley–Shubik index of this system.

TABLE 11.6 Nassau County Board of Supervisors, 1982

Supervisor From	Population	Number of Votes	Banzhaf Power Index	
Quota			65	72
Hempstead (presiding)	738,517	30	30	26
Hempstead		28	26	22
North Hempstead	218,624	15	18	18
Oyster Bay	305,750	22	22	18
Glen Cove	24,618	6	2	2
Long Beach	43,073	7	6	6
Totals	1,320,582	108	104	92

◆ **20.** A five-member committee has the following voting system. The chairperson can pass or block any motion that she supports or opposes, provided that at least one other member is on her side. Show that this voting system is equivalent to the weighted voting system [4: 3, 1, 1, 1, 1].

21. Calculate the Banzhaf index for the weighted voting system in Exercise 20.

22. Find weighted voting systems that are equivalent to

(a) a committee of three faculty members and the dean. To pass a measure, at least two faculty members and the dean must vote "yes."
(b) a committee of three faculty members, the dean, and the provost. To pass a measure, two faculty, the dean, and the provost must vote "yes."

◆ **23.** A four-member faculty committee and a three-member administration committee vote separately on each issue. The measure passes if it receives the support of a majority of each of the committees. Show that this system is not equivalent to a weighted voting system.

24. Calculate the Banzhaf index of the voting system in Exercise 23. Who is more powerful according to the Banzhaf model, a faculty member or an administrator?

25. Determine the Shapley–Shubik index of the system in Exercise 23. Who is more powerful according to the Shapley–Shubik model, a faculty member or an administrator?

◆ **26.** Explain why a voting system in which no voter has veto power must have at least three minimal winning coalitions.

◆ **27.** How many *distinct* (nonequivalent) voting systems with four voters can you find? Systems that have dummies don't count. The challenge is to find all nine.

28. A corporation has four shareholders and a total of 100 shares. The quota for passing a measure is the votes of shareholders owning 51 or more shares. The number of shares owned are as follows:

A	48 shares
B	23 shares
C	22 shares
D	7 shares

There is also an investor, *E*, who is interested in buying shares but does not own any shares at present. Sales of fractional shares are not permitted.

(a) List the winning coalitions and compute the number of extra votes for each. Make a separate list of the losing coalitions, and compute the number of votes that would be needed to make the coalition winning.
(b) How many shares can A sell to B without causing any of the winning coalitions listed in part (a) to lose or any of the losing coalitions in part (a) to win?
(c) How many shares can A sell to D without changing the sets of winning or losing coalitions?
(d) How many shares can A sell to E without changing the winning coalitions? Since E is now a dummy, he must remain a dummy after the trade.
(e) How many shares can D sell—without changing the set of winning coalitions—to $A, B, C,$ or E? Again, it is conceivable that D would be able to sell more to one stockholder than to another.
(f) How many shares can D sell to $A, B, C,$ or E without becoming a dummy?
(g) How many shares can B sell to C without changing the set of winning coalitions?

29. Which of the following voting systems is equivalent to the voting system in use by the corporation in Exercise 28?

(a) [3: 1, 1, 1, 1]
(b) [3: 2, 1, 1, 1]
(c) [5: 3, 1, 1, 1]
(d) [5: 3, 2, 1, 1]

30. Determine the Banzhaf and Shapley–Shubik power indices for the corporation in Exercise 28.

31. A nine-member committee has a chairperson and eight ordinary members. A motion can pass if and only if it has the support of the chairperson and at least two other members, or if it has the support of all eight ordinary members.

(a) Find an equivalent weighted voting system.
(b) Determine the Banzhaf power index.
(c) Determine the Shapley–Shubik power index.
(d) Compare the results of parts (b) and (c): Do the power indices agree on how power is shared in this committee?

32. The New York City Board of Estimate consists of the mayor, the comptroller, the city council president, and the presidents of each of the five boroughs. It employed a voting system in which the city officials each had 2 votes and the borough presidents each had 1; the quota to pass a measure was 6. This voting system was declared unconstitutional by the U.S. Supreme Court in 1989 (*Morris* v. *Board of Estimate*).

(a) Describe the minimal winning coalitions.
(b) Determine the Banzhaf power index.

33. Here is a proposed weighted voting system for the New York City Board of Estimate that is based on the populations of the boroughs (see Exercise 32):

$$[71: 35, 35, 35, 11.3, 7.3, 9.6, 6.0, 1.8]$$

Find a simpler system of weights that yields an equivalent voting system.

34. The United Nations Security Council has 5 permanent members—China, France, Russia, the United Kingdom, and the United States—and 10 other members that serve two-year terms. To resolve a dispute not involving a member of the Security Council, 9 votes, including the votes of each of the permanent members, are required. (Thus, each permanent member has veto power.)

◆ **(a)** Show that this voting system is equivalent to the weighted voting system in which each permanent member has 7 votes, each ordinary member has 1 vote, and the quota is 39.
(b) Compute the Banzhaf index for the Security Council.
◆ **(c)** The Security Council originally had 5 permanent members and 6 members who served two-year terms. Each permanent member had veto power, and 6 votes were required to resolve an issue. Devise an equivalent weighted voting system and compute its Banzhaf index. Do you think that the addition of 4 more nonpermanent members involved a significant loss of power by the permanent members?

35. Find the minimal winning coalitions of the weighted voting system [7 : 3, 3, 3, 1, 1, 1] and determine the Banzhaf index.

36. A new weight-1 voter joins the system of Exercise 35. Again, describe the minimal winning coalitions and determine the Banzhaf power index. Does the presence of this new voter increase or decrease the share of power of each weight-1 voter?

37. Compute the Shapley–Shubik power index for the systems in Exercises 35 and 36. How does the addition of the new voter affect the power of the other three weight-1 voters?

◆ **38.** An alumni committee consists of 3 rich alumni and 12 recent graduates. To pass a measure, a majority, including at least 2 of the rich alumni, must approve. Is this equivalent to a weighted voting system? If so, find the weights and a quota; if not, explain why not.

◆ **39.** Show that if a state uses the district system to choose its electors in a two-candidate presidential election, as Maine and Nebraska do, then some electoral permutations are impossible. Give an example of an impossible permutation. How would this affect the calculation of the Shapley–Shubik power index?

WRITING PROJECTS

1. The most important weighted voting system in the United States is the Electoral College (see Spotlight 11.1). Three alternate methods to elect the president of the United States have been proposed:

- *Direct election.* The Electoral College would be abolished, and the candidate receiving a plurality of the votes would be elected. Most versions of this system include a runoff election or a vote in the House of Representatives in cases where no candidate receives more than 40% of the vote.
- *District system.* This system could be adopted by individual states without amending the Constitution or passing a federal law. It is now in use by two states, Maine and Nebraska. In each congressional district, and in the District of Columbia, the candidate receiving the plurality would select one elector. Furthermore, in each state, including the District of Columbia, the candidate receiving the plurality would receive two electors. In effect, the unit rule would be retained for the District of Columbia and for states with a single congressional district. Larger states might have electors representing both parties.
- *Proportional system.* Each state and the District of Columbia would have fractional electoral votes assigned to each candidate in proportion to the number of popular votes received. Under this system, Governor Bush, who received 4,567,429 popular votes out of 10,965,822 cast in California in 2000, would have received

  $$\frac{4{,}567{,}429}{10{,}965{,}822} \times 54 \text{ electoral votes} = 22.4918 \text{ electoral votes}$$

  Vice President Gore would have received

  $$\frac{5{,}861{,}203}{10{,}965{,}822} \times 54 \text{ electoral votes} = 28.8629 \text{ electoral votes}$$

  The Green Party received 418,707 votes in California and would be entitled to 2.06188 electoral votes. There were four other parties that received between them less than one electoral vote. Obviously, no actual electors would be chosen.

Should the present electoral college, operating under the unit rule, be replaced by one of these systems? Reference: *The Presidential Election Game*, by Steven Brams, which contains useful references to Senate hearings on electoral college reform.

2. Write an essay on weighted voting in the Council of Ministers of the European Community. Compute the Banzhaf and Shapley–Shubik indices for the system as it was in 1958. In later years, the number of member nations increased significantly, and you may want to use the power index calculator, available on the Web at www.whfreeman.com/fapp. If they differ significantly in their allocation of power, which index represents the true balance of power best?

3. California has 10.22% of the votes in Electoral College, but according to Spotlight 11.4 that state has more than 11% of the power in the Electoral College, as measured by either of our power indices. Discuss the appropriateness of each power index as a measure of voting power in the Electoral College. Is the disproportionate power of California in the Electoral College a problem that the United States should address? Assume that California has acquired additional congressional seats as a result of migration. Calculate the Banzhaf index when California has 65, 75, and 100 electors. In each case take the electoral votes that are to be awarded to California from other states.

What would happen if all states, except California, adopted the district system for choosing electors? See Writing Project 1 for a discussion of the district system.

SUGGESTED READINGS

BRAMS, STEVEN J. *Game Theory and Politics,* 2nd Ed., Dover Publications, New York, 2004.

Iannucci v. *Board of Supervisors of Washington County* 20 N.Y. 2d 244, 251, 229 N.E. 2d 195, 198, 282 N.Y.S. 2d 502, 507 (1967). This code will help a law librarian find this case for you. It opened a "mathematical quagmire."

FELSENTHAL, DAN S., and MOSHÉ MACHOVER, *The Measurement of Voting Power: Theory and Practice, Problems and Paradoxes,* Edward Elgar, Cheltenham, U.K., 1998. This book has a detailed analysis of the Council of Ministers of the European Community, and a thorough treatment of the power indices mentioned in this chapter.

TAYLOR, ALAN D. *Mathematics and Politics: Strategy, Voting Power, and Proof,* Springer-Verlag, New York, 1995. Chapter 4 covers weighted voting systems and their analysis using the Shapley–Shubik and Banzhaf power indices. It has no mathematical prerequisites, but it does include carefully written logical arguments that must be carefully read.