Handout 4 - Dilation and Lamé Constants

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The purpose of this handout is to show you an alternate way of writing the relationship between stresses and strains. This relationship is very edifying in terms of a developing a feel for how normal and shear stresses relate to strains and each other. First, we need to define a term called *dilation*, which represents the relative change in the volume (or area) of a body. We will look at the definition in 2-d, because it's easier, using figure 1. First, let's look at the change in area of the body:

$$A_0 = \Delta x \Delta y \tag{1}$$
$$A = (\Delta x + \Delta u)(\Delta y + \Delta v) \tag{2}$$
$$\Delta A = A - A_0 \tag{3}$$
$$\Delta A = \Delta x \Delta y + \Delta u \Delta y + \Delta v \Delta x + \Delta u \Delta v - \Delta x \Delta y \tag{4}$$
$$\Delta A = \Delta u \Delta y + \Delta v \Delta x + \Delta u \Delta v \tag{5}$$

Next, the dilation (e) is defined as follows, with the superscript indicating that this is specifically for 2-d:

$$e^{2d} = \frac{\Delta A}{A_0} = \frac{\Delta u \Delta y + \Delta v \Delta x + \Delta u \Delta v}{\Delta x \Delta y} \tag{6}$$
$$e^{2d} = \frac{\Delta u}{\Delta x}\cancelto{1}{\frac{\Delta y}{\Delta y}} + \frac{\Delta v}{\Delta y}\cancelto{1}{\frac{\Delta x}{\Delta x}} + \cancelto{0}{\frac{\Delta u \Delta v}{\Delta x \Delta y}} \tag{7}$$
$$e^{2d} = \lim_{\Delta u, \Delta v, \Delta x, \Delta y \to 0} \left(\frac{\Delta u}{\Delta x} + \frac{\Delta v}{\Delta y} \right) = \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} \tag{8}$$
$$e^{2d} = \underline{\varepsilon_{xx} + \varepsilon_{yy}} \tag{9}$$

The third term in equation (7) is zero because the multiplication of two small values is insignificant compared to the other terms. The same procedure can be followed in 3-d in order to obtain the dilation:

$$e = \frac{\Delta V}{V_o} = \underline{\varepsilon_{xx} + \varepsilon_{yy} + \varepsilon_{zz}} \tag{10}$$

So why is the dilation important? Let's revisit the stress-strain relationship discussed in class:

$$\sigma_x = \frac{E}{(1+\nu)(1-2\nu)} \left[\varepsilon_x (1-\nu) + \varepsilon_y \nu + \varepsilon_z \nu \right] \tag{11}$$

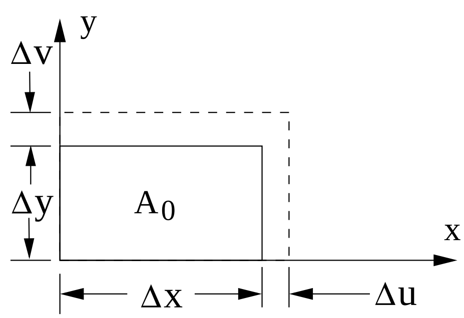

Figure 1: Deformed two-dimensional body

We can add a term $(\varepsilon_x\nu - \varepsilon_x\nu)$ to this equation (11) and then simplify:

$$\sigma_x = \frac{E}{(1+\nu)(1-2\nu)}\left[\varepsilon_x - \varepsilon_x\nu + \underbrace{\varepsilon_x\nu - \varepsilon_x\nu}_{0} + \varepsilon_y\nu + \varepsilon_z\nu\right] \tag{12}$$

$$\sigma_x = \frac{E}{(1+\nu)(1-2\nu)}\left(\varepsilon_x\nu + \varepsilon_y\nu + \varepsilon_z\nu\right) + \frac{E}{(1+\nu)(1-2\nu)}\left(\varepsilon_x - 2\varepsilon_x\nu\right) \tag{13}$$

$$\sigma_x = \underbrace{\frac{E\nu}{(1+\nu)(1-2\nu)}}_{\lambda}\underbrace{(\varepsilon_x + \varepsilon_y + \varepsilon_z)}_{e} + \frac{E}{(1+\nu)(1-2\nu)}\varepsilon_x\left(1-2\nu\right) \tag{14}$$

$$\sigma_x = \lambda e + \underbrace{\frac{E}{1+\nu}}_{2(G \text{ or } \mu)}\varepsilon_x \tag{15}$$

This procedure can be repeated for the other two equations. The three equations are written as follows:

$$\sigma_x = \lambda e + 2\mu\varepsilon_x \tag{16}$$

$$\sigma_y = \lambda e + 2\mu\varepsilon_y \tag{17}$$

$$\sigma_z = \lambda e + 2\mu\varepsilon_z \tag{18}$$

The terms λ and μ are the first and second Lamé constants, respectively. The Lamé constants are material properties that are related to the elastic modulus and Poisson ratio. The second Lamé constant is identical to the modulus of rigidity (G).

The shear stresses can be written in terms of the second Lamé constant:

$$\tau_{xy} = 2\mu\varepsilon_{xy} = \mu\gamma_{xy} \tag{19}$$

$$\tau_{xz} = 2\mu\varepsilon_{xz} = \mu\gamma_{xz} \tag{20}$$

$$\tau_{yz} = 2\mu\varepsilon_{yz} = \mu\gamma_{yz} \tag{21}$$

Finally, these equations can again be written in matrix-vector form:

$$\begin{bmatrix}\sigma_x\\ \sigma_y\\ \sigma_z\\ \tau_{xy}\\ \tau_{xz}\\ \tau_{yz}\end{bmatrix} = \begin{bmatrix}\lambda+2\mu & \lambda & \lambda & 0 & 0 & 0\\ \lambda & \lambda+2\mu & \lambda & 0 & 0 & 0\\ \lambda & \lambda & \lambda+2\mu & 0 & 0 & 0\\ 0 & 0 & 0 & 2\mu & 0 & 0\\ 0 & 0 & 0 & 0 & 2\mu & 0\\ 0 & 0 & 0 & 0 & 0 & 2\mu\end{bmatrix}\begin{bmatrix}\varepsilon_x\\ \varepsilon_y\\ \varepsilon_z\\ \varepsilon_{xy}\\ \varepsilon_{xz}\\ \varepsilon_{yz}\end{bmatrix} \tag{22}$$

The importance of this form of the equations is that this is a much simpler form, from which we can learn a lot about what causes stresses.

1. Normal stresses in the three principal directions are all influenced equally by a common term: the change in volume (λe) and a unique term influenced strains in the same direction as the stress and the rigidity of the body ($2\mu\varepsilon_i$).

2. Any change in volume necessarily induces stresses in all three directions.

3. The shear and normal stress are both influenced by an identical term, that is related to the rigidity modulus and the corresponding strain. Essentially, normal and shear stresses have the same relationship to strain, except that normal stresses include the effect of changes in volume. This validates the discussion in class about normal stresses leading to changes in volume and shear stresses leading only to changes in shape.