Chapter 15

Computing Points and Tangents on Bézier Surface Patches

This chapter presents an algorithm for computing points and tangents on a tensor-product rational Bézier surface patch that has $O(n^2)$ time complexity.

A rational Bézier curve in $\mathbf{R}^3$ is defined

$$\mathbf{p}(t) = \Pi\left(\mathbf{P}(t)\right) \tag{15.1}$$

with

$$\mathbf{P}(t) = (\mathbf{P}_x(t), \mathbf{P}_y(t), \mathbf{P}_z(t), \mathbf{P}_w(t)) = \sum_{i=0}^{n} \mathbf{P}_i B_i^n(t) \tag{15.2}$$

where $\mathbf{P}_i = w_i(x_i, y_i, z_i, 1)$ and the projection operator Π is defined $\Pi(x, y, z, w) = (x/w, y/w, z/w)$. We will use upper case bold-face variables to denote four-tuples (homogeneous points) and lower case bold-face for triples (points in R^3).

The point and tangent of this curve can be found using the familiar construction

$$\mathbf{P}(t) = (1-t)\mathbf{Q}(t) + t\mathbf{R}(t) \tag{15.3}$$

with

$$\mathbf{Q}(t) = \sum_{i=0}^{n-1} \mathbf{P}_i B_i^{n-1}(t) \tag{15.4}$$

and

$$\mathbf{R}(t) = \sum_{i=1}^{n} \mathbf{P}_i B_{i-1}^{n-1}(t) \tag{15.5}$$

where line $\mathbf{q}(t)$—$\mathbf{r}(t) \equiv \Pi\left(\mathbf{Q}(t)\right)$—$\Pi\left(\mathbf{R}(t)\right)$ is tangent to the curve, as seen in Figure 15.1. As a sidenote, the correct magnitude of the derivative of $\mathbf{p}(t)$ is given by

$$\frac{d\mathbf{p}(t)}{dt} = n\frac{\mathbf{R}_w(t)\mathbf{Q}_w(t)}{((1-t)\mathbf{Q}_w(t) + t\mathbf{R}_w(t))^2}\left[\mathbf{r}(t) - \mathbf{q}(t)\right] \tag{15.6}$$

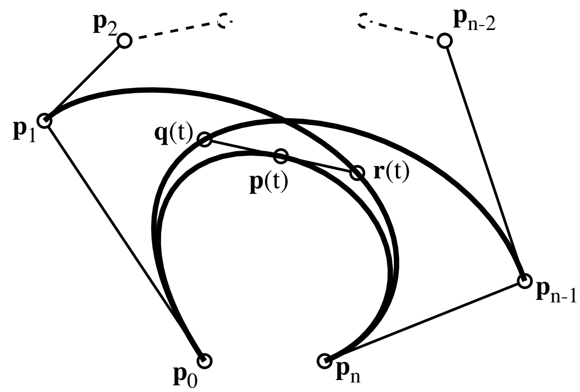

Figure 15.1: Curve example

The values $\mathbf{Q}(t)$ and $\mathbf{R}(t)$ can be found using the modified Horner's algorithm for Bernstein polynomials, involving a pseudo–basis conversion

$$\frac{\mathbf{Q}(t)}{(1-t)^{n-1}} = \hat{\mathbf{Q}}(u) = \sum_{i=0}^{n-1} \hat{\mathbf{Q}}_i u^i \tag{15.7}$$

where $u = \frac{t}{1-t}$ and $\hat{\mathbf{Q}}_i = \binom{n-1}{i}\mathbf{P}_i$, $i = 0, 1, \ldots, n-1$. Assuming the curve is to be evaluated several times, we can ignore the expense of precomputing the $\hat{\mathbf{Q}}_i$, and the nested multiplication

$$\hat{\mathbf{Q}}(u) = [\cdots[[\hat{\mathbf{Q}}_{n-1}u + \hat{\mathbf{Q}}_{n-2}]u + \hat{\mathbf{Q}}_{n-3}]u + \ldots \hat{\mathbf{Q}}_1]u + \hat{\mathbf{Q}}_0 \tag{15.8}$$

can be performed with $n-1$ multiplies and adds for each of the four x, y, z, w coordinates. It is not necessary to post-multiply by $(1-t)^{n-1}$, since

$$\Pi\left(\mathbf{Q}(t)\right) = \Pi\left((1-t)^{n-1}\hat{\mathbf{Q}}(u)\right) = \Pi\left(\hat{\mathbf{Q}}(t)\right). \tag{15.9}$$

Therefore, the point $\mathbf{P}(t)$ and its tangent direction can be computed with roughly $2n$ multiplies and adds for each of the four x, y, z, w coordinates.

This method has problems near $t = 1$, so it is best for $.5 \le t \le 1$ to use the form

$$\frac{\mathbf{Q}(t)}{t^{n-1}} = \sum_{i=0}^{n-1} \hat{\mathbf{Q}}_{n-i-1} u^i \tag{15.10}$$

with $u = \frac{1-t}{t}$.

A tensor product rational Bézier surface patch is defined

$$\mathbf{p}(s,t) = \Pi\left(\mathbf{P}(s,t)\right) \tag{15.11}$$

where

$$\mathbf{P}(s,t) = \sum_{i=0}^{m}\sum_{j=0}^{n} \mathbf{P}_{ij} B_i^m(s) B_j^n(t). \tag{15.12}$$

We can represent the surface $\mathbf{p}(s,t)$ using the following construction:

$$\mathbf{P}(s,t) = (1-s)(1-t)\mathbf{P}^{00}(s,t) + s(1-t)\mathbf{P}^{10}(s,t) + (1-s)t\mathbf{P}^{01}(s,t) + st\mathbf{P}^{11}(s,t) \tag{15.13}$$

where

$$\mathbf{P}^{00}(s,t) = \sum_{i=0}^{m-1}\sum_{j=0}^{n-1} \mathbf{P}_{ij} B_i^{m-1}(s) B_j^{n-1}(t), \tag{15.14}$$

$$\mathbf{P}^{10}(s,t) = \sum_{i=1}^{m}\sum_{j=0}^{n-1} \mathbf{P}_{ij} B_{i-1}^{m-1}(s) B_j^{n-1}(t), \tag{15.15}$$

$$\mathbf{P}^{01}(s,t) = \sum_{i=0}^{m-1}\sum_{j=1}^{n} \mathbf{P}_{ij} B_i^{m-1}(s) B_{j-1}^{n-1}(t), \tag{15.16}$$

$$\mathbf{P}^{11}(s,t) = \sum_{i=1}^{m}\sum_{j=1}^{n} \mathbf{P}_{ij} B_{i-1}^{m-1}(s) B_{j-1}^{n-1}(t). \tag{15.17}$$

The tangent vector $\mathbf{p}_s(s,t)$ is parallel with the line

$$\Pi\left((1-t)\mathbf{P}^{00}(s,t) + t\mathbf{P}^{01}(s,t)\right) — \Pi\left((1-t)\mathbf{P}^{10}(s,t) + t\mathbf{P}^{11}(s,t)\right) \tag{15.18}$$

and the tangent vector $\mathbf{p}_t(s,t)$ is parallel with

$$\Pi\left((1-s)\mathbf{P}^{00}(s,t) + s\mathbf{P}^{10}(s,t)\right) — \Pi\left((1-s)\mathbf{P}^{01}(s,t) + s\mathbf{P}^{11}(s,t)\right). \tag{15.19}$$

The Horner algorithm for a tensor product surface emerges by defining

$$\frac{\mathbf{P}^{kl}(s,t)}{(1-s)^{m-1}(1-t)^{n-1}} = \hat{\mathbf{P}}^{kl}(u,v) = \sum_{i=k}^{m+k-1}\sum_{j=l}^{n+l-1} \hat{\mathbf{P}}_{ij}^{kl} u^i v^j; \quad k,l = 0,1 \tag{15.20}$$

where $u = \frac{s}{1-s}$, $v = \frac{t}{1-t}$, and $\hat{\mathbf{P}}_{ij}^{kl} = \binom{m-1}{i-k}\binom{n-1}{j-l}\mathbf{P}_{ij}$. The n rows of these four bivariate polynomials can each be evaluated using $m-1$ multiplies and adds per x, y, z, w component, and the final evaluation in t costs $n-1$ multiplies and adds per x, y, z, w component.

Thus, if $m = n$, the four surfaces $\mathbf{P}^{00}(s,t)$, $\mathbf{P}^{01}(s,t)$, $\mathbf{P}^{10}(s,t)$, and $\mathbf{P}^{11}(s,t)$ can each be evaluated using n^2-1 multiplies and n^2-1 adds for each of the four x, y, z, w components, a total of $16n^2-16$ multiplies and $16n^2-16$ adds.

If one wishes to compute a grid of points on this surface which are evenly spaced in parameter space, the four surfaces $\mathbf{P}^{00}(s,t)$, $\mathbf{P}^{01}(s,t)$, $\mathbf{P}^{10}(s,t)$, and $\mathbf{P}^{11}(s,t)$ can each be evaluated even more quickly using forward differencing.