Runge-Kutta-Fehlberg Method (RKF45)

One way to guarantee accuracy in the solution of an I.V.P. is to solve the problem twice using step sizes h and $h/2$ and compare answers at the mesh points corresponding to the larger step size. But this requires a significant amount of computation for the smaller step size and must be repeated if it is determined that the agreement is not good enough.

The Runge-Kutta-Fehlberg method (denoted RKF45) is one way to try to resolve this problem. It has a procedure to determine if the proper step size h is being used. At each step, two different approximations for the solution are made and compared. If the two answers are in close agreement, the approximation is accepted. If the two answers do not agree to a specified accuracy, the step size is reduced. If the answers agree to more significant digits than required, the step size is increased.

Each step requires the use of the following six values:

$$
\begin{aligned}
k_1 &= hf(t_k, y_k), \\
k_2 &= hf\left(t_k + \frac{1}{4}h, y_k + \frac{1}{4}k_1\right), \\
k_3 &= hf\left(t_k + \frac{3}{8}h, y_k + \frac{3}{32}k_1 + \frac{9}{32}k_2\right), \\
k_4 &= hf\left(t_k + \frac{12}{13}h, y_k + \frac{1932}{2197}k_1 - \frac{7200}{2197}k_2 + \frac{7296}{2197}k_3\right), \\
k_5 &= hf\left(t_k + h, y_k + \frac{439}{216}k_1 - 8k_2 + \frac{3680}{513}k_3 - \frac{845}{4104}k_4\right), \\
k_6 &= hf\left(t_k + \frac{1}{2}h, y_k - \frac{8}{27}k_1 + 2k_2 - \frac{3544}{2565}k_3 + \frac{1859}{4104}k_4 - \frac{11}{40}k_5\right).
\end{aligned}
$$

(28)

Then an approximation to the solution of the I.V.P. is made using a Runge-Kutta method of order 4:

(29)
$$
y_{k+1} = y_k + \frac{25}{216}k_1 + \frac{1408}{2565}k_3 + \frac{2197}{4101}k_4 - \frac{1}{5}k_5,
$$

where the four function values f_1, f_3, f_4, and f_5 are used. Notice that f_2 is not used in formula (29). A better value for the solution is determined using a Runge-Kutta method of order 5:

(30)
$$
z_{k+1} = y_k + \frac{16}{135}k_1 + \frac{6656}{12,825}k_3 + \frac{28,561}{56,430}k_4 - \frac{9}{50}k_5 + \frac{2}{55}k_6.
$$

The optimal step size sh can be determined by multiplying the scalar s times the current step size h. The scalar s is

(31)
$$
s = \left(\frac{\mathrm{tol}\, h}{2|z_{k+1} - y_{k+1}|}\right)^{1/4} \approx 0.84\left(\frac{\mathrm{tol}\, h}{|z_{k+1} - y_{k+1}|}\right)^{1/4}
$$

Table 9.10   RKF45 Solution to $y' = 1 + y^2$, $y(0) = 0$

k	t_k	RK45 approximation y_k	True solution, $y(t_k) = \tan(t_k)$	Error $y(t_k) - y_k$
0	0.0	0.0000000	0.0000000	0.0000000
1	0.2	0.2027100	0.2027100	0.0000000
2	0.4	0.4227933	0.4227931	−0.0000002
3	0.6	0.6841376	0.6841368	−0.0000008
4	0.8	1.0296434	1.0296386	−0.0000048
5	1.0	1.5574398	1.5774077	−0.0000321
6	1.1	1.9648085	1.9647597	−0.0000488
7	1.2	2.5722408	2.5721516	−0.0000892
8	1.3	3.6023295	3.6021024	−0.0002271
9	1.35	4.4555714	4.4552218	−0.0003496
10	1.4	5.7985045	5.7978837	−0.0006208

where tol is the specified error control tolerance.

The derivation of formula (31) can be found in advanced books on numerical analysis. It is important to learn that a fixed step size is not the best strategy even though it would give a nicer-appearing table of values. If values are needed that are not in the table, polynomial interpolation should be used.

Example 9.12.   Compare RKF45 and RK4 solutions to the I.V.P.

$$y' = 1 + y^2 \quad \text{with} \quad y(0) = 0 \quad \text{on} \quad [0, 1.4].$$

An RKF45 program was used with the value tol $= 2 \times 10^{-5}$ for the error control tolerance. It changed the step size automatically and generated the 10 approximations to the solution in Table 9.10. An RK4 program was used with the a priori step size of $h = 0.1$, which required the computer to generate 14 approximations at the equally spaced points in Table 9.11. The approximations at the right endpoint are

$$y(1.4) \approx y_{10} = 5.7985045 \quad \text{and} \quad y(1.4) \approx y_{14} = 5.7919748$$

and the errors are

$$E_{10} = -0.0006208 \quad \text{and} \quad E_{14} = 0.0059089$$

for the RKF45 and RK4 methods, respectively. The RKF45 method has the smaller error.   ∎

Table 9.11   RK4 Solution to $y' = 1 + y^2$, $y(0) = 0$

k	t_k	RK4 approximation y_k	True solution, $y(t_k) = \tan(t_k)$	Error $y(t_k) - y_k$
0	0.0	0.0000000	0.0000000	0.0000000
1	0.1	0.1003346	0.1003347	0.0000001
2	0.2	0.2027099	0.2027100	0.0000001
3	0.3	0.3093360	0.3093362	0.0000002
4	0.4	0.4227930	0.4227932	0.0000002
5	0.5	0.5463023	0.5463025	0.0000002
6	0.6	0.6841368	0.6841368	0.0000000
7	0.7	0.8422886	0.8422884	-0.0000002
8	0.8	1.0296391	1.0296386	-0.0000005
9	0.9	1.2601588	1.2601582	-0.0000006
10	1.0	1.5574064	1.5574077	0.0000013
11	1.1	1.9647466	1.9647597	0.0000131
12	1.2	2.5720718	2.5721516	0.0000798
13	1.3	3.6015634	3.6021024	0.0005390
14	1.4	5.7919748	5.7978837	0.0059089