Edmund Landau: The Master Rigorist

Edmund Yehezkel Landau was born in Berlin in 1877; his father was the well-known gynecologist Leopold Landau. He began his education at the French Gymnasium (high school) in Berlin and soon thereafter devoted himself entirely to mathematics. Among his teachers was Ferdinand Lindemann (1852–1939), who in 1882 proved the transcendence of π—the fact that π cannot be the root of a polynomial equation with integer coefficients—thereby settling the age-old problem of "squaring the circle" (see p. 181). From the beginning Landau was interested in analytic number theory—the application of analytic methods to the study of integers. In 1903 he gave a simplified proof of the Prime Number Theorem, which had first been proved seven years earlier after having defied some of the greatest minds of the nineteenth century.[1] In 1909, when he was only 31, Landau was appointed professor of mathematics at the University of Göttingen, the world-renowned center of mathematical research up until World War II; he succeeded Hermann Minkowski (1864–1909), known for his four-dimensional interpretation of Einstein's theory of relativity, who had died at the age of forty-five. Landau published over 250 papers and wrote several major works in his field, among them *Handbook of the Theory and Distribution of the Prime Numbers* (in two volumes, 1909) and *Lectures on Number Theory* (in three volumes, 1927).

Landau was one of eight distinguished scholars who were invited to talk at the ceremonies inaugurating the Hebrew University of Jerusalem in 1925. From atop Mount Scopus overlooking the Holy City, he spoke about "Solved and Unsolved Problems in Elementary Number Theory"—a rather unusual subject for such a festive occasion. He accepted the university's invitation to occupy its first chair of mathematics, teaching himself Hebrew expressly for this purpose. He joined the university in 1927 but shortly afterward returned to Germany to resume his duties at Göttingen. His brilliant career, however, was soon to come to an end: when the Nazis came to power in 1933 he—along with all Jewish professors at German universities—was forced to resign his position. His sudden death in 1938 saved him from the fate awaiting the Jewish community in Germany.

Landau embodied the ultimate image of the pure mathematician. He viewed any practical applications to mathematics with disdain and avoided the slightest reference to them, dismissing them as *Schmieröl* (grease); among the "practical applications" was geometry, which he entirely shunned from his exposition. In his lectures and written work, definitions, theorems, and proofs followed in quick succession, without the slightest hint at the motivation behind them. His goal was absolute and uncompromising rigor. His assistant, who always attended his lectures, was instructed to interrupt him if the professor omitted the slightest detail.[2]

To the student of higher mathematics Landau is best known for his two textbooks, *Grundlagen der Analysis* (Foundations of analysis, 1930) and *Differential and Integral Calculus* (1934).[3] The former opens with two prefaces, one intended for the student and the other for the teacher. The preface for the student begins thus:

1. Please don't read the preface for the teacher.

2. I will ask of you only the ability to read English and to think logically—no high school mathematics, and certainly no higher mathematics.

3. Please forget everything you have learned in school; for you haven't learned it.

4. The multiplication table will not occur in this book, not even the theorem,

$$2 \cdot 2 = 4,$$

but I would recommend, as an exercise, that you define

$$2 = 1 + 1,$$

$$4 = (((1 + 1) + 1) + 1),$$

and then prove the theorem.

The preface for the teacher ends with these words:

My book is written, as befits such easy material, in merciless telegram style ("Axiom," "Definition," "Theorem," "Proof,"' occasionally "Preliminary Remark") . . . . I hope that I have written this book in such a way that a normal student can read it in two days. And then (since he already knows the formal rules from school) he may forget its contents.

While it is not clear whom Landau may have considered a "normal student," it is hard to believe that an average student, or even a mathematics professor, could master in two days the 301 theorems of the book, written in almost hieroglyphic form in the book's 134 pages (fig. 91).

His textbook on the calculus is just 372 pages long—a far cry from today's thousand-page texts. Not a single illustration graces the book—after all, illustrations would imply that geometric concepts are being used, and geometry was *schmieröl*. Again the preface sets the tone for the entire work. First Landau refers to his *Grundlagen*, which he says received "tolerant and even some friendly reviews." He goes on to say that "a reader whose main interest lies in the applications of the calculus... should not make this book his choice." "My task," he says, is that of "bringing out into the open the definitions and theorems which are often implicitly assumed and which serve as the mortar when the whole structure is being built up with all the right floors in the right places."

True to its word, the book is structured in a terse definition-theorem-proof style, with an occasional example following a theorem. Definition 25 introduces the derivative of a function, followed immediately by two theorems: that differentiability implies continuity, and that there exist everywhere-continuous, nowhere-differentiable functions. This last theorem is due to the German mathematician Karl Theodor Wilhelm Weierstrass (1815–1897), whose lifelong goal was to rid analysis of any vestige of intuitiveness, and whose rigorous approach served as a model to Landau. Landau gives as an example the function

$$f(x) = \lim_{m\to\infty} \sum_{i=0}^{m} \{4^i x\}/4^i,$$

where $\{x\}$ is the distance of x to its nearest integer, and then proves that $f(x)$ is everywhere continuous but nowhere differentiable; the proof takes up nearly five pages.[4]

Of special interest to us here is the chapter on the trigonometric functions. It begins thus:

Theorem 248:

$$\sum_{m=0}^{\infty} \frac{(-1)^m}{(2m+1)!} x^{2m+1}$$

converges everywhere.

Theorem 280: *If* $\mathfrak{f}(1)$ *and* $\mathfrak{f}(1+1)$ *are defined, then*

$$\sum_{n=1}^{1+1} \mathfrak{f}(n) = \mathfrak{f}(1) \dotplus \mathfrak{f}(1+1).$$

Proof: By Theorems 278 and 277, we have

$$\sum_{n=1}^{1+1} \mathfrak{f}(n) = \sum_{n=1}^{1} \mathfrak{f}(n) \dotplus \mathfrak{f}(1+1) = \mathfrak{f}(1) \dotplus \mathfrak{f}(1+1).$$

Theorem 281: *If* $\mathfrak{f}(n)$ *is defined for* $n \leqq x+y$, *then*

$$\sum_{n=1}^{x+y} \mathfrak{f}(n) = \sum_{n=1}^{x} \mathfrak{f}(n) \dotplus \sum_{n=1}^{y} \mathfrak{f}(x+n).$$

Proof: Fix x, and let $\mathfrak{M}$ be the set of all y for which this holds.

I) If $\mathfrak{f}(n)$ is defined for $n \leqq x+1$, then we have by Theorems 278 and 277 that

$$\sum_{n=1}^{x+1} \mathfrak{f}(n) = \sum_{n=1}^{x} \mathfrak{f}(n) \dotplus \mathfrak{f}(x+1) = \sum_{n=1}^{x} \mathfrak{f}(n) \dotplus \sum_{n=1}^{1} \mathfrak{f}(x+n).$$

Hence 1 belongs to $\mathfrak{M}$.

II) Let y belong to $\mathfrak{M}$. If $\mathfrak{f}(n)$ is defined for $n \leqq x+(y+1)$, then we have by Theorem 278 (applied to $x+y$ instead of x) that

$$\sum_{n=1}^{x+(y+1)} \mathfrak{f}(n) = \sum_{n=1}^{(x+y)+1} \mathfrak{f}(n) = \sum_{n=1}^{x+y} \mathfrak{f}(n) \dotplus \mathfrak{f}((x+y)+1)$$

$$= \left(\sum_{n=1}^{x} \mathfrak{f}(n) \dotplus \sum_{n=1}^{y} \mathfrak{f}(x+n)\right) \dotplus \mathfrak{f}(x+(y+1))$$

$$= \sum_{n=1}^{x} \mathfrak{f}(n) \dotplus \left(\sum_{n=1}^{y} \mathfrak{f}(x+n) \dotplus \mathfrak{f}(x+(y+1))\right),$$

which by Theorem 278 (applied to y instead of x, and to $\mathfrak{f}(x+n)$ instead of $\mathfrak{f}(n)$) is

$$= \sum_{n=1}^{x} \mathfrak{f}(n) \dotplus \sum_{n=1}^{y+1} \mathfrak{f}(x+n).$$

Hence $y+1$ belongs to $\mathfrak{M}$, and Theorem 281 is proved.

Theorem 282: *If* $\mathfrak{f}(n)$ *and* $\mathfrak{g}(n)$ *are defined for* $n \leqq x$, *then*

$$\sum_{n=1}^{x} (\mathfrak{f}(n) \dotplus \mathfrak{g}(n)) = \sum_{n=1}^{x} \mathfrak{f}(n) \dotplus \sum_{n=1}^{x} \mathfrak{g}(n).$$

Proof: Let $\mathfrak{M}$ be the set of all x for which this holds.

I) If $\mathfrak{f}(1)$ and $\mathfrak{g}(1)$ are defined, then

$$\sum_{n=1}^{1} (\mathfrak{f}(n) \dotplus \mathfrak{g}(n)) = \mathfrak{f}(1) \dotplus \mathfrak{g}(1) = \sum_{n=1}^{1} \mathfrak{f}(n) \dotplus \sum_{n=1}^{1} \mathfrak{g}(n).$$

Hence 1 belongs to $\mathfrak{M}$.

Fig. 91. A page from Edmund Landau's *Foundations of Analysis* (1930).

(This, of course, is the power series $x - x^3/3! + x^5/5! - + \cdots$). This is followed by

Definition 59:

$$\sin x = \sum_{m=0}^{\infty} \frac{(-1)^m}{(2m+1)!} x^{2m+1}.$$

sin is to be read "sine."

After defining $\cos x$ in a similar manner there come several theorems establishing the familiar properties of these functions. Then,

Theorem 258:

$$\sin^2 x + \cos^2 x = 1.$$

Proof:

$$\begin{aligned} 1 &= \cos 0 = \cos(x - x) = \cos x \cos(-x) - \sin x \sin(-x) \\ &= \cos^2 x + \sin^2 x. \end{aligned}$$

Thus, out of the blue and without any mention by name, the most famous theorem of mathematics is introduced: the Pythagorean Theorem.[5]

Today, when textbooks fiercely compete with one another and must sell well in order to justify their publication, it is doubtful that Landau's texts would find a wide audience. In prewar European universities, however, higher education was the privilege of a very few. Moreover, a professor had total authority to teach his class the way he chose, including the choice of a textbook. Most professors followed no text at all but lectured from their own notes, and it was up to the student to supplement these notes with other material. In this atmosphere Landau's texts were held in high esteem as offering a true intellectual challenge to the serious student.

Notes and Sources

1. The theorem says that the average density of the prime numbers—the number of primes below a given integer x, divided by x—approaches $1/\ln x$ as $x \to \infty$. The theorem was first conjectured by Gauss in 1792, when he was fifteen years old. It was first proved in 1896 by Jacques Salomon Hadamard (1865–1963) of France and de la Vallée-Poussin (1866–1962) of Belgium, working independently.

2. Constance Reid, *Courant in Göttingen and New York: The Story of an Improbable Mathematician* (New York: Springer-Verlag, 1976),

pp. 25–26 and 126–127. See also "In Memory of Edmund Landau: Glimpses from the Panorama of Number Theory and Analysis," in *Edmund Landau: Collected Works*, ed. L. Mirsky et al. (Essen: Thales Verlag, 1985), pp. 25–50.

3. Both works appeared in English translation by Chelsea Publishing Company, New York: *Foundations of Analysis: The Arithmetic of Whole, Rational, Irrational and Complex Numbers*, trans. F. Steinhardt (1951), and *Differential and Integral Calculus*, trans. Melvin Hausner and Martin Davis (1950). The excerpts given above are from the English translations.

4. In his preface Landau defends this approach thus: "Some mathematicians may think it unorthodox to give as the second theorem after the definition of the derivative, Weierstrass' theorem... To them I would say that while there are very good mathematicians who have never learned any proof of that theorem, it can do the beginner no harm to learn the simplest proof to date right from his textbook."

5. The number π is introduced a few lines farther down as the smallest positive solution of the equation $\cos x = 0$. This "universal constant" is then denoted by π; there is no mention whatsoever of the numerical value of this constant, nor its relation to the circle.

Go to Chapter 15