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Automorphic Sheaves and Eisenstein Series

THESIS SUBMITTED FOR THE DEGREE "DOCTOR OF PHILOSOPHY"

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Contents

Introduction

0.1. Let X be a smooth, complete and geometrically connected curve over $\mathbb{F}_q$ of genus $g \geq 1$. The purpose of the geometric Langlands program initiated by V.Drinfeld in [2] and by G.Laumon in [10] is to associate to any n-dimensional ℓ-adic local system E on X, a sheaf Aut_E on the moduli stack Bun_n of n-bundles on X, which is a Hecke eigen-sheaf with respect to E.

The construction of automorphic sheaves which was suggested in [2] and developed in [10] and [11] is inspired by a construction of automorphic functions for $GL(n)$ due to Shalika and Piatetski-Shapiro, and is based on considering a geometric analog of Whittaker functions (see [4], Sect.1–3 for a detailed discussion). Let us briefly review it.

We embed the stack Bun_n into $\mathcal{S}h_n$–the moduli stack of coherent sheaves of generic rank n on X. This is an open embedding and our task will be to construct a sheaf $\mathcal{S}_E$ on $\mathcal{S}h_n$ whose restriction to Bun_n will produce the sheaf Aut_E. As we shall see (cf. Theorem 2), in order for Aut_E to be an E-Hecke eigen-sheaf, it is natural to require that $\mathcal{S}_E$ satisfies the modified Hecke property with respect to E.

For an integer k let $\mathcal{S}h'_k$ (resp., ${}^0\mathcal{S}h'_k$) denote the stack that classifies pairs $(M_k, s : \Omega^{\otimes k-1} \to M_k)$, where M_k is an object of $\mathcal{S}h_k$ and s is a regular map (resp., s is an embedding of sheaves), see Sect. 2.1.1. We have an open embedding $j_k :{}^0 \mathcal{S}h'_k \to \mathcal{S}h'_k$. Serre's duality implies that the categories of sheaves on $\mathcal{S}h'_k$ and on ${}^0\mathcal{S}h'_{k+1}$ are equivalent by means of a Fourier transform functor Four_k.

Let now $\mathcal{S}_E$ be a sheaf on $\mathcal{S}h_n$; we shall denote by $\mathcal{S}'_E$ its pull-back to ${}^0\mathcal{S}h'_n$. Let us now apply to $\mathcal{S}'_E$ the following procedure: we make a Fourier transform to get a sheaf on $\mathcal{S}h'_{n-1}$ and then restrict it to ${}^0\mathcal{S}h'_{n-1}$ and so on. After $n-1$ iterations we will get a sheaf on ${}^0\mathcal{S}h'_1$. It follows from Laumon's work (Corollary 4.3 of [11]) that if $\mathcal{S}_E$ satisfies the modified Hecke property with respect to E, the resulting sheaf on ${}^0\mathcal{S}h'_1$ must be a pull-back of some explicit sheaf $\mathcal{L}_E$ on $\mathcal{S}h_0$, which we shall call Laumon's sheaf (cf. Definition 3).

This phenomenon is a geometric counterpart of the Casselman-Shalika formula. It is the sheaf $\mathcal{L}_E$ that encodes the input of the local system E and its construction in terms of E is local with respect to X.

Suppose now being given a local system E and let us now try to reconstruct the sheaf $\mathcal{S}_E$. We shall start with $\mathcal{L}_E$ and pull it back to ${}^0\mathcal{S}h'_1$. However, in order to apply to it the Fourier transform and get a sheaf on ${}^0\mathcal{S}h'_2$, we must choose its prolongation onto the whole of $\mathcal{S}h'_1$.

If we use the extension by zero and iterate the procedure $n-1$ times, we will get a sheaf on ${}^0\mathcal{S}h'_n$ that has "lost" all its degenerate Fourier coefficients (a precise statement on the level of functions is proven in [4]).

It is, therefore, natural to use the Goresky-MacPherson extension on each step of the construction. We conjecture, following Laumon, that for any local system E the resulting sheaf on ${}^0\mathcal{S}h'_n$ is a pull-back of a sheaf on $\mathcal{S}h_n$.

The first case in which this result has been proven is when $n = 2$ and the local system E is geometrically irreducible (this is a statement equivalent to the main theorem in [2]). Since $n = 2$, we have to apply the Goresky-MacPherson extension only once and it turns out that in this case it coincides with the extension by zero, due to a theorem by Deligne. In this work we will give another proof of Drinfeld's theorem and we will prove also that the resulting sheaf Aut_E on Bun_2 is an E-Hecke eigen-sheaf.

Let now E be a direct sum $E = E_1 \oplus ... \oplus E_n$ of 1-dimensional pairwise non-isomorphic local systems. In this case, Laumon's sheaf $\mathcal{L}_E$ also splits into a direct sum (cf. Corollary 3) and let us construct the sheaf $\mathcal{S}'_E$ on ${}^0\mathcal{S}h'_n$ according to the procedure described above.

We claim, that it is a pull-back of a sheaf $\mathcal{S}_E$ on $\mathcal{S}h_n$, which can alternatively be described as the geometric Eisenstein series corresponding to $E_1,...,E_n$ (cf. Definition 1). This is a true statement, but we will not prove it in this paper. We will only identify certain direct summands of the two sheaves (cf. Main Theorem B), and this information would be enough to derive some fundamental properties of the geometric Eisenstein series.[1] This and a new proof of Drinfeld's theorem are the main results of this work.

0.2. Let us now describe the contents of the paper.

In Sect.1.1 of Chapter 1 we introduce, following [9], the geometric Eisenstein series. As an input we use n pairwise non-isomorphic 1-dimensional local systems and $\mathrm{Eis}_{E_1,...,E_n}$ is defined as a direct sum of the sheaves $\mathrm{Eis}^{d_1,...,d_n}_{E_1,...,E_n}$ (cf. Definition 1) where $(d_1, ..., d_n)$ run over the coweights of $GL(n)$.

Main theorem A states that $\mathrm{Eis}^{d_1,...,d_n}_{E_1,...,E_n}$ are irreducible perverse sheaves that satisfy a geometric version of the functional equation. [2]

In Sect.1.2 we will recall the definition of the modified Hecke functors (they were first considered by G.Laumon in [11] and on the level of functions by M.Kapranov in [7]). We will show that the sheaf $\mathrm{Eis}_{E_1,...,E_n}$ satisfies the modified Hecke property with respect to the n-dimensional local system $E_1 \oplus ... \oplus E_n$;

[1] A proof of the more general result mentioned above will be given elsewhere.

[2] It was shown in [9] that on the level of functions $\mathrm{Eis}_{E_1,...,E_n}$ corresponds to the classical Eisenstein series multiplied by the value at 1 of a suitable product of L-functions.

this will imply, in particular, that $\mathrm{Eis}_{E_1,\dots,E_n}$ is a Goresky-MacPherson extension of its restriction to Bun_n.

In Sect.1.3 we will show that any sheaf on Bun_n, whose Goresky-MacPherson extension to $\mathcal{S}h_n$ has the modified Hecke property with respect to an n-dimensional local system, is a Hecke eigen-sheaf with respect to this local system. [3] We will prove also a general statement (Theorem 4) that in order to check that a sheaf on Bun_n is a Hecke eigen-sheaf, it is enough to do so only for the first Hecke functor; the proof of this result uses ideas of Drinfeld's approach to the definition of the convolution product of the Hecke functors (cf. [13]).

In Sect.2.1 of Chapter 2 we will introduce the "fundamental diagram" and formulate Main Theorem B, that identifies for some weights $(d_1, \dots, d_n)$ the pull-back of $\mathrm{Eis}^{d_1,\dots,d_n}_{E_1,\dots,E_n}$ to ${}^0\mathcal{S}h'_n$ with a direct summand of the sheaf $\mathcal{S}'_E$, whose construction we have described above. We will then show how to deduce Main Theorem A from Main Theorem B.

The proof of Main Theorem B occupies Sect.2.2 and 2.3. It is much simpler when $n = 2$, since in that case we can describe explicitly the Goresky-MacPherson extensions $j_{1!*}$ of the pull-backs to ${}^0\mathcal{S}h'_1$ of all the direct summands of $\mathcal{L}_{E_1\oplus\dots\oplus E_n}$.

The main technical difficulty that shows up when $n \geq 3$ is that we cannot avoid considering the Fourier transform functor from the category of sheaves on $\mathcal{S}h'_k$ to that on ${}^0\mathcal{S}h'_{k+1}$ over the locus of $\mathcal{S}h_k$, where the dimensions of the fibers of the vector bundle $\mathcal{S}h'_k$ jump, or equivalently, where the map $\pi^\vee_k : {}^0\mathcal{S}h'_{k+1} \to \mathcal{S}h_k$ is non-representable.

Chapter 3 is devoted to the case of a 2-dimensional geometrically irreducible local system.

In Sect.3.1 we will formulate Main Theorem C and present a "simple" proof of Drinfeld's theorem. The main idea of our approach is that one can deduce that the sheaf $\mathcal{S}'_E$ is a pull-back of some sheaf on $\mathcal{S}h_2$ from the fact that it is irreducible and from the fact that the Euler-Poincaré characteristics of its stalks are constant over a sufficiently big sub-stack of ${}^0\mathcal{S}h'_2$.

To prove the required property of the Euler-Poincaré characteristics of the stalks of $\mathcal{S}'_E$, we show that they are equal to the Euler-Poincaré characteristics of the stalks of the sheaf $\mathcal{S}'_{E_1\oplus E_2}$, where E_1 and E_2 are two arbitrary non-isomorphic

[3] We conjecture that the converse is also true: if a perverse sheaf on Bun_n is a Hecke-eigen-sheaf with respect to a n-dimensional local system E, then its Goresky-MacPherson extension on the whole of $\mathcal{S}h_n$ satisfies the modified Hecke property with respect to E. This statement follows from a conjectural description of the intersection cohomology sheaf on some explicit algebraic stack.

to each other 1-dimensional local systems on X. This is done using Theorem 6 and Proposition 8.

Theorem 6, which is proven in Sect.3.2 generalizes a theorem by Deligne (cf. [2], Appendix) and it is a deep result that reflects the fact, that the sheaf Aut_E corresponding to a geometrically irreducible local system E is cuspidal. [4]

In Sect.3.3 we will complete the proof of Main Theorem C by establishing the Hecke property of the resulting sheaf $\mathrm{Aut}_E := \mathcal{S}_E|_{\mathrm{Bun}_2}$. This is done by combining Theorem 4 with Corollary 4.3 of [11].

0.3 **Generalizations.** We will mention two possible generalizations of the results of this work.

Let G be an arbitrary reductive group and let E be a local system on X with respect to the maximal torus of ${}^LG$. Using Drinfeld's compactification of the stack Bun_B (cf. [3]), we can define the sheaves Eis_E^μ indexed by elements μ of the weight lattice of ${}^LG$.

It follows like in [8], that in the case of $G = GL(n)$ one reproduces the usual Eisenstein series that were discussed above.

In this set-up one can prove the Hecke property of $\bigoplus\limits_{\mu} \mathrm{Eis}_E^\mu$ (cf. [5]); the proof being a generalization of the second proof of Theorem 2 for $n = 2$. Moreover, in this way one reproves the main result (Theorem 4.3) of [13] (the situation here is analogous to the one of Theorem 13.2 of [3]).

Assume now that E is regular, i.e. that $w(E)$ is not isomorphic to E for any element w of the Weyl group. In this case one can prove the functional equation by reducing it to the case of rank=1.

We conjecture by analogy with the case of $GL(n)$, that for E regular, the sheaves Eis_E^μ are perverse and irreducible, but we have no idea at the moment how to prove that.

This will be a subject of a future publication.

Let us return now to the case of $GL(n)$ and let E be an n-dimensional geometrically irreducible local system. In a forthcoming paper by the authors of [4], we will show that to prove the existence of an automorphic sheaf $\mathcal{S}_E$ attached to E, it is sufficient to know a generalization of Deligne's theorem (cf. proof of

[4]In fact, a more general result than Theorem 6 is true: the maps

$$j_{1!} \circ \pi_0^{\vee *}(\mathcal{L}_E) \to j_{1!*} \circ \pi_0^{\vee *}(\mathcal{L}_E) \to j_{1*} \circ \pi_0^{\vee *}(\mathcal{L}_E)$$

are isomorphisms not only over $\mathcal{S}h_1'^{d;\leq 1}$, but over $\mathcal{S}h_1'^{d;\leq t}$ provided that d is large enough comparatively to t. A proof of this fact will appear in a subsequent publication.

Theorem 6) on each step of the fundamental diagram. If we recall the proof of Deligne's theorem given in [2], we shall see that this amounts to knowing the Langlands' correspondence in the opposite direction for $GL(k)$, $k < n$.

0.4 **Conventions.** Throughout the paper we will be working with algebraic stacks in the smooth topology and with perverse sheaves on them (cf. [12]).

For example, when we write: "consider the stack that classifies pairs $M_1 \hookrightarrow M_2$ with M_1 (resp. M_2) being a coherent sheaf on X of generic rank i_1 and of degree d_1 (resp., of gen. rk. i_2 and of degree d_2)", the reader should keep in mind that what we mean is the stack that corresponds to a functor that to any scheme S associates the category, whose objects are pairs $M_1 \to M_2$ of coherent sheaves on $S \times X$ that are S-flat and such that for any closed point $s \in S$ we have: $M_1|_{s\times X} \to M_2|_{s\times X}$ is an embedding and the conditions on the generic rank and on the degree of $M_i|_{s\times X}$, $i = 1, 2$ hold. Morphisms from an object $M_1 \hookrightarrow M_2$ to an object $M_1' \hookrightarrow M_2'$ are by definition isomorphisms $M_1 \to M_1'$ and $M_2 \to M_2'$ such that the corresponding square becomes commutative.

When we write $M \in \mathcal{Y}$, where $\mathcal{Y}$ is some stack, we always mean that M is an object of the corresponding category for an arbitrary parameterizing scheme S.

All stacks in this paper will be unions of open sub-stacks of the form $\mathcal{Y}/\mathcal{G}$, where $\mathcal{Y}$ is a scheme and $\mathcal{G}$ is a smooth group-scheme over a base which is also a scheme. A perverse sheaf on a stack $\mathcal{Y}/\mathcal{G}$ is the same as a perverse sheaf on $\mathcal{Y}$ which is $\mathcal{G}$-equivariant (appropriately shifted).

Throughout the paper the word "sheaf" will mean an object of a suitable derived category (the emphasis is that it is not necessarily perverse). However, the existence of such a derived category will never be used: any complex will appear only through its cohomologies (in the perverse sense).

If $f : \mathcal{Y}_1 \to \mathcal{Y}_2$ is a map between algebraic stacks, we have the functors $f^!$ and f^* defined on the derived category. When f is representable, we have also the functors $f_!$ and f_*. When f is non-representable, the functors $f_!$ and f_* are "sick": they may have an infinite cohomological dimension, etc. We will never use them for non-representable morphisms except in the case when $\mathcal{Y}_1 \simeq \mathcal{Y}_1'/\mathcal{A}$, where the map $\mathcal{Y}_1' \to \mathcal{Y}_2$ is representable and $\mathcal{A}$ is a smooth unipotent group scheme over $\mathcal{Y}_2$ (in this situation there is no problem to define these functors). If, moreover, $\mathcal{Y}_1'$ is itself a smooth unipotent group-scheme (resp., a principle homogeneous space with respect to a smooth unipotent group scheme) over $\mathcal{Y}_2$ and the action of $\mathcal{A}$ on $\mathcal{Y}_1'$ comes from a homomorphism of the corresponding group-schemes, we will say that the map $\mathcal{Y}_1 \to \mathcal{Y}_2$ is a generalized (non-representable) vector (resp., affine) fibration.

For a fixed curve X, we denote by Bun_n (resp., $\mathcal{S}h_n$) the moduli stack of n-bundles on X (resp., the moduli stack of coherent sheaves of generic rank n on X). Both these stacks are smooth. By $\mathcal{S}h_n^d$ we will denote the connected component of the stack $\mathcal{S}h_n$ that corresponds to coherent sheaves of degree d. We will denote by $\mathcal{S}h_n^{d;\leq t}$ (resp., by $\mathcal{S}h_n^{d;t}$) the open (resp., locally closed) sub-stack of $\mathcal{S}h_n^d$ that corresponds to those coherent sheaves whose maximal torsion sub-sheaf has length at most t (exactly t).

For two strings of integers $(d_1, ..., d_k)$ and $(i_1, ..., i_k)$ we will denote by $\mathcal{F}l_{i_1,...,i_k}^{d_1,...,d_k}$ the stack that classifies n-tuples $(0 \hookrightarrow M_1 \hookrightarrow M_2 \hookrightarrow ... \hookrightarrow M_k)$, where M_1,...,M_k are coherent sheaves on X with M_j/M_{j-1} being of generic rank i_j and of degree d_j. We have a map denoted $p_{i_1,...,i_k}^{d_1,...,d_k}$ from $\mathcal{F}l_{i_1,...,i_k}^{d_1,...,d_k}$ to the stack $\mathcal{S}h_{i_1+i_2+...+i_k}^{d_1+d_2+...+d_k}$ that sends an n-tuple as above to $M_k \in \mathcal{S}h_{i_1+i_2+...+i_k}^{d_1+d_2+...+d_k}$. It is easy to see that this map is representable an proper. In addition, we have a map denoted $q_{i_1,...,i_k}^{d_1,...,d_k}$ from $\mathcal{F}l_{i_1,...,i_k}^{d_1,...,d_k}$ to the direct product $\mathcal{S}h_{i_1}^{d_1} \times \mathcal{S}h_{i_2}^{d_2} \times ... \times \mathcal{S}h_{i_k}^{d_k}$ that sends an object $(0 \hookrightarrow M_1 \hookrightarrow M_2 \hookrightarrow ... \hookrightarrow M_k) \in \mathcal{F}l_{i_1,...,i_k}^{d_1,...,d_k}$ to $M_1 \times M_2/M_1 \times ... \times M_k/M_{k-1}$.

It is easy to check that the map $q_{i_1,...,i_k}^{d_1,...,d_k}$ is a generalized (non-representable) affine fibration. This implies, in particular, that the stack $\mathcal{F}l_{i_1,...,i_k}^{d_1,...,d_k}$ is smooth.

We will denote by Pic the Picard of X in the stack-theoretical sense (it identifies to the quotient of the Picard-variety by the trivial action of G_m). By Pic^d we will denote the connected component of Pic that corresponds to line bundles of degree d. We will denote by Ω the canonical line bundle on X and we will choose its square root $\Omega^{\frac{1}{2}} \in \mathrm{Pic}^{g-1}$. We let "det" denote the natural maps $\mathcal{S}h_n \to \mathrm{Pic}$ (note that these maps are smooth for every $n \geq 1$).

In this paper we will work for definiteness over $\mathbb{F}_q$ as the ground field. However, all our discussions apply also in the case of a ground field of char=0 ($\overline{\mathbb{Q}}_\ell$-adic sheaves will have to be replaced by holonomic D-modules).

We will fix a square root of q in $\overline{\mathbb{Q}}_\ell$ and define using it the sheaf $\overline{\mathbb{Q}}_\ell(\frac{1}{2})$ over $\mathrm{Spec}(\mathbb{F}_q)$. The Fourier transform functor is normalized so that it maps perverse sheaves to perverse sheaves and preserves weights.

Chapter 1

Basic properties of Eisenstein series

1.1 Laumon's definition of Eisenstein series

1.1.1

For a partition $n = 1 + 1 + ... + 1$ and a weight $\overline{d^n} = (d_1, ..., d_n) \in \mathbb{Z}^n$ with $\deg(\overline{d^n}) := \sum_i d_i = d$ consider the diagram

$$\begin{array}{ccccc} \mathcal{F}l_{1,...,1}^{d_1,...,d_n} & \xrightarrow{q_{1,...,1}^{d_1,...,d_n}} & \displaystyle\mathop{\times}_{i=1}^{n} \mathcal{S}h_i^{d_i} & \xrightarrow{\det^{\times n}} & \displaystyle\mathop{\times}_{i=1}^{n} \mathrm{Pic}^{d_i} \\ {\scriptstyle p_{1,...,1}^{d_1,...,d_n}}\downarrow & & & & \\ \mathcal{S}h_n^d & & & & \end{array}$$

Let $E_1, E_2, ..., E_n$ be 1-dimensional local systems on X. Throughout the paper we will be assuming that $H^0(X, E_i \otimes E_j^\vee) = 0$ for $i \neq j$.

Following [9], we define the geometric Eisenstein series sheaf $\mathrm{Eis}_{E_1,...,E_n}^{d_1,...,d_n}$ as follows:

Let

$$2\rho = (2\rho_1, ..., 2\rho_n) = (n-1, n-3, ..., 3-n, 1-n) \in \mathbb{Z}^n$$

be the sum of the positive roots of $GL(n)$ and for each i let $E_i^{d_i}$ denote the 1-dimensional local system on Pic^{d_i} that corresponds to E_i via the geometric abelian class field theory.

Definition 1.

$$\begin{aligned}\mathrm{Eis}^{d_1,\ldots,d_n}_{E_1,\ldots,E_n} &:= p^{d_1-(g-1)2\rho_1,\ldots,d_n-(g-1)2\rho_n}_{1,\ldots,1}{}_! \circ p^{d_1-(g-1)2\rho_1,\ldots,d_n-(g-1)2\rho_n}_{1,\ldots,1}{}^* \circ \\ &\circ \det{}^{\times n*}\Big(\boxtimes_{i=1}^{n} E_i^{d_i-(g-1)2\rho_i}\Big) \bigotimes_{i=1}^{n} \big(E_i^{(g-1)2\rho_i}|_{\Omega^{\frac{1}{2}2\rho_i}} \otimes [-2\rho_i\cdot d_i](-\rho_i\cdot d_i)\big) \\ &\otimes (\overline{\mathbb{Q}}_\ell[1](\frac{1}{2}))^{\otimes\frac{(g-1)n(n+1)(2n+1)}{2}},\end{aligned}$$

(we used the fact that $1+2^2+3^2+\ldots+n^2=\frac{n(n+1)(2n+1)}{2}$).

Remark 1. We have introduced the shift by ρ in the definition of Eisenstein series in order to simplify the form of the functional equation. It is analogous to the ρ-shift in the definition of Verma modules.

We will denote by $\mathrm{Eis}_{E_1,\ldots,E_n}$ the direct sum $\bigoplus_{d_1,\ldots,d_n} \mathrm{Eis}^{d_1,\ldots,d_n}_{E_1,\ldots,E_n}$ and by $\mathrm{Eis}^{d}_{E_1,\ldots,E_n}$ the restriction of $\mathrm{Eis}_{E_1,\ldots,E_n}$ to $\mathcal{S}h^d_n$.

Lemma 1. *Let $E_1^\vee,\ldots,E_n^\vee$ be 1-dimensional local systems dual to $E_1,\ldots,E_n$ respectively. Then $\mathbb{D}(\mathrm{Eis}^{d_1,\ldots,d_n}_{E_1,\ldots,E_n}) \simeq \mathrm{Eis}^{d_1,\ldots,d_n}_{E_1^\vee,\ldots,E_n^\vee}$.*

Proof. We only have to show that the cohomological shift and Tate's twist in the definition of Eisenstein series were chosen correctly. This is an easy computation. □

1.1.2

The following theorem is one of our main results.

Main Theorem A

(1) The sheaves $\mathrm{Eis}^{d_1,\ldots,d_n}_{E_1,\ldots,E_n}$ are perverse, irreducible and pairwise non-isomorphic for different $(d_1,\ldots,d_n)$.

(2) The functional equation holds:

Let σ be an element of the Weyl group of $GL(n)$, i.e. a permutation on the set $\{1,\ldots n\}$. Then

$$\mathrm{Eis}^{d_1,\ldots,d_n}_{E_1,\ldots,E_n} \simeq \mathrm{Eis}^{d_{\sigma(1)},\ldots,d_{\sigma(n)}}_{E_{\sigma(1)},\ldots,E_{\sigma(n)}}.$$

The proof of this theorem will be given in Sect. 2.1. In this chapter we will use it to derive some basic properties of the geometric Eisenstein series.

1.1.3

Let $n = n_1 + ... + n_k$ be a partition and let us fix for each i, $1 \leq i \leq k$ a weight $\overline{d^{n_i}} = (d_1^i, ..., d_{n_i}^i)$ of the group $GL(n_i)$ and a collection $E_1^i, ..., E_{n_i}^i$ of 1-dimensional local systems on X.

Let $\overline{d^n} = (\overline{d^{n_1}}, ..., \overline{d^{n_k}})$ be the corresponding weight of $GL(n)$:

$$\overline{d^n} = (d_1^1, ..., d_{n_1}^1, d_1^2, ...d_{n_2}^2, ..., d_1^k, ..., d_{n_k}^k).$$

Consider the correspondence:

$$\begin{array}{ccc} \mathcal{F}l_{n_1,...,n_k}^{\deg(\overline{d^{n_1}}),...,\deg(\overline{d^{n_k}})} & \xrightarrow{q_{n_1,...,n_k}^{\deg(\overline{d^{n_1}}),...,\deg(\overline{d^{n_k}})}} & \times_{i=1}^{k} \mathcal{S}h_{n_i}^{\deg(\overline{d^{n_i}})} \\ p_{n_1,...,n_k}^{\deg(\overline{d^{n_1}}),...,\deg(\overline{d^{n_k}})} \downarrow & & \\ \mathcal{S}h_n^{\deg(\overline{d^n})} & & \end{array}$$

Let $2\rho^{(n_1,...,n_k)}$ be the sum of those positive roots of $GL(n)$ who are contained in the unipotent radical of the parabolic subgroup of $GL(n)$ that corresponds to the partition $n = n_1 + ... + n_k$ as above. It has the form

$$\underbrace{2\rho_1^{(n_1,...,n_k)}, ..., 2\rho_1^{(n_1,...,n_k)}}_{n_1}, \underbrace{2\rho_2^{(n_1,...,n_k)}, ..., 2\rho_2^{(n_1,...,n_k)}}_{n_2}, ..., \underbrace{2\rho_k^{(n_1,...,n_k)}, ..., 2\rho_k^{(n_1,...,n_k)}}_{n_k}.$$

The following statement is almost immediate from the definitions (cf. [9]):

Proposition 1. *We have an isomorphism*

$$\mathrm{Eis}_{E_1^1,...,E_{n_k}^k}^{\overline{d^n}} \simeq p_{n_1,...,n_k\,!}^{\deg(\overline{d^{n_1}}),...,\deg(\overline{d^{n_k}})} \circ q_{n_1,...,n_k}^{\deg(\overline{d^{n_1}}),...,\deg(\overline{d^{n_k}})\,*}\Big(\boxtimes_{i=1}^{k} \mathrm{Eis}_{E_1^i,...,E_{n_i}^i}^{\overline{d^{n_i}}}\Big) \otimes$$
$$\bigotimes_{i=1}^{k} \Big((E_1^i \otimes ... \otimes E_{n_i}^i)^{(g-1)2\rho_i^{(n_1,...,n_k)}}\Big|_{\Omega^{\frac{1}{2}(g-1)2\rho_i^{(n_1,...,n_k)}}}\Big)$$

up to a cohomological shift and Tate's twist that depend only on the integers $n_1, ..., n_k$ and the genus g.

Proof. In the diagram below the square is Cartesian

$$\begin{array}{ccccc}
\mathcal{F}l^{\overline{d^n}}_{1,...,1} & \longrightarrow & \mathop{\times}\limits_{i=1}^{k} \mathcal{F}l^{\overline{d^{n_i}}}_{1,...,1} & \longrightarrow & \mathop{\times}\limits_{i=1}^{k} (\mathop{\times}\limits_{j=1}^{n_i} \mathcal{S}h_1^{d^i_j}) \\
\downarrow & & \Big\downarrow {\scriptstyle \mathop{\times}\limits_{i=1}^{k} p^{\overline{d^{n_i}}}_{1,...,1}} & & \\
\mathcal{F}l^{\deg(\overline{d^{n_1}}),...,\deg(\overline{d^{n_k}})}_{n_1,...,n_k} & \xrightarrow{q^{\deg(\overline{d^{n_1}}),...,\deg(\overline{d^{n_k}})}_{n_1,...,n_k}} & \mathop{\times}\limits_{i=1}^{k} \mathcal{S}h^{\deg(\overline{d^{n_i}})}_{n_i} & & \\
{\scriptstyle p^{\deg(\overline{d^{n_1}}),...,\deg(\overline{d^{n_k}})}_{n_1,...,n_k}} \Big\downarrow & & & & \\
\mathcal{S}h^{\deg(\overline{d^n})}_n & & & &
\end{array}$$

and the compositions $\mathcal{F}l^{\overline{d^n}}_{1,...,1} \to \mathcal{S}h^{\deg(\overline{d^n})}_n$ and $\mathcal{F}l^{\overline{d^n}}_{1,...,1} \to \mathcal{S}h_1^{d^1_1} \times ... \times \mathcal{S}h_1^{d^k_{n_k}}$ that result from the diagram coincide with the maps $p^{\overline{d^n}}_{1,...,1}$ and $q^{\overline{d^n}}_{1,...,1}$ respectively.

This implies the assertion, since $\rho = \rho^{(n_1,...,n_k)} + (\rho(GL(n_1)), ..., \rho(GL(n_k)))$. □

Corollary 1. *The assertion of Main Theorem A(2) for $n = 2$ implies Main Theorem A(2) for all n.*

Proof. It is enough to prove the assertion of Main Theorem A(2) for σ being a simple reflection. So, let us assume that σ acts as a transposition of two elements $(i, i+1) \in (1, ..., n)$.

Consider the partition $n = (i-1) + 2 + (n-i-1)$ and let $E^1_1, ..., E^1_{i-1}$ be equal to $E_1, ..., E_{i-1}$ correspondingly, $E^2_1 = E_i$, $E^2_2 = E_{i+1}$, $E^3_j = E_{i+1+j}$ for $1 \le j \le n-i-1$. Let us also define $\overline{d^{i-1}} = (d_1, ..., d_{i-1})$, $\overline{d^{n-i-1}} = (d_{i+2}, ..., d_n)$.

According to Proposition 1, $\mathrm{Eis}^{\overline{d^n}}_{E_1,...E_i,E_{i+1},...,E_n}$ and $\mathrm{Eis}^{\overline{d^n}}_{E_1,...E_{i+1},E_i,...,E_n}$ identify with the images under the functor

$$p^{\deg(\overline{d^{i-1}}),d_i+d_{i+1},\deg(\overline{d^{n-i-1}})}_{i-1,2,n-i-1}{}_! \circ q^{\deg(\overline{d^{i-1}}),d_i+d_{i+1},\deg(\overline{d^{n-i-1}})*}_{i-1,2,n-i-1}$$

of the sheaves

$$(\mathrm{Eis}^{\overline{d^{i-1}}}_{E_1,...,E_{i-1}} \boxtimes \mathrm{Eis}^{(d_i,d_{i+1})}_{E_i,E_{i+1}} \boxtimes \mathrm{Eis}^{\overline{d^{n-i-1}}}_{E_{i+2},...,E_n})$$

and

$$(\mathrm{Eis}^{\overline{d^{i-1}}}_{E_1,...,E_{i-1}} \boxtimes \mathrm{Eis}^{(d_{i+1},d_i)}_{E_{i+1},E_i} \boxtimes \mathrm{Eis}^{\overline{d^{n-i-1}}}_{E_{i+2},...,E_n})$$

respectively, up to tensoring with a 1-dimensional vector space, which is canonically the same in both cases.

Now, by Main Theorem A(2) for $n = 2$ we have:

$$\mathrm{Eis}^{(d_{i+1},d_i)}_{E_{i+1},E_i} \simeq \mathrm{Eis}^{(d_i,d_{i+1})}_{E_i,E_{i+1}},$$

which implies the assertion of Main Theorem A(2).

□

1.1.4

Assuming Main Theorem A(2) (for $n = 2$) we obtain the following local finiteness property of $\mathrm{Eis}_{E_1,\ldots,E_n}$:

Corollary 2. *The stack $\mathcal{S}h^d_n$ is a union of open sub-stacks, such that over each of them only finitely many of the $Eis^{d_1,\ldots,d_n}_{E_1,\ldots,E_n}$ are non-zero.*

Proof. For an integer d' consider the open sub-stack in $\mathcal{S}h^d_n$ that corresponds to those sheaves $M \in \mathcal{S}h^d_n$ who admit no coherent sub-sheaves of generic rank 1 of degree $\geq d' - \rho_1$. We claim that $\mathrm{Eis}^{d_1,\ldots,d_n}_{E_1,\ldots,E_n}$ vanishes on this sub-stack if $d_i \geq d'$ for some i.

Indeed, $\mathrm{Eis}^{d_1,\ldots,d_n}_{E_1,\ldots,E_n}$ clearly vanishes if $d_1 \geq d'$, since the preimage of our sub-stack in $\mathcal{F}l^{d_1-\rho_1,\ldots,d_n-\rho_n}_{1,\ldots,1}$ is empty in this case. The assertion for other i's follows from the functional equation (Main Theorem A(2)).

□

1.1.5

Consider the map $m : \mathcal{S}h^d_n \times \mathrm{Pic}^c \to \mathcal{S}h^{d+nc}_n$ given by

$$M \times L \to M \otimes L.$$

For $L_0 \in \mathrm{Pic}^c$ we will denote by m_{L_0} the restriction of m to $\mathcal{S}h^d_n \times L_0$.

The next assertion is an immediate consequence of the definitions:

Lemma 2. *We have:*

$$m^*(\mathrm{Eis}^{\overline{d^n}}_{E_1,\ldots,E_n}) \simeq \mathrm{Eis}^{\overline{d^n}-c(1,\ldots,1)}_{E_1,\ldots,E_n} \boxtimes (\bigotimes_{i=1}^{n} E_i)^c$$

1.2 The modified Hecke property of Eisenstein series

1.2.1

Consider the open sub-stacks ${}^{rss}\mathcal{S}h_0^c \subset^r \mathcal{S}h_0^c$ of $\mathcal{S}h_0^c$ that correspond to regular semi-simple and regular sheaves respectively.

We have a smooth map $X^{(c)} \to {}^r\mathcal{S}h_0^c$ and ${}^{rss}\mathcal{S}h_0^c$ identifies with $X^{(c)} - \Delta/G_m^c$, where Δ is the divisor of coinciding points.

We will denote by ${}^{rss}\mathcal{F}l^{\overbrace{1,\dots,1}^{c}}_{\underbrace{0,\dots,0}_{c}}$ the preimage of ${}^{rss}\mathcal{S}h_0^c$ in $\mathcal{F}l_{0,\dots,0}^{1,\dots,1}$. The restriction of the map $p_{0,\dots,0}^{1,\dots,1}$ to ${}^{rss}\mathcal{F}l_{0,\dots,0}^{1,\dots,1}$ is an S^c-Galois étale cover and we have a Cartesian square:

$$\begin{array}{ccc} X^c - \Delta & \longrightarrow & {}^{rss}\mathcal{F}l_{0,\dots,0}^{1,\dots,1} \\ {\scriptstyle \mathrm{sym}^c}\downarrow & & {\scriptstyle p_{0,\dots,0}^{1,\dots,1}}\downarrow \\ X^{(c)} - \Delta & \longrightarrow & {}^{rss}\mathcal{S}h_0^c \end{array}$$

(here sym^c denotes the natural map $X^c \to X^{(c)}$.

For a local system E on X we define Springer's sheaf $\mathcal{S}pr_E^c$ on $\mathcal{S}h_0^c$:

Definition 2.

$$\mathcal{S}pr_E^c := p_{0,\dots,0!}^{1,\dots,1} \circ q_{0,\dots,0}^{1,\dots,1*} \circ \det{}^{\times c\,*}(E \boxtimes \dots \boxtimes E),$$

where we have denoted by E the corresponding sheaf on $\mathcal{S}h_0^1 \simeq X/G_m$.

When E is the trivial 1-dimensional local system on X, we will denote the corresponding Springer sheaf simply by $\mathcal{S}pr$.

Lemma 3. (1) *The sheaf $\mathcal{S}pr_E^c$ is perverse and it coincides with the Goresky-MacPherson extension of its restriction to ${}^{rss}\mathcal{S}h_0^c$.*

(2) *The sheaf $\mathcal{S}pr_E^c$ carries a natural S^c-action.*

Proof. The first point follows from the fact that the map $p_{0,\dots,0}^{1,\dots,1}$ is small.

Consider now the pull-back of $\mathcal{S}pr_E^c$ to $X^{(c)} - \Delta$. It identifies to the sheaf $\mathrm{sym}_!^c(E^{\boxtimes c})|_{X^{(c)}-\Delta}$. According to the first point of the lemma, it is enough to show that $\mathrm{sym}_!^c(E^{\boxtimes c})|_{X^{(c)}-\Delta}$ carries a natural S^c-action, but this follows from the fact that the sheaf $E^{\boxtimes c}$ on X^c is S^c-equivariant.

□

Definition 3. For a local system E on X, we define Laumon's perverse sheaf $\mathcal{L}_E^c$ on $\mathcal{S}h_0^c$ as $\mathrm{Hom}_{S^c}(\mathrm{triv}, \mathcal{S}pr_E^c)$, where "triv" denotes the trivial representation of the symmetric group S^c.

It is easy to see, that the pull-back of $\mathcal{L}_E^c$ to $X^{(c)}$ identifies with the sheaf $E^{(c)}$ and Lemma 3(1) implies that $\mathcal{L}_E^c$ is the Goresky-MacPherson extension of its restriction to ${}^{rss}\mathcal{S}h_0^c$. Therefore, the sheaf $\mathcal{L}_E^c$ is irreducible (resp., semi-simple) if E is irreducible (resp., semi-simple).

1.2.2

Fix two non-negative integers c_1, c_2 and consider the diagram:

$$\begin{array}{ccc} \mathcal{F}l_{0,0}^{c_1,c_2} & \xrightarrow{q_{0,0}^{c_1,c_2}} & \mathcal{S}h_0^{c_1} \times \mathcal{S}h_0^{c_2} \\ {\scriptstyle p_{0,0}^{c_1,c_2}} \downarrow & & \\ \mathcal{S}h_0^{c_1+c_2} & & \end{array}$$

Given two local systems E' and E'' on X we define the sheaf $\mathcal{L}_{E',E''}^{c_1,c_2}$ on $\mathcal{S}h_0^{c_1+c_2}$ as

$$p_{0,0}^{c_1,c_2}{}_! \circ q_{0,0}^{c_1,c_2 *}(\mathcal{L}_{E'}^{c_1} \boxtimes \mathcal{L}_{E''}^{c_2}).$$

The basic properties of the sheaves $\mathcal{L}_E^c$ and $\mathcal{L}_{E',E''}^{c_1,c_2}$ are summarized in the following proposition:

Proposition 2. (1) *If E is 1-dimensional,* $\mathcal{L}_E^c \simeq \det^*(E^c)$.
(2) $q_{0,0}^{c_1,c_2}{}_! \circ p_{0,0}^{c_1,c_2 *}(\mathcal{L}_E^{c_1+c_2}) \simeq \mathcal{L}_E^{c_1} \boxtimes \mathcal{L}_E^{c_2}$.
(3) *Let E be a direct sum $E = E' \oplus E''$. Then* $\mathcal{L}_E^c = \bigoplus_{0 \leq c_1 \leq c} \mathcal{L}_{E',E''}^{c_1,c-c_1}$
(4) $\mathcal{L}_{E',E''}^{c_1,c_2} \simeq \mathcal{L}_{E'',E'}^{c_2,c_1}$
(5) *Let E''' be another local system on X. Then for $c_1 + c_2 = c$:*

$$\mathrm{Hom}_{\mathcal{S}h_0^c}(\mathcal{L}_{E'''}^c, \mathcal{L}_{E',E''}^{c_1,c_2}) \simeq \mathrm{Hom}_{\mathcal{S}h_0^{c_1}}(\mathcal{L}_{E'''}^{c_1}, \mathcal{L}_{E'}^{c_1}) \otimes \mathrm{Hom}_{\mathcal{S}h_0^{c_2}}(\mathcal{L}_{E'''}^{c_2}, \mathcal{L}_{E''}^{c_2}).$$

Proof. The first point is obvious and the second one is Theorem 4.1 in [11].

The map $p_{0,0}^{c_1,c_2}$ is a small map, which is finite over ${}^{rss}\mathcal{S}h_0^c$. Therefore, each $\mathcal{L}_{E',E''}^{c_1,c_2}$ is a Goresky-MacPherson extension of its restriction to ${}^{rss}\mathcal{S}h_0^c$ and it is enough to prove points (3) and (4) of the proposition for the pull-backs of the corresponding sheaves to $X^{(c)} - \Delta$.

We have a Cartesian square

$$\begin{array}{ccc} X^{(c_1)} \times X^{(c_2)} - \Delta & \longrightarrow & {}^{rss}\mathcal{F}l_{0,0}^{c_1,c_2} \\ \downarrow & & \downarrow \\ X^{(c_1+c_2)} - \Delta & \longrightarrow & {}^{rss}\mathcal{S}h_0^c \end{array}$$

The pull-back of $q_{0,0}^{c_1,c_2 *}(\mathcal{L}_{E'}^{c_1} \boxtimes \mathcal{L}_{E''}^{c_2})$ to $X^{(c_1)} \times X^{(c_2)} - \Delta$ identifies with $E'^{c_1} \boxtimes E''^{c_2}$.

This makes the assertion of point (4) obvious and point (3) follows from the fact that the sheaf $(E' \oplus E'')^c$ on $X^{(c)}$ identifies with the direct sum over c_1 between 0 and c of the direct images of the sheaves $E'^{c_1} \boxtimes E''^{c-c_1}$ on $X^{(c_1)} \times X^{(c-c_1)}$.

The fifth point follows from the second and the third one by adjointness (we will use the facts that the map $p_{0,0}^{c_1,c_2}$ is proper and that the map $q_{0,0}^{c_1,c_2}$ is smooth):

$$\mathrm{Hom}_{\mathcal{S}h_0^c}(\mathcal{L}_{E'''}^c, \mathcal{L}_{E',E''}^{c_1,c_2}) \simeq \mathrm{Hom}_{\mathcal{F}l_{0,0}^{c_1,c_2}}(p_{0,0}^{c_1,c_2 *}(\mathcal{L}_{E'''}^c), q_{0,0}^{c_1,c_2 *}(\mathcal{L}_{E'}^{c_1}) \boxtimes \mathcal{L}_{E''}^{c_2})) \simeq$$
$$\mathrm{Hom}_{\mathcal{S}h_0^{c_1} \times \mathcal{S}h_0^{c_2}}(q_{0,0}^{c_1,c_2}{}_! \circ p_{0,0}^{c_1,c_2 *}(\mathcal{L}_{E'''}^c), \mathcal{L}_{E'}^{c_1} \boxtimes \mathcal{L}_{E''}^{c_2}) \simeq$$
$$\mathrm{Hom}_{\mathcal{S}h_0^{c_1} \times \mathcal{S}h_0^{c_2}}(\mathcal{L}_{E'''}^{c_1} \boxtimes \mathcal{L}_{E'''}^{c_2}, \mathcal{L}_{E'}^{c_1} \boxtimes \mathcal{L}_{E''}^{c_2}) \simeq \mathrm{Hom}_{\mathcal{S}h_0^{c_1}}(\mathcal{L}_{E'''}^{c_1}, \mathcal{L}_{E'}^{c_1}) \otimes \mathrm{Hom}_{\mathcal{S}h_0^{c_2}}(\mathcal{L}_{E'''}^{c_2}, \mathcal{L}_{E''}^{c_2})$$

□

Corollary 3. *Assume that E is a sum of 1-dimensional local systems E_i. Then*

$$\mathcal{L}_E^c \simeq \bigoplus_{c_1+\ldots+c_n=c} \mathcal{L}_{E_1,\ldots E_n}^{c_1,\ldots,c_n},$$

where

$$\mathcal{L}_{E_1,\ldots,E_n}^{\overline{c^n}} := p_{0,\ldots,0}^{c_1,\ldots,c_n}{}_! \circ q_{0,\ldots,0}^{c_1,\ldots,c_n *} \det{}^{\times n}(E_1^{c_1} \boxtimes \ldots \boxtimes E_n^{c_n}).$$

Moreover, the perverse sheaves $\mathcal{L}_E^{c_1,\ldots,c_n}$ are irreducible and pairwise non-isomorphic provided that $\mathrm{Hom}(E_i, E_j) = 0$ *for $i \neq j$.*

1.2.3

For $c \geq 0$ consider the correspondence:

$$\begin{array}{ccc} \mathcal{F}l_{n,0}^{d-c,c} & \xrightarrow{p_{n,0}^{d-c,c}} & \mathcal{S}h_n^d \\ {\scriptstyle q_{n,0}^{d-c,c}} \downarrow & & \\ \mathcal{S}h_n^{d-c} \times \mathcal{S}h_0^c & & \end{array}$$

We have two functors

$$\mathcal{K} \to q_{n,0}^{d-c,c}{}_! \circ p_{n,0}^{d-c,c*}(\mathcal{K})[c(n+1)](\frac{c(n+1)}{2}) \text{ and}$$

$$\mathcal{K} \to q_{n,0}^{d-c,c}{}_* \circ p_{n,0}^{d-c,c!}(\mathcal{K})[-c(n+1)](-\frac{c(n+1)}{2})$$

from the category of sheaves on $\mathcal{S}h_n^d$ to that on $\mathcal{S}h_n^{d-c} \times \mathcal{S}h_0^c$. We shall call them the modified Hecke functors.

Theorem 1.

$$q_{n,0}^{d-c,c}{}_! \circ p_{n,0}^{d-c,c*}(\mathrm{Eis}_{E_1,\dots,E_n}^{\overline{d^n}})[c(n+1)](\frac{c(n+1)}{2}) \simeq \bigoplus_{\overline{c^n}} \mathrm{Eis}_{E_1,\dots,E_n}^{\overline{d^n}-\overline{c^n}} \boxtimes \mathcal{L}_{E_1,\dots,E_n}^{\overline{c^n}} \simeq$$

$$\simeq q_{n,0}^{d-c,c}{}_* \circ p_{n,0}^{d-c,c!}(\mathrm{Eis}_{E_1,\dots,E_n}^{\overline{d^n}})[-c(n+1)](-\frac{c(n+1)}{2}),$$

where $\overline{c^n}$ runs over n-tuples $(c_1,\dots,c_n)$ that satisfy $0 \le c_i \le d_i$.

Proof. Let d be $\deg(\overline{d^n}) = \deg(\overline{d^n} - (g-1)2\rho)$ and consider the Cartesian product of $\mathcal{F}l_{1,\dots,1}^{\overline{d^n}-(g-1)2\rho}$ with $\mathcal{F}l_{n,0}^{d-c,c}$ over $\mathcal{S}h_n^d$:

$$\begin{array}{ccc}
\mathcal{F}l_{n,0}^{d-c,c} \underset{\mathcal{S}h_n^d}{\times} \mathcal{F}l_{1,\dots,1}^{\overline{d^n}-(g-1)2\rho} & \xrightarrow{'p_{n,0}^{d-c,c}} & \mathcal{F}l_{1,\dots,1}^{\overline{d^n}-(g-1)2\rho} \\
{}'p_{1,\dots,1}^{\overline{d^n}-(g-1)2\rho} \downarrow & & p_{1,\dots,1}^{\overline{d^n}-(g-1)2\rho} \downarrow \\
\mathcal{F}l_{n,0}^{d-c,c} & \xrightarrow{p_{n,0}^{d-c,c}} & \mathcal{S}h_n^d \\
q_{n,0}^{d-c,c} \downarrow & & \\
\mathcal{S}h_n^{d-c} \times \mathcal{S}h_0^c & &
\end{array}$$

The sheaf $q_{n,0}^{d-c,c}{}_! \circ p_{n,0}^{d-c,c*}(\mathrm{Eis}_{E_1,\dots,E_n}^{\overline{d^n}})$ identifies with

$$q_{n,0}^{d-c,c}{}_! \circ {}'p_{1,\dots,1}^{\overline{d^n}-(g-1)2\rho}{}_! \circ {}'p_{n,0}^{d-c,c*} \circ (\det{}^{\times n})^*(E_1^{d_1-(g-1)2\rho_1} \boxtimes \dots \boxtimes E_n^{d_n-(g-1)2\rho_n})$$

tensored by

$$\bigotimes_{i=1}^{n} (E_i^{(g-1)2\rho_i}|_{\Omega^{\frac{1}{2}2\rho_i}}[-2\rho_i \cdot d_i](-\rho_i \cdot d_i)) \otimes (\overline{\mathbb{Q}}_\ell[1](\frac{1}{2}))^{\otimes \frac{(g-1)n(n+1)(2n+1)}{2}}.$$

The stack $\mathcal{F}l_{n,0}^{d-c,c} \underset{\mathcal{S}h_n^d}{\times} Fl_{1,\dots,1}^{\overline{d^n}-(g-1)2\rho}$ admits a stratification by locally closed substacks $U_{\overline{c^n}}$ parameterized by n-tuples $\overline{c^n}$ with $0 \leq c_i \leq d_i$:

We say that a point

$$(0 \to M \to M' \to T \to 0, M_1' \hookrightarrow \dots \hookrightarrow M_n' = M')$$

is a point of $U_{\overline{c^n}}$ if the filtration $T_1 \hookrightarrow \dots \hookrightarrow T_n = T$ on T given by the images of M_i''s satisfies:

$$\text{length}(T_i/T_{i-1}) = c_i.$$

We have $U_{\overline{c^n}'} \subset \overline{U_{\overline{c^n}}}$ if $\Sigma_{j=1}^i c_j' \leq \Sigma_{j=1}^i c_j$ for each $1 \leq i \leq n-1$.

We denote by $g_{\overline{c^n}}$ the restriction of the composition $q_{n,0}^{d-c,c} \circ {}'p_{1,\dots,1}^{\overline{d^n}-(g-1)2\rho}$ to $U_{\overline{c^n}}$ and by $\mathcal{T}_{\overline{c^n}}$ the restriction to $U_{\overline{c^n}}$ of the sheaf

$${}'p_{2,0}^{d-c,c*} \circ (\det{}^{\times n})^*(E_1^{d_1-(g-1)2\rho_1} \boxtimes \dots \boxtimes E_n^{d_n-(g-1)2\rho_n})$$

tensored by $\overset{n}{\underset{i=1}{\otimes}} (E_i^{(g-1)2\rho_i}|_{\Omega^{\frac{1}{2}2\rho_i}}[-2\rho_i \cdot d_i](-\rho_i \cdot d_i)) \otimes (\overline{\mathbb{Q}}_\ell[1](\frac{1}{2}))^{\otimes \frac{(g-1)n(n+1)(2n+1)}{2}}$.

Lemma 4.

$$g_{\overline{c^n}!}(\mathcal{T}_{\overline{c^n}}) \simeq \text{Eis}_{E_1,\dots,E_n}^{\overline{d^n}-\overline{c^n}} \boxtimes \mathcal{L}_{E_1,\dots,E_n}^{\overline{c^n}}[-c(n+1)](-\frac{c(n+1)}{2}).$$

Proof of the lemma

We have a map

$$\widetilde{g}_{\overline{c^n}} : U_{\overline{c^n}} \to \mathcal{F}l_{1,\dots,1}^{\overline{d^n}-\overline{c^n}-(g-1)2\rho} \times \mathcal{F}l_{0,\dots 0}^{\overline{c^n}}$$

that sends a pair

$$(0 \to M \to M' \to T \to 0, M_1' \hookrightarrow \dots \hookrightarrow M_n' = M')$$

as above to

$$(M \cap M_1' \hookrightarrow M \cap M_2' \hookrightarrow \dots \hookrightarrow M, 0 \hookrightarrow T_1 \hookrightarrow \dots \hookrightarrow T_n = T).$$

It is easy to prove by induction that this map is a (non-representable) generalized vector fibration of relative dimension $c_1 + 2c_2 + \dots + nc_n$.

Note, that

$$c_1 + 2c_2 + \dots + nc_n = \frac{c(n+1) - (2\rho, \overline{c^n})}{2}.$$

We have $g_{\overline{c^n}} = (p_{1,\dots,1}^{\overline{d^n}-(g-1)2\rho} \times p_{0,\dots,0}^{\overline{c^n}}) \circ \widetilde{g}_{\overline{c^n}}$.
Moreover, $\mathcal{T}_{\overline{c^n}}$ identifies with

$$\widetilde{g}_{\overline{c^n}}{}^* \circ (q_{1,\dots,1}^{d_1-(g-1)2\rho_1,\dots,d_n-(g-1)2\rho_n} \times q_{0,\dots,0}^{c_1,\dots,c_n})^*$$
$$(\det{}^{\times n *}(E_1^{d_1-(g-1)2\rho_1} \boxtimes \dots \boxtimes E_n^{d_n-(g-1)2\rho_n}) \boxtimes \det{}^{\times n *}(E_1^{c_1} \boxtimes \dots \boxtimes E_n^{c_n}))$$

tensored by

$$\bigotimes_{i=1}^{n} (E_i^{(g-1)2\rho_i}|_{\Omega^{\frac{1}{2}2\rho_i}}[-2\rho_i \cdot d_i](-\rho_i \cdot d_i)) \otimes (\overline{\mathbb{Q}}_\ell[1](\frac{1}{2}))^{\otimes \frac{(g-1)n(n+1)(2n+1)}{2}}.$$

Therefore, by the projection formula,

$$g_{\overline{c^n}!}(\mathcal{T}_{\overline{c^n}}) \simeq p_{1,\dots,1}^{\overline{d^n}-(g-1)2\rho}{}_! \circ \det{}^{\times n *}(E_1^{d_1-(g-1)2\rho_1} \boxtimes \dots \boxtimes E_n^{d_n-(g-1)2\rho_n}) \boxtimes$$
$$\boxtimes\, p_{0,\dots,0!}^{\overline{c^n}} \circ \det{}^{\times n *}(E_1^{c_1} \boxtimes \dots \boxtimes E_n^{c_n})[-2\frac{c(n+1)-(2\rho,\overline{c^n})}{2}](-\frac{c(n+1)-(2\rho,\overline{c^n})}{2}),$$

tensored by

$$\bigotimes_{i=1}^{n} (E_i^{(g-1)2\rho_i}|_{\Omega^{\frac{1}{2}2\rho_i}}[-2\rho_i \cdot d_i](-\rho_i \cdot d_i)) \otimes (\overline{\mathbb{Q}}_\ell[1](\frac{1}{2}))^{\otimes \frac{(g-1)n(n+1)(2n+1)}{2}},$$

which identifies with $\mathrm{Eis}_{E_1,\dots,E_n}^{\overline{d^n}-\overline{c^n}} \boxtimes \mathcal{L}_{E_1,\dots,E_n}^{\overline{c^n}}[-c(n+1)](\frac{-c(n+1)}{2})$.
□(Lemma)

The fact that all the sheaves $\mathrm{Eis}_{E_1,\dots,E_n}^{\overline{d^n}-\overline{c^n}} \boxtimes \mathcal{L}_{E_1,\dots,E_n}^{\overline{c^n}}$ on $\mathcal{S}h_n^{d-c} \times \mathcal{S}h_0^c$ are perverse (Main Theorem A(1)) implies the degeneration of the Cousin spectral sequence that computes the direct image

$$q_{n,0}^{d-c,c}{}_! \circ {}'p_{1,\dots,1}^{\overline{d^n}-(g-1)2\rho}{}_! \circ {}'p_{2,0}^{d-c,c*} \circ (\det{}^{\times n})^*(E_1^{d_1} \boxtimes \dots \boxtimes E_n^{d_n})$$

with respect to the stratification by the $U_{\overline{c^n}}$'s.

Therefore, we get an increasing filtration on $q_{n,0}^{d-c,c}{}_! \circ p_{n,0}^{d-c,c*}(\mathrm{Eis}_{E_1,\dots,E_n}^{\overline{d^n}})[c(n+1)](\frac{c(n+1)}{2})$ parameterized by the partially ordered set of $\overline{c^n}$'c (the order is that of adjunction of the $U_{\overline{c^n}}$'s), with successive quotients being $\mathrm{Eis}_{E_1,\dots,E_n}^{\overline{d^n}-\overline{c^n}} \boxtimes \mathcal{L}_{E_1,\dots,E_n}^{\overline{c^n}}$.

However, we can consider $q_{n,0}^{d-c,c}{}_! \circ p_{n,0}^{d-c,c*}(\mathrm{Eis}_{E_n,E_{n-1},\dots,E_1}^{d_n,d_{n-1},\dots,d_1})[c(n+1)](\frac{c(n+1)}{2})$, which on the one hand is isomorphic to $q_{n,0}^{d-c,c}{}_! \circ p_{n,0}^{d-c,c*}(\mathrm{Eis}_{E_1,\dots,E_n}^{\overline{d^n}})[c(n+1)](\frac{c(n+1)}{2})$, according to Main Theorem A(2) and to Proposition 2(4), and on the other hand acquires an decreasing filtration with the same successive quotients.

Since these successive quotients are irreducible and pairwise non-isomorphic (Main Theorem A(1)), the above filtrations degenerate canonically into direct sums.

This establishes the isomorphism

$$q_{n,0}^{d-c,c}{}_{!} \circ p_{n,0}^{d-c,c*}(\mathrm{Eis}^{\overline{d^n}}_{E_1,\dots,E_n})[c(n+1)](\frac{c(n+1)}{2}) \simeq \bigoplus_{\overline{c^n}} \mathrm{Eis}^{\overline{d^n}}_{E_1,\dots,E_n} \boxtimes \mathcal{L}^{\overline{c^n}}_{E_1,\dots,E_n}.$$

The second isomorphism stated in the theorem follows by applying the Verdier duality. □

Corollary 4. *We have:*

$$q_{n,0}^{d-c,c}{}_{!} \circ p_{n,0}^{d-c,c*}(\mathrm{Eis}^{d}_{E_1,\dots,E_n})[c(n+1)](\frac{c(n+1)}{2}) \simeq \mathrm{Eis}^{d-c}_{E_1\oplus\dots\oplus E_n} \boxtimes \mathcal{L}^{c}_{E_1\oplus\dots\oplus E_n}.$$

1.2.4

Proposition 3. *Each* $\mathrm{Eis}^{d_1,\dots,d_n}_{E_1,\dots,E_n}$ *is a Goresky-MacPherson of its restriction to* Bun_n.

Proof. Since $\mathbb{D}(\mathrm{Eis}^{d_1,\dots,d_n}_{E_1,\dots,E_n}) \simeq \mathrm{Eis}^{d_1,\dots,d_n}_{E_1^\vee,\dots,E_n^\vee}$, it is enough to prove that the restriction of $\mathrm{Eis}^{d_1,\dots,d_n}_{E_1,\dots,E_n}$ to $\mathcal{S}h_n^{d;c}$ lives in cohomological degrees < 0 for $c > 0$.

Let us denote temporarily by V the preimage in $\mathcal{F}l_{n,0}^{d-c,c}$ of $\mathrm{Bun}_n^{d-c} \times \mathcal{S}h_0^c$ under the projection $q_{n,0}^{d-c,c}$.

Obviously, V over $\mathrm{Bun}_n^{d-c} \times \mathcal{S}h_0^c$ is a smooth representable vector bundle and let $\mathrm{Bun}_n^{d-c} \times \mathcal{S}h_0^c \simeq V_0 \subset V$ denote the zero-section.

We have a Cartesian square:

$$\begin{array}{ccc} V_0 & \longrightarrow & V \\ \downarrow & & \downarrow{\scriptstyle p_{n,0}^{d-c,c}} \\ \mathcal{S}h_n^{d;c} & \longrightarrow & \mathcal{S}h_n^d \end{array}$$

The restriction of the map $p_{n,0}^{d-c,c} : \mathcal{F}l_{n,0}^{d-c,c} \to \mathcal{S}h_n^d$ to V is smooth of relative dimension $n \cdot c$. Therefore, it is enough to show, that $p_{n,0}^{d-c,c!}(\mathrm{Eis}^{d_1,\dots,d_n}_{E_1,\dots,E_n})[-n \cdot c]|_{V_0}$ is concentrated in cohomological dimensions < 0.

However, since $p_{n,0}^{d-c,c!}(\mathrm{Eis}^{d_1,\dots,d_n}_{E_1,\dots,E_n})|_V$ is a G_m-equivariant sheaf on a vector bundle,

$$p_{n,0}^{d-c,c!}(\mathrm{Eis}^{d_1,\dots,d_n}_{E_1,\dots,E_n})|_{V_0} \simeq q_{n,0}^{d-c,c}{}_{*}(p_{n,0}^{d-c,c!}(\mathrm{Eis}^{d_1,\dots,d_n}_{E_1,\dots,E_n})|_V),$$

the latter by Theorem 1 identifies with

$$\bigoplus_{\overline{c^n}} \mathrm{Eis}^{\overline{d^n}-\overline{c^n}}_{E_1,\dots,E_n} \boxtimes \mathcal{L}^{\overline{c^n}}_{E_1,\dots,E_n}[(n+1)c],$$

up to Tate's twist.

Therefore,

$$p^{d-c,c!}_{n,0}(\mathrm{Eis}^{d_1,\dots,d_n}_{E_1,\dots,E_n})[-n\cdot c]|_{V_0} \simeq \bigoplus_{\overline{c^n}} \mathrm{Eis}^{\overline{d^n}-\overline{c^n}}_{E_1,\dots,E_n} \boxtimes \mathcal{L}^{\overline{c^n}}_{E_1,\dots,E_n}[c],$$

up to Tate's twist, which implies the assertion.

□

In the course of the proof of Proposition 3 we have obtained the following result:

Corollary 5. *The pull-back of* $\mathrm{Eis}^{d_1,\dots,d_n}_{E_1,\dots,E_n}|_{\mathcal{S}h^{d;c}_n}$ *under the natural map*

$$\mathcal{S}h^{d-c}_n \times \mathcal{S}h^c_0 \to \mathcal{S}h^{d;c}_n$$

identifies with $\bigoplus_{\overline{c^n}} \mathrm{Eis}^{\overline{d^n}-\overline{c^n}}_{E_1,\dots,E_n} \boxtimes \mathcal{L}^{\overline{c^n}}_{E_1,\dots,E_n}[c(1-n)](\frac{c(n-1)}{2})$.

1.3 The Hecke property of Eisenstein series

1.3.1

Let us denote by the same character the restriction of the perverse sheaf $\mathrm{Eis}^{d_1,\dots,d_n}_{E_1,\dots,E_n}$ to $\mathrm{Bun}_n \subset \mathcal{S}h_n$. We will deduce from Theorem 1 the usual Hecke-property of the sheaves $\mathrm{Eis}^{\overline{d^n}}_{E_1,\dots,E_n}$ on Bun_n.

Let Mod^1_n be the stack that classifies triples $(x \in X, M \hookrightarrow M')$, where M and M' are two n-bundles on X with M'/M being a torsion sheaf of length 1 supported at x.

We have the maps $\mathfrak{p}, \mathfrak{q} : \mathrm{Mod}^1_n \to \mathrm{Bun}_n$ that send a triple as above to M' and M respectively and a map $\mathrm{supp} : \mathrm{Mod}^1_n \to X$ that sends $(x \in X, M \hookrightarrow M')$ to x.

We have a correspondence:

$$\begin{array}{ccc} \mathrm{Mod}^1_n & \xrightarrow{\mathfrak{p}} & \mathrm{Bun}_n \\ {\scriptstyle \mathfrak{q}\times\mathrm{supp}}\downarrow & & \\ \mathrm{Bun}_n \times X & & \end{array}$$

The functor $(\mathfrak{q} \times \text{supp})_! \circ \mathfrak{p}^*[n-1](\frac{n-1}{2})$ from the category of sheaves on on Bun_n to the category of sheaves on $\text{Bun}_n \times X$ is called the (usual) first Hecke functor.

Theorem 2.

$$(\mathfrak{q} \times \text{supp})_! \circ \mathfrak{p}^*(\text{Eis}_{E_1,\dots,E_n}^{d_1,\dots,d_n}|_{\text{Bun}_n})[n-1](\frac{n-1}{2}) \simeq \bigoplus_{i=1}^{n} \text{Eis}_{E_1,\dots,E_n}^{d_1,\dots,d_i-1,\dots,d_n}|_{\text{Bun}_n} \boxtimes E_i.$$

1.3.2

The proof of Theorem 2 is based on the following general result:

Proposition 4. *Let $\mathcal{Y}$ be an algebraic stack and let $a : \mathcal{E} \to \mathcal{Y}$ be a vector-bundle over it. Let $\mathbb{P}a : \mathbb{P}\mathcal{E} \to \mathcal{Y}$ denote the corresponding projectivized bundle.*

Let now $\mathcal{K}$ be a perverse sheaf on $\mathcal{E}$ with the following additional properties:

(a) $\mathcal{K}$ is equivariant with respect to the G_m action on $\mathcal{E}$.

(b) $\mathcal{K}$ is the Goresky-MacPherson extension of its restriction to the complement of the zero section.

(c) $a_!(\mathcal{K})[1]$ is a perverse sheaf on $\mathcal{Y}$.

Then $\mathbb{P}a_!(\mathcal{K}') \simeq a_!(\mathcal{K})[1](1)$, where $\mathcal{K}'$ is the perverse sheaf on $\mathbb{P}\mathcal{E}$ corresponding to $\mathcal{K}$.

Proof of the Proposition

Let ${}^0\mathcal{E} \xrightarrow{j} \mathcal{E}$ denote embedding of the complement to the zero section. The assumption on $\mathcal{K}$ implies that

$$H^i(a_! \circ j_! \circ j^*(\mathcal{K})) = 0$$

except for $i = 0$ or $i = 1$. This in turn implies that $H^i(\mathbb{P}a_!(\mathcal{K}')) = 0$ except for $i = 0$.

Let now $\widetilde{\mathcal{E}}$ denote the blow-up of $\mathcal{E}$ along the zero section and let $\widetilde{j}$ (resp., b) denote the embedding of ${}^0\mathcal{E}$ into $\widetilde{\mathcal{E}}$ (resp., the projection $\widetilde{\mathcal{E}} \to \mathcal{E}$).

The fact that $H^i(\mathbb{P}a_!(\mathcal{K}')) = 0$ for $i \neq 0$ implies that

$$j_{!*} \circ j^*(\mathcal{K}) \simeq b_! \circ \widetilde{j}_{!*} \circ j^*(\mathcal{K}).$$

Therefore,

$$\mathbb{P}a_!(\mathcal{K}') \simeq (a \circ b)_! \circ \widetilde{j}_{!*} \circ j^*(\mathcal{K})[1](1) \simeq a_!(\mathcal{K})[1](1).$$

□(Proposition)

Proof of Theorem 2.

We have a smooth map $\mathrm{Bun}_n \times X \to \mathcal{S}h_n \times \mathcal{S}h_0^1$ and the stack Mod_n^1 identifies with the projectivization of the pull-back of the vector bundle $\mathcal{F}l'^{,1}_{n,0}$ over $\mathcal{S}h_n \times \mathcal{S}h_0^1$ to $\mathrm{Bun}_n \times X$.

The composition

$$\mathrm{Bun}_n^d \times X \underset{\mathcal{S}h_n^d \times \mathcal{S}h_0^1}{\times} \mathcal{F}l_{n,0}^{d,1} \to \mathcal{F}l_{n,0}^{d,1} \xrightarrow{p_{n,0}^{d,1}} \mathcal{S}h_n^{d+1}$$

is a smooth map. Therefore, the pull-back of $\mathrm{Eis}_{E_1,\ldots,E_n}^{d_1,\ldots,d_n}$ to $\mathrm{Bun}_n^d \times X \underset{\mathcal{S}h_n^d \times \mathcal{S}h_0^1}{\times} \mathcal{F}l_{n,0}^{d,1}$ is a G_m-equivariant perverse sheaf which, moreover, is a Goresky-McPherson extension of its restriction to the complement of the zero-section, by Proposition 3.

According to Theorem 1,

$$q_{n,o!}^{d,1} \circ p_{n,o}^{d,1*}(\mathrm{Eis}_{E_1,\ldots,E_n}^{d_1,\ldots,d_n})[n+1](\frac{n+1}{2}) \simeq \bigoplus_{i=1}^{n} \mathrm{Eis}_{E_1,\ldots,E_n}^{d_1,\ldots,d_i-1,\ldots,d_n}|_{\mathrm{Bun}_n} \boxtimes E_i$$

and the assertion of the theorem follows by applying Proposition 4.

□

1.3.3

Let us now present another proof of Theorem 2 for $n = 2$, which is independent of Theorem 1, and therefore of Main Theorem A.

Second Proof of Theorem 2 for $n = 2$.

Consider the Cartesian product $\mathrm{Mod}_2^1 \underset{\mathcal{S}h_2^{d_1+d_2}}{\times} \mathcal{F}l_{1,1}^{d_1,d_2}$.

This is a smooth stack that classifies quadruples

$$(x, M \hookrightarrow M' \hookleftarrow L)$$

where $M', M \in \mathrm{Bun}_2$, $\deg(M') = d_1 + d_2$, and $L \in \mathrm{Pic}^{d_1}$. We have a map $\mathrm{pr} : \mathrm{Mod}_2^1 \underset{\mathcal{S}h_2^{d_1+d_2}}{\times} \mathcal{F}l_{1,1}^{d_1,d_2} \to \mathcal{F}l_{1,1}^{d_1-1,d_2} \times X$ that sends

$$(x, M \hookrightarrow M' \hookleftarrow L) \to (L(-x) \to M, x) \in \mathcal{F}l_{1,1}^{d_1-1,d_2} \times X$$

Consider now the closed embedding of stacks:

$$t : \mathcal{F}l_{1,1}^{d_1,d_2-1} \times X \to \mathcal{F}l_{1,1}^{d_1-1,d_2} \times X$$

that sends $(x, L \hookrightarrow M)$ to $(x, L(-x) \hookrightarrow M)$.

$$\begin{array}{ccccc}
\mathcal{F}l_{1,1}^{d_1-1,d_2} \times X & \xleftarrow{\mathrm{pr}} & \mathrm{Mod}_2^1 \underset{\mathcal{S}h_2^{d_1+d_2}}{\times} \mathcal{F}l_{1,1}^{d_1,d_2} & \xrightarrow{\mathfrak{p}'} & \mathcal{F}l_{1,1}^{d_1,d_2} \\
\downarrow{\scriptstyle q_{1,1}^{d_1-1,d_2}\times \mathrm{id}} & & \downarrow & & \downarrow{\scriptstyle q_{1,1}^{d_1,d_2}} \\
\mathrm{Bun}_2 \times X & \xleftarrow{\mathfrak{q}\times \mathrm{supp}} & \mathrm{Mod}_2^1 & \xrightarrow{\mathfrak{p}} & \mathrm{Bun}_2
\end{array}$$

(The left square in this diagram is NOT Cartesian)

Lemma 5.

$$\mathrm{pr}_! \circ \mathfrak{p}'^*(\det \times \det)^*(E_1^{d_1} \boxtimes E_2^{d_2}) \simeq$$

$$\simeq (\det \times \det)^*(E_1^{d_1-1} \boxtimes E_2^{d_2}) \boxtimes E_1 \oplus t_* \circ (\det \times \det)^*(E_1^{d_1} \boxtimes E_2^{d_2-1}) \boxtimes E_2[-2](-1)$$

The assertion of the theorem immediately follows from the lemma.

□(Theorem)

Proof of the lemma

The map pr is proper and semi-small. It is an isomorphism over the complement of $t(\mathcal{F}l_{1,1}^{d_1,d_2-1} \times X)$ in $\mathcal{F}l_{1,1}^{d_1-1,d_2} \times X$, and

$$(\mathcal{F}l_{1,1}^{d_1,d_2-1} \times X) \underset{(\mathcal{F}l_{1,1}^{d_1-1,d_2} \times X)}{\times} \mathrm{Mod}_2^1 \underset{\mathcal{S}h_2^{d_1+d_2}}{\times} \mathcal{F}l_{1,1}^{d_1,d_2}$$

is a projectivization of a 2-dimensional bundle over $\mathcal{F}l_{1,1}^{d_1-1,d_2} \times X$.

This implies the lemma.

□(Lemma)

1.3.4

We define the sheaf $\mathrm{Aut}_{E_1\oplus\ldots\oplus E_n}$ on Bun_n to be the restriction of the sheaf $\mathrm{Eis}_{E_1,\ldots,E_n}$ to $\mathrm{Bun}_n \subset \mathcal{S}h_n$.

Corollary 6. *The sheaf* $\mathrm{Aut}_{E_1\oplus\ldots\oplus E_n}$ *satisfies:*

$$(\mathfrak{q} \times \mathrm{supp})_! \circ \mathfrak{p}^*(\mathrm{Aut}_{E_1\oplus\ldots\oplus E_n})[n-1](\frac{n-1}{2}) \simeq \mathrm{Aut}_{E_1\oplus\ldots\oplus E_n} \boxtimes (E_1 \oplus \ldots \oplus E_n).$$

We will now show that Corollary 6 implies that $\mathrm{Aut}_{E_1\oplus\ldots\oplus E_n}$ is a Hecke eigensheaf with respect to the n-dimensional local system $E_1 \oplus \ldots \oplus E_n$.

1.3.5

Consider the stack Mod_n^m that classifies pairs $(M \hookrightarrow M')$, where M and M' are two rank-n bundles on X with $\deg(M') = \deg(M) + m$, $m \geq 0$. Obviously, Mod_n^m identifies with the open sub-stack $p_{n,0}^{;m-1}(\mathrm{Bun}_n) \subset \mathcal{F}l_{n,0}^{;m}$.

We will denote by ${}_n\mathfrak{q}_n^m \times {}_0\mathfrak{q}_n^m$ the projection $\mathrm{Mod}_n^m \to \mathrm{Bun}_n \times \mathcal{S}h_0^m$, which is obtained by restricting the map $q_{n,o}^{;m}$ to Mod_n^m and by $\mathfrak{p}_n^m$ the restriction to Mod_n^m of the projection $p_{n,o}^{;m}$. [1]

We will also denote by supp_n^m the composition map

$$\mathrm{Mod}_n^m \xrightarrow{{}_0\mathfrak{q}} \mathcal{S}h_0^m \xrightarrow{\mathrm{supp}_0^m} X^{(m)}$$

(supp_0^m is a version of the Harish-Chandra map).

Theorem 3. For a local system E on X let Aut_E be a perverse sheaf on Bun_n satisfying:

$$({}_n\mathfrak{q}_n^1 \times \mathrm{supp}_n^1)_! \circ \mathfrak{p}_n^{1*}(\mathrm{Aut}_E)[n-1](\frac{n-1}{2}) \simeq \mathrm{Aut}_E \boxtimes E.$$

Then there is an isomorphism that preserves the S^m-action:

$$({}_n\mathfrak{q}_n^m \times \mathrm{supp}_n^m)_! \circ (\mathfrak{p}_n^m \times {}_0\mathfrak{q}_n^m)^*(\mathrm{Aut}_E \boxtimes \mathcal{S}pr^m)(\overline{\mathbb{Q}}_\ell[1](\frac{1}{2}))^{\otimes m(n-1)} \simeq \mathrm{Aut}_E \boxtimes \mathrm{sym}_!^m(E^{\boxtimes m}).$$

Proof. Let us denote by $\widetilde{\mathrm{Mod}}_n^m$ the fibered product

$$\widetilde{\mathrm{Mod}}_n^m := \mathrm{Mod}_n^m \underset{\mathcal{S}h_0^m}{\times} \mathcal{F}l_{0,\ldots,0}^{1,\ldots,1}.$$

In other words, $\widetilde{\mathrm{Mod}}_n^m$ is the stack that parameterizes strings of length m of n-bundles on X:

$$(M \hookrightarrow M_1 \hookrightarrow \ldots \hookrightarrow M_m = M'),$$

where M_i/M_{i-1} is a torsion sheaf on X of length 1.

Let us denote by cov the natural map $\widetilde{\mathrm{Mod}}_n^m \to \mathrm{Mod}_n^m$ and by $\widetilde{\mathfrak{p}}_n^m$ and ${}_n\widetilde{\mathfrak{q}}_n^m$ the compositions $\widetilde{\mathfrak{p}}_n^m \circ \mathrm{cov}$ and ${}_n\widetilde{\mathfrak{q}} \circ \mathrm{cov}$ from $\widetilde{\mathrm{Mod}}_n^m$ to Bun_n and by ${}_0\widetilde{\mathfrak{q}}$ the natural map $\widetilde{\mathrm{Mod}}_n^m \to \mathcal{F}l_{0,\ldots,0}^{1,\ldots,1}$. We will denote by $\widetilde{\mathrm{supp}}_n^m$ the composition

$$\mathrm{supp}_0^{1\times m} \circ q_{0,\ldots,0}^{1,\ldots,1} \circ {}_0\widetilde{\mathfrak{q}} : \widetilde{\mathrm{Mod}}_n^m \to X^m.$$

[1] In Sect. 1.3.1, the maps ${}_n\mathfrak{q}_n^1, \mathfrak{p}_n^1 : \mathrm{Mod}_n^1 \to \mathrm{Bun}_n$ were denoted by $\mathfrak{q}$ and $\mathfrak{p}$ respectively and the map supp_n^1–by supp.

$$
\begin{array}{ccccc}
\mathrm{Bun}_n \times X^m & \xleftarrow{{}_n\widetilde{\mathfrak{q}}_n^m \times \widetilde{\mathrm{supp}}_n^m} & \widetilde{\mathrm{Mod}}_n^m & \xrightarrow{\widetilde{\mathfrak{p}}_n^m \times {}_0\widetilde{\mathfrak{q}}_n^m} & \mathrm{Bun}_n \times \mathcal{F}l_{0,\dots,0}^{1,\dots,1} \\
\Big\downarrow \scriptstyle{\mathrm{id} \times \mathrm{sym}^m} & & \Big\downarrow \scriptstyle{\mathrm{cov}} & & \Big\downarrow \scriptstyle{\mathrm{id} \times p_{0,\dots,0}^{1,\dots,1}} \\
\mathrm{Bun}_n \times X^{(m)} & \xleftarrow{{}_n\widetilde{\mathfrak{q}}_n^m \times \mathrm{supp}_n^m} & \mathrm{Mod}_n^m & \longrightarrow & \mathrm{Bun}_n \times \mathcal{S}h_0^m
\end{array}
$$

In the above commutative diagram the left square is not Cartesian. However, in the diagram below both left and right squares are Cartesian and the stacks in the top row are S^m-Galois étale covers of the stacks in the bottom row:

$$
\begin{array}{ccccc}
\mathrm{Bun}_n \times (X^m - \Delta) & \longleftarrow & \widetilde{\mathrm{supp}}_n^{m-1}(X^m - \Delta) & \longrightarrow & \mathrm{Bun}_n \times {}^{rss}\mathcal{F}l_{0,\dots,0}^{1,\dots,1} \\
\Big\downarrow \scriptstyle{\mathrm{id} \times \mathrm{sym}^m} & & \Big\downarrow \scriptstyle{\mathrm{cov}} & & \Big\downarrow \scriptstyle{\mathrm{id} \times p_{0,\dots,0}^{1,\dots,1}} \\
\mathrm{Bun}_n \times (X^{(m)} - \Delta) & \longleftarrow & \mathrm{supp}_n^{m-1}(X^{(m)} - \Delta) & \longrightarrow & \mathrm{Bun}_n \times {}^{rss}\mathcal{S}h_0^m
\end{array}
$$

Lemma 6. *We have:*

$$({}_n\widetilde{\mathfrak{q}}_n^m \times \widetilde{\mathrm{supp}}_n^m)_! \circ \widetilde{\mathfrak{p}}_n^{m*}(\mathrm{Aut}_E)[m(n-1)](\frac{m(n-1)}{2}) \simeq \mathrm{Aut}_E \boxtimes (E^{\boxtimes m}).$$

Proof of the lemma

We have a Cartesian square:

$$
\begin{array}{ccccc}
\widetilde{\mathrm{Mod}}_n^m & \longrightarrow & \mathrm{Mod}_n^1 & \xrightarrow{\widetilde{\mathfrak{p}}_n^1} & \mathrm{Bun}_n \\
\Big\downarrow & & \Big\downarrow \scriptstyle{{}_n\widetilde{\mathfrak{q}}_n^1 \times \widetilde{\mathrm{supp}}_n^1} & & \\
\widetilde{\mathrm{Mod}}_n^{m-1} \times X & \xrightarrow{\widetilde{\mathfrak{p}}_n^{m-1} \times \mathrm{id}} & \mathrm{Bun}_n \times X & &
\end{array}
$$

such that the composition map in the top row is $\widetilde{\mathfrak{p}}_n^m$. The assertion follows now from the assumption on Aut_E by induction on m.

□(Lemma)

Therefore, by the projection formula,

$$
\begin{aligned}
&({}_n\mathfrak{q}_n^m \times \mathrm{supp}_n^m)_! \circ (\mathfrak{p}_n^m \times {}_0\mathfrak{q}_n^m)^*(\mathrm{Aut}_E \boxtimes \mathcal{S}pr^m) \simeq \\
&\simeq (\mathrm{id} \times \mathrm{sym}^m)_! \circ ({}_n\widetilde{\mathfrak{q}}_n^m \times \widetilde{\mathrm{supp}}_n^m)_! \circ \widetilde{\mathfrak{p}}_n^{m*}(\mathrm{Aut}_E) \\
&\mathrm{Aut}_E \boxtimes \mathrm{sym}_!^m(E^{\boxtimes m})[m(1-n)](\frac{m(n-1)}{2})
\end{aligned}
$$

The two S^m-actions on the right-hand side (one of them comes from the S^m-action on $\mathcal{S}pr^m$ by functoriality and another comes from the action on $\mathrm{sym}_!^m(E^{\boxtimes m})$) coincide, since they do over $\mathrm{Bun}_n \times (X^{(m)} - \Delta)$. This implies the assertion of the theorem.

□

1.3.6

For i between 1 and n, let $\mathcal{H}_n^i$ be the i-th Hecke-correspondence:

$$\begin{array}{ccc} \mathcal{H}^i & \xrightarrow{{}^p h_n^i} & \mathrm{Bun}_n \\ {\scriptstyle {}^q h_n^i}\downarrow & & \\ \mathrm{Bun}_n \times X & & \end{array}$$

We have: $\mathcal{H}_n^1 = \mathrm{Mod}_n^1$.

Theorem 4. Let E be a local system on X and let Aut_E be a perverse sheaf on Bun_n satisfying:

$$({}_n\mathfrak{q}_n^1 \times \mathrm{supp}_n^1)_! \circ \mathfrak{p}_n^{1*}(\mathrm{Aut}_E)[n-1](\frac{(n-1)}{2}) \simeq \mathrm{Aut}_E \boxtimes E.$$

Then Aut_E is a Hecke eigen-sheaf with respect to E, i.e.

$${}^q h^i_{n!} \circ {}^p h_n^{i*}(\mathrm{Aut}_E)[i(n-i)](\frac{i(n-i)}{2}) \simeq \mathrm{Aut}_E \boxtimes \Lambda^i(E).$$

Proof. Take $m = i$ and consider the closed sub-stack ${}^{nil}\mathcal{S}h_0^m \simeq \mathcal{S}h_0^m \underset{X^{(m)}}{\times} X$, where the map $X \to X^{(m)}$ is the diagonal embedding. Consider also the stack ${}^{nil}\mathrm{Mod}_n^m := \mathrm{Mod}_n^m \underset{X^{(m)}}{\times} X$.

Let us denote by ${}^{nil}\mathcal{S}pr^m$ the restriction of the sheaf $\mathcal{S}pr^m$ to ${}^{nil}\mathcal{S}h_0^m$ and by ${}^{nil}\mathfrak{p}_n^m \times {}^{nil}_0\mathfrak{q}_n^m$ (resp., ${}^{nil}_n\mathfrak{q}_n^m \times {}^{nil}\mathrm{supp}^m$) the projection from ${}^{nil}\mathrm{Mod}_n^m$ to $\mathrm{Bun}_n \times {}^{nil}\mathcal{S}h_0^m$ (resp., to $\mathrm{Bun}_n \times X$) obtained by restricting the projection $\mathfrak{p}_n^m \times {}_0\mathfrak{q}_n^m$ (resp., ${}_n\mathfrak{q}_n^m \times \mathrm{supp}^m$).

According to Theorem 3 we have an isomorphism of S^m-sheaves:

$$({}^{nil}_n\mathfrak{q}_n^m \times {}^{nil}\mathrm{supp}^m)_! \circ ({}^{nil}\mathfrak{p}_n^m \times {}^{nil}_0\mathfrak{q}_n^m)^*({}^{nil}\mathcal{S}pr^m)[m(n-1)](\frac{m(n-1)}{2}) \simeq \mathrm{Aut}_E \boxtimes E^{\otimes m}.$$

Consider now the closed embedding $e^m : X/GL(m) \to {}^{nil}\mathcal{S}h_0^m \hookrightarrow \mathcal{S}h_0^m$ given by $x \to V \otimes \mathcal{O}/\mathcal{O}(-x)$, where V is an m-dimensional vector space.

It is well-known that

$$\mathrm{Hom}_{S^m}(\mathrm{sign}, {}^{nil}\mathcal{S}pr) \simeq e_!^m(\overline{\mathbb{Q}}_{\ell X/GL(m)})[-m(m-1)](\frac{-m(m-1)}{2}),$$

where "sign" denotes the sign representation of S^m.

We have an embedding $e'^m : \mathcal{H}_n^m \to {}^{nil}\mathrm{Mod}_n^m$ such that the compositions ${}^{nil}\mathfrak{p}_n^m \circ e'^m$ and $({}^{nil}_n\mathfrak{q}_n^m \times {}^{nil}\mathrm{supp}^m) \circ e'^m$ identify with the maps ${}^p h_n^i$ and ${}^q h_n^i$ respectively.

Moreover, we have a Cartesian square:

$$\begin{array}{ccc} \mathcal{H}_n^m & \xrightarrow{e'^m} & {}^{nil}\mathrm{Mod}_n^m \\ \downarrow & & \downarrow {\scriptstyle {}^{nil}\mathfrak{p}_n^m \times_0^{nil} \mathfrak{q}_n^m} \\ \mathrm{Bun}_n \times X/GL(m) & \xrightarrow{\mathrm{id} \times e^m} & \mathrm{Bun}_n \times {}^{nil}\mathcal{S}h_0^m \end{array}$$

Therefore,

$$\mathrm{Hom}_{S^m}(\mathrm{sign}, ({}^{nil}\mathfrak{p}_n^m \times_0^{nil} \mathfrak{q}_n^m)^*(\mathrm{Aut}_E \boxtimes {}^{nil}\mathcal{S}pr)) \simeq e'^m_! \circ {}^p h_n^{i\,*}(\mathrm{Aut}_E)(\overline{\mathbb{Q}}_\ell[1](\frac{1}{2}))^{\otimes -m(m-1)}$$

and we obtain that

$$\mathrm{Aut}_E \boxtimes \Lambda^m(E) \simeq \mathrm{Hom}_{S^m}(\mathrm{sign}, \mathrm{Aut}_E \boxtimes E^{\otimes m}) \simeq$$

$$\mathrm{Hom}_{S^m}(\mathrm{sign}, ({}^{nil}_n\mathfrak{q}_n^m \times {}^{nil}\mathrm{supp}^m)_! \circ ({}^{nil}\mathfrak{p}_n^m \times {}^{nil}_0\mathfrak{q}_n^m)^*({}^{nil}\mathcal{S}pr^m))(\overline{\mathbb{Q}}_\ell[1](\frac{1}{2}))^{\otimes m(n-1)} \simeq$$

$$\mathrm{Hom}_{S^m}(\mathrm{sign}, ({}^{nil}_n\mathfrak{q}_n^m \times {}^{nil}\mathrm{supp}^m)_! \circ e'^m_! \circ {}^p h_n^{i\,*}(\mathrm{Aut}_E))(\overline{\mathbb{Q}}_\ell[1](\frac{1}{2}))^{\otimes m(n-m)} \simeq$$

$$\simeq {}^q h_n^{i\,*} \circ {}^p h_n^{i\,*}(\mathrm{Aut}_E)[m(n-m)](\frac{m(n-m)}{2})$$

□

Corollary 7. *Let $E_1,\ldots,E_n$ be 1-dimensional local systems on X with $H(X, E_i \otimes E_j^{-1}) = 0$ for $i \neq j$. Then the sheaf $\mathrm{Aut}_{E_1 \oplus \ldots \oplus E_n}$ is a Hecke eigen-sheaf with respect to the local system $E_1 \oplus \ldots \oplus E_n$.*

This is a combination of Theorem 4 and Corollary 6.

1.3.7

Corollary 8. *Let E be a n-dimensional local system on X, which is irreducible, but which splits as a direct sum of 1-dimensional local systems after an extension of scalars $\mathbb{F}_q \to \overline{\mathbb{F}_q^m}$. Then there exists a perverse sheaf Aut_E on $\mathcal{S}h_n$, which is a Hecke-eigen-sheaf with respect to E.*

Proof. Over $X \underset{\mathbb{F}_q^m}{\times} \mathbb{F}_q$, the local system E splits as a sum $E \simeq E_1 \oplus ... \oplus E_n$, where E_i's are geometrically non-isomorphic, and there exists a permutation σ, such that

$$F^*(E_i) \simeq E_{\sigma(i)},$$

where F denotes the geometric Frobenius.

We set the sheaf Aut_E on $\mathrm{Bun}_n \underset{\mathbb{F}_q^m}{\times} \mathbb{F}_q$ to be equal to $\mathrm{Eis}_{E_1,...,E_n}|_{\mathrm{Bun}_n}$.

For each weight $(d_1, ..., d_n)$ choose an isomorphism

$$\mathrm{Eis}^{d_1,...,d_n}_{E_1,...,E_n} \simeq \mathrm{Eis}^{d_{\sigma(1)},...,d_{\sigma(n)}}_{E_{\sigma(1)},...,E_{\sigma(n)}},$$

which exists according to Main Theorem A(2).

We have

$$F^*(\mathrm{Eis}^{d_1,...,d_n}_{E_1,...,E_n}) \simeq \mathrm{Eis}^{d_1,...,d_n}_{E_{\sigma(1)},...,E_{\sigma(n)}},$$

where F is the geometric Frobenius acting on $\mathrm{Bun}_n \underset{\mathbb{F}_q^m}{\times} \mathbb{F}_q$ and we define the $\mathbb{F}_q$-structure on Aut_E by:

$$F^*(\mathrm{Aut}_E) \simeq F^*(\underset{d_1,...,d_n}{\oplus} \mathrm{Eis}^{d_1,...,d_n}_{E_1,...,E_n}) \simeq \underset{d_1,...,d_n}{\oplus} \mathrm{Eis}^{d_1,...,d_n}_{E_{\sigma(1)},...,E_{\sigma(n)}} \simeq$$

$$\simeq \underset{d_1,...,d_n}{\oplus} \mathrm{Eis}^{d_{\sigma^{-1}(1)},...,d_{\sigma^{-1}(n)}}_{E_1,...,E_n} \simeq \underset{d_1,...,d_n}{\oplus} \mathrm{Eis}^{d_1,...,d_n}_{E_1,...,E_n} \simeq \mathrm{Aut}_E\,.$$

For each weight $d_1, ..., d_n$ we can choose isomorphisms of Theorem 2 to make them compatible with isomorphisms

$$\mathrm{Eis}^{d_1,...,d_n}_{E_1,...,E_n} \simeq \mathrm{Eis}^{d_{\sigma(1)},...,d_{\sigma(n)}}_{E_{\sigma(1)},...,E_{\sigma(n)}}\,.$$

2

This makes the isomorphisms:

$$(\mathfrak{q} \times \mathrm{supp})_* \circ \mathfrak{p}^*(\mathrm{Aut}_E)[n-1](\frac{n-1}{2}) \simeq \mathrm{Aut}_E \boxtimes E$$

of sheaves over $\mathrm{Bun}_n \underset{\mathbb{F}_q^m}{\times} \mathbb{F}_q$ compatible with the $\mathbb{F}_q$-structure, which implies the assertion of the corollary in view of Theorem 4.

□

[2] One can show that isomorphisms of Main Theorem A(2) and of Theorem 1.3.1 are canonical and compatible with each other. To prove this fact one has to compare the isomorphisms established in Theorem 1, Main Theorem B and Corollary 4.3 of [11].

Chapter 2

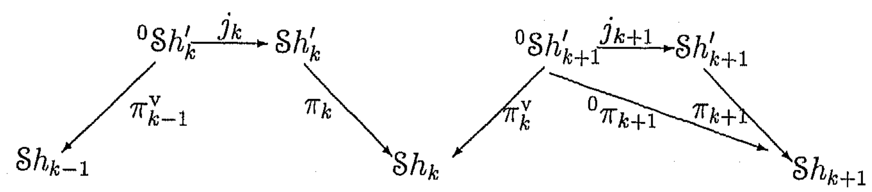
Eisenstein series via the fundamental diagram

2.1 Generalized Whittaker model for Eisenstein series

2.1.1

For an integer n consider the stack $\mathcal{S}h'_n$ (resp. ${}^0\mathcal{S}h'_n$) that classifies pairs $(M_n, s : \Omega^{\otimes n-1} \to M_n)$ (resp. $(M_n, s : \Omega^{\otimes n-1} \hookrightarrow M_n)$), where M_n is a coherent sheaf of generic rank n on X. We have an obvious open embedding $j_n : {}^0\mathcal{S}h'_n \to \mathcal{S}h'_n$. In addition to the natural projection $\pi_n : \mathcal{S}h'_n \to \mathcal{S}h_n$ we have the projection $\pi^{\vee}_{n-1} : {}^0\mathcal{S}h'_n \to \mathcal{S}h_{n-1}$ that sends a pair $(M_n, s : \Omega^{\otimes n-1} \hookrightarrow M_n) \in {}^0\mathcal{S}h'_n$ to $M_n/\operatorname{Im}(s) \in \mathcal{S}h_{n-1}$.

We have the following picture, which we will call the fundamental diagram:



Recall, that $\mathcal{S}h^{d;\leq t}_n$ (resp., $\mathcal{S}h^{d;t}_n$) denotes the open sub-stack of $\mathcal{S}h_n$ that corresponds to sheaves of degree d with torsion of length not greater that t (resp., the locally closed sub-stack $\mathcal{S}h_n$ that corresponds to sheaves of degree d with torsion of length exactly t). Analogously, ${\mathcal{S}h'_n}^{d;\leq t}$ (resp. ${}^0{\mathcal{S}h'_n}^{d;\leq t}$) will denote the preimage of $\mathcal{S}h_n^{d;\leq t}$ in $\mathcal{S}h'_n$ (resp., in ${}^0\mathcal{S}h'_n$). We have:

$$ {\pi^{\vee}_{n-1}}^{-1}(\mathcal{S}h_{n-1}^{d;\leq t}) \subset {}^0{\mathcal{S}h'_n}^{d+(n-1)(2g-2);\leq t}. $$

For an integer c we will denote by ${}_c\mathcal{S}h_k$ the open sub-stack of $\mathcal{S}h_k$ that corresponds to sheaves $M \in \mathcal{S}h_k$ with the following property:

$$ \mathrm{Ext}^1(L, M) = 0, \forall L \in \mathrm{Pic}^{c'} \text{ with } c' \leq c. $$

We will denote by ${}_c\mathcal{S}h'_k$ (resp. ,by ${}^0_c\mathcal{S}h'_k$) the preimage of ${}_c\mathcal{S}h_k$ under the projection π_k (resp., ${}^0\pi_k$). We have:

$$ {}^0_c\mathcal{S}h'_k \subset {\pi^{\vee}_{k-1}}^{-1}({}_c\mathcal{S}h_{k-1}). $$

Over ${}_{(k-1)(2g-2)}\mathcal{S}h_k$ the stacks ${}_{(k-1)(2g-2)}\mathcal{S}h'_k$ and ${\pi_k^{\vee}}^{-1}({}_{(k-1)(2g-2)}\mathcal{S}h_k) \subset {}^0\mathcal{S}h'_{k+1}$ are mutually dual vector bundles and there is the Fourier transform functor, denoted Four_k from the category of sheaves on ${}_{(k-1)(2g-2)}\mathcal{S}h'_k$ to the category of sheaves on ${\pi_k^{\vee}}^{-1}({}_{(k-1)(2g-2)}\mathcal{S}h_k)$.

We define the functor $comp_k$ from the category of **perverse** sheaves on ${}^0_0\mathcal{S}h'_1$ to the category of perverse sheaves on ${}_{(k-1)(2g-2)}{}^0\mathcal{S}h'_k$ by

$$ comp_k(\mathcal{K}) := (\mathrm{Four}_k \circ j_{k!*}(comp_{k-1}(\mathcal{K})))|_{{}_{(k-1)(2g-2)}{}^0\mathcal{S}h'_k}. $$

By definition, $comp_0$ is the identity functor.

2.1.2

Let us consider the functor $comp_{!k}$ from the category of all sheaves on ${}^0_0\mathcal{S}h'_1$ to that on ${}_{(k-1)(2g-2)}{}^0\mathcal{S}h'_k$ defined by

$$ comp_{!k}(\mathcal{K}) := (\mathrm{Four}_k \circ j_{k!}(comp_{!k-1}(\mathcal{K})))|_{{}_{(k-1)(2g-2)}{}^0\mathcal{S}h'_k}. $$

It was shown in [4] that if E is an n-dimensional local system on X, the restriction of the sheaf

$$ \mathcal{S}'_E := comp_{!n-1}(\pi_0^{\vee *}(\mathcal{L}^{d_1,\ldots,d_n}_{E_1,\ldots,E_n})) $$

to ${}^0{\mathcal{S}h'_n}^{d+n(n-1)(g-1);\leq 0}$ gives rise to a function on the set of $\mathbb{F}_q$-points of this stack that coincides with the one coming from the Whittaker model. In other words, the corresponding function has the same non-degenerate Fourier coefficients as the automorphic function corresponding to the local system E, provided that the latter exists.

However, when the local system E is reducible, i.e. when $E \simeq \bigoplus_{i=1}^{n} E_i$, the degenerate Fourier coefficients of the function corresponding to $\mathcal{S}'_E$ are wrong; in particular, it is not a pull-back of any function on $\mathrm{Bun}_n(\mathbb{F}_q)$.

The next theorem is a statement in the direction that when E is a direct sum as above, if we use the functor $comp_{n-1}$ instead of $comp_{!n-1}$, we get a right sheaf on ${}_{(n-1)(2g-2)}{}^0\mathcal{S}h'^{\,d}_n$. We will prove this for $n = 2$ and will establish a partial result for any n.

2.1.3

Let now E_1,...,E_n be 1-dimensional local systems on X satisfying

$$\mathrm{Hom}(E_i, E_j) = 0 \text{ for} i \neq j.$$

Let $\overline{d^n} = (d_1, ..., d_n)$ be a string of n non-negative integers and let d denote their sum. Let $\mathfrak{c}_n$ denote the integer $\max(d_i) + (n-2)(2g-2)$.

Main Theorem B

Over ${\pi^{\vee}_{n-1}}^{-1}({}^0_{\mathfrak{c}_n}\mathcal{S}h^{d+(g-1)(n-1)(n-2)}_{n-1}) \cap {}_{(n-1)(2g-2)}{}^0\mathcal{S}h'^{\,d+(g-1)n(n-1)}_n$ there is an isomorphism:

$$\begin{aligned}&{}^0\pi^*_n(\mathrm{Eis}^{d_1+(n-1)(g-1),...,d_n+(n-1)(g-1)}_{E_1,...,E_n})[d-(g-1)n^2](\frac{-(g-1)n^2}{2}) \simeq \\ &\simeq comp_{n-1} \circ \pi^{\vee *}_0(\mathcal{L}^{d_1,...,d_n}_{E_1,...,E_n}[d]) \bigotimes_{i=1}^{n} (E_i^{(n-1)(g-1)}|_{\Omega^{\otimes \frac{n-1}{2}}}).\end{aligned}$$

When $n = 2$, we will prove a prove a little stronger statement:

Main Theorem B for $n = 2$

Over ${\pi^{\vee}_1}^{-1}({}_{d_2}\mathcal{S}h^d_1)$ there is an isomorphism:

$$\begin{aligned}&{}^0\pi^*_2(\mathrm{Eis}^{d_1+g-1,d_2+g-1}_{E_1,E_2})[-(g-1)4+d](-2g+2) \simeq \\ &\simeq comp_1(\pi^{\vee *}_0(\mathcal{L}^{d_1,d_2}_{E_1,E_2}[d]) \otimes ((E_1 \otimes E_2)^{g-1}|_{\Omega^{\frac{1}{2}}}).\end{aligned}$$

2.1.4

We will now show how Main Theorem B implies Main Theorem A. First, we will consider the case of $n=2$, where the proof is much simpler.

Proof of Main Theorem A(1) and (2) for $n=2$.

It is enough to show that the assertion of the theorem holds for $\mathrm{Eis}^{d_1,d_2}_{E_1,E_2}$ restricted to every open sub-stack $U \subset \mathcal{S}h_2^{d+2g-2}$ of finite type. The assumtion on U implies that it is contained in ${}_c\mathcal{S}h_2^{d+2g-2;\leq t}$ for some c and t.

We claim, that we can assume that U is contained in

$$\mathcal{S}h_2^{d+2g-2;\leq \min(d_1,d_2)-(2g-2)} \cap_{2g-1} \mathcal{S}h_2^{d+2g-2}.$$

Indeed, by Lemma 2, the statement for U is equivalent to the statement for $m_{L_0}(U)$ for any line bundle L_0 and if we choose L_0 so that

$$\min(d_1,d_2)+\deg(L_0) > t+2g-2 \text{ and } c+2\deg(L_0) > 2g-1,$$

we will have:

$$m_{L_0}(U) \subset \mathcal{S}h_2^{d+2g-2+2\deg(L_0);\leq \min(d_1+\deg(L_0),d_2+\deg(L_0))-(2g-2)} \cap_{2g-1} \mathcal{S}h_2^{d+2g-2+2\deg(L_0)}.$$

It is easy to see that for any t, the two open subs-tacks ${}_t\mathcal{S}h_1^d$ and $\mathcal{S}h_1^{d,\leq t-(2g-2)}$ of $\mathcal{S}h_1^d$ coincide. Therefore, the restriction of the map ${}^0\pi_2$ to $\pi_1^{\vee -1}({}_{\max(d_1,d_2)}\mathcal{S}h_1'^{\,d-2g+2})$ is smooth, surjective and has connected fibers over

$$\mathcal{S}h_2^{d+2g-2;\leq \min(d_1,d_2)-(2g-2)} \cap_{2g-1} \mathcal{S}h_2^{d+2g-2},$$

as it is easy to deduce from the Riemann-Roch theorem.

To prove Main Theorem A(1) and (2) it suffices to check, therefore, that ${}^0\pi_2^*(\mathrm{Eis}^{d_1,d_2}_{E_1,E_2})[d-(g-1)n^2]$ and ${}^0\pi_2^*(\mathrm{Eis}^{d_2,d_1}_{E_2,E_1})[d-(g-1)n^2]$ are perverse, irreducible and isomorphic to one another over $\pi_1^{\vee -1}({}_{\max(d_1,d_2)}\mathcal{S}h_1'^{\,d-2g+2})$.

The assertion of Main Theorem A(1) follows now from Main Theorem B (for $GL(2)$) combined with Corollary 3, since the functors $j_{1!*}$ and Four_1 preserve the properties of being perverse and irreducible. Main Theorem A(2) follows by combining Main Theorem B with Proposition 2(4).

□

Corollary 9. *Assume that* $d > 4g-4$. *Then over* ${}_{2g-2}{}^0\mathcal{S}h_2'^{\,d+(2g-2)}$ *we have an isomorphism:*

$${}^0\pi_2^*(\mathrm{Eis}^{d+2g-2}_{E_1,E_2})[-4g+4+d](-2g+2) \simeq comp_1 \circ \pi_0^{\vee *}(\mathcal{L}^d_{E_1\oplus E_2}[d]) \otimes ((E_1\otimes E_2)^{g-1}|_{\Omega^{\frac{1}{2}}}).$$

Proof. ¿From the definition of ${}_{2g-2}\mathcal{S}h_2$ it follows that $\mathrm{Eis}^{d_1+g-1,d_2+g-1}_{E_1,E_2}$ vanishes on this sub-stack if $d_1 > d$. Therefore, using Main Theorem A(2), we obtain:

$$\mathrm{Eis}^{d+2g-2}_{E_1,E_2}|_{{}_{2g-2}\mathcal{S}h_2^{d+2g-2}} \simeq \bigoplus_{d_1+d_2=d, 0\leq d_1 \leq d} \mathrm{Eis}^{d_1+g-1,d_2+g-1}_{E_1,E_2}|_{{}_{2g-2}\mathcal{S}h_2^{d+2g-2}}.$$

Therefore, it suffices to show, that for $d_1, d_2 \geq 0$, over ${}_{2g-2}{}^0\mathcal{S}h_2'^{\,d+(2g-2)}$ the two sheaves: ${}^0\pi_2^*(\mathrm{Eis}^{d_1+g-1,d_2+g-1}_{E_1,E_2})[-4g+4+d](-2g+2)$ and $comp_1 \circ \pi_0^{\vee *}(\mathcal{L}^{d_1,d_2}_{E_1,E_2}[d]) \otimes ((E_1 \otimes E_2)^{g-1}|_{\Omega^{\frac{1}{2}}})$ are isomorphic. Without restricting the generality, we can assume, that $d_1 \geq d_2$, therefore, $d_1 > 2g-2$.

The two sheaves above are perverse and irreducible, by Main Theorem A(1). Therefore, it is enough to show that they are isomorphic over a non-empty open subs-tack of ${}_{2g-2}{}^0\mathcal{S}h_2'^{\,d+(2g-2)}$. It is easy to see, that the intersection

$${}_{2g-2}{}^0\mathcal{S}h_2'^{\,d+(2g-2)} \cap \pi_1^{\vee -1}({}_{d-(2g-2)}\mathcal{S}h_1^d)$$

is non-empty and the needed isomorphism follows from Main Theorem B for $n = 2$.

□

2.1.5

We will now prove Main Theorem A for any n.

For an integer c' let $\mathcal{U}^{d'}(c')$ denote the maximal open sub-stack of $\mathcal{S}h_n^{d'+(g-1)n(n-1)}$ over which the projection ${}^0\pi_n$ restricted to

$${\pi_{n-1}^{\vee}}^{-1}({}^0_{c'}\mathcal{S}h_{n-1}^{d'+(g-1)(n-1)(n-2)})$$

is smooth and surjective.

The proof of Main Theorem A is based on the following assertion:

Lemma 7. *Let U be an open sub-stack of finite type in $\mathcal{S}h_n^{d+(g-1)n(n-1)}$. Then for any c there exists a line bundle L_0 of sufficiently large degree such that $m_{L_0}(U) \subset \mathcal{U}^{d+n\deg(L_0)}(c+\deg(L_0))$.*

Proof. For every sub-stack $U \subset \mathcal{S}h_n^{d+(g-1)n(n-1)}$ of finite type, there exists an integer c_0 such that $U \subset {}_{c_0}\mathcal{S}h_n^{d+(g-1)n(n-1)}$. If we choose L_0 such that $c_0 + \deg(L_0) > (n-1)(2g-2)$, then $m_{L_0}(U) \subset {}_{c_0+\deg(L_0)}\mathcal{S}h_n^{d+n\deg(L_0)}$ and the map ${}^0\pi_n$ will be smooth over $m_{L_0}(U)$.

For every integer $c' \leq c$, consider the stack $\mathcal{A}_{c'}$ that classifies pairs $s : M \to L_{c'} \otimes \Omega$, where $M \in U$, $L_{c'} \in \mathrm{Pic}^{c'}$ and s is surjection (since U is of finite type, only for finitely many c''s the stack $\mathcal{A}_{c'}$ will be non-empty). For each c' we have a projection $\mathcal{A}_{c'} \to \mathcal{S}h_{n-1}^{d+(g-1)n(n-1)-c'-(2g-2)}$ that sends a triple as above to the kernel of s. The union of the images of $\mathcal{A}_{c'}$'s in $\mathcal{S}h_{n-1}$ is contained in a open sub-stack of finite type.

For every L_0 consider the fibered product

$$\mathcal{B}_{c'}(L_0) := {}^0\mathcal{S}h'_{n-1}{}^{d+(g-1)n(n-1)-c'-(2g-2)+(n-1)(\deg(L_0)-(2g-2))} \underset{\mathcal{S}h_{n-1}^{d+(g-1)n(n-1)-c'-(2g-2)}}{\times} \mathcal{A}_{c'},$$

where the map

$${}^0\mathcal{S}h'_{n-1}{}^{d+(g-1)n(n-1)-c'-(2g-2)+(n-1)(\deg(L_0)-(2g-2))} \to \mathcal{S}h_{n-1}^{\mathcal{S}h_{n-1}^{d+(g-1)n(n-1)-c'-(2g-2)}}$$

used in the definition of $\mathcal{A}_{c'}$ sends a pair $s : \Omega^{\otimes n-2} \to M'$ to $M' \otimes L_0^{-1} \otimes \Omega$. This is a stack that classifies triples:

$$(\Omega^{\otimes n-1} \hookrightarrow M' \otimes L_0, 0 \to M' \to M \to L_c \otimes \Omega \to 0),$$

where $M \in U$ and $L_c \in \mathrm{Pic}^c$.

Consider now the fibered product:

$${}^0\mathcal{S}h'_n{}^{d+(g-1)n(n-1)+n\deg(L_0)} \underset{\mathcal{S}h_n^{d+(g-1)n(n-1)+n\deg(L_0)}}{\times} U,$$

constructed using the map $U \hookrightarrow \mathcal{S}h_n^{d+(g-1)n(n-1)} \xrightarrow{m_{L_0}} \mathcal{S}h_n^{d+(g-1)n(n-1)+n\deg(L_0)}$.

We have a natural map:

$$\mathcal{B}_{c'}(\mathcal{L}_0) \to {}^0\mathcal{S}h'_n{}^{d+(g-1)n(n-1)+n\deg(L_0)} \underset{\mathcal{S}h_n^{d+(g-1)n(n-1)+n\deg(L_0)}}{\times} U,$$

that sends a triple

$$(\Omega^{\otimes n-1} \hookrightarrow M' \otimes L_0, 0 \to M' \to M \to L_c \otimes \Omega \to 0) \in \mathcal{B}_{c'}(\mathcal{L}_0)$$

as above to

$$(\Omega^{\otimes n-1} \hookrightarrow M' \hookrightarrow M \otimes L_0, M).$$

¿From Serre's duality it follows that $m_{L_0}(U)$ is contained in the image under ${}^0\pi_n$ of $\pi_{n-1}^{\vee}{}^{-1}({}_{c+\deg(L_0)}{}^0\mathcal{S}h_{n-1}{}^{d+n\deg(L_0)+(g-1)(n-1)(n-2)})$ if and only if the image of $\mathcal{B}_{c'}(\mathcal{L}_0)$ under the above map when intersected with every fiber of the projection

$$ {}^0\mathcal{S}h'_n{}^{d+(g-1)n(n-1)+n\deg(L_0)} \underset{{}^0\mathcal{S}h_n^{d+(g-1)n(n-1)+n\deg(L_0)}}{\times} U \to U $$

is a proper sub-variety there.

However, by Riemann-Roch, when the degree of L_0 is large, the dimension of $\mathcal{B}_{c'}(\mathcal{L}_0)$ grows as $const_1 + (n-1)\deg(L_0)$ and the dimension of every fiber of the above map grows as $const_2 + n\deg(L_0)$, the second constant being uniform on U, since it is of finite type.

This proves the required statement.

□

2.1.6

Proof of Main Theorem A(1)

Proof. It is enough to prove that the sheaf $\mathrm{Eis}_{E_1,\ldots,E_n}^{d_1,\ldots,d_n}$ is perverse and irreducible when restricted to every open sub-stack $U \subset \mathcal{S}h_n^{d+(g-1)n(n-1)}$ of finite type.

Using Lemma 2, we deduce that $\mathrm{Eis}_{E_1,\ldots,E_n}^{d_1,\ldots,d_n}|_U$ is perverse and irreducible if and only if $\mathrm{Eis}_{E_1,\ldots,E_n}^{d_1+\deg(L_0),\ldots,d_n+\deg(L_0)}|_{m_{L_0}(U)}$ is for any L_0. Let us choose L_0 as in the lemma above with $c = \mathfrak{c}_n$.

Since the map ${}^0\pi_n$ from

$$ \pi_{n-1}^{\vee}{}^{-1}({}_{\mathfrak{c}_n+\deg(L_0)}{}^0\mathcal{S}h_{n-1}{}^{d+(g-1)(n-1)(n-2)+(n-1)\deg(L_0)}) \cap_{(n-1)2g-2} \mathcal{S}h'_n{}^{d+(g-1)n(n-1)+n\deg(L_0)} $$

to $\mathcal{S}h_n^{d+(g-1)n(n-1)+n\deg(L_0)}$ is smooth and surjective over $m_{L_0}(U)$, it is enough to verify that the pull-back of $\mathrm{Eis}_{E_1,\ldots,E_n}^{d_1+\deg(L_0),\ldots,d_n+\deg(L_0)}[d-(g-1)n^2]$ to

$$ \pi_{n-1}^{\vee}{}^{-1}({}_{\mathfrak{c}_n+\deg(L_0)}{}^0\mathcal{S}h_{n-1}{}^{d+(g-1)(n-1)(n-2)+(n-1)\deg(L_0)}) \cap_{(n-1)(2g-2)} \mathcal{S}h'_n{}^{d+(g-1)n(n-1)+n\deg(L_0)} $$

is perverse and irreducible. The latter follows from Main Theorem B, since the sheaf $comp_{n-1}(\pi_0^{\vee *}(\mathcal{L}_{E_1,\ldots,E_n}^{d_1,\ldots,d_n})[d])$ is perverse and irreducible.

□

The assertion of Main Theorem A(2) has been already established, by combining Main Theorem A(2) for $n = 2$ and Corollary 1. Alternatively, we can repeat the argument used to prove Main theorem A(2) for $GL(2)$.

Corollary 10. *Let $E_1, ..., E_n$, $d_1, ..., d_n$ and $\mathfrak{c}_n$ be as in Main Theorem B. Then the isomorphism*

$$
{}^{0}\pi_n{}^{*}(\mathrm{Eis}_{E_1,...,E_n}^{d_1+(n-1)(g-1),...,d_n+(n-1)(g-1)})[d-(g-1)n^2](\frac{-(g-1)n^2}{2}) \simeq
$$
$$
\simeq comp_{n-1} \circ \pi_0^{\vee *}(\mathcal{L}_{E_1,...,E_n}^{d_1,...,d_n}[d]) \bigotimes_{i=1}^{n} (E_i^{(n-1)(g-1)}|_{\Omega^{\frac{n-1}{2}}}).
$$

holds over ${}_{(n-1)(2g-2)}{}^{0}\mathcal{S}h'^{\,d+n(n-1)(g-1)}_n$, *whenever the intersection*

$$
\pi_{n-1}^{\vee}{}^{-1}({}^{0}_{\mathfrak{c}_n}\mathcal{S}h_{n-1}^{d+(g-1)(n-1)(n-2)}) \cap {}_{(n-1)(2g-2)}{}^{0}\mathcal{S}h'^{\,d+(g-1)n(n-1)}_n
$$

is non-empty.

The proof follows from the fact that both the right hand side and the left hand side of the equation are irreducible perverse sheaves that are isomorphic over the above intersection.

Remark 2. In the case of $n = 2$, we have established in Corollary 9, that the isomorphism as in Corollary 10 holds for any $d_1, ..., d_n$ provided that d is large enough. This is true for any n, though will not prove it here.

2.2 Proof of Main Theorem B, first step

2.2.1

The proof of Main Theorem B will be based on considering several new geometric objects.

For a pair of integers (d_1, d_2) with $d_1 + d_2 = d$, consider the fibered product:

$$
\mathcal{S}h_1'^{\,d} \underset{\mathcal{S}h_1^d}{\times} \mathcal{F}l_{1,0}^{d_1,d_2}.
$$

This is a stack that classifies triples

$$
(M_1' \hookrightarrow M_1 \in \mathcal{F}l_{1,0}^{d_1,d_2}, s : \mathcal{O} \to M_1)
$$

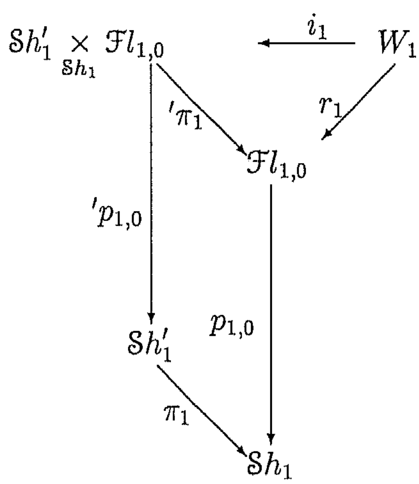
and it is a (non-smooth) vector bundle over $\mathcal{F}l_{1,0}^{d_1,d_2}$ which is smooth over the preimage of ${}_0\mathcal{S}h_1^d$ in $\mathcal{F}l_{1,0}^{d_1,d_2}$.

Let $W_1^{d_1,d_2}$ denote the closed sub-stack of $\mathcal{S}h_1'^{\,d} \underset{\mathcal{S}h_1^d}{\times} \mathcal{F}l_{1,0}^{d_1,d_2}$ that corresponds to those triples $(M_1' \hookrightarrow M_1, s)$ as above for which s factors as

$$s : \mathcal{O} \to M_1' \hookrightarrow M_1.$$

The stack $W_1^{d_1,d_2}$ is a smooth vector bundle over the preimage of ${}_{d_2}\mathcal{S}h_1^d$ under the projection $\mathcal{F}l_{1,0}^{d_1,d_2} \xrightarrow{p_{1,0}^{d_1,d_2}} \mathcal{S}h_1^d$.

We will denote by r_1 the projection from $W_1^{d_1,d_2}$ to $\mathcal{F}l_{1,0}^{d_1,d_2}$ and by i_1 the embedding of W^{d_1,d_2} into $\mathcal{S}h_1'^{\,d} \underset{\mathcal{S}h_1^d}{\times} \mathcal{F}l_{1,0}^{d_1,d_2}$.



Let E' (resp., E'') be a 1-dimensional (resp., an $m-1$-dimensional) local system on X.

We define the sheaf $\mathcal{F}1_{E',E''}^{d_1,d_2}$ on $\mathcal{S}h_1'^{\,d}$ as:

$${}'p_{1,0}^{d_1,d_2}{}_! \circ i_{1!} \circ r_1^* \circ q_{1,0}^{d_1,d_2 *}(\det{}^*(E'^{d_1}) \boxtimes \mathcal{L}_{E''}^{d_2})[d].$$

2.2.2

Lemma 8. $j_1^*(\mathcal{F}1_{E',E''}^{d_1,d_2}) \simeq \pi_0^{\vee *}(\mathcal{L}_{E',E''}^{d_1,d_2})[d]$.

Proof. Let ${}^0W_1^{d_1,d_2}$ be the preimage of ${}^0\mathcal{S}h'_1$ under the map ${}'p_{1,0}^{d_1,d_2} \circ i_1$.

We have a map ${}^0W_1^{d_1,d_2} \to \mathcal{F}l_{0,0}^{d_1,d_2}$ that sends a triple

$$(s : \mathcal{O} \hookrightarrow M'_1 \hookrightarrow M_1)$$

to

$$0 \to M'_1/\operatorname{Im}(s) \to M_1/\operatorname{Im}(s) \to M_1/M'_1 \to 0,$$

such that the square

$$\begin{array}{ccc} {}^0W_1^{d_1,d_2} & \longrightarrow & \mathcal{F}l_{0,0}^{d_1,d_2} \\ {\scriptstyle {}'p_{1,0}^{d_1,d_2}\circ i_1}\downarrow & & \downarrow{\scriptstyle q_{0,0}^{d_1,d_2}} \\ {}^0\mathcal{S}h_1'^{\,d} & \xrightarrow{\pi_0^\vee} & \mathcal{S}h_0^d \end{array}$$

is Cartesian.

Moreover, the map

$$(\det \times \mathrm{id}) \circ q_{1,0}^{d_1,d_2} \circ r_1 :{}^0 W_1^{d_1,d_2} \to \operatorname{Pic}^{d_1} \times \mathcal{S}h_0^{d_2}$$

coincides with the composition

$${}^0W_1^{d_1,d_2} \to \mathcal{F}l_{0,0}^{d_1,d_2} \xrightarrow{q_{0,0}^{d_1,d_2}} \mathcal{S}h_0^{d_1} \times \mathcal{S}h_0^{d_2+\ldots+d_n} \xrightarrow{\det\times\mathrm{id}} \operatorname{Pic}^{d_1} \times \mathcal{S}h_0^{d_2},$$

and the assertion follows by base change. □

The proof of Main Theorem B is based on the following result.

Proposition 5. (1) *Over* ${}_{d_1+(m-2)(2g-2)}\mathcal{S}h_1'^{\,d}$, *we have an isomorphism:*

$$\mathcal{F}_1{}_{E',E''}^{d_1,d_2} \simeq j_{1!} \circ j_1^*(\mathcal{F}_1{}_{E',E''}^{d_1,d_2}).$$

(2) *Over* ${}_{d_2}\mathcal{S}h_1'^{\,d}$, *we have an isomorphism:*

$$\mathcal{F}_1{}_{E',E''}^{d_1,d_2} \simeq j_{1!*} \circ j_1^*(\mathcal{F}_1{}_{E',E''}^{d_1,d_2}).$$

2.2.3

Before giving a proof of Proposition 5, let us show how it implies Main Theorem B for $n=2$.

Proof of Main Theorem B for $n=2$

According to Proposition 5(2), we must compute the Fourier transform of the sheaf ${}'p_{1,0}^{d_1,d_2}{}_! \circ i_{1!} \circ r_1^* \circ q_{1,0}^{d_1,d_2 *}(\det^*(E'^{d_1}) \boxtimes \det^*(E''^{d_2}))[d]$ on ${}_{d_2}\mathcal{S}h_1'^{\,d}$.

Consider the vector bundle ${}^0\mathcal{S}h_2'^{\,d+2g-2} \underset{\mathcal{S}h_1^d}{\times} \mathcal{F}l_{1,0}^{d_1,d_2}$ over $\mathcal{F}l_{1,0}^{d_1,d_2}$; when working over $p_{1,0}^{d_1,d_2-1}({}_{d_2}\mathcal{S}h_1^d) \subset \mathcal{F}l_{1,0}^{d_1,d_2}$ it identifies with with the dual of $\mathcal{S}h_1'^{\,d} \underset{\mathcal{S}h_1^d}{\times} \mathcal{F}l_{1,0}^{d_1,d_2}$. Let ${}''p_{1,0}^{d_1,d_2}$ denote the projection ${}^0\mathcal{S}h_2'^{\,d+2g-2} \underset{\mathcal{S}h_1^d}{\times} \mathcal{F}l_{1,0}^{d_1,d_2,\leq t} \to {}^0\mathcal{S}h_2'^{\,d+2g-2}$ and let $\widetilde{\mathrm{Four}}_1$ denote the relative Fourier transform functor from the category of perverse sheaves on $\mathcal{S}h_1'^{\,d} \underset{\mathcal{S}h_1^d}{\times} \mathcal{F}l_{1,0}^{d_1,d_2}$ to that on ${}^0\mathcal{S}h_2'^{\,d+2g-2} \underset{\mathcal{S}h_1^d}{\times} \mathcal{F}l_{1,0}^{d_1,d_2}$.

According to the standard properties of the Fourier transform functor,

$$\mathrm{Four}_1 \circ {}'p_{1,0}^{d_1,d_2}{}_! \circ i_{1!} \circ r_1^* \circ q_{1,0}^{d_1,d_2 *}(\det^*(E'^{d_1}) \boxtimes \det^*(E''^{d_2})) \simeq$$
$${}''p_{1,0}^{d_1,d_2}{}_! \circ \widetilde{\mathrm{Four}}_1 \circ i_! \circ r_1^* \circ q_{1,0}^{d_1,d_2 *}(\det^*(E'^{d_1}) \boxtimes \det^*(E''^{d_2}))$$

Let $i_1' : W_1'^{d_1,d_2} \to {}^0\mathcal{S}h_2'^{\,d+2g-2} \underset{\mathcal{S}h_1^d}{\times} \mathcal{F}l_{1,0}^{d_1,d_2}$ denote the embedding of the orthogonal complement to W_1 inside ${}^0\mathcal{S}h_2'^{\,d+2g-2} \underset{\mathcal{S}h_1^d}{\times} \mathcal{F}l_{1,0}^{d_1,d_2}$ and let $r_1' : W_1'^{d_1,d_2} \to \mathcal{F}l_{1,0}^{d_1,d_2}$ denote its projection onto $\mathcal{F}l_{1,0}^{d_1,d_2}$.

We have:

$$\widetilde{\mathrm{Four}}_1 \circ i_{1!} \circ r_1^* \circ q_{1,0}^{d_1,d_2 *}(\det^*(E'^{d_1}) \boxtimes \mathcal{L}_{E''}^{d_2}) \simeq$$
$$i'_{1!} \circ r_1'^* \circ q_{1,0}^{d_1,d_2 *}(\det^*(E'^{d_1}) \boxtimes \det^*(E''^{d_2}))[d_2-d_1+g-1](\frac{d_2-d_1+g-1}{2})$$

and therefore $\mathrm{Four}_1(\mathcal{F}1_{E',E''}^{d_1,d_2})$ identifies with

$${}''p_{1,0}^{d_1,d_2}{}_! \circ i'_{1!} \circ r_1'^* \circ q_{1,0}^{d_1,d_2 *}(\det^*(E'^{d_1}) \boxtimes \det^*(E''^{d_2}))(\overline{\mathbb{Q}}_\ell[1](\frac{1}{2}))^{\otimes d_2-d_1+g-1}[d].$$

Lemma 9. *We have a natural map:*

$$W_1'^{d_1,d_2} \to \mathcal{F}l_{1,1}^{d_1,d_2+2g-2} \underset{\mathcal{S}h_2^{d+2g-2}}{\times} {}^0\mathcal{S}h_2'^{\,d+2g-2},$$

which is an isomorphism over the preimage of ${}_{d_2}{}^0\mathcal{S}h_2'^{\,d+2g-2}$ *such that:*

(1) *The map* $''pr_{1,0}^{d_1,d_2} \circ i_1' : W_1'^{d_1,d_2} \to {}^0\mathcal{S}h_2'^{\,d+2g-2}$ *coincides with the composition*

$$W_1'^{d_1,d_2} \to \mathcal{F}l_{1,1}^{d_1,d_2+2g-2} \underset{\mathcal{S}h_2^{d+2g-2}}{\times} {}^0\mathcal{S}h_2'^{\,d+2g-2} \to {}^0\mathcal{S}h_2'^{\,d+2g-2}.$$

(2) *The map*

$$(\det \times \det) \circ q_{1,0}^{d_1,d_2} \circ r_1' : W_1'^{d_1,d_2} \to \mathrm{Pic}^{d_1} \times \mathrm{Pic}^{d_2}$$

coincides with the composition

$$W_1'^{d_1,d_2} \to \mathcal{F}l_{1,1}^{d_1,d_2+2g-2} \underset{\mathcal{S}h_2^{d+2g-2}}{\times} {}^0\mathcal{S}h_2'^{\,d+2g-2} \to \mathcal{F}l_{1,1}^{d_1,d_2+2g-2} \to$$
$$\to \mathcal{S}h_1^{d_1} \times \mathcal{S}h_1^{d_2+2g-2} \to \mathrm{Pic}^{d_1} \times \mathrm{Pic}^{d_2+2g-2} \xrightarrow{m_{\Omega^{-1}}} \mathrm{Pic}^{d_1} \times \mathrm{Pic}^{d_2}$$

According to this lemma, we obtain by base change:

$$\mathrm{Four}_1 \circ j_{1!*} \circ \pi_0^{\vee *}(\mathcal{L}_{E_1,E_2}^{d_1,d_2}[d]) \simeq$$
$$\simeq {}^0\pi_2{}^*(\mathrm{Eis}_{E_1,E_2}^{d_1+g-1,d_2+g-1}) \otimes ((E_1 \otimes E_2)^{1-g}|_{\Omega^{-\frac{1}{2}}})[d-(4g-4)](2-2g).$$

□

2.2.4

We will deduce Proposition 5 from the following result:

Lemma 10. *Let E' and E'' be a 1-dimensional and an $m-1$-dimensional local systems on X respectively satisfying: $H^0(X, E'^{\vee} \otimes E'') = 0$. Then:*

(1) *For any $d' \leq d$ the sheaf*

$$p_{1,0}^{d',d-d'}{}_! \circ q_{1,0}^{d',d-d'}{}^*(\det{}^*(E'^{d'}) \boxtimes \mathcal{L}_{E''}^{d-d'})$$

vanishes over ${}_{d'+(m-2)(2g-2)}\mathcal{S}h_1^d$.

(2) *If $d' > g-1$, the sheaf $p_{1,0}^{d',d-d'}{}_! \circ q_{1,0}^{d',d-d'}{}^*(\det{}^*(E'^{d'}) \boxtimes \mathcal{L}_{E''}^{d-d'})$ over the whole of $\mathcal{S}h_1^d$ has cohomologies in degrees $< d$.*

Proof. It is easy to reduce the assertion of the lemma to the case when E' is a trivial local system (we do this in order to simplify the notation).

Consider the stratification of $\mathcal{S}h_1^d$ by the locally closed sub-stacks $\mathcal{S}h_1^{d;\leq t}$. We have:

$$_{d'+(m-2)(2g-2)}\mathcal{S}h_1^d = \bigcup_{0\leq t<d-d'-(m-1)(2g-2)} \mathcal{S}h_1^{d;\leq t}.$$

We stratify $\mathcal{F}l_{1,0}^{d',d-d'}$ by locally closed sub-stacks $Y^{t,t'}$, $\max(t-d+d',0)\leq t'\leq t$, where a pair $L'\hookrightarrow L\in\mathcal{F}l_{1,0}^{d',d-d'}$ belongs to $Y^{t,t'}$ if $L\in\mathcal{S}h_1^{d;t}$ and $L'\in\mathcal{S}h_1^{d';t'}$.

By restricting the map $p_{1,0}^{d',d-d'}$ to $Y^{t,t'}$ we obtain a map $\beta_{t,t'}:Y^{t,t'}\to\mathcal{S}h_1^{d;t}$. It is enough to check that the sheaf obtained by applying the functor $\beta_{t,t'!}$ to the restriction of $q_{1,0}^{d',d-d'*}(\det^*(E'^{d'})\boxtimes\mathcal{L}_{E''}^{d-d'})$ to $Y^{t,t'}$ vanishes over $_{d'+(m-2)(2g-2)}\mathcal{S}h_1^d$ (resp., has cohomologies in degrees $<d$ over $\mathcal{S}h_1^d$, if $d'>g-1$).

We have a natural map $\epsilon:\mathcal{S}h_1^{d;t}\to\mathcal{S}h_0^t$. Consider now the following fibered products:

$$\mathcal{S}h_1^{d;t}\underset{\mathcal{S}h_0^t}{\times}\mathcal{F}l_{0,0}^{t',t-t'}$$

and

$$\widetilde{Y}^{t,t'}:=\mathcal{S}h_1^{d;t}\underset{\mathcal{S}h_0^t}{\times}\mathcal{F}l_{0,0}^{t',t-t'}\times X^{(d-d'-(t-t'))}\underset{\mathcal{S}h_0^{t-t'}\times\mathcal{S}h_0^{d-d'-(t-t')}}{\times}\mathcal{F}l_{0,0}^{t-t',d-d'-(t-t')}$$

(we used the natural map $X^{(d-d'-(t-t'))}\to\mathcal{S}h_0^{d-d'-(t-t')}$ to define the second fibered product) and let us denote by $\gamma_{t,t'}$ the map $\widetilde{Y}^{t,t'}\to\mathcal{F}l_{0,0}^{t-t',d-d'-(t-t')}$.

We have a map $\alpha_{t,t'}:Y^{t,t'}\to\widetilde{Y}^{t,t'}$ that sends a pair $(L'\hookrightarrow L)\in Y^{t,t'}$ to

$$(L,0\to T'\to T\to T/T'\to 0)\in\mathcal{S}h_1^{d;t}\underset{\mathcal{S}h_0^t}{\times}\mathcal{F}l_{0,0}^{t',t-t'},(L'/T'\hookrightarrow L/T)\in X^{(d-d'-(t-t'))},$$

$$(0\to T/T'\to L/L'\to(L/T)/(L'/T')\to 0)\in\mathcal{F}l_{0,0}^{t-t',d-d'-(t-t')}.$$

It is easy to see that $\alpha_{t,t'}$ is an affine fibration of relative dimension $t-t'$ and that the restriction of $q_{1,0}^{d',d-d'*}(\det^*(E'^{d'})\boxtimes\mathcal{L}_{E''}^{d-d'})$ to $Y^{t,t'}$ identifies with

$$\alpha_{t,t'}^*\circ\gamma_{t,t'}^*\circ p_{0,0}^{t-t',d-d'-(t-t')*}(\mathcal{L}_{E''}).$$

The map $\beta_{t,t'!}$ is a composition of $\alpha_{t,t'}$ and the projection

$$\widetilde{Y}^{t,t'}\to\mathcal{S}h_1^{d;t}\underset{\mathcal{S}h_0^t}{\times}\mathcal{F}l_{0,0}^{t',t-t'}\times X^{(d-d'-(t-t'))}\to\mathcal{S}h_1^{d;t}\underset{\mathcal{S}h_0^t}{\times}\mathcal{F}l_{0,0}^{t',t-t'}\to\mathcal{S}h_1^{d;t}.$$

According to the projection formula and to Proposition 2(2) we obtain that the direct image of $\alpha^*_{t,t'} \circ \gamma^*_{t,t'} \circ p^{t-t',d-d'-(t-t')*}_{0,0}(\mathcal{L}_{E''})$ on $\mathcal{S}h^{d;t}_1 \underset{\mathcal{S}h^t_0}{\times} \mathcal{F}l^{t',t-t'}_{0,0} \times X^{(d-d'-(t-t'))}$ identifies with

$$'q^{t',t-t'*}_{0,0}(\mathcal{L}^{t-t'}_{E''}) \boxtimes E''^{(d-d'-(t-t'))}[-2(t-t')](t'-t),$$

where we have denoted by $'q^{t',t-t'}_{0,0}$ the composition map

$$\mathcal{S}h^{d;t}_1 \underset{\mathcal{S}h^t_0}{\times} \mathcal{F}l^{t',t-t'}_{0,0} \to \mathcal{F}l^{t',t-t'}_{0,0} \xrightarrow{q^{t',t-t'}_{0,0}} \mathcal{S}h^{t'}_0 \times \mathcal{S}h^{t-t'}_0 \to \mathcal{S}h^{t-t'}_0.$$

Now, over ${}_{d'+(m-2)(2g-2)}\mathcal{S}h^d_1$, we have:

$d-d'-(t-t') \geq d-d'-t > d-d'-(d-d'-(m-1)(2g-2)) = (m-1)(2g-2)$,

hence the hyper-cohomology $\mathbb{H}(X^{(d-d'-(t-t')}, E''^{(d-d'-(t-t')})$ vanishes, since E'' being non-trivial implies that

$$\mathbb{H}(X^{(d-d'-(t-t')}, E''^{(d-d'-(t-t')}) \simeq \Lambda^{(d-d'-(t-t')}(H^1(X, E''))$$

and $d-d'-(t-t') > (m-1)(2g-2) = \dim(H^1(X, E''))$.

Therefore, the direct image of the restriction of $q^{d',d-d'*}_{1,0}(\det{}^*(E'^{d'}) \boxtimes \mathcal{L}^{d-d'}_{E''})$ to $Y^{t,t'}$ under the map $Y^{t,t'} \to \mathcal{S}h^{d;t}_1 \underset{\mathcal{S}h^t_0}{\times} \mathcal{F}l^{t',t-t'}_{0,0}$ vanishes if $\mathcal{S}h^{d;t}_1 \in {}_{d'+(m-2)(2g-2)}\mathcal{S}h^d_1$, which implies the first point of the lemma.

To prove the second point of the lemma, note that

$$\beta_{t,t'!}(q^{d',d-d'*}_{1,0}(\det{}^*(E'^{d'}) \boxtimes \mathcal{L}^{d-d'}_{E''})|_{Y^{t,t'}})$$

identifies with the pull-back under $\epsilon : \mathcal{S}h^{d;t} \to \mathcal{S}h^t_0$ of the sheaf $\mathcal{L}^{t',t-t'}_{E',E''}$ tensored by $\Lambda^{(d-d'-(t-t')}(H^1(X, E''))[-(d-d'-(t-t'))-2(t-t')](t'-t)$.

The map ϵ is smooth of relative dimension $(g-1)-t$ and the sheaf $\mathcal{L}^{t',t-t'}_{E',E''}$ on $\mathcal{S}h^t_0$ is perverse. Therefore,

$$\beta_{t,t'!}(q^{d',d-d'*}_{1,0}(\det{}^*(E'^{d'}) \boxtimes \mathcal{L}^{d-d'}_{E''})|_{Y^{t,t'}})$$

lives in the cohomological degree $(d-d'-(t-t')) + 2(t-t') + (g-1) - t \leq (g-1) - d' + d < d$.

□

2.2.5

Proof of Proposition 5.

In order to simplify the notation, we will again assume that E' is a trivial local system.

When working over ${}_{d_2}\mathcal{S}h_1^d$, the map r_1 is smooth. Therefore over ${}_{d_2}\mathcal{S}h_1'^{\,d}$

$$\mathbb{D}(\mathcal{F}1^{d_1,d_2}_{E',E''}) \simeq \mathcal{F}1^{d_1,d_2}_{E'^{\vee},E''^{\vee}}$$

and to prove the proposition it is enough to show, that

$${}'p^{d_1,d-d_1}_{1,0}{}_! \circ i_{1!} \circ r_1^* \circ q^{d_1,d-d_1 *}_{1,0}(\det{}^*(E_1^{d_1}) \boxtimes \mathcal{L}^{d-d_1}_{E_2\oplus\ldots\oplus E_n})$$

vanishes when restricted to ${}_{d_1+(m-2)(2g-2)}\mathcal{S}h_1'^{\,d} - {}_{d_1+(m-2)(2g-2)}{}^0\mathcal{S}h_1'^{\,d}$ and has cohomologies in degrees $< d$ when restricted to ${}_{d_2}\mathcal{S}h_1'^{\,d} - {}^0_{d_2}\mathcal{S}h_1'^{\,d}$.

For any c, the stack ${}_c\mathcal{S}h_1'^{\,d} - {}^0_c\mathcal{S}h_1'^{\,d}$ is a union of locally closed sub-stacks $Z_{\tilde{t}}$, $0 \leq \tilde{t} < d-(c+2g-2)$, where $Z_{\tilde{t}}$ is a stack that classifies pairs

$$(L \in {}_c\mathcal{S}h_1^d, s : \mathcal{O} \to L)$$

with s being surjective on a torsion sub-sheaf $\tilde{T} \hookrightarrow L$ of length $\tilde{t}$.

We have a natural map $Z_{\tilde{t}} \to {}_{c-\tilde{t}}\mathcal{S}h_1^{d-\tilde{t}}$ that sends a pair $(L \in {}_c\mathcal{S}h_1^d, s : \mathcal{O} \to L)$ as above to

$$L/\tilde{T} \in {}_c\mathcal{S}h_1^{d-\tilde{t}} \subset {}_{c-\tilde{t}}\mathcal{S}h_1^{d-\tilde{t}}$$

and a map

$$Z_{\tilde{t}} \underset{{}_c\mathcal{S}h_1'^{\,d}}{\times} W_1 \to \mathcal{F}l^{d_1-\tilde{t},d_2}_{1,0}$$

that sends a triple

$$(L' \hookrightarrow L, s : \mathcal{O} \to L') \in Z_{\tilde{t}} \underset{{}_c\mathcal{S}h_1'^{\,d}}{\times} W_1$$

to

$$(L'/\tilde{T} \hookrightarrow L/\tilde{T}) \in \mathcal{F}l^{d_1-\tilde{t},d_2}_{1,0}.$$

We have a Cartesian square:

$$\begin{array}{ccc} Z_{\tilde{t}} \underset{{}_c\mathcal{S}h_1'^{\,d}}{\times} W_1 & \longrightarrow & {}_{c-\tilde{t}}\mathcal{F}l^{d_1-\tilde{t},d_2}_{1,0} \\ \downarrow & & \downarrow {\scriptstyle p^{d_1-\tilde{t},d-d_1}_{1,0}} \\ Z_{\tilde{t}} & \longrightarrow & {}_{c-\tilde{t}}\mathcal{S}h_1^{d-\tilde{t}} \end{array}$$

and the restriction of the sheaf

$$'p_{1,0}^{d_1,d-d_1}{}_! \circ i_{1!} \circ r_1^* \circ q_{1,0}^{d_1,d-d_1 *}(\det{}^*(E'^{d_1}) \boxtimes \mathcal{L}_{E''}^{d_2})$$

to $Z_{\tilde{t}}$ identifies with the pull-back under the map $Z_{\tilde{t}} \to {}_{c-\tilde{t}}\mathcal{S}h_1^{d-\tilde{t}}$ of the sheaf

$$p_{1,0}^{d_1-\tilde{t},d_2}{}_! \circ q_{1,0}^{d_1-\tilde{t},d_2 *}(\det{}^*(E'^{d_1-\tilde{t}}) \boxtimes \mathcal{L}_{E''}^{d_2})$$

on ${}_{c-\tilde{t}}\mathcal{S}h_1^{d-\tilde{t}}$.

Therefore, for $c = d_1+(m-2)(g-1)$ this restriction vanishes by Lemma 10(1), which proves the first point of the proposition.

To prove the second point of the lemma, let us take $c = d_2$. We will have:

$$d_1 - \tilde{t} > 2g - 2 \geq g - 1,$$

hence the sheaf

$$p_{1,0}^{d_1-\tilde{t},d_2}{}_! \circ q_{1,0}^{d_1-\tilde{t},d_2 *}(\det{}^*(E'^{d_1-\tilde{t}}) \boxtimes \mathcal{L}_{E''}^{d_2})$$

on $\mathcal{S}h_1^{d-\tilde{t}}$ has cohomologies in degrees $< d$, by Lemma 10(2). This implies the second point of the proposition ,since the map $Z_{\tilde{t}} \to \mathcal{S}h_1^{d-\tilde{t}}$ is smooth of relative dimension 0.

□

2.2.6

The following result will be an important technical point in the proof of Main Theorem C.

Proposition 6. *Let d satisfy $d > 4g-4+2t$ and let E_1 and E_2 be non-isomorphic 1-dimensional local systems. Then for any point of $\mathcal{S}h_1'^{\,d;\leq t} - {}^0\mathcal{S}h_1'^{\,d;\leq t}$ the Euler-Poincaré characteristic of the stalk of $j_{1!*} \circ \pi_0^{\vee *}(\mathcal{L}_{E_1,E_2})$ vanishes.*

Proof. We will again assume, that E' is the constant local system.

Let us divide all pairs $(d_1, d_2), d_1 + d_2 = d$ into 3 non-intersecting sets:

$$d_2 \leq 2g - 2 + t; d_1 \leq 2g - 2 + t; \min(d_1, d_2) > 2g - 2 + t.$$

Proposition 5(1) and (2) implies, that if a pair (d_1, d_2) belongs to the third set,

$$j_{1!*} \circ \pi_0^{\vee *}(\mathcal{L}_{E_1,E_2}^{d_1,d_2})|\mathcal{S}h_1'^{\,d;\leq t} - {}^0\mathcal{S}h_1'^{\,d;\leq t} = 0.$$

We will show that the Euler-Poincaré characteristic of the stalk of $\underset{(d_1,d_2)}{\oplus} j_{1!*} \circ \pi_0^{\vee *}(\mathcal{L}_{E_1,E_2}^{d_1,d_2})$ vanishes at every point of $\mathcal{S}h_1'^{\,d;\leq t} - {}^0\mathcal{S}h_1'^{\,d;\leq t}$ when (d_1, d_2) run through the first set or the second one.

If (d_1, d_2) belongs to the first set, then $d_1 > 2g - 2 + t$ and according to Proposition 5(1), we can compute $j_{1!*} \circ \pi_0^{\vee *}(\mathcal{L}_{E_1,E_2}^{d_1,d_2}[d])$ over $\mathcal{S}h_1'^{\,d;\leq t}$ through

$$'p_{1,0}^{d_1,d_2}{}_! \circ i_{1!} \circ r_1^* \circ q_{1,0}^{d_1,d_2 *}(\det{}^*(E'^{d_1}) \boxtimes \det{}^*(E''^{d_2}))[d].$$

As we saw in the proof of Proposition 5 the stalk of

$$'p_{1,0}^{d_1,d_2}{}_! \circ i_{1!} \circ r_1^* \circ q_{1,0}^{d_1,d_2 *}(\det{}^*(E'^{d_1}) \boxtimes \det{}^*(E''^{d_2}))$$

at every point of $\mathcal{S}h_1'^{\,d;\leq t} - {}^0\mathcal{S}h_1'^{\,d;\leq t}$ is isomorphic to the stalk of

$$p_{1,0}^{d_1-\tilde{t},d_2}{}_! \circ q_{1,0}^{d_1-\tilde{t},d_2 *}(\det{}^*(E'^{d_1-\tilde{t}}) \boxtimes \det{}^*(E''^{d_2}))$$

at some point of $\mathcal{S}h_1{}^{d-\tilde{t};t'}$ for some $\tilde{t}$ and t', up to tensoring by a 1-dimensional vector space.

Following the proof of Lemma 10, we see that each

$$p_{1,0}^{d_1-\tilde{t},d_2}{}_! \circ q_{1,0}^{d_1-\tilde{t},d_2 *}(\det{}^*(E'^{d_1-\tilde{t}}) \boxtimes \det{}^*(E''^{d_2}))$$

is a successive extension of sheaves $\mathcal{T}(t, t'') \otimes \mathbb{H}^{\cdot}(X^{(d_2-(t'-t''))}, (E_2 \otimes E_1^{\vee})^{(d_2-(t'-t''))})$, where $\mathcal{T}(t, t'')$ is a sheaf on $\mathcal{S}h_1{}^{d-\tilde{t};t'}$ independent of d_2 and where t'' runs between $t' - d_2$ and t'.

For (d_1, d_2) running through the first set for a point of $\mathcal{S}h_1{}^{d-\tilde{t};t'}$ we have:

$$\chi(\underset{(d_1,d_2)}{\oplus} p_{1,0}^{d_1-\tilde{t},d_2}{}_! \circ q_{1,0}^{d_1-\tilde{t},d_2 *}(\det{}^*(E'^{d_1-\tilde{t}}) \boxtimes \det{}^*(E''^{d_2})) =$$
$$= \sum_{t''} \chi(\mathcal{T}(t', t'')) \cdot \chi(\sum_{t'-t'' \leq d_2 \leq 2g-2+t} \mathbb{H}^{\cdot}(X^{(d_2-(t'-t''))}, (E_2 \otimes E_1^{\vee})^{(d_2-(t'-t''))}))$$

However,

$$\chi(\sum_i H^{\cdot}(X^{(i)}, (E_2 \otimes E_1^{\vee})^{(i)})) = \chi(\Lambda^{\cdot}(H^1(X, E_2 \otimes E_1)[-1])) = 0,$$

which implies the assertion.

The statement for (d_1, d_2) running through the second set, follows from what we have proven for the first set by interchanging the roles of (d_1, E_1) and (d_2, E_2) using Proposition 2(4).

□

Remark 3. In fact, Proposition 5 allows to calculate completely the stalks of $j_{1!*} \circ \pi_0^{\vee *}(\mathcal{L}_{E_1,E_2})$ studied above.

Corollary 11. *Let E_1 and E_2 be as in Proposition 6. For $d > 4g-2$, the stalks of the sheaves* $\mathrm{Four}_1 \circ j_{1!*} \circ \pi_0^{\vee *}(\mathcal{L}_{E_1,E_2})$ *and* $\mathrm{Four}_1 \circ j_{1!} \circ \pi_0^{\vee *}(\mathcal{L}_{E_1,E_2})$ *have an equal Euler-Poincaré characteristic at every point of* $\pi_1^{\vee -1}(\mathcal{S}h_1'^{\,d;\leq 1})$.

Proof. Consider the cone of the arrow

$$j_{1!} \circ \pi_0^{\vee *}(\mathcal{L}_{E_1,E_2}) \to j_{1!*} \circ \pi_0^{\vee *}(\mathcal{L}_{E_1,E_2}).$$

Over $\mathcal{S}h_1^{d;\leq 0}$, this a complex of sheaves supported on the zero section of the vector bundle $\mathcal{S}h_1'^{d;\leq 0}$ with a zero pointwise Euler-Poincaré characteristic. Therefore, the Euler-Poincaré characteristic of the stalks of its Fourier transform also vanishes.

Over every point of $\mathcal{S}h_1^{d;1}$, the above cone is a G_m-equivariant sheaf supported on a 1-dimensional sub-space of the fiber of $\mathcal{S}h_1'^{d;\leq 1}$ at this point. In this case it also easy to see that vanishing of the pointwise Euler-Poincaré characteristics implies the same fact after the Fourier transform.

□

2.3 Proof of Main Theorem B

2.3.1

To prove Main theorem B for any n we have to introduce several new geometric objects, that generalize those considered in Sect. 2.2.1.

Consider the fibered product

$$\mathcal{S}h_k'^{\,d+k(k-1)(g-1)} \underset{\mathcal{S}h_k^{d+k(k-1)(g-1)}}{\times} \mathcal{F}l_{k,0}^{d_1+\ldots+d_k+k(k-1)(g-1),d_{k+1}+\ldots+d_n}.$$

This is a stack that classifies triples

$$(M_k' \hookrightarrow M_k \in \mathcal{F}l_{k,0}^{d_1+\ldots+d_k+k(k-1)(g-1),d_{k+1}+\ldots+d_n}, s : \Omega^{\otimes k-1} \to M_k)$$

and it is a (non-smooth) vector bundle over $\mathcal{F}l_{k,0}^{d_1+\ldots+d_k+k(k-1)(g-1),d_{k+1}+\ldots+d_n}$ which is smooth over the preimage of ${}_{(k-1)(2g-2)}\mathcal{S}h_k^{d+k(k-1)(g-1)}$ under the projection

$$p_{k,0}^{d_1+\ldots+d_k+k(k-1)(g-1),d_{k+1}+\ldots+d_n} : \mathcal{F}l_{k,0}^{d_1+\ldots+d_k+k(k-1)(g-1),d_{k+1}+\ldots+d_n} \to \mathcal{S}h_k^{d+k(k-1)(g-1)}.$$

Let $W_k^{d_1+\ldots+d_k+k(k-1)(g-1),d_{k+1}+\ldots+d_n}$ denote the closed sub-stack of

$$\mathcal{S}h'^{\,d+k(k-1)(g-1)}_k \underset{\mathcal{S}h_k^{d+k(k-1)(g-1)}}{\times} \mathcal{F}l_{k,0}^{d_1+\ldots+d_k+k(k-1)(g-1),d_{k+1}+\ldots+d_n}$$

that corresponds to those triples $(M'_k \hookrightarrow M_k, s)$ as above for which s factors as

$$s : \Omega^{k-1} \to M'_k \hookrightarrow M_k.$$

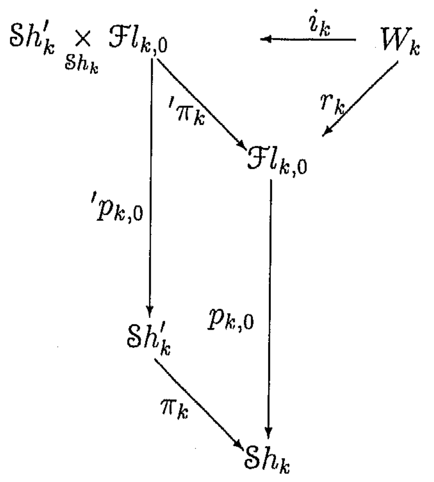
We will denote by r_k the projection

$$W_k^{d_1+\ldots+d_k+k(k-1)(g-1),d_{k+1}+\ldots+d_n} \to \mathcal{F}l_{k,0}^{d_1+\ldots+d_k+k(k-1)(g-1),d_{k+1}+\ldots+d_n}$$

and by i_k the embedding of $W_k^{d_1+\ldots+d_k+k(k-1)(g-1),d_{k+1}+\ldots+d_n}$ into

$$\mathcal{S}h'^{\,d+k(k-1)(g-1)}_k \underset{\mathcal{S}h_k^{d+k(k-1)(g-1)}}{\times} \mathcal{F}l_{k,0}^{d_1+\ldots+d_k+k(k-1)(g-1),d_{k+1}+\ldots+d_n}.$$

The map r_k is representable and it realizes $W_k^{d_1+\ldots+d_k+k(k-1)(g-1),d_{k+1}+\ldots+d_n}$ as a non-smooth vector bundle over the base $\mathcal{F}l_{k,0}^{d_1+\ldots+d_k+k(k-1)(g-1),d_{k+1}+\ldots+d_n}$.



For each k, $1 \leq k \leq n-1$ we introduce a sheaf $\mathcal{F}_{k\,E_1,\dots,E_n}^{\,d_1,\dots,d_n}$ on $\mathcal{S}h_k'^{\,d+k(k-1)(g-1)}$:

$$\mathcal{F}_{k\,E_1,\dots,E_n}^{\,d_1,\dots,d_n} := {}'p_{k,0}^{\sum_{i=1}^{k} d_i+k(k-1)(g-1),\ \sum_{i=k+1}^{n} d_i}{}_! \circ i_{k!} \circ r_k^* \circ q_{k,0}^{\sum_{i=1}^{k} d_i+k(k-1)(g-1),\ \sum_{i=k+1}^{n} d_i}{}_*$$
$$(\mathrm{Eis}_{E_1,\dots,E_k}^{d_1,\dots,d_k} \boxtimes \mathcal{L}_{E_{k+1},\dots,E_n}^{d_{k+1},\dots,d_n}) \otimes$$
$$\otimes \left((E_1 \otimes \dots \otimes E_k)^{-(g-1)(k-1)}|_{\Omega^{\otimes -\frac{k-1}{2}}}\right) \otimes (\overline{\mathbb{Q}}_\ell[1](\frac{1}{2}))^{\otimes d_1+\dots+d_k-k^2(g-1)+k(d_{k+1}+\dots+d_n)}$$

The sheaf $\mathcal{F}_{1\,E_1,\dots,E_n}^{\,d_1,\dots,d_n}$ of Sect. 2.2.1 is a direct summand of $\mathcal{F}_{1\,E_1,E_2\oplus\dots\oplus E_n}^{\,d_1,d-d_1}$ and as in Lemma 8, we have:

Lemma 11. $j_1^*(\mathcal{F}_{1\,E_1,\dots,E_n}^{\,d_1,\dots,d_n}) \simeq \pi_0^{\vee *}(\mathcal{L}_{E_1,\dots,E_n}^{d_1,\dots,d_n})[d]$.

2.3.2

We will deduce Main Theorem B from the following result:

Proposition 7. (1) *The canonical map*

$$j_{1!} \circ j_1^*(\mathcal{F}_{1\,E_1,\dots,E_n}^{\,d_1,\dots,d_n}) \to \mathcal{F}_{1\,E_1,\dots,E_n}^{\,d_1,\dots,d_n}$$

is an isomorphism over ${}_{d_1+(n-2)(2g-2)}\mathcal{S}h_1'^{\,d}$.

(2) *For each k we have a natural map:*

$$j_{k+1!} \circ \mathrm{Four}_k(\mathcal{F}_{k\,E_1,\dots,E_n}^{\,d_1,\dots,d_n}) \to \mathcal{F}_{k+1\,E_1,\dots,E_n}^{\,d_1,\dots,d_n}$$

which is an isomorphism over ${}_{d_{k+1}+(n-2)(2g-2)}\mathcal{S}h_k'^{\,d}$.

(3) *We have a canonical map:*

$$\mathrm{Four}_{n-1}(\mathcal{F}_{k+1\,E_1,\dots,E_n}^{\,d_1,\dots,d_n}) \to$$
$${}^0\pi_n^*(\mathrm{Eis}_{E_1,\dots,E_n}^{d_1,\dots,d_n})[d-n^2](-\frac{n^2}{2}) \otimes \left((E_1 \otimes \dots \otimes E_n)^{-(n-1)(g-1)}|_{\Omega^{\otimes -\frac{n-1}{2}}}\right)$$

which is an isomorphism over ${\pi_{n-1}^{\vee}}^{-1}\left({}_{d_n+(n-1)(2g-2)}{}^0\mathcal{S}h_{n-1}^{\,d+(g-1)(n-1)(n-2)}\right)$.

Proof of Main Theorem B.

Remark 4. The proof given below uses the machinery of weights and relies on the decomposition theorem. There exists also a more natural, purely geometric argument, in the spirit of Langlands classification. We will present it elsewhere.

It is known that every 1-dimensional local system on a smooth projective and geometrically connected curve over $\mathbb{F}_q$ is pure. By making a Tate twist, we can assume that for each i, $0 \leq i \leq n$, E_i is pure of weight 0.

Points (1) and (2) of Proposition 7 imply by induction that $\forall k$, $1 \leq k \leq n-1$

(a) Over ${}_{\mathfrak{c}_n}\mathcal{S}h'^{\,d+k(k-1)(g-1)}_k$, the sheaf $\mathcal{F}_k{}^{d_1,\dots,d_n}_{E_1,\dots,E_n}$ lives in cohomological dimensions ≤ 0.

(b) Over ${}_{\mathfrak{c}_n}\mathcal{S}h'^{\,d+k(k-1)(g-1)}_k$, the sheaf $\mathcal{F}_k{}^{d_1,\dots,d_n}_{E_1,\dots,E_n}$ has weights ≤ 0.

(c) The only sub-quotient of weight 0 of $h^0(\mathcal{F}_k{}^{d_1,\dots,d_n}_{E_1,\dots,E_n})$ over ${}_{\mathfrak{c}_n}\mathcal{S}h'^{\,d+k(k-1)(g-1)}_k$ is

$$comp_{k-1}(\pi_0^{\vee *}(\mathcal{L}^{d_1,\dots,d_n}_{E_1,\dots,E_n})[d])$$

(this last points relies on the decomposition theorem, [1]).

Therefore the same is true for the restriction of $\mathrm{Four}_{n-1}(\mathcal{F}_{n-1}{}^{d_1,\dots,d_n}_{E_1,\dots,E_n})$ to

$$\pi^{\vee}_{n-1}{}^{-1}({}^{0}_{\mathfrak{c}_n}\mathcal{S}h^{d+(g-1)(n-1)(n-2)}_{n-1}),$$

where it identifies with

$${}^{0}\pi_n^*(\mathrm{Eis}^{d_1,\dots,d_n}_{E_1,\dots,E_n})[d-n^2](-\frac{n^2}{2}) \otimes ((E_1 \otimes \dots \otimes E_n)^{-(n-1)(g-1)}|_{\Omega^{\otimes -\frac{n-1}{2}}}),$$

according to Proposition 7(3).

However, the sheaf $\mathrm{Eis}^{d_1,\dots,d_n}_{E_1,\dots,E_n}$ is pure of weight 0 and

$$\mathbb{D}(\mathrm{Eis}^{d_1,\dots,d_n}_{E_1,\dots,E_n}) \simeq \mathrm{Eis}^{d_1,\dots,d_n}_{E_1^{\vee},\dots,E_n^{\vee}},$$

and so is the pull-back of $\mathrm{Eis}^{d_1,\dots,d_n}_{E_1,\dots,E_n}[d-n^2](-\frac{n^2}{2})$ to ${}_{(n-1)(2g-2)}{}^{0}\mathcal{S}h'^{\,d+n(n-1)(2g-2)}_n$, since the map π_n is smooth over ${}_{(n-1)(2g-2)}\mathcal{S}h^{d+n(n-1)(2g-2)}_n$.

Therefore, the pull-back of $\mathrm{Eis}^{d_1,\dots,d_n}_{E_1,\dots,E_n}[d-n^2](\frac{n^2}{2})$ to the intersection

$$\pi^{\vee}_{n-1}{}^{-1}({}_{\mathfrak{c}_n}\mathcal{S}h^{d+(n-1)(n-2)(g-1)}_{n-1} \cap {}_{(n-1)(2g-2)}{}^{0}\mathcal{S}h'^{\,d+n(n-1)(2g-2)}_n,$$

is perverse and pure of weight 0.

By point (c) above, over this stack we have an isomorphism

$${}^{0}\pi_n^*(\mathrm{Eis}^{d_1,\dots,d_n}_{E_1,\dots,E_n})[d-n^2](-\frac{n^2}{2}) \otimes ((E_1 \otimes \dots \otimes E_n)^{-(n-1)(g-1)}|_{\Omega^{\otimes -\frac{n-1}{2}}}) \simeq$$
$$comp_{n-1}(\pi_0^{\vee *}(\mathcal{L}^{d_1,\dots,d_n}_{E_1,\dots,E_n})[d]).$$

□

2.3.3

Our present aim is to prove Proposition 7. Note, that point (1) is a consequence of Proposition 5, since $\mathcal{F}1_{E_1,\dots,E_n}^{d_1,\dots,d_n}$ is a direct summand of $\mathcal{F}1_{E_1,E_2\oplus\dots\oplus E_n}^{d_1,d-d_1}$.

Consider now the fibered product

$$^0\mathcal{S}h'^{\,d+k(k+1)(g-1)}_{k+1} \underset{\mathcal{S}h_k^{d+k(k-1)(g-1)}}{\times} \mathcal{F}l_{k,0}^{d_1+\dots+d_k+k(k-1)(g-1),d_{k+1}+\dots+d_n}$$

and let $''p_{k,0}^{d_1+\dots+d_k+k(k-1)(g-1),d_{k+1}+\dots+d_n}$ denote the base change map from this stack to $\mathcal{S}h'^{\,d+k(k-1)(g-1)}_k$.

Over the preimage of $_{(k-1)(2g-2)}\mathcal{S}h_k^{d+k(k-1)(g-1)}$ under the projection

$$\mathcal{F}l_{k,0}^{d_1+\dots+d_k+k(k-1)(g-1),d_{k+1}+\dots+d_n} \overset{q_{k,0}^{d_1+\dots+d_k+k(k-1)(g-1),d_{k+1}+\dots+d_n}}{\longrightarrow} \mathcal{S}h_k^{d+k(k-1)(g-1)},$$

the stacks

$$^0\mathcal{S}h'^{\,d+k(k+1)(g-1)}_{k+1} \underset{\mathcal{S}h_k^{d+k(k-1)(g-1)}}{\times} \mathcal{F}l_{k,0}^{d_1+\dots+d_k+k(k-1)(g-1),d_{k+1}+\dots+d_n}$$

and

$$\mathcal{S}h'^{\,d+k(k-1)(g-1)}_k \underset{\mathcal{S}h_k^{d+k(k-1)(g-1)}}{\times} \mathcal{F}l_{k,0}^{d_1+\dots+d_k+k(k-1)(g-1),d_{k+1}+\dots+d_n}$$

become mutually dual vector bundles.

We will denote by $\widetilde{\mathrm{Four}_k}$ the Fourier transform functor from the category of sheaves on

$$_{(k-1)(2g-2)}\mathcal{S}h'^{\,d+k(k-1)(g-1)}_k \underset{\mathcal{S}h_k^{d+k(k-1)(g-1)}}{\times} \mathcal{F}l_{k,0}^{d_1+\dots+d_k+k(k-1)(g-1),d_{k+1}+\dots+d_n}$$

to that on

$$_{k(2g-2)}{}^0\mathcal{S}h'^{\,d+k(k+1)(g-1)}_{k+1} \underset{\mathcal{S}h_k^{d+k(k-1)(g-1)}}{\times} \mathcal{F}l_{k,0}^{d_1+\dots+d_k+k(k-1)(g-1),d_{k+1}+\dots+d_n}.$$

Let finally $W'^{\,d_1+\dots+d_k+k(k-1)(g-1),d_{k+1}+\dots+d_n}_k$ denote the stack that classifies those triples

$$(0 \to \Omega^{\otimes k} \to M_{k+1} \to M_k \to 0, M'_k \hookrightarrow M_{k+1}),$$

with $M_k \in \mathcal{S}h_k^{d+k(k-1)(g-1)}$ for which the composition $M'_k \to M_k$ is an embedding, i.e.

$$M'_k \hookrightarrow M_k \in \mathcal{F}l_{k,0}^{d_1+\ldots+d_k+k(k-1)(g-1),d_{k+1}+\ldots+d_n}.$$

We have a tautological open embedding of $W'^{\sum_{i=1}^{k} d_i+k(k-1)(g-1),\ \sum_{i=k+1}^{n} d_i}_k$ into

$$\mathcal{S}h'^{\ d+k(k+1)(g-1)}_{k+1} \underset{\mathcal{S}h_{k+1}^{d+k(k+1)(g-1)}}{\times} \mathcal{F}l_{k,1}^{\sum_{i=1}^{k} d_i+k(k-1)(g-1),\ \sum_{i=k+1}^{n} d_i+k(2g-2)};$$

we shall denote this embedding by $\widetilde{j}_{k+1}$.

We shall denote by i'_k and r_k the projections from $W'^{\sum_{i=1}^{k} d_i+k(k-1)(g-1),\ \sum_{i=k+1}^{n} d_i}_k$ to

$${}^0\mathcal{S}h'^{\ d+k(k+1)(g-1)}_{k+1} \underset{\mathcal{S}h_{k}^{d+k(k-1)(g-1)}}{\times} \mathcal{F}l_{k,0}^{d_1+\ldots+d_k+k(k-1)(g-1),d_{k+1}+\ldots+d_n} \text{and to}$$

$\mathcal{F}l_{k,0}^{d_1+\ldots+d_k+k(k-1)(g-1),d_{k+1}+\ldots+d_n}$ respectively.

The following assertion follows from the standard properties of the Fourier transform functor.

Lemma 12. (1) *Let $\mathcal{K}$ be a sheaf on*

$${}_{(k-1)(2g-2)}\mathcal{S}h'^{\ d+k(k-1)(g-1)}_{k} \underset{\mathcal{S}h_{k}^{d+k(k-1)(g-1)}}{\times} \mathcal{F}l_{k,0}^{d_1+\ldots+d_k+k(k-1)(g-1),d_{k+1}+\ldots+d_n}.$$

Then

$$\mathrm{Four}_k \circ' p_{k,0\ !}^{\sum_{i=1}^{k} d_i+k(k-1)(g-1),\ \sum_{i=k+1}^{n} d_i}(\mathcal{K}) \simeq'' p_{k,0\ !}^{\sum_{i=1}^{k} d_i+k(k-1)(g-1),\ \sum_{i=k+1}^{n} d_i} \circ \widetilde{\mathrm{Four}_k}(\mathcal{K}).$$

(2) *Let $\mathcal{K}'$ be a sheaf on $\mathcal{F}l_{k,0}^{d_1+\ldots+d_k+k(k-1)(g-1),d_{k+1}+\ldots+d_n}$. Then*

$$\widetilde{\mathrm{Four}_k} \circ i_{k!} \circ r_k^*(\mathcal{K}') \simeq i'_{k!} \circ r_k'^{*}(\mathcal{K}')(\overline{\mathbb{Q}}_\ell[1](\tfrac{1}{2}))^{\otimes -d_1-\ldots-d_k+k^2+d_{k+1}+\ldots+d_n}.$$

2.3.4

Consider now a map

$$\mathcal{S}h'^{\,d+k(k+1)(g-1)}_{k+1} \times_{\mathcal{S}h^{d+k(k+1)(g-1)}_{k+1}} \mathcal{F}l_{k,1}^{\sum_{i=1}^{k} d_i+k(k-1)(g-1),\sum_{k+1}^{m} d_i} \to \mathcal{S}h'^{\,\sum_{k+1}^{n} d_i}_{1},$$

given by sending $(s:\Omega^{\otimes k}\to M_{k+1}, M'_k\hookrightarrow M_{k+1})$ with $M_{k+1}\in \mathcal{S}h^{d+k(k+1)(g-1)}_{k+1}$ to $\Omega^{\otimes -k}\otimes(M_{k+1}/M'_k)\in \mathcal{S}h'^{\,d_{k+1}+\ldots+d_n}_{1}$.

Lemma 13. *We have a Cartesian square:*

$$\begin{array}{ccc}
W'^{\,\sum_{i=1}^{k} d_i+k(k-1)(g-1),\ \sum_{i=k+1}^{n} d_i+k(2g-2)}_{k} & \longrightarrow & {}^{0}\mathcal{S}h'^{\,\sum_{i=k+1}^{n} d_i}_{1} \\
\Big\downarrow \tilde{j}_{k+1} & & \Big\downarrow j_1 \\
\mathcal{S}h'^{\,d+k(k+1)(g-1)}_{k+1} \times_{\mathcal{S}h^{d+k(k+1)(g-1)}_{k+1}} \mathcal{F}l_{k,1}^{\sum_{i=1}^{k} d_i+k(k-1)(g-1),\ \sum_{i=k+1}^{n} d_i+k(2g-2)} & \longrightarrow & \mathcal{S}h'^{\,\sum_{i=k+1}^{n} d_i}_{1}
\end{array}$$

Proof. The preimage in

$$\mathcal{S}h'^{\,d+k(k+1)(g-1)}_{k+1} \times_{\mathcal{S}h^{d+k(k+1)(g-1)}_{k+1}} \mathcal{F}l_{k,1}^{d_1+\ldots+d_k+k(k-1)(g-1),d_{k+1}+\ldots+d_n+k(2g-2)}$$

of the open sub-stack

$${}^{0}\mathcal{S}h'^{\,d_{k+1}+\ldots+d_n}_{1} \hookrightarrow \mathcal{S}h'^{\,d_{k+1}+\ldots+d_n}_{1}$$

corresponding to those triples

$$(0\to\Omega^{\otimes k}\to M_{k+1}\to M_k\to 0, M'_k\hookrightarrow M_{k+1})$$

for which $\Omega^{\otimes k}$ maps injectively into the quotient M_{k+1}/M'_k, which is equivalent to the condition that M'_k embeds into M_k, the latter being the definition of $W'^{\,d_1+\ldots+d_k+k(k-1)(g-1),d_{k+1}+\ldots+d_n}_{k}$.

□

2.3.5

Proof of Proposition 7(2).

Let

$$'p_{k,1}^{\sum_{i=1}^{k} d_i+k(k-1)(g-1),\ \sum_{i=k+1}^{n} d_i+k(2g-2)} \quad \text{and} \quad 'q_{k,1}^{\sum_{i=1}^{k} d_i+k(k-1)(g-1),\ \sum_{i=k+1}^{n} d_i+k(2g-2)}$$

denote the projections of

$$\mathcal{S}h'^{\,d+k(k+1)(g-1)}_{k+1} \underset{\mathcal{S}h_{k+1}^{d+k(k+1)(g-1)}}{\times} \mathcal{F}l_{k,1}^{\sum_{i=1}^{k} d_i+k(k-1)(g-1),\ \sum_{i=k+1}^{n} d_i+k(2g-2)}$$

to $\mathcal{S}h'^{\,d+k(k+1)(g-1)}_{k+1}$ and to $\mathcal{S}h_k^{\sum_{i=1}^{k} d_i+k(k-1)(g-1)} \times \mathcal{S}h'^{\,\sum_{k+1}^{n} d_i}_1$ respectively.

Proposition 1 implies, that $\mathcal{F}_{k+1}{}^{d_1,\dots,d_n}_{E_1,\dots,E_n}$ identifies with

$$'p_{k,1}^{\sum_{i=1}^{k} d_i+k(k-1)(g-1),\ \sum_{i=k+1}^{n} d_i+k(2g-2)}{}_! \circ {'q}_{k,1}^{\sum_{i=1}^{k} d_i+k(k-1)(g-1),\ \sum_{i=k+1}^{n} d_i+k(2g-2)}{}^*$$
$$(\mathrm{Eis}^{d_1,\dots,d_k}_{E_1,\dots,E_k} \boxtimes \mathcal{F}_1{}^{d_{k+1},\dots,d_n}_{E_{k+1},\dots,E_n}) \otimes$$
$$\bigotimes_{i=1}^{k} (E_i^{(g-1)(1-k)}|_{\Omega^{\otimes \frac{1-k}{2}}}) \bigotimes_{i=k+1}^{n} (E_i^{k(2g-2)}|_{\Omega^{\otimes k}}) \otimes (\overline{\mathbb{Q}}_\ell[1](\frac{1}{2}))^{\otimes k(\sum_{i=k+1}^{n} d_i)}.$$

The image under the map $'q_{k,1}^{\sum_{i=1}^{k}+k(k-1)(g-1),\ \sum_{i=k+1}^{n} d_i+k(2g-2)}$ of

$$'p_{k,1}^{d_1+\dots+d_k+k(k-1)(g-1), d_{k+1}+\dots+d_n+k(2g-2)-1}({}_{d_{k+1}+(n-2)(2g-2)}\mathcal{S}h'^{\,d+k(k+1)(2g-2)}_k)$$

belongs to the open sub-stack

$$\mathcal{S}h_k^{\sum_{i=1}^{k} d_i+k(k-1)(g-1)} \times_{d_{k+1}+(n-k-2)(g-1)} \mathcal{S}h'^{\,\sum_{k+1}^{n} d_i}_1,$$

since the fact that $M_{k+1} \in {}_{d_{k+1}+(n-2)(2g-2)}\mathcal{S}h_{k+1}$ implies that

$$(M_{k+1}/M'_k) \otimes \Omega^{\otimes -k} \in {}_{d_{k+1}+(n-k-2)(2g-2)}\mathcal{S}h_1.$$

Using Proposition 5(1) and Lemma 13, over ${}_{d_{k+1}+(n-2)(2g-2)}\mathcal{S}h'^{\,d+k(k+1)(2g-2)}_{\substack{1\\k}}$ we can rewrite the expression for $\mathcal{F}_{k+1}{}^{d_1,\dots,d_n}_{E_1,\dots,E_n}$ as

$$'p_{k,1}^{\sum_{i=1}^{k} d_i+k(k-1)(g-1),\ \sum_{i=k+1}^{n} d_i+k(2g-2)}{}_! \circ \widetilde{j}_{k+1!} \circ \widetilde{j}^*_{k+1} \circ {}'q_{k,1}^{d_1+\dots+d_k+k(k-1)(g-1),d_{k+1}+\dots+d_n+k(2g-2)*}$$
$$(\mathrm{Eis}^{d_1,\dots,d_k}_{E_1,\dots,E_k}) \boxtimes \mathcal{F}_1{}^{d_{k+1},\dots,d_n}_{E_{k+1},\dots,E_n})$$
$$\bigotimes_{i=1}^{k} (E_i^{(g-1)(1-k)}|_{\Omega^{\otimes\frac{1-k}{2}}}) \bigotimes_{i=k+1}^{n} (E_i^{k(2g-2)}|_{\Omega^{\otimes k}}) \otimes (\overline{\mathbb{Q}}_\ell[1](\frac{1}{2}))^{\otimes k(\sum_{i=k+1}^{n} d_i)}$$

On the other hand,we can compute the sheaf $\mathrm{Four}_k(\mathcal{F}_k{}^{d_1,\dots,d_n}_{E_1,\dots,E_n})$ using Lemma 12 and we will obtain that $\mathrm{Four}_k(\mathcal{F}_k{}^{d_1,\dots,d_n}_{E_1,\dots,E_n})$ identifies with

$$''p_{k,0}^{\sum_{i=1}^{k} d_i+k(k-1)(g-1),\ \sum_{i=k+1}^{n} d_i}{}_! \circ i'_{k!} \circ r'^*_k \circ q_{k,0}^{\sum_{i=1}^{k} d_i+k(k-1)(g-1),\ \sum_{i=k+1}^{n} d_i}{}^*(\mathrm{Eis}^{d_1,\dots,d_k}_{E_1,\dots,E_k} \boxtimes \mathcal{L}^{d_{k+1},\dots,d_n}_{E_{k+1},\dots,E_n})$$
$$\otimes ((E_1 \otimes \dots \otimes E_k)^{-(g-1)(k-1)}|_{\Omega^{\otimes -\frac{k-1}{2}}}) \otimes (\overline{\mathbb{Q}}_\ell[1](\frac{1}{2}))^{\otimes(k+1)(\sum_{i=k+1}^{n} d_i)}$$

Observe that the maps

$$j_k \circ {}''p_{k,0}^{d_1+\dots+d_k+k(k-1)(g-1),d_{k+1}+\dots+d_n} \circ i'_k \text{ and } {}'p_{k,1}^{d_1+\dots+d_k+k(k-1)(g-1),d_{k+1}+\dots+d_n} \circ \widetilde{j}_{k+1}$$

from $W'^{d_1+\dots+d_k+k(k-1)(g-1),d_{k+1}+\dots+d_n+k(2g-2)}_k$ to $\mathcal{S}h'^{\,d+k(k+1)(g-1)}_{k+1}$ coincide.

In addition, the composition of the projection $q_{k,0}^{\sum_{i=1}^{k} d_i+k(k-1)(g-1),\ \sum_{i=k+1}^{n} d_i} \circ r'_k$ with the automorphism $\mathrm{id} \times m_{\Omega^{\otimes k}}$ of the stack

$$\mathcal{S}h_k^{\sum_{i=1}^{k} d_i+k(k-1)(g-1)} \times \mathcal{S}h_0^{\sum_{i=k+1}^{n} d_i}$$

coincides with the composition

$$(\mathrm{id} \times \pi_0^\vee) \circ {}'q_{k,1}^{\sum_{i=1}^{k} d_i+k(k-1)(g-1),\ \sum_{i=k+1}^{n} d_i+k(2g-2)} \circ \widetilde{j}_{k+1}$$

that maps

$$W'^{\sum_{i=1}^{k} d_i+k+k(k-1)(g-1),\ \sum_{i=k+1}^{n} d_i+k(2g-2)}_k \to \mathcal{S}h_k^{\sum_{i=1}^{k} d_i+k(k-1)(g-1)} \times \mathcal{S}h_0^{\sum_{i=k+1}^{n} d_i}.$$

Therefore, using Lemma 11, we obtain that the sheaf

$$r_k'^* \circ q_{k,0}^{\sum_{i=1}^{k} d_i+k(k-1)(g-1),\ \sum_{i=k+1}^{n} d_i}{}_*(\mathrm{Eis}_{E_1,\dots,E_k}^{d_1,\dots,d_k} \boxtimes \mathcal{L}_{E_{k+1},\dots,E_n}^{d_{k+1},\dots,d_n})$$

on $W_k'^{\sum_{i=1}^{k} d_i+k+k(k-1)(g-1),\ \sum_{i=k+1}^{n} d_i+k(2g-2)}$ identifies with

$$\widetilde{j}_{k+1}^* \circ {}'q_{k,1}^{\sum_{i=1}^{k} d_i+k(k-1)(g-1),\ \sum_{i=k+1}^{n} d_i+k(2g-2)}{}_*$$

$$(\mathrm{Eis}_{E_1,\dots,E_k}^{d_1,\dots,d_k} \boxtimes \mathcal{F}_1{}_{E_{k+1},\dots,E_n}^{d_{k+1},\dots,d_n}) \bigotimes_{i=k+1}^{n} (E_i^{k(2g-2)}|_{\Omega^{\otimes k}}) \otimes (\overline{\mathbb{Q}}_\ell[1](\frac{1}{2}))^{\otimes -(\sum_{i=k+1}^{n} d_i)}.$$

Hence, the sheaf $\mathrm{Four}_k(\mathcal{F}_k{}_{E_1,\dots,E_n}^{d_1,\dots,d_n})$ identifies with

$${}'p_{k,1}^{\sum_{i=1}^{k} d_i+k(k-1)(g-1),\ \sum_{i=k+1}^{n} d_i+k(2g-2)}{}_! \circ \widetilde{j}_{k+1!} \circ \widetilde{j}_{k+1}^* \circ {}'q_{k,1}^{\sum_{i=1}^{k} d_i+k(k-1)(g-1),\ \sum_{i=k+1}^{n} d_i+k(2g-2)}{}_*$$

$$(\mathrm{Eis}_{E_1,\dots,E_k}^{d_1,\dots,d_k}) \boxtimes \mathcal{F}_1{}_{E_{k+1},\dots,E_n}^{d_{k+1},\dots,d_n})$$

$$\bigotimes_{i=1}^{k} (E_i{}^{(g-1)(1-k)}_{\Omega^{\otimes \frac{1-k}{2}}}) \bigotimes_{i=k+1}^{n} (E_i^{k(2g-2)}|_{\Omega^{\otimes k}}) \otimes (\overline{\mathbb{Q}}_\ell[1](\frac{1}{2}))^{\otimes k(\sum_{i=k+1}^{n} d_i)}.$$

Comparing this with the expression for $\mathcal{F}_{k+1}{}_{E_1,\dots,E_n}^{d_1,\dots,d_n}$ obtained above concludes the proof of the proposition.

□

2.3.6

Proof of Proposition 7(3)

As in the proof of Proposition 7(2),

$${}^0\pi_n^*(\mathrm{Eis}_{E_1,\dots,E_n}^{d_1,\dots,d_n}) \otimes ((E_1 \otimes \dots \otimes E_n)^{-(n-1)(g-1)}|_{\Omega^{\otimes -\frac{n-1}{2}}})[d-n^2](-\frac{n^2}{2})$$

identifies with

$${}'p_{n-1,1}^{d-d_n+(n-1)(n-2)(g-1),d_n+(n-1)(2g-2)}{}_! \circ {}'q_{n-1,1}^{d-d_n+(n-1)(n-2)(g-1),d_n+(n-1)(2g-2)*}$$

$$(\mathrm{Eis}_{E_1,\dots,E_{n-1}}^{d_1,\dots,d_{n-1}} \boxtimes \mathcal{F}_1{}_{E_n}^{d_n}) \bigotimes_{i=1}^{n-1} (E_i{}^{(g-1)(2-n)}_{\Omega^{\otimes \frac{2-n}{2}}})(E_n|\Omega^{\otimes n-1}) \otimes (\overline{\mathbb{Q}}_\ell[1](\frac{1}{2}))^{\otimes (n-1)d_n}$$

and $\mathrm{Four}_{n-1}(\mathcal{F}_{n-1})$ identifies with

$$'p_{n-1,1}^{d-d_n+(n-1)(n-2)(g-1),d_n+(n-1)(2g-2)}{}_! \circ {'q}_{n-1,1}^{d-d_n+(n-1)(n-2)(g-1),d_n+(n-1)(2g-2)*} \circ \tilde{j}_{n!} \circ \tilde{j}_n^*$$

$$(\mathrm{Eis}_{E_1,\dots,E_{n-1}}^{d_1,\dots,d_{n-1}} \boxtimes \mathcal{F}_1{}_{E_n}^{d_n}) \bigotimes_{i=1}^{n-1} (E_i^{(g-1)(2-n)}|_{\Omega^{\otimes \frac{2-n}{2}}})(E_n|_{\Omega^{\otimes n-1}}) \otimes (\overline{\mathbb{Q}}_\ell[1](\frac{1}{2}))^{\otimes(n-1)d_n}$$

The assertion follows now from the following lemma:

Lemma 14. *The open embedding*

$$j_n : W_{n-1} \to \mathcal{S}h_n'^{\,d+n(n-1)(g-1)} \underset{\mathcal{S}h_n^{d+n(n-1)(g-1)}}{\times} \mathcal{F}l_{n-1,1}^{d-d_n+n(n-1)(g-1),d_n}$$

is an isomorphism over the preimage of ${}_{d_n+(n-2)(2g-2)}\mathcal{S}h_{n-1}^{d+(n-1)(n-2)(g-1)}$ *in*

$${}^0\mathcal{S}h_n'^{\,d+n(n-1)(g-1)} \underset{\mathcal{S}h_n^{d+n(n-1)(g-1)}}{\times} \mathcal{F}l_{n-1,1}^{d-d_n+n(n-1)(g-1),d_n}.$$

Proof. We have to prove that if

$$0 \to \Omega^{\otimes n-1} \to M_n \to M_{n-1} \to 0$$

is a short exact sequence with $\mathrm{Ext}^1(L_c, M_{n-1}) = 0$ for any $L_c \in \mathrm{Pic}^c$ with $c \geq d_n + (n-2)(2g-2)$, then for any embedding $M'_{n-1} \hookrightarrow M_n$, with $\deg(M'_{n-1}) = \deg(M_{n-1}) - d_n$, the composition $M'_{n-1} \to M_n \to M_{n-1}$ is also an embedding.

Suppose that this is not so and let K denote a maximal sub-sheaf in M'_{n-1} which does not intersect the kernel of the map $M'_{n-1} \to M_{n-1}$. Let us introduce the notation $L_1 := M_{n-1}/K$, $L'_1 = M'_{n-1}/K$ and $M_2 := M_n/K$. It is easy to see, that M'_{n-1}/K is a line bundle and that

$$\deg(M_{n-1}/K) - \deg(M'_{n-1}/K) = d_n.$$

The kernel of the map $L'_1 \to L_1$ embeds into $\Omega^{\otimes n-1}$, therefore it has a degree $\leq (n-1)(2g-2)$. Therefore, L_1 has a torsion sub-sheaf of length $\geq \deg(L'_1) - (n-1)(2g-2)$ and hence L_1 maps surjectively onto a line bundle $\widetilde{L}$ with $\deg(\widetilde{L}) \leq \deg(L_1) - \deg(L'_1) + (n-1)(2g-2) = d_n + (n-1)(2g-2)$. But for such $\widetilde{L}$ there always exists a line bundle L_c of degree $c = d_n + (n-2)(2g-2)$ with $\mathrm{Ext}^1(L_c, \widetilde{L}) \neq 0$.

However, the maps

$$\mathrm{Ext}^1(L_c, M_{n-1}) \to \mathrm{Ext}^1(L_c, L_1) \to \mathrm{Ext}^1(L_c, \widetilde{L})$$

are surjective, hence $\mathrm{Ext}^1(L_c, M_{n-1}) \neq 0$ and this is a contradiction. □

Chapter 3

Automorphic sheaves attached to a 2-dimensional geometrically irreducible local system

3.1 A "simple" proof of Drinfeld's theorem

3.1.1

Let E be a 2-dimensional geometrically irreducible local system on X. Our goal in this chapter is to present a proof of the following theorem:

Main Theorem C

There exists an irreducible perverse sheaf Aut_E on Bun that satisfies:

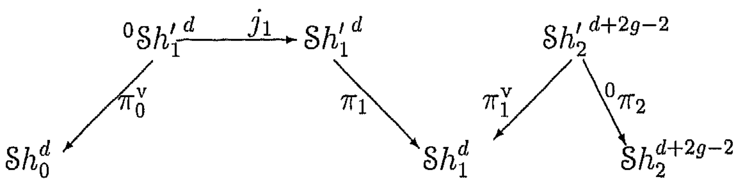
(1) $m^*(\mathrm{Aut}_E) \simeq \mathrm{Aut}_E \boxtimes \Lambda^2(E)$

(2) $(\mathfrak{q} \times \mathrm{supp})_! \circ \mathfrak{p}^*(\mathrm{Aut}_E)[1](\frac{1}{2}) \simeq \mathrm{Aut}_E \boxtimes E$.

Our strategy will be to define first a sheaf $\mathcal{S}'_E$ over ${}^0\mathcal{S}h'^d_2$ for d large enough and to prove that it is a pull-back under ${}^0\pi_2$ of a sheaf $\mathcal{S}_E$ on $\mathcal{S}h_2$. The sheaf Aut_E will then be defined as the restriction of $\mathcal{S}_E$ to $\mathrm{Bun}_2 \subset \mathcal{S}h_2$.

3.1.2

Recall now the "fundamental diagram" introduced in Sect. 2.1.1:



Let d satisfy $d > 4g-4$. For a 2-dimensional local system E' on X we define the sheaf $\mathcal{S}'_{E'}{}^d$ on $\pi_1^{\vee-1}({}_0\mathcal{S}h_1^{d-2g+2}) \subset \mathcal{S}h'^{\,d}_2$ as follows:

Definition 4.

$$\mathcal{S}'_{E'}{}^d := \mathrm{Four} \circ j_{1!*} \circ \pi_0^{\vee *}(\mathcal{L}_{E'}^{d-2g+2}[d]).$$

Remark 5. In the case when $E' = E_1 \oplus E_2$ with E_1 and E_2 being geometrically non-isomorphic 1-dimensional local systems on X, we have shown in Corollary 9 that there exists a perverse sheaf $\mathcal{S}_{E'}{}^d := \mathrm{Eis}^d_{E_1,E_2}$ on $\mathcal{S}h_2^d$ that satisfies

$${}^0\pi_2^*(\mathcal{S}_{E'}{}^d)[d-4g+4](-2g+2) \simeq \mathcal{S}'_{E'}{}^d$$

over ${}_{2g-2}{}^0\mathcal{S}h'^{\,d}_2$. The next theorem states that the same is true for $\mathcal{S}'_E{}^d$ when E is geometrically irreducible.

Theorem 5. Let E be geometrically irreducible and let d satisfy: $d > 6g-4$. There exists an irreducible perverse sheaf $\mathcal{S}^d_E$ over $\mathcal{S}h^d_2$ such that the sheaves ${}^0\pi_2^*(\mathcal{S}^d_E)[d-4g+4](-2g+2)$ and $\mathcal{S}'_E{}^d$ are isomorphic over ${}^{\,0}_{2g-2}\mathcal{S}h'^{\,d}_2$.

A statement equivalent to Theorem 5 was proven by V.Drinfeld in [2]. Below we will give another proof, where we will derive Theorem 5 from the following two assertions:

Theorem 6. Let d satisfy $d > 4g-3$. The fact that E is geometrically irreducible implies that the canonical maps

$$j_{1!} \circ \pi_0^{\vee *}(\mathcal{L}_E) \to j_{1!*} \circ \pi_0^{\vee *}(\mathcal{L}_E) \to j_{1*} \circ \pi_0^{\vee *}(\mathcal{L}_E)$$

are isomorphisms over $\mathcal{S}h'^{\,d;\leq 1}_1$.

Proposition 8. *Let d satisfy $d > 4g-2$ and let E' and E'' be any 2-dimensional local systems on X. Then the stalks of the two sheaves*

$$\mathrm{Four} \circ j_{1!} \circ \pi_0^{\vee *}(\mathcal{L}^d_{E'}) \text{ and } \mathrm{Four} \circ j_{1!} \circ \pi_0^{\vee *}(\mathcal{L}^d_{E''})$$

have equal Euler-Poincaré characteristics at every point of $\pi_1^{\vee-1}(\mathcal{S}h'^{\,d;\leq 1}_1)$.

3.1.3

The following simple observation will be a key point in the proof of Theorem 5.

Lemma 15. *Let $\mathcal{K}$ be an irreducible perverse sheaf on a smooth algebraic stack $\mathcal{Y}$. Assume that the Euler-Poincaré characteristics of the stalks of $\mathcal{K}$ are the same for all points of $\mathcal{Y}$. Then $\mathcal{K}$ is locally constant.*

Proof of Lemma 15

By definition, we may assume that $\mathcal{Y}$ is a smooth algebraic variety. Let ${}^0\mathcal{Y} \subset \mathcal{Y}$ denote an open sub-variety, where $\mathcal{K}$ is locally constant.

It is enough to show that $\mathcal{K}$ can be continued as a local system to the generic point of every component of $\mathcal{Y} - {}^0\mathcal{Y}$ of codimension 1. Therefore, we are reduced to the case when $\mathcal{Y}$ is a spectrum of a regular local ring of dimension 1 and let Φ and Ψ denote the functors of vanishing and nearby cycles respectively.

By the assumption, $\chi(\Psi(\mathcal{K})) = \chi(\mathcal{K}|_{\mathcal{Y}-{}^0\mathcal{Y}})$, therefore $\chi(\Phi(\mathcal{K})) = 0$. Since the functor Φ is exact, this implies that $\Phi(\mathcal{K}) = 0$.

□

Proof of Theorem 5

Pick E_1 and E_2 to be two 1-dimensional local systems on X satisfying $H^0(X, E_1 \otimes E_2^\vee) = 0$.

By combining Theorem 6, Proposition 8 and Corollary 11, we learn that the Euler-Poincaré characteristics of the stalks of ${}^0\pi_2^*(\mathrm{Eis}^d_{E_1,E_2})[d-4g+4]$ are equal to those of ${\mathcal{S}'_E}^d$ over ${\pi_1^\vee}^{-1}(\mathcal{S}h_1^{d-2g+2;\leq 1})$ and hence over ${\pi_1^\vee}^{-1}(\mathcal{S}h_1^{d-2g+2;\leq 1}) \cap {}^0_{2g-2}\mathcal{S}{h'_2}^{d;\leq 0}$.

Let U denote the support of $\mathrm{Eis}^d_{E_1,E_2}$ on $\mathrm{Bun}_2 \cap_{2g-2}\mathcal{S}h_2^d$ and let U^0 denote a smooth non-empty open sub-stack of U, where $\mathrm{Eis}^d_{E_1,E_2}$ is locally constant.[1] We will denote by $\widetilde{U}$ (resp., by $\widetilde{U^0}$) the preimage of U (resp., of U^0) in ${\pi_1^\vee}^{-1}(\mathcal{S}h_1^{d-2g+2;\leq 1}) \cap {}^0_{2g-2}\mathcal{S}{h'_2}^{d;\leq 0}$.

The fact that the Euler-Poincaré characteristics of the stalks of ${}^0\pi_2^*(\mathrm{Eis}^d_{E_1,E_2})[d-4g+4]$ and of ${\mathcal{S}'_E}^d$ coincide implies that the support of ${\mathcal{S}'_E}^d$ on ${\pi_1^\vee}^{-1}(\mathcal{S}h_1^{d-2g+2;\leq 1}) \cap {}^0_{2g-2}\mathcal{S}{h'_2}^{d;\leq 0}$ coincides with $\widetilde{U}$.

Therefore, ${\mathcal{S}'_E}^d$ is an irreducible perverse sheaf on $\widetilde{U}$ and it is enough to show that its restriction to $\widetilde{U^0}$ is a pull-back of some sheaf on U^0, since the map ${}^0_{2g-2}\mathcal{S}{h'_2}^d \to {}_{2g-2}\mathcal{S}h_2^d$ is smooth.

[1] In fact, one can show that $\mathrm{Eis}^d_{E_1,E_2}$ is supported at the generic point of Bun_2.

We claim that it is enough to show that ${\mathcal{S}'_E}^d$ is locally constant over $\widetilde{U^0}$. Indeed, ${\mathcal{S}'_E}^d$ is equivariant with respect to the action of G_m on ${\mathcal{S}h'_2}^d$ by dilations and the fibers of the map $\widetilde{U^0}/G_m \to U^0$ are connected and simply connected, since $\mathcal{S}h_1^d - \mathcal{S}h_1^{d,\leq 1}$ has codimension 2 in $\mathcal{S}h_1^d$ and the fibers of the projection ${}^0\pi_2$ over U^0 are vector spaces with removed zero.

However, by the definition of U^0, the Euler-Poincaré characteristics of the stalks of ${}^0\pi_2^*(\mathrm{Eis}^d_{E_1,E_2})[d-4g+4]$ are constant over $\widetilde{U^0}$, hence the same is true for ${\mathcal{S}'_E}^d$. The assertion of the theorem follows now from Lemma 15.

□(Theorem 5)

3.2 Proofs of Theorem 6 and of Proposition 8

3.2.1

The proof of Theorem 6 relies on the following two geometric observations:

Lemma 16. *Let $\mathcal{Y}$ be an algebraic stack and let $a : \mathcal{E} \to \mathcal{Y}$ be an (n-dimensional, $n \geq 2$) vector-bundle over it. Let $\mathbb{P}E$, $\mathbb{P}a$, $\mathcal{K}$, $\mathcal{K}'$ be as in Proposition 4. The following three conditions are equivalent:*

(a) $j_!(\mathcal{K}) \simeq j_{!*}(\mathcal{K}) \simeq j_*(\mathcal{K})$

(b) $a_! \circ j_!(\mathcal{K}) = a_* \circ j_*(\mathcal{K}) = 0$

(c) $\mathbb{P}a_!(\mathcal{K}') = 0$.

Lemma 17. *Let $a : \mathcal{Y}_1 \to \mathcal{Y}_2$ be a representable proper map between algebraic stacks. Let $j : {}^0\mathcal{Y}_1 \to \mathcal{Y}_1$ be an open embedding such that the map of the complement of ${}^0\mathcal{Y}_1$ in $\mathcal{Y}_1$ to $\mathcal{Y}_2$ is a finite morphism. Let $\mathcal{K}$ be a perverse sheaf on ${}^0\mathcal{Y}_1$. Assume that*

$$(a \circ j)_!(\mathcal{K}) = (a \circ j)_*(\mathcal{K}) = 0.$$

Then $j_!(\mathcal{K}) \simeq j_{!*}(\mathcal{K}) \simeq j_*(\mathcal{K})$.

Proof of Lemma 16

It is well-known that (b) and (c) are equivalent and that $a_! \circ j_*(\mathcal{K}) = a_* \circ j_!(\mathcal{K}) = 0$, which proves that (a) implies (b).

Assume now (b). According to Proposition 4, $a_! \circ j_{!*}(\mathcal{K}) = 0$. Since $\mathcal{K}$ is G_m-equivariant, the functors $a_!$ and inverse image with compact supports onto the zero-section produce isomorphic objects when applied to $\mathcal{K}$.

This proves that $j_{!*}(\mathcal{K}) \simeq j_*(\mathcal{K})$. The isomorphism $j_!(\mathcal{K}) \simeq j_{!*}(\mathcal{K})$ follows by duality.

□

Proof of Lemma 17

It is sufficient to prove that the cone of the map $j_!(\mathcal{K}) \to j_*(\mathcal{K})$ vanishes. This cone is supported on $\mathcal{Y}_1 - {}^0\mathcal{Y}_1$ and since the restriction of the map a to $\mathcal{Y}_1 - {}^0\mathcal{Y}_1$ is finite, it is sufficient to prove that the above cone vanishes after applying the functor $a_! = a_*$ (the map a is proper).

We have:

$$a_!(\mathrm{Cone}(j_!(\mathcal{K}) \to j_*(\mathcal{K}))) \simeq \mathrm{Cone}(a_! \circ j_!(\mathcal{K}) \to a_* \circ j_*(\mathcal{K})) = 0.$$

□

3.2.2

Proof of Theorem 6

Let us first prove the assertion of Theorem 6 over $\pi_1^{-1}(\mathcal{S}h_1^{d;\leq 0})$.

The closed sub-stack $\mathcal{S}h_1'^{\,d;\leq 0} - {}^0\mathcal{S}h_1'^{\,d;\leq 0}$ is the zero section of the vector bundle $\mathcal{S}h_1'^{\,d,\leq 0}$. We will use Lemma 16 to deduce the required assertion from a result by Deligne.

Consider the projectivization

$$\mathbb{P}\pi_1 : \mathbb{P}\mathcal{S}h_1'^{\,d;\leq 1} \to \mathcal{S}h_1^{d;\leq 1}$$

of the vector bundle $\mathcal{S}h_1'^{\,d;\leq 1}$ over $\mathcal{S}h_1^{d;\leq 1}$.

Over $\mathcal{S}h_1^{d;\leq 0} \simeq \mathrm{Pic}^d$, the stack $\mathbb{P}\mathcal{S}h_1'^{\,d;\leq 1}$ identifies with $X^{(d)}/G_m$. Under this identification, the projection $\mathbb{P}\pi_1$ goes over to the Abel-Jacoby map $X^{(d)}/G_m \to \mathrm{Pic}^d$ and the sheaf on $\mathbb{P}\mathcal{S}h_1'^{\,d;\leq 1}$ corresponding to the sheaf $\pi_0^\vee(\mathcal{L}^d)$ on $\mathcal{S}h_1^{d;\leq 0}$ goes over to $E^{(d)}$.

Deligne's theorem ([2], Appendix), states that the direct image of $E^{(d)}$ under the Abel-Jacoby map vanishes if $d > 4g - 4$ and E is geometrically irreducible. This proves our the assertion over the preimage of $\mathcal{S}h_1^{d;\leq 0}$.

To treat the general case, let us denote by $\mathbb{P}j_1 : {}^0\mathbb{P}\mathcal{S}h_1'^{\,d;\leq 1} \to \mathbb{P}\mathcal{S}h_1'^{\,d;\leq 1}$ the open embedding of the image of ${}^0\mathcal{S}h_1'^{\,d;\leq 1}$ under the natural projection.

The map of $\mathbb{P}\mathcal{S}h_1'^{\,d;\leq 1} - {}^0\mathbb{P}\mathcal{S}h_1'^{\,d,\leq 1}$ to $\mathcal{S}h_1^{d,\leq 1}$ is finite and by combining Lemma 16 and Lemma 17 we see that it is enough to check that the functors $(\mathbb{P}\pi_1 \circ \mathbb{P}j_1)_!$ and $(\mathbb{P}\pi_1 \circ \mathbb{P}j_1)_*$ applied to the sheaf on ${}^0\mathbb{P}(\mathcal{S}h_1'^{\,d,\leq 1})$ corresponding to $\pi_0^{\vee *}(\mathcal{L}_E^d)$ produce zero. The latter is equivalent to

$$\pi_{1!} \circ j_{1!}(\pi_0^{\vee *}(\mathcal{L}_E^d)) = \pi_{1*} \circ j_{1*}(\pi_0^{\vee *}(\mathcal{L}_E^d)) = 0.$$

Consider now the complement $\mathcal{S}h_1^{d;1} = \mathcal{S}h_1^{d;\leq 1} - \mathcal{S}h_1^{d;\leq 0}$.

There is a map

$$\mathcal{S}h_1^{d;1} \underset{\mathcal{S}h_1^{d;\leq 1}}{\times} {}^0\mathcal{S}h'^{d;\leq 1}_1 \to {}^0\mathcal{S}h'^{d-1,\leq 0}_1$$

that sends a pair

$$(0 \to T \to L' \to L \to 0, \mathcal{O} \hookrightarrow L') \to (\mathcal{O} \hookrightarrow L) \in {}^0\mathcal{S}h'^{d-1,\leq 0}_1$$

and a map

$$\mathcal{S}h_1^{d;1} \underset{\mathcal{S}h_1^{d;\leq 1}}{\times} {}^0\mathcal{S}h'^{d;\leq 1}_1 \to \mathcal{F}l_{0,0}^{1,d-1}$$

that sends a pair

$$(0 \to T \to L' \to L \to 0, \mathcal{O} \hookrightarrow L')$$

to

$$(0 \to T \to L'/\mathcal{O} \to L/\mathcal{O} \to 0) \in \mathcal{F}l_{0,0}^{1,d-1}.$$

The map

$$\kappa : \mathcal{S}h_1^{d;1} \underset{\mathcal{S}h_1^{d;\leq 1}}{\times} {}^0\mathcal{S}h'^{d;\leq 1}_1 \to (\mathcal{S}h_1^{d;1} \underset{\mathcal{S}h_1^{d-1;\leq 0}}{\times} {}^0\mathcal{S}h'^{d-1;\leq 0}_1) \underset{\mathcal{S}h_0^1 \times \mathcal{S}h_0^{d-1}}{\times} \mathcal{F}l_{0,0}^{1,d-1}$$

is a representable smooth affine fibration of relative dimension 1.

$$\begin{array}{ccccc}
\mathcal{S}h_1^{d;1} \underset{\mathcal{S}h_1^{d;\leq 1}}{\times} {}^0\mathcal{S}h'^{d;\leq 1}_1 & & & & \\
\downarrow{\scriptstyle \kappa} & & & & \\
(\mathcal{S}h_1^{d;1} \underset{\mathcal{S}h_1^{d-1;\leq 0}}{\times} {}^0\mathcal{S}h'^{d-1;\leq 0}_1) \underset{\mathcal{S}h_0^1 \times \mathcal{S}h_0^{d-1}}{\times} \mathcal{F}l_{0,0}^{1,d-1} & \xrightarrow{'(\epsilon \times \pi_0^\vee)} & \mathcal{F}l_{0,0}^{1,d-1} & \xrightarrow{p_{0,0}^{1,d-1}} & \mathcal{S}h_0^d \\
\downarrow{\scriptstyle '(q_{0,0}^{1,d-1})} & & \downarrow{\scriptstyle q_{0,0}^{1,d-1}} & & \\
\mathcal{S}h_1^{d;1} \underset{\mathcal{S}h_1^{d-1;\leq 0}}{\times} {}^0\mathcal{S}h'^{d-1;\leq 0}_1 & \xrightarrow{\epsilon \times \pi_0^\vee} & \mathcal{S}h_0^1 \times \mathcal{S}h_0^{d-1} & & \\
\downarrow{\scriptstyle \mathrm{id} \times {}^0\pi_1} & & & & \\
\mathcal{S}h_1^{d;1} & & & &
\end{array}$$

(In the diagram above, ϵ is the map $\mathcal{S}h_1^{d;1} \to \mathcal{S}h_0^1$ as in the proof of Lemma 10.)

Therefore

$$\pi_{1!} \circ j_{1!} \circ \pi_0^{\vee *}(\mathcal{L}_E^d)|_{\mathcal{S}h_1^{d;1}} \simeq (\mathrm{id} \times {}^0\pi_1)_! \circ {}'(q_{0,0}^{1,d-1})_! \circ \kappa_! \circ \kappa^* \circ {}'(\epsilon \times \pi_0^\vee)^* p_{0,0}^{1,d-1*}(\mathcal{L}_E^d) \simeq$$
$$\simeq (\mathrm{id} \times {}^0\pi_1)_! \circ (\epsilon \times \pi_0^\vee)^* \circ q_{0,0}^{1,d-1}{}_! \circ p_{0,0}^{1,d-1*}(\mathcal{L}_E^d)[-2](-1),$$

which by Proposition 2(2) identifies with

$$(\mathrm{id} \times {}^0\pi_1)_! \circ (\epsilon \times \pi_0^\vee)^*(\mathcal{L}_E^1 \boxtimes \mathcal{L}_E^{d-1})[-2](-1).$$

Using the projection formula, the latter identifies with the tensor product of $\epsilon^*(\mathcal{L}_E^1)[-2](-1)$ and the pull-back under the map $\mathcal{S}h_1^{d;1} \to \mathcal{S}h_1^{d-1;0}$ of the sheaf ${}^0\pi_{1!} \circ \pi_0^{\vee *}(\mathcal{L}_E^{d-1})$.

However, we have already shown that ${}^0\pi_{1!} \circ \pi_0^{\vee *}(\mathcal{L}_E^{d-1})$ vanishes when restricted to $\mathcal{S}h_1^{d-1,\leq 0}$, since $d-1 > 4g-4$, by the assumption.

The vanishing result for $\pi_{1*} \circ j_{1*}(\pi_0^{\vee *}(\mathcal{L}_E))$ follows by Verdier's duality, since $\mathbb{D}(\mathcal{L}_E) = \mathcal{L}_{E^\vee}$.

□

3.2.3

We will deduce Proposition 8 from the following lemma.

Lemma 18. *Let E' and E'' be two 2-dimensional local systems on X. Then for any variety mapping to $\mathcal{S}h_0^d$ the pull-backs of $\mathcal{L}_{E'}^d$ and of $\mathcal{L}_{E''}^d$ are isomorphic locally in the étale topology (by this we mean that the corresponding projective systems of sheaves with torsion coefficients can be chosen to have locally isomorphic members).*

Proof. Let us consider the analogous sheaves $\mathcal{L}_{E'}$ and $\mathcal{L}_{E''}$ constructed out of local systems E' and E'' with torsion coefficients. It is enough to exhibit an étale cover of the stack $\mathcal{S}h_0^d$ such that the pull-backs of $\mathcal{L}_{E'}$ and of $\mathcal{L}_{E''}$ to it become isomorphic.

Let now $\delta : X' \to X$ be an étale Galois cover and consider the open sub-stack $\mathcal{S}h_0^d(X, X')$ in $Sh_0^d(X')$ that corresponds to those torsion sheaves $T \in \mathcal{S}h_0^d(X')$ for which the support of T does not intersect with the support of $\sigma(T)$ for any element $\sigma \in \mathrm{Gal}(X'/X)$.

The natural map $\delta^d : \mathcal{S}h_0^d(X, X') \to \mathcal{S}h_0^d(X)$ given by

$$T \in Sh_0^d(X') \to \delta_*(T) \in \mathcal{S}h_0^d(X)$$

is étale and surjective. This implies the assertion, since if $\delta^*(E') \simeq \delta^*(E'')$,

$$\delta^{d^*}(\mathcal{L}_{E'}) \simeq \mathcal{L}_{\delta^*(E')}|_{\mathcal{S}h_0^d(X,X')} \simeq \mathcal{L}_{\delta^*(E'')}|_{\mathcal{S}h_0^d(X,X')} \simeq \delta^{d^*}(\mathcal{L}_{E''}).$$

□

The proof will use the following result, which follows from a theorem by Deligne [6]:

Proposition 9. *Let $\mathcal{K}_1$ and $\mathcal{K}_2$ be two constructible complexes on a complete variety whose cohomology sheaves are isomorphic locally in the étale topology (cf. Lemma 18 above). Then the hyper-cohomologies of these complexes have equal Euler-Poincaré characteristics.*

3.2.4

Proof of Proposition 8. [2]

Let $\mathcal{K}_{E'}$ and $\mathcal{K}_{E''}$ denote the pull-backs to the stack ${}^0\mathbb{P}\mathcal{S}h_1'^{d;\leq 1}$ of the sheaves $\mathcal{L}_{E'}^d$ and $\mathcal{L}_{E''}^d$ on $\mathcal{S}h_0^d$.

We claim, that it is enough to show that for any map $\phi : \mathcal{Y} \to \mathbb{P}\mathcal{S}h_1'^{d;\leq 1}$, where $\mathcal{Y}$ is a complete variety, the Euler-Poincaré characteristics of the hyper-cohomologies

$$\mathbb{H}(\mathcal{Y}, \phi^* \circ \mathbb{P}j_{1!}(\mathcal{K}_{E'})) \text{ and of } \mathbb{H}(\mathcal{Y}, \phi^* \circ \mathbb{P}j_{1!}(\mathcal{K}_{E''}))$$

are equal.

Indeed, for any G_m-equivariant sheaf $\mathcal{K}$ on a vector space $\mathcal{E}$, the stalks of $\mathrm{Four}(\mathcal{K})$ at $e^\vee \in \mathcal{E}^\vee$ are glued from $\mathcal{K}|_0$, $\mathbb{H}(\mathbb{P}\mathcal{E}, \mathcal{K}')$ and $\mathbb{H}(\mathcal{H}_{e^\vee}, \mathcal{K}'|_{\mathcal{H}_{e^\vee}})$, where $\mathcal{K}'$ is the corresponding sheaf on $\mathbb{P}\mathcal{E}$ and $\mathcal{H}_{e^\vee}$ is the hyper-plane in $\mathbb{P}\mathcal{E}$ corresponding to $e^\vee$.

To prove the required property of $\mathcal{K}_{E'}$ and of $\mathcal{K}_{E''}$ it is enough to find a stack $\widetilde{\mathbb{P}\mathcal{S}h}_1'^{d;\leq 1}$ with a proper map to $\mathbb{P}\mathcal{S}h_1'^{d;\leq 1}$ such that $\mathbb{P}j_{1!}(\mathcal{K}_{E'})$ and $\mathbb{P}j_{1!}(\mathcal{K}_{E''})$ are direct images of two locally isomorphic sheaves on $\widetilde{\mathbb{P}\mathcal{S}h}_1'^{d;\leq 1}$.

Let $\widetilde{\mathbb{P}\mathcal{S}h}_1'^{d;\leq 1}$ be the stack that classifies pairs $L_0 \hookrightarrow L$, where L_0 is a coherent sheaf of generic rank 1 and with an identification $\det(L_0) = \mathcal{O}$ and where $L \in \mathcal{S}h_1^{d;\leq 1}$. The stack $\widetilde{\mathbb{P}\mathcal{S}h}_1'^{d;\leq 1}$ is a sub-stack of $\mathcal{F}l_{1,0}^{0;d}$, therefore we have a projection

[2] I am grateful to G.Laumon, who has found a mistake in the previous version of the proof of this proposition.

$\widetilde{\mathbb{P}\mathcal{S}h_1'^{d;\leq 1}} \to \mathcal{S}h_1^{0;\leq 1}$. Let ${}^0\widetilde{\mathbb{P}\mathcal{S}h_1'^{d;\leq 1}} \overset{\widetilde{\mathbb{P}j_1}}{\hookrightarrow} \widetilde{\mathbb{P}\mathcal{S}h_1'^{d;\leq 1}}$ denote the open embedding of preimage of $\mathcal{S}h_1^{0;\leq 0}$ in $\widetilde{\mathbb{P}\mathcal{S}h_1'^{d;\leq 1}}$.

Consider the fibered product $\widetilde{\mathbb{P}\mathcal{S}h_1'^{d;\leq 1}} \underset{\mathcal{S}h_1^0}{\times} \mathcal{S}h_1'^{\,0}$. We have a natural proper map

$$\widetilde{\mathbb{P}\mathcal{S}h_1'^{d;\leq 1}} \underset{\mathcal{S}h_1^0}{\times} \mathcal{S}h_1'^{\,0} \to \mathcal{S}h_1'^{\,d;\leq 1}.$$

It is easy to see that $\widetilde{\mathbb{P}\mathcal{S}h_1'^{d;\leq 1}} \underset{\mathcal{S}h_1^0}{\times} \mathcal{S}h_1'^{\,0}$ is a line bundle over $\widetilde{\mathbb{P}\mathcal{S}h_1'^{d;\leq 1}}$ and hence we get a map $\widetilde{\mathbb{P}\mathcal{S}h_1'^{d;\leq 1}} \to \mathbb{P}\mathcal{S}h_1'^{\,d;\leq 1}$ with ${}^0\widetilde{\mathbb{P}\mathcal{S}h_1'^{d;\leq 1}}$ mapping isomorphically onto ${}^{00}\mathbb{P}\mathcal{S}h_1'^{\,d;\leq 1} \subset \mathbb{P}\mathcal{S}h_1'^{\,d;\leq 1}$.

Let $\widetilde{\mathcal{K}}_{E'}$ (resp., $\widetilde{\mathcal{K}}_{E''}$) denote the pull-back of $\mathcal{L}^d_{E'}$ (resp., of $\mathcal{L}^d_{E''}$) under the natural projection $\widetilde{\mathbb{P}\mathcal{S}h_1'^{d;\leq 1}} \to \mathcal{S}h_0^d$. The sheaves $\widetilde{\mathcal{K}}_{E'}$ and $\widetilde{\mathcal{K}}_{E'}$ are locally isomorphic and so are the sheaves $\widetilde{\mathbb{P}j_1}_! \circ \widetilde{\mathbb{P}j_1}^*(\widetilde{\mathcal{K}}_{E'})$ and $\widetilde{\mathbb{P}j_1}_! \circ \widetilde{\mathbb{P}j_1}^*(\widetilde{\mathcal{K}}_{E''})$. However, the direct image of $\widetilde{\mathbb{P}j_1}_! \circ \widetilde{\mathbb{P}j_1}^*(\widetilde{\mathcal{K}}_{E'})$ (resp., of $\widetilde{\mathbb{P}j_1}_! \circ \widetilde{\mathbb{P}j_1}^*(\widetilde{\mathcal{K}}_{E'})$) onto $\mathbb{P}\mathcal{S}h_1'^{\,d;\leq 1}$ identifies with $\mathbb{P}j_{1!}(\mathcal{K}_{E'})$ (resp., with $\mathbb{P}j_{1!}(\mathcal{K}_{E''})$), which implies the assertion.

□(Proposition 8)

Remark 6. When working over a field of characteristic zero, the proof of Proposition 8 becomes considerably simpler. In particular, we can prove the assertion not only over $\pi_1^{\vee -1}(\mathcal{S}h_1'^{\,d;\leq 1})$, but rather over $\pi_1^{\vee -1}(\mathcal{S}h_1'^{\,d;\leq t})$ provided that $d > 2g-2+t$:

Indeed, we have an analog Proposition 18 that asserts that Euler-Poincaré characteristics of $\mathcal{L}_{E'}$ and of $\mathcal{L}_{E''}$ are equal at all points of $\mathcal{S}h_0^d$. The assertion of Proposition 8 follows now from the fact that for any two complexes of holonomic D-modules with regular singularities on any variety (not necessarily complete (!)), that have equal pointwise Euler-Poincaré characteristics, the Euler-Poincaré characteristics of their hyper-cohomologies are equal as well.

3.3 Proof of Main Theorem C

3.3.1

For $d > 6g-4$ we have constructed the sheaf $\mathcal{S}^d_E$ on $\mathcal{S}h_2^d$. We define the sheaf Aut^d_E for $d > 6g-4$ to be the restriction of $\mathcal{S}^d_E$ to Bun_2.

To prove Main Theorem C we have to define the sheaves Aut^d_E for all $d \in \mathbb{Z}$ and to establish the Hecke property. This will be done in two steps:

Step 1.

We will prove that for $d > 6g-4$ over ${}_{2g-2}\mathcal{S}h_2^d \cap \mathrm{Bun}_2$ we have:

$$(\mathfrak{q} \times \mathrm{supp})_! \circ \mathfrak{p}^*(\mathrm{Aut}_E^{d+1})[1](\frac{1}{2}) \simeq \mathrm{Aut}_E^d \boxtimes E.$$

Step 2.

We will prove that for $d > 6g-4$ and $c \geq 0$, over $(\mathrm{Bun}_2 \cap_{2g-2}\mathcal{S}h_2^d) \times \mathrm{Pic}^c$

$$m^*(\mathrm{Aut}_E^{d+2c}) \simeq \mathrm{Aut}_E^d \boxtimes (\Lambda^2(E))^c$$

Let us now show how Step 1 and Step 2 imply Main Theorem C.

Proof of Main Theorem C

First of all, since the sheaves $\mathcal{S}_E^d$ are irreducible and the map m is smooth, the assertion of Step 2 implies that the isomorphism

$$m^*(\mathcal{S}_E^{d+2c}) \simeq \mathcal{S}_E^d \boxtimes (\Lambda^2(E))^c$$

$d > 6g-4$ and $c \geq 0$ holds over the whole of $\mathcal{S}h_2^d \times \mathrm{Pic}^c$.

For any $d \in \mathbb{Z}$ pick a line bundle L_0 such that $d + 2\deg(L_0) > 6g-4$. We define Aut_E^d as

$$\mathrm{Aut}_E^d := m_{L_0}^*(\mathrm{Aut}_E^{d+2\deg(L_0)}) \otimes ((\Lambda^2(E^\vee))^{\deg(L_0)}|_{L_0}).$$

The assertion of Step 2 implies that this definition is independent of L_0.

Moreover, the second Hecke property

$$m^*(\mathrm{Aut}_E^{d+2c}) \simeq \mathrm{Aut}_E^d \boxtimes (\Lambda^2(E))^c$$

is automatically satisfied.

To prove the first Hecke property it is enough to establish the isomorphism

$$(\mathfrak{q} \times \mathrm{supp})_! \circ \mathfrak{p}^*(\mathrm{Aut}_E^{d+1})[1](\frac{1}{2}) \simeq \mathrm{Aut}_E^d \boxtimes E$$

over $\mathrm{Bun}_2 \cap_{d'} \mathcal{S}h_2^d$ for each d and d'.

For fixed d and d' pick again L_0 in such a way that $d + 2\deg(L_0) > 6g-4$ and $d' + \deg(L_0) > 2g-2$.

We have: $m_{L_0}(\mathrm{Bun}_2 \cap_{d'} \mathcal{S}h_2^d) \subset \mathrm{Bun}_2 \cap_{2g-2} \mathcal{S}h_2^{d+2\deg(L_0)}$ and

$$(\mathfrak{q} \times \mathrm{supp})_! \circ \mathfrak{p}^*(\mathrm{Aut}_E^{d+1})|_{\mathrm{Bun}_2 \cap_{d'} \mathcal{S}h_2^d}[1](\frac{1}{2})$$

is isomorphic to

$$m^*_{L_0}(\mathfrak{q}\times\mathrm{supp})_!\circ\mathfrak{p}^*(\mathrm{Aut}_E^{d+1+2\deg(L_0)})|_{\mathrm{Bun}_2\cap_{2g-2}\mathcal{S}h_2^{d+2\deg(L_0)}}\otimes((\Lambda^2(E^\vee))^{\deg(L_0)}|_{L_0})[1](\frac{1}{2}).$$

The latter, according to the assertion of Step 1, identifies with

$$(m_{L_0}\times\mathrm{id})^*(\mathrm{Aut}_E^{d+2\deg(L_0)}\boxtimes E)\otimes((\Lambda^2(E^\vee))^{\deg(L_0)}|_{L_0})\simeq\mathrm{Aut}_E^d\boxtimes E,$$

which implies the required statement.

□

3.3.2

Proof of Step 1

Consider the fibered product ${}^0\mathcal{S}h_2'^{\,d}\underset{\mathcal{S}h_2^d}{\times}\mathcal{F}l_{2,0}^{d,1}$. Let us denote by ${q'_{2,0}}^{d,1}$ the base change map ${}^0\mathcal{S}h_2'^{\,d}\underset{\mathcal{S}h_2^d}{\times}\mathcal{F}l_{2,0}^{d,1}\to{}^0\mathcal{S}h_2'^{\,d}\times\mathcal{S}h_0^1$ and by ${p'_{2,0}}^{d,1}$ the map ${}^0\mathcal{S}h_2'^{\,d}\underset{\mathcal{S}h_2^d}{\times}\mathcal{F}l_{2,0}^{d,1}\to{}^0\mathcal{S}h_2'^{\,d+1}$ that sends a pair

$$(\Omega\hookrightarrow M,\ 0\to M\to M'\to T\to 0)$$

to $\Omega\hookrightarrow M'$.

We have a commutative diagram:

$$\begin{array}{ccccc}
{}^0\mathcal{S}h_2'^{\,d}\times\mathcal{S}h_0^1 & \xleftarrow{{q'_{2,0}}^{d,1}} & {}^0\mathcal{S}h_2'^{\,d}\underset{\mathcal{S}h_2^d}{\times}\mathcal{F}l_{2,0}^{d,1} & \xrightarrow{{p'_{2,0}}^{d,1}} & {}^0\mathcal{S}h_2'^{\,d+1}\\
{\scriptstyle {}^0\pi_2\times\mathrm{id}}\downarrow & & \downarrow & & {\scriptstyle {}^0\pi_2}\downarrow\\
\mathcal{S}h_2^d\times\mathcal{S}h_0^1 & \xleftarrow{q_{2,0}^{d,1}} & \mathcal{F}l_{2,0}^{d,1} & \xrightarrow{p_{2,0}^{d,1}} & \mathcal{S}h_2^{d+1}
\end{array}$$

(note, that the right square is NOT Cartesian)

We will use the following result due to G.Laumon ([11], Corollary 4.3):

Proposition 10. *The sheaf* $\mathrm{Four}\circ j_{1!}\circ\pi_0^{\vee*}(\mathcal{L}_E^{d+1-2g+2})$ *on* ${}^0\mathcal{S}h_2'^{\,d+1}$ *satisfies:*

$${q'_{2,0}}^{d,1}{}_!\circ{p'_{2,0}}^{d,1*}(\mathrm{Four}\circ j_{1!}\circ\pi_0^{\vee*}(\mathcal{L}_E^{d+1-2g+2}))[3](\frac{3}{2})\simeq\mathrm{Four}\circ j_{1!}\circ\pi_0^{\vee*}(\mathcal{L}_E^{d-2g+2})\boxtimes\mathcal{L}_E^1$$

over $\pi_1^{\vee-1}({}_0\mathcal{S}h_1^{d-2g+2})$.

According to Theorem 6, we see that over $\pi_1^{\vee -1}({}_0\mathcal{S}h_1^{d-2g+2})$

$$q'^{\,d,1}_{2,0!} \circ p'^{\,d,1*}_{2,0}(\mathcal{S}'^{\,d+1}_E)[3](\frac{3}{2}) \simeq \mathcal{S}'^{\,d}_E \boxtimes \mathcal{L}^1_E.$$

We have: $p_{2,0}^{d,1}(q_{2,0}^{d,1^{-1}}({}_{2g-2}\mathcal{S}h_2^d)) \subset {}_{2g-2}\mathcal{S}h_2^{d+1}$, and by applying Theorem 5 we obtain that over $\pi_1^{\vee -1}({}_0\mathcal{S}h_1^{d-2g+2}) \cap {}^0_{2g-2}\mathcal{S}h'^{\,d+1}_2$

$$({}^0\pi_2 \times \mathrm{id})^*(q_{2,0!}^{d,1} \circ p_{2,0}^{d,1*}(\mathcal{S}_E^{d+1}))[3](\frac{3}{2}) \simeq ({}^0\pi_2 \times \mathrm{id})^*(\mathcal{S}_E^d \boxtimes E).$$

Since over $\mathrm{Bun}_2 \cap_{2g-2}\mathcal{S}h^d$ the map ${}^0\pi_2 : \pi_1^{\vee -1}({}_0\mathcal{S}h_1^{d-2g+2}) \to \mathcal{S}h_2^d$ is smooth and has connected fibers, the above isomorphism implies that

$$q_{2,0!}^{d,1} \circ p_{2,0}^{d,1*}(\mathcal{S}_E^{d+1})[3](\frac{3}{2}) \simeq \mathcal{S}_E^d \boxtimes \mathcal{L}_E.$$

The assertion of Step 1 follows now by applying Proposition 4 with $\mathcal{Y} = \mathrm{Bun}_2 \cap_{2g-2}\mathcal{S}h^d$ and $\mathcal{E}$ being the restriction to $\mathrm{Bun}_2 \cap_{2g-2}\mathcal{S}h^d$ of the vector bundle $\mathcal{F}l_{2,0}^{d,1}$.
□

3.3.3

Let m_X be the map $\mathrm{Bun}_2 \times X \to \mathrm{Bun}_2$. The pull-back m_X^* is the second Hecke functor for $GL(2)$.

We claim, that the proof Theorem 4 presented in Sect. 1.3.6 applies to deduce from the assertion of Step 1 that for $d > 6g-4$ over $\mathrm{Bun}_2 \cap_{2g-2}\mathcal{S}h_2^d$,

$$m_X^*(\mathrm{Aut}_E^{d+2}) \simeq \mathrm{Aut}_E^d \boxtimes \Lambda^2(E).$$

The only modification required is to replace all the relevant stacks by their open sub-stacks that correspond to the preimage of $\mathrm{Bun}_2 \cap_{2g-2}\mathcal{S}h_2^d$ under the q-projection.

Therefore, the inverse image under the map $\mathrm{Bun}_2^d \times X^{(c)} \to \mathrm{Bun}_2$ of Aut_E^{d+2c} identifies with $\mathrm{Aut}_E \boxtimes (\Lambda^2(E))^{(c)}$, which in turn implies by the geometric class field theory, that for $c \geq 0$,

$$m * (\mathrm{Aut}_E^{d+2c}) \simeq \mathrm{Aut}^d \boxtimes (\Lambda^2(E))^c.$$

This completes the proof of Main Theorem C.

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