Multiplicity Assignments for Algebraic Soft-Decoding of Reed-Solomon Codes

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Abstract — **A soft-decision decoding algorithm for Reed-Solomon codes was recently proposed in [2]. This algorithm converts probabilities observed at the channel output into algebraic interpolation conditions, specified in terms of a multiplicity matrix M. Koetter-Vardy [2] show that the probability of decoding failure is given by $\Pr\{\mathcal{S}_M \le \Delta(M)\}$, where $\mathcal{S}_M$ is a random variable and $\Delta(M)$ is a known function of M. They then compute the multiplicity matrix M that maximizes the *expected value* of $\mathcal{S}_M$. Here, we attempt to *directly minimize* the overall probability of decoding failure $\Pr\{\mathcal{S}_M \le \Delta(M)\}$. First, we recast this optimization problem into a geometrical framework. Using this framework, we derive a simple modification to the KV algorithm which results in a provably better multiplicity assignment. Alternatively, we approximate the distribution of $\mathcal{S}_M$ by a Gaussian distribution, and develop an iterative algorithm to minimize $\Pr\{\mathcal{S}_M \le \Delta(M)\}$ under this approximation. This leads to coding gains of about 0.20 dB for RS codes of length 255 and up to 0.75 dB for RS codes of length 15, as compared to the Koetter-Vardy algorithm.**

I. Introduction

A breakthrough in algebraic coding theory was achieved by Sudan and Guruswami-Sudan [1], who developed a list-decoding algorithm for Reed-Solomon codes based on algebraic interpolation and factorization techniques. This was later extended to a soft-decision decoding algorithm for RS codes by Koetter and Vardy [2]. The soft-decoding algorithm of [2] converts probabilities observed at the channel output into algebraic interpolation conditions, specified in terms of a multiplicity matrix M. It is shown in [2] that the probability of decoding failure is given by $\Pr\{\mathcal{S}_M \le \Delta(M)\}$, where $\mathcal{S}_M$ is a random variable whose distribution depends on M and on the channel observations, while $\Delta(M)$ is a known function of M. Koetter-Vardy [2] derive an efficient multiplicity assignment scheme that maximizes the *mean* of $\mathcal{S}_M$ for a fixed $\Delta(M)$, and show that this approach is optimal as the length n of the Reed-Solomon code becomes large.

However, for each fixed n, maximizing the mean of $\mathcal{S}_M$ may be suboptimal. Our goal herein is to devise multiplicity assignment schemes that directly minimize the probability $\Pr\{\mathcal{S}_M \le \Delta(M)\}$ of decoding failure. To this end, we first recast the problem into a geometrical framework. We then prove that the optimal multiplicity assignment must lie on a tangent from a certain point to a certain sphere in $\mathbb{R}^{qn}$. Using this result, we derive a simple modification to the KV algorithm [2] which leads to a provably better multiplicity assignment. Alternatively, we approximate the distribution of $\mathcal{S}_M$ by a Gaussian distribution, and then show how to minimize $\Pr\{\mathcal{S}_M \le \Delta(M)\}$ directly under this approximation.

II. Geometrical Framework

Let us think of the $q \times n$ multiplicity matrix M as a point in the Euclidean space $\boldsymbol{\mathcal{V}} = \mathbb{R}^{qn}$. Note that $\Delta(M)$ depends on M only through its *cost* $\mathcal{C}(M) = {}^1\!/\!_2 \sum_{i=1}^{q} \sum_{j=1}^{n} m_{i,j}(m_{i,j}+1)$. We prove that the set of all points $M \in \boldsymbol{\mathcal{V}}$ with a given cost $\mathcal{C}(M) = \mathcal{C}$ is a sphere of radius $r_{\mathcal{C}}^2 = 2\mathcal{C} + \frac{qn}{4}$ centered about the point $(-{}^1\!/\!_2, -{}^1\!/\!_2, \ldots, -{}^1\!/\!_2) \in \boldsymbol{\mathcal{V}}$. We call it the *cost sphere.* Let Δ denote the constant value of $\Delta(M)$ for all points M that lie on the cost sphere. Our key result is the following theorem.

Theorem 1. *A multiplicity matrix M that minimizes the probability of decoding failure $\Pr\{\mathcal{S}_M \le \Delta(M)\}$ lies on a tangent from the point $\xi = (\frac{\Delta}{n}, \frac{\Delta}{n}, \ldots, \frac{\Delta}{n}) \in \boldsymbol{\mathcal{V}}$ to the cost sphere.*

Generally, the KV multiplicity matrix M_{KV}, that maximizes the mean of $\mathcal{S}_M$, does not lie on a tangent from ξ to the cost sphere. We identify a specific path in $\mathbb{R}^{qn}$ that moves M_{KV} to the desired tangent, while decreasing the probability of decoding failure at each step. The performance of the resulting multiplicity assignment scheme is shown in Figure 1.

III. Gaussian Approximation

As shown in [2], the random variable $\mathcal{S}_M$ is a sum of n independent random variables. Therefore, the distribution of $\mathcal{S}_M$ is well approximated by the Gaussian distribution for large n. Given M, the mean $\mu(\mathcal{S}_M)$ and the variance $\sigma^2(\mathcal{S}_M)$ of $\mathcal{S}_M$ can be easily computed from the channel observations. Thus, under the Gaussian approximation, our goal is to minimize $\Pr\{\mathcal{S}_M \le \Delta(M)\} \simeq Q\big((\Delta(M) - \mu(\mathcal{S}_M))/\sigma(\mathcal{S}_M)\big)$, where $Q(\cdot)$ is the Gaussian tail-function. Using Lagrange multipliers, we show that this is equivalent to solving a system of qn nonlinear equations. We then develop a fast iterative method to solve this system of equations. We also derive an approximate analytic solution, whose performance essentially coincides with that of exact Lagrange optimization. The resulting coding gain for the $(255, 239)$ RS code is about 0.17 dB, while for short RS codes, the gain is about 0.75 dB (see Figure 1).

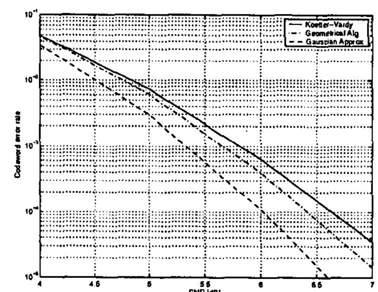

Figure 1. *Soft-decoding of* $(15, 11)$ RS *code on AWGN*

References

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[2] R. Koetter and A. Vardy, "Algebraic soft-decision decoding of Reed-Solomon codes," *IEEE Trans. Inform. Theory*, submitted for publication, September 2001.

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