Selected topics in the history of the two-dimensional biharmonic problem

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This review article gives a historical overview of some topics related to the classical 2D biharmonic problem. This problem arises in many physical studies concerning bending of clamped thin elastic isotropic plates, equilibrium of an elastic body under conditions of plane strain or plane stress, or creeping flow of a viscous incompressible fluid. The object of this paper is both to elucidate some interesting points related to the history of the problem and to give an overview of some analytical approaches to its solution. This review article contains 758 references. [DOI: 10.1115/1.1521166]

CONTENTS

"*Ceux qui, les premiers, ont signalé ces nouveaux instruments, n'existeront plus et seront complétement oubliés; à moins que quelque géomètre archéologue ne ressuscite leurs noms. Eh! qu'importe, d'ailleurs, si la science a marché*!" (G Lamé [1]).

1 INTRODUCTION

There are a great many physical problems concerning bending of clamped thin elastic isotropic plates, equilibrium of an elastic body under conditions of plane strain or plane stress, or creeping flow of a very viscous incompressible fluid, which can be formulated in terms of the two-dimensional (2D) biharmonic equation for one scalar function with prescribed values of the function and its normal derivative at the boundary. Using the words of Jeffery [2] (p 265), all these problems "seem to be a branch of mathematical physics in which knowledge comes by the patient accumulation of special solutions rather than by the establishment of great general propositions." Nevertherless, the biharmonic problem was and still is challenging in various divisions of the linear theory of elasticity, low-Reynolds-number hydrodynamics, structural engineering, and mathematics.

In structural engineering, for example, a thin plate riveted to a rigid frame along its edge and subjected to normal pressure is one of the most popular elements. The questions of how thick the plate has to be in order to withstand the applied pressure, and where the maximum stresses will be, are of vital interest when designing any engineering structure. Already Barre de Saint-Venant in his extensive comments to the French translation of Clebsch [3] pointed out on p 777:

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> C'est un problème sur l'importance duquel au point de vue des applications il convient d'appeler l'attention des géomètres-physiciens, ainsi que sur la méthode au moyen de laquelle on réussira peut-être à trouver la solution pour d'autres formes que la circulaire.

In the beginning of his talk read before the Spring Meeting of the 43rd Session of the Institution of Naval Architects, March 19, 1902, Russian naval architect Lieutenant Ivan G Bubnov (or Boobnoff, according to the French spelling of his name in the publication) noticed [4] (p 15):

> I do not know of any question in the theory of elasticity which should interest the naval architect to the same extent as that of the flexion of thin plating. Indeed, the whole ship from keel to upper deck, consists of plates, which are to fulfil the most varied purposes and to withstand all kinds of stresses. Owing to this, naval architects cannot be satisfied with approximate and rough practical formulas, which may be regarded as sufficient by engineers of other branches of engineering profession, and they are bound to examine and solve this question in detail.

On the other hand, the biharmonic problem provides a number of interesting questions in mathematics as to the solvability of certain functional equations in the complex plane, convergence of series of the non-orthogonal systems of complex eigenfunctions, and the uniqueness of the solution for specific domains with corner points under general boundary conditions imposed on the function and its normal derivative. Besides, it represents an excellent testing problem for checking already existing and developing new numerical methods.

Typical examples of the engineering, mathematical, and historical approaches to the various biharmonic problems were provided by Biezeno [5] in the general opening lecture read on April 23, 1924 at the First International Congress on Applied Mechanics in Delft, in the Presidential addresses by Love [6] and Dixon [7] to the London Mathematical Society, and in several talks [4,8–13] delivered at the sessions of the Institution of Naval Architects.

The historical aspect of the biharmonic problem also presents an interest. Already Maxwell deplored the growth of a "narrow professional spirit" amongst scientists, and suggested that it was the duty of scientists to preserve their acquaintance with literary and historical studies. Thus, the "undue specialization in Science" with which we are often charged today is no new thing. In addition to fascinating historical introductions in the textbooks by Love [14–18], Lorenz [19], and Westergaard [20], there also exist excellent books by Todhunter and Pearson [21,22] and Timoshenko [23][1] specially devoted to the history of theory of elasticity and strength of materials that contain a lot of interesting results on the 2D biharmonic problem.

Numerious data of the solution of the biharmonic problem are widely scattered over the literature on the theory of elasticity, theory of plates, and creeping flow of a viscous fluid. The literature of the subject is far too big for an adequate treatment within a reasonably finite number of pages. This task is partially addressed in the widely known treatises and courses on theory of elasticity and theory of plates. Even an incomplete list of monographs and textbooks on theory of elasticity and theory of plates published in various countries and in various languages contains several dozen titles. One can mention (in alphabetical order) widely known treatises and courses by Barber [24], Biezeno and Grammel [25], Ciarlet [26], Coker and Filon [27], Föppl and Föppl [28], Frocht [29], Girkmann [30], Gould [31], Green and Zerna [32], Hahn [33], Happel and Brenner [34], Love [14–18][2], Lur'e [35], Milne-Thomson [36], Muskhelishvili [37–39], Richards [40], Sokolnikoff [41], Southwell [42], Timoshenko [43,44], Timoshenko and Goodier [45,46], Timoshenko and Woinowsky-Krieger [47], Villaggio [48], Wang [49], along with less known (or, at least, less available at present time) textbooks and monographs by Agarev [50], Babuška, Rektorys and Vyčichlo [51], Belluzzi [52], Boresi and Chong [53], Bricas [54], Burgatti [55], Butty [56,57], Filonenko-Borodich [58–61], Galerkin [62], Godfrey [63], Grinchenko [64], l'Hermite [65], Hlitčijev [66], Huber [67], Kolosov [68], Leibenzon [69,70], Lecornu [71], Little [72], Lorenz [19], Mansfield [73], Marcolongo [74], Morozov [75], Nádai [76], Novozhilov [77,78], Panc [79], Papkovich [80], Savin [81–84], Segal' [85], Solomon [86], Stiglat and Wippel [87], Szilard [88], Teodorescu [89,90], Timoshenko [91–93], Uflyand [94], and Westergaard [20]. All these books provide extensive references and surveys of many other articles and books related to the 2D biharmonic problem in various domains.

There exist review articles [95–122] written in the course of the twentieth century. Further, papers written in the thirties of the last century were reviewed in full detail in *Zentralblatt für Mechanik* published from 1934–1941. (It should be noted that the editorial board of several first volumes was really international since it consisted of A Betz (Göttingen), CB Biezeno (Delft), JM Burgers (Delft), R Grammel (Stuttgart), E Hahn (Nancy), Th von Kármán (Aachen, Pasadena), T Levi-Civita (Rome), EL Nicolai (Leningrad), L Prandtl (Göttingen), GI Taylor (Cambridge), and SP Timoshenko (Ann Arbor). The later volumes, due to the political issues of that time, were edited only by German editors W Flügge and O Neugebauer from Göttingen.) Later publications in this area are extensively reviewed in *Applied Mechanics Reviews* published since 1948 (with SP Timoshenko, Th von Kármán, and LH Donnell as founders and the first editors). Moreover, some mathematical review journals such as *Jahrbuch über die Fortschritte der Mathematik* published from 1875–1942, *Zentralblatt für Mathematik und ihre Grenzgebiete* (published since 1931), and *Mathematical Reviews* (published

[1]Note the number of reviews of this book in various journals.

[2]There was a favorite saying among graduate students and professors in an earlier era that "All you really need is Love;" see [31], p 107.

since 1940) also contain sections devoted to the 2D biharmonic equation in various mathematical and mechanical problems.

The main goal of the present article is to elucidate some interesting points in the historical development of the 2D biharmonic problem. It is rather a discourse on those aspects of the problem with which I myself have had contact over recent years. My choice of topics therefore has a very personal bias, for a special attention is paid to the lesser known aspects of mutual interaction of the pure mathematical and engineering approaches to solving the problems in several typical canonical domains. For many books and papers which are in an unfamiliar language, eg, Russian, I have given the English translation of the title. The title of periodicals, however, remained in the original language (in English transliteration). In such cases, whenever possible, I tried to add the references to English or German reviews (however, full search of the review journals was not my goal). In addition, I have given birth and death dates of authors, where this information was available to me.

2 STATEMENT OF THE BIHARMONIC PROBLEM

The classical biharmonic problem, as stated in [123,124], consists of finding a continuous function U, with continuous partial derivatives of the first four orders, which satisfies the homogeneous biharmonic equation

$$\Delta\Delta U=0 \tag{1}$$

at every point inside the domain S and has the prescribed values of the function and its outward normal derivative,

$$U=f(l), \quad \frac{\partial U}{\partial n}=g(l) \tag{2}$$

on its boundary L. Here and in what follows Δ denotes the 2D Laplacian operator.

In the classical theory of thin plates, the differential equation describing the deflection w of the middle surface of an elastic isotropic flat plate of uniform thickness h reads as:

$$D\Delta\Delta w=p \tag{3}$$

where the constant $D=Eh^3/12(1-\nu^2)$ is called the flexural rigidity of the plate (with E and ν being Young's modulus and Poisson's [95] ratio, respectively), p is the load per unit area of the plate. Two boundary conditions imposed on the function w and its first, second, or third normal derivatives must also be satisfied. In various engineering structures (bulkheads of a ship, for example) the edges of the plate are firmly clamped, or attached to angle irons which allow no side motions. The deflection w must vanish at the edge; and, in addition, the tangent plane at every point of the edge must remain fixed when the plate is bent:

$$w=0, \quad \frac{\partial w}{\partial n}=0 \tag{4}$$

at the contour L.

In the theory of elasticity, the determination of stresses in an infinite prism with the surface loads being the same along the generating line of the prism (the state of plane strain) or thin plate under thrust in its own plane (the state of plane stress) reduces to the solution of the 2D biharmonic problem. Under assumptions of plane strain or plane stress in the (x,y)-plane when no body forces are present, the normal stresses X_x, Y_y and shear stress $X_y=Y_x$ inside the domain S must satisfy the system of two equations of static equilibrium

$$\frac{\partial X_x}{\partial x}+\frac{\partial X_y}{\partial y}=0, \quad \frac{\partial X_y}{\partial x}+\frac{\partial Y_y}{\partial y}=0 \tag{5}$$

(Traditionally in the theory of elasticity there exist several notations for stresses, see Note A in Love [18] and Sommerfeld [125] (Section 8). For a 2D stress field among them are:

- the normal stresses X_x, Y_y and shear stress $X_y=Y_x$, introduced by Kirchhoff [126] and used in the textbooks by, *eg*, Love [14–18], Timoshenko [91], Muskhelishvili [37–39], and Papkovich [80];
- the normal stresses σ_x, σ_y and shear stress $\tau_{xy}=\tau_{yx}=\tau$, introduced by Föppl [127] and used by, eg, von Kármán [128], Timoshenko [43], and Timoshenko and Goodier [45,46]; now they are generally acceptable, especially in technical literature;
- the normal stresses N_1, N_2 and shear stress T, introduced by Lamé [1,129];
- the normal stresses p_{xx}, p_{yy} and shear stress $p_{xy}=p_{yx}$, introduced by Saint-Venant [130] and used by Rankine [131].)

A possible solution of system (5) may be expressed in the following manner:

$$X_x=\frac{\partial^2\chi}{\partial y^2}, \quad Y_y=\frac{\partial^2\chi}{\partial x^2}, \quad X_y=-\frac{\partial^2\chi}{\partial x\partial y} \tag{6}$$

by means of single auxiliary function $\chi(x,y)$, called the (Airy) *stress function*. The governing equation for defining χ must represent the condition of the compatibility of deformations in the elastic body in accordance with Hooke's law, and it is written as the biharmonic equation

$$\Delta\Delta\chi=0 \tag{7}$$

If the force $(X_l dl, Y_l dl)$ acts on an element dl of the boundary contour L, then the boundary conditions for the function χ can be written as

$$\frac{d}{dl}\left(\frac{\partial\chi}{\partial y}\right)=X_l, \quad \frac{d}{dl}\left(\frac{\partial\chi}{\partial x}\right)=-Y_l \tag{8}$$

corresponding to the system of normal and shear forces applied at the boundary L which maintain the body in equilibrium.

Filon [132], in his memoir received by the Royal Society on June 12, 1902, introduced the notion of what was subsequently called by Love [16,18] (Section 94) the *generalized plane stress* of a thin elastic sheet. This considers the mean of the displacement and stress components through the thickness of the sheet; see also [133]. For these mean components the stress equations are of the same form as the equations for 2D strain or stress and, consequently, the relations (6) and (7) hold well.

Two-dimensional creeping flow of a viscous incompressible fluid can also be described in terms of the biharmonic problem. If the motion is assumed to be so slow that the inertial terms involving the squares of the velocities may be omitted compared with the viscous terms, the stream function ψ satisfies the 2D biharmonic equation

$$\Delta\Delta\psi=0 \tag{9}$$

This type of flow is also called the low-Reynolds-number flow [34] or slow viscous flow [134]. It is also named the Stokes flow after the famous Stokes' [135] memoir devoted to estimation of the frictional damping of the motion of a spherical pendulum blob due to air resistance.

The velocity components u and v in the Cartesian (x,y) coordinates are expressed as

$$u=\frac{\partial\psi}{\partial y}, \quad v=-\frac{\partial\psi}{\partial x} \tag{10}$$

If a flow in a cavity S is produced by applying a tangential velocity $U_t(l)$ along its boundary contour L, then the boundary conditions for the stream function are:

$$\psi=0, \quad \frac{\partial\psi}{\partial n}=U_t(l) \tag{11}$$

The biharmonic equations express, in the most general and most concise manner, the necessary relations of numerical analysis to a very extensive class of mechanical phenomena. It remains now to discover the proper treatment of all these problems in order to derive their complete solutions and to consider their applications.

Comparing Eqs. (3), (7), and (9) one may conclude that three independent mechanical phenomena are found to be expressible in an identical mathematical form: the homogeneous or inhomogeneous 2D biharmonic equation with nonzero or zero boundary conditions for the functions themselves and their first normal derivatives. This analogy was pointed out a long time ago and since then it has been used widely. The stream function ψ and the Airy stress function χ as functions of two variables define surfaces, certain geometrical properties of which are a measure of quantities of interest in the particular mechanical problem. For example, the component slopes of the stream-function surface are proportional to component velocities, while the curvatures of the Airy surface give the elastic stresses. The stream and Airy functions define mathematical surfaces, whereas the deflection of a clamped plate presents a real physical surface, which may be studied quantitatively.

The analogy between the slow 2D motion of incompressible viscous fluid between rigid boundaries and transverse flexure of an elastic plate clamped along the same boundaries was first pointed out by Lord Rayleigh [136] and mentioned by Sommerfeld [137] and in the classical textbook of Lamb [138].

Klein and Wieghardt [139] pointed out the analogy between the deflection of a clamped plate and the 2D Airy function in following words:

> Um die Spannungsverteilung zu finden, welche in einer einfach zusammenhängenden, homogenen elastisch-isotropen Platte durch ein am Rande angreifendes Gleichgewichtssystem von Kräften erzeugt wird, konstruiere man zunächst diejenige, bis auf eine beliebig hinzuzufügende Ebene völlig bestimmte abwickelbare Fläche, welche Spannungsfläche der durch das Kraftsystem definierten Streifenfolge ist, und sodann diejenige Fläche, welche sich über dem Plattenrand überall ohne Knick an diese abwickelbare Fläche anschliesst und über dem Inneren der Platte überall die Differentialgleichung $\nabla\nabla z=0$ befriedigt. Ist $z=F(x,y)$ diese Fläche, so sind die gesuchten Spannungen selbst durch die Gleichungen (6) gegeben.

Wieghardt[3] was the first who used [141] the analogy between the deflection of a clamped plate and the 2D Airy function to study experimentally the distribution of stresses in some elastic structures. (This study had already been reported [142] at the Aachen *Bezirksverein Deutscher Ingenieure* on May 1905.)

Since these pioneering works, these analogies were used in many further studies [143–154]. For example, Southwell [153] used a problem in the bending of plates to resolve Stokes's paradox in fluid motion, while Richards [154] presented interesting tables and figures of correspondence between analogous quantities for flexure and extension of a plate and fluid flow for several typical geometrical regions including stress concentration problems.

2.1 Derivation of biharmonic equation in theory of thin plates

Equation (3) has been known since 1811 (before establishing the general laws of the theory of elasticity) and its derivation was connected with the names of the French scientists Lagrange (1736–1813), Sophie Germain (1776–1831), Navier (1785–1836), and Poisson (1781–1840). In 1808 the French *Institut* (Academy of Sciences) proposed as a subject for a prize:

> De donner la théorie mathématique des vibration des surfaces élastiques, et de la comparer à l'expérience.

The prize (of a medal of one kilogram of gold) was offered by the Emperor Napoleon who, being deeply impressed by Chladni's experiments on sand figures on a vibrating plate, had added 6000 francs to the 3000 francs of the standard award, see Chladni [155]. In fact, the subject was proposed three times with dates for receiving the essays of candidates by October 1, 1811, 1813, and 1815.

Most mathematicians did not attempt to solve the problem, because Lagrange had said that the mathematical methods available were inadequate to solve it. Sophie Germain, however, spent a lot of time attempting to derive a theory of

[3] Trefftz [140] published a short obituary note for Karl Wieghardt (1874–1924), who was a student of Felix Klein, and after his dissertation in Göttingen in 1903, worked as professor at various technical schools in Germany.

elasticity, competing and collaborating with some of the most eminent mathematicians and physicists, namely, Navier and Poisson. Germain was the only entrant in the contest in 1811, but her work did not win the award. She had not derived her hypothesis from principles of physics, nor could she have done so at the time because she had not had training in analysis and the calculus of variations. In her first essay, the right hand side of Eq. (3) contains the erroneous term $\partial^6 w/\partial x^4 \partial y^2 + \partial^6 w/\partial x^2 \partial y^4$. Lagrange in 1811, who was one of the judges in the contest, corrected the errors in a referee's note (published posthumously [156]) and came up with an equation that he believed might describe Chladni's patterns. Only at the third attempt did Germain gain the prize (she did not attend the award ceremony, however), and later on the winning essay was published [157]; see also [158]. (Considering the contemporary state of knowledge in elasticity and differential geometry Truesdell [159] came to rather negative conclusions with regard to Germain's contribution to the theory of elasticity.)

The fascinating story of the derivation of Eq. (3), full of controversies and discussions between Germain, Fourier, Navier, and Poisson, is presented in the book [160], and the review articles [161–164], and it was also briefly addressed in the books [23] (Section 29), and [21,165].

2.2 Airy stress function in 2D elasticity

The reduction of the 2D elastic problems under plane strain or plane stress conditions to the statement of the biharmonic problem is usually associated with the name of the Astronomer Royal George Biddel Airy (1801–1892) who during his long life occupied positions of Lucasian Professor at Cambridge, President of the Royal Society of London, and President of the British Association for Advance Sciences. In his paper [166], which was received by the Royal Society on 6 November 1862, and read on 11 December 1862, Airy considered a flexure of a finite rectangular beam as a 2D problem in the theory of elasticity. Because of the prevailing tradition of the Royal Society at that time, the extended abstract of the paper was published separately [167]. In fact, the results were reported before as a talk [168] at the 32nd meeting of the British Association for Advancement of Sciences held at Cambridge in October 1862.

The reason why the Astronomer Royal (Airy occupied this position from 1835 until 1881!) and well-known scholar in mathematics and astronomy (he improved the orbital theory of Venus and the Moon, made a mathematical study of the rainbow and computed the density of the Earth by swinging a pendulum at the top and bottom of a deep mine), and fluid dynamics (the theory of waves and tides) considered elastic problems was partially explained by Academician Krylov in the preface to the first Russian edition of Muskhelishvili [37]. In the English translation [39] (p XVIII), it reads:

> One might usefully remember that the stress function itself was introduced into the theory of elasticity by the famous Astronomer Royal Sir James [*sic*, in Russian only G.] Biddel Airy who, I believe, was director of Greenwich Observatory for more than 50 years. At the beginning of the 1860's, he built for the Observatory a new large meridian line with a telescope having an 8-inch object lens. He had to count with the flexure of the telescope under the weight of the lens and ocular and of other devices, a fact which had caused errors up to 2 arc seconds at the Paris Observatory, errors which are inadmissible in such accurate observations with significant measurements in decimal seconds.

Airy introduced one function F and represented the stresses (or "strains," as he called them) in the beam, as the solution of the equilibrium equations

$$\frac{d}{dx}p_{xx}+\frac{d}{dy}p_{xy}=0$$
$$\frac{d}{dx}p_{xy}+\frac{d}{dy}p_{yy}+g=0 \tag{12}$$

with account of gravity forces, as

$$p_{xx}=\frac{d^2F}{dy^2},\quad p_{yy}=\frac{d^2F}{dx^2}-gy,\quad p_{xy}=-\frac{d^2F}{dxdy} \tag{13}$$

Here, Rankine's [131] notations p_{xx}, p_{xy}, p_{yy} for the pressure parallel to x, the shearing force and the pressure parallel to y, respectively, instead of Airy's unusual notations L, M, $-Q$, are used. Next, g represents the gravity force and total derivatives (d) instead of now traditional partial (∂) ones are used.

Airy considered a few practically important cases: a beam clamped by one end, a beam under its own weight supported on two piers and unloaded, centrally loaded or eccentrically loaded, a beam fixed at both ends, and a beam fixed at one end and supported at the other. For all these cases, the author assumed an expression for F containing a sufficient number of terms of powers and products of x and y. Tables were given, showing the values of principal stresses at selected points, and diagrams were added showing the direction of stresses at every point of the beam in each of these cases.

The Secretary of the Royal Society George Gabriel Stokes (1819–1903), then age 43, sent the paper for review to James Clerk Maxwell (1831–1879), then age 31, on December 18, 1862, and to William JM Rankine (1820–1872), then age 42, on December 31, 1863. Rankine raised no objections to Airy's paper, and in his report, dated January 26, 1863, observed that "the introduction of that function F leads to remarkably clear, simple, and certain methods of solving problems respecting the strains in the interior of beams," and concluded that the paper is "theoretically interesting, and practically useful, in the highest degree, and well worthy of being published in the Transactions."

In contrast, Maxwell [169] considered the paper very carefully, and he noticed that the 2D equations of equilibrium (12) can be also satisfied by choosing a more general than (13) representation:

$$p_{xx}=\frac{d^2F}{dy^2}+Y(y),\quad p_{yy}=\frac{d^2F}{dx^2}+X(x)-gy,$$

$$p_{xy}=-\frac{d^2F}{dxdy} \tag{14}$$

Next, Maxwell considered two cases: 1) a very thin lamina free from the pressure along the z-axis, 2) a very thin plank unable to expand in the z direction, (plane stress and strain, respectively, in modern terminology). For both of them, by applying the laws of elasticity connecting displacements and stresses, Maxwell obtained the additional equation

$$\frac{d}{dy}\int p_{xx}dx-2p_{xy}+\frac{d}{dx}\int p_{yy}dy-ghx=0 \tag{15}$$

where $h=-\nu$ or $h=1/(1-h)$ for the first and second case, respectively.

Maxwell found that Airy's solution $F=(r-x)(3sy^2-2y^3)/2s^2$ for the case of the beam clamped at $x=0$ does not satisfy this equation, and he suggested his own expression for the stress p_{xx}. However, Maxwell positively estimated Airy's approximate solution:

> If any one can work out the *exact* solutions, he will have performed a mathematical feat, but I do not think he will have added anything to our practical knowledge of the forces in a beam not near the ends or the points where pressures are applied. For all such points the formulas obtained in this paper are quite satisfactory and as far as I know they are new.

(In his covering letter to Stokes, Maxwell [170] was more cautious: "I am not enough up in the literature of the subject to say whether it is quite new. I have not Lamé's Leçons to refer to and there may be something of the kind in the Journal de L'École Polytechnique.")

At the same time, Maxwell gave the following comment: "The objection which I have to the method of investigation is that the conditions arising from the elasticity of the beam are not taken account of at all or even mentioned."

In the letter February 22, 1863 to Stokes (who was also the Editor of the *Philosophical Transactions of the Royal Society of London*), having been sent Maxwell's report, Airy responded:

> This remark astonishes me. The elasticity and its law, are the foundations of every one of my applications of the new theory. ⟨ . . . ⟩ In every instance, the value of the function [F] is found from a process which rests ENTIRELY on the theory of elasticity. ⟨ . . . ⟩ If Professor Maxwell on further consideration should see reason for making other remarks, I shall be delighted to see them in the form of Appendix to the paper, if approved by the President and Council of the Royal Society.

Stokes, in his letter of February 26, 1863 to Airy answered:

> I have not as yet myself read your paper, and therefore cannot fully enter into the report and your letter; but unless I greatly mistake I catch his [Maxwell's] meaning—that your investigation takes account of *systems of forces only* not entering into displacements; that your result is therefore necessarily FROM THE VERY PRINCIPLE OF THE PROCESS, indeterminate.

In the response letter of February 27, 1863, Airy wrote: "Pray send my letter for Prof Maxwell's reading, if you think there is nothing in it which he can take the slightest umbrage." In his reply dated March 18, Stokes mentioned that "I have not yet written to Prof Maxwell about your paper, because there was no hurry about it."

Later Stokes sent these letters to Maxwell who responded on June 9, 1863 in length [171]:

> Dear Stokes
>
> I have received your letter and that of the Astronomer Royal. Perhaps I ought to have explained more distinctly what I meant by the conditions arising from elasticity.
>
> There are three separate subjects of investigations in the theory of Elastic Solids.
>
> 1st Theory of Internal Forces or Stresses their resolution and composition and the conditions of equilibrium of an element.
>
> 2nd Theory of Displacements or Strains their resolution and composition and the equation of continuity (if required).
>
> 3rd Theory of Elasticity or the relations between systems of stresses and systems of strains in particular substances.
>
> Airy's conclusions are all deducible from the conditions of equilibrium of the *Forces* or *Stresses* for although he has introduced into his calculation considerations arising from the observed uniformly varying strain and stress between the top and bottom of the beam (see top of p 3 of his letter and his paper Art 15), yet I have shown that on his own principles these assumptions are not required, for the results may be gotten from the conditions near the end of my report namely that the pressure all over the surface is zero. Now we know that without any theory of elasticity and any application of elastic principles which tells us no more than this may in a mathematical paper be treated as an episode an illustration or instructive consideration but not a necessary part of the investigation (just as many mechanical experiments help us to see the truth of principles which we can establish otherwise).
>
> What I meant by the conditions arising from the elasticity of the beam may perhaps be more accurately described as "Conditions arising from the beam having been once an unstrained solid free from stress."
>
> That is, the stresses must be accounted for by displacements of an elastic solid from a state in which there were no forces in action.
>
> I think what you and the author intend is that I

> should state the result of the above assumption instead of that of the paper. The mode of getting complete solutions I have only partially worked out. It depends on expanding the applied forces in Fouriers series the terms are of the form $A\sin(nx+b)e^{\pm ny}$. I shall send you the note or appendix when I can write it I hope before Thursday.

(Airy's paper had reached revised proofs by June 1863 and it did not contain an appendix by Maxwell.)

Nevertheless, it was none other than Maxwell [172] who referred to "important simplification of the theory of the equilibrium of stress in two dimensions by means of the stress function" and suggested the name "Airy's Function of Stress." The governing equation for defining this function F Maxwell presented not in form (15), but in the explicit form of the biharmonic equation:

$$\Delta\Delta F=0, \tag{16}$$

which can be obtained from (15) by substitution (13) in the absence of body forces. This equation represents the condition of the compatibility of deformations in the elastic body in two dimensions expressed in terms of the stresses.

Later Maxwell [173,174] suggested the geometrical interpretation of the stress function:

> If a plane sheet is in equilibrium under the action of internal stress of any kind, then a quantity, which we shall call Airy's Function of Stress, can always be found, which has the following properties.
>
> At each point of the sheet, let a perpendicular be erected proportional to the function of stress at that point, so that the extremities of such perpendiculars lie in a certain surface, which we may call the surface of stress. In the case of a plane frame the surface of stress is a plane-faced polyhedron, of which the frame is the projection. On another plane, parallel to the sheet, let a perpendicular be erected of height of unity, and from the extremity of this perpendicular let a line be drawn normal to the tangent plane at a point of the surface of stress, and meeting the plane at a certain point.
>
> Thus, if points be taken in the plane sheet, corresponding points may be found by this process in the other plane, and if both points are supposed to move, two corresponding lines will be drawn, which have the following property: The resultant of the whole stress exerted by the part of the sheet on the right hand side of the line on the left hand side, is represented in direction and magnitude by the line joining the extremities of the corresponding line in the other figure. In the case of a plane frame, the corresponding figure is the reciprocal diagram described above.
>
> From this property the whole theory of the distribution of stress in equilibrium in two dimensions may be deduced. ⟨...⟩ These equations are especially useful in the cases in which we wish to determine the stresses in uniform beams. The distribution of stress in such cases is determined, as in all other cases, by the elastic yielding of the material; but if this yielding is small and the beam uniform, the stress at any point will be the same, whatever be the actual value of the elasticity of the substance.
>
> Hence the coefficients of elasticity disappear from the ultimate value of the stresses.
>
> In this way, I have obtained values for the stresses in a beam supported in a specific way, which differ only by small quantities from the values obtained by Airy, by a method involving certain assumptions, which were introduced in order to avoid the consideration of elastic yielding.

Thus, already in 1870 Maxwell anticipated the Lévy [175]–Michell [176] theorem on independence of stress in 2D elasticity upon elastic moduli. However, until his premature death in 1879 Maxwell did not publish any more on this subject.

It seems instructive to consider a reception that the Airy stress function had received among early researchers in several leading scientific countries of that time. In England, in the fundamental treatise by Todhunter and Pearson [22] which dealt in detail with even minor contributions in the theory of elasticity, Airy's paper was discussed only briefly in Section 666, occupying only a half page! The biharmonic Eq. (16) was not explicitly mentioned. In contrast, in a rather popular at the time textbook by Ibbetson [177], Airy's studies were reproduced on 10 pages. But at the last moment Ibbetson added a short note in small letters

> 307 *bis* **Important Addition and Correction**. The solutions of the problems suggested in the last two Articles were given—as has already been stated—on the authority of a paper by the late Astronomer Royal, published in a report of the British Association. I now observe, however—when the printing of the Articles and engraving of the Figures is already completed—that they cannot be accepted as true solutions, inasmuch as they do not satisfy the general Eq. (164) of Section 303. It is perhaps as well that they should be preserved as a warning to the student against the insidious and comparatively rare error of choosing a solution which satisfies completely all the boundary conditions, without satisfying the fundamental conditions of strain, and which is therefore of course not a solution at all.

Love in all editions of his famous treatise [14–18] besides historical introduction chapter mentioned Airy's name only once, in connection with the more general Maxwell approach based upon the 3D stress functions. However, in [16–18] (Section 144) Love expressed the displacement components corresponding to plane strain in terms of Airy's stress function. Filon [132] chose a similar way for considering in the Cartesian coordinates several benchmark problems for an infinite elastic layer or long rectangle. At the same time, Michell mentioned at the beginning of his important paper [176] that "Airy did not consider the differential equation satisfied by his function," and constructed the representation of the stress function in polar coordinates. Later, Michell [178,179] used it to study some elementary distribution of stress in an elastic plane and wedge.

In Germany, Venske [180] and Klein and Wieghardt [139] were the first scientists who attracted attention to the Airy stress function, and this approach was widely developed in dissertations of Timpe [181] and Wieghardt [141]. Sommerfeld [182,183] considered some specific problems for an elastic layer by means of the stress function. Later this subject has received sufficient attention in review articles [95,98,184].

In Italy at the turn of the nineteenth-twentieth centuries there was a strong group of mathematicians, Almansi [185,186], Boggio [187–189], (see, also, recollections [190] written more than half a century later!), Levi-Cevita [191,192], Lauricella [193], and Volterra [194], who were interested mainly in solution of the classical biharmonic problem (1), (2) for some canonical domains. These results were summarized in [74,196–198].

In France, Mathieu [123] studied general mathematical properties of the biharmonic functions, the uniqueness of the solution of the classical biharmonic problem (1), (2) for general domains, the Green's function, and, finally, the solution [199–201] of the basic problem for a rectangular prism. Goursat [202] presented a complex representation of the solution of Eq. (1) which finally led to the effective method of complex variables for solving the biharmonic problem. Lévy [175] introduced a system different from (7) for the stress components N_1, N_2, T (in Lamé's notations)

$$\frac{\partial N_1}{\partial x}+\frac{\partial T}{\partial y}=0, \quad \frac{\partial T}{\partial x}+\frac{\partial N_2}{\partial y}=0,$$

$$\frac{\partial^2(N_1+N_2)}{\partial x^2}+\frac{\partial^2(N_1+N_2)}{\partial y^2}=0 \tag{17}$$

The third equation expresses the continuity of the body under deformation. Thus the sum of the normal stresses N_1+N_2 represents a harmonic stress function. The whole system (17) is often called the "Maurice Lévy equations" for the 2D elastic problems. It is readily seen by means of substitution (6) that the third equation in (17) is reduced to (16). Based upon this system, Mesnager [203,204] presented the stress distribution in specific rectangular and wedge geometries in the form of finite polynomials.

In Russia, Abramov, Kolosov (or Kolossoff, according to the French spelling of his name in some publications) and Gersevanov in their dissertations [205–207] used the Airy stress function to solve various specific problems. A systematic usage of the solution of the biharmonic problem in rectangular and polar coordinates was given in the textbook by Timoshenko [91].

3 GENERAL METHODS OF SOLUTION FOR AN ARBITRARY DOMAIN

3.1 General representations of solution of the biharmonic equation

Searching for more simple representations of the displacement vector and the stress tensor from the general Lamé and Beltrami-Michell equations of the theory of elasticity has a longstanding fascinating history which is represented in [117,208]. However, one must remember that the main difficulty in solving the biharmonic problem consists of satisfying the prescribed boundary conditions. According to Golovin [209] (p 378) a similar opinion had already been expressed by Kirchhoff and Riemann; see also Muskhelishvili [39] (Section 105) for further discussion.

3.1.1 Representations of solutions of the biharmonic equation in Cartesian coordinates

Joseph Valentin Boussinesq (1842–1929) [210] considered in detail several forms of the general solution of the biharmonic equation, mainly for the 3D case, but of course, all these results can be easily transformed into the case of two dimensions. The main idea of Boussinesq entailed the usage of simpler harmonic functions (the 'potentials' in his terminology) in order to obtain general solutions of the biharmonic equation. First of all, it is obvious that any harmonic function $\phi(x,y)$ with $\Delta\phi=0$ automatically satisfies the biharmonic equation $\Delta\Delta\phi=0$. Next, Boussinesq proved that if ϕ_1, ϕ_2 and ψ are harmonic functions, then the functions $x\phi_1$, $y\phi_2$, $(x^2+y^2)\psi$ are biharmonic. Finally, combining these types of solutions, he established that the functions $x\phi_1+\psi$, $y\phi_2+\psi$, $\phi_1+(x^2+y^2-a^2)\partial\psi/\partial x$, $\phi_2+(x^2+y^2-a^2)\partial\psi/\partial y$, with a an arbitrary constant, are biharmonic functions. Boussinesq widely used these combinations to solve the now famous problem of normal loading of an elastic halfspace (or a halfplane). Similar representations were independently obtained by Almansi [186]; see also [74,211–213] for further mathematical details.

Biezeno and Grammel [25] presented an extensive collection of these types of biharmonic functions useful for considering the biharmonic problem in some canonical domains. They pointed out that for any analytic function $f(z)$ with $z=x+\mathrm{i}y$, the real functions $\mathrm{Re}\, f(x\pm\mathrm{i}y)$ and $\mathrm{Im}\, f(x\pm\mathrm{i}y)$ are harmonic. This circumstance considerably simplifies the search for suitable biharmonic functions. An extensive listing of functions which can be used for solution of the biharmonic equation is given in [214,215].

Papkovich [80] and Biezeno and Grammel [25] simulteneously and independently posed an interesting question about the number of independent harmonic functions that are needed to represent an arbitrary 2D biharmonic function. They established that *any* biharmonic function can be represented by means of *two* arbitrary independent harmonic functions in one of the following forms $x\phi_1+\psi_1$, $y\phi_2+\psi_2$, $(x^2+y^2)\phi_3+\psi_4$, with ϕ_i and ψ_i being harmonic functions. In fact, the similar question has been addressed by Chaplygin already in 1904, see [216].

Love [14] see, also, [18], (Section 144), was the first author who addessed an important question of determination of the components of displacement u and v via a biharmonic stress function χ. On the other hand, Papkovich [80,217] established the relation between the biharmonic Airy stress function χ and harmonic functions ϕ_0, ϕ_1, ϕ_2 in his famous general solution for the displacement components u and v.

3.1.2 *Representations of solutions of the biharmonic equation in polar coordinates*

Although Clebsch [3,218] and Venske [180] constructed solutions of the biharmonic Eq. (1) written in the polar coordinates r, θ in form of Fourier series while looking for the (nonaxisymmetric) Green's function of a clamped circular plate, John H Michell (1863–1940) [176] (p 111) was the first author who presented, without any derivation, the general form of solution of the biharmonic Eq. (1) as

$$\begin{aligned}\phi=&A_0r^2+B_0r^2(\ln r-1)+C_0\ln r+D_0\theta\\&+(A_1r+B_1r^{-1}+B_1'\theta r+C_1r^3+D_1r\ln r)\cos\theta\\&+(E_1r+F_1r^{-1}+F_1'\theta r+G_1r^3+H_1r\ln r)\sin\theta\\&+\sum_{n=2}^{\infty}(A_nr^n+B_nr^{-n}+C_nr^{n+2}+D_nr^{-n+2})\cos n\theta\\&+\sum_{n=2}^{\infty}(E_nr^n+F_nr^{-n}+G_nr^{n+2}+H_nr^{-n+2})\sin n\theta\end{aligned} \tag{18}$$

with arbitrary constants $A_0, \ldots, H_n$. Later, a similar solution was derived by Timpe [181,219]. Timoshenko [43,91] added to the solution (18) the term $d_0r^2\theta$ and he, Coker and Filon [27], and Papkovich [80] discussed in full details the mechanical meaning of each term in this representation.

Filonenko-Borodich [59] pointed out the possibility of existence of the terms $\theta r^2\ln r$, $\theta\ln r$, $r\ln r\theta\cos\theta$, $r\ln r\theta\sin\theta$ in the general solution of the biharmonic equation, but he explained that these terms lead to multivalued stresses if the coordinates' origin is located inside the body. They can be important for the elastic dislocation theory [220]. It is interesting to note that 20 years later exactly the same solutions were derived [221] that led to short comments [222–224]. These latter authors have mentioned not only the English translation of Filonenko-Borodich [61], but also a rather forgotten paper by Sonntag [225]. It should be mentioned that Filonenko-Borodich [59] himself attributed these additional solutions to Biezeno and Grammel [25]. These authors presented the most complete set of solutions of the biharmonic Eq. (1) which also include the terms $r^\lambda\cos\lambda\theta$, $r^{\lambda+2}\cos\lambda\theta$, $\cos(\lambda\ln r)\cosh\lambda\theta$, $r^2\cos(\lambda\ln r)\cosh\lambda\theta$ (and corresponding terms with sin and sinh) with an arbitrary value of λ.

3.2 Green's function for the biharmonic problem

The French mathematician Emile Leonard Mathieu (1835-1890)[4] was the first who addressed in full the mathematical properties of the biharmonic equation in a singly connected 2D domain S enclosed by a contour L. In an elaborate memoir, Mathieu [123] developed the theory of the so-called "second" potential, obtained the analogy of Green's formulas for biharmonic functions, proved some theorems concerning the existence and uniqueness of the solution of the biharmonic Eq. (1) with either boundary condition (2) or with prescribed values of the function and its Laplace operator at the boundary. Although his four main theorems describe the properties of the 3D biharmonic functions, similar results for the 2D case were also stated.

First, Mathieu established the generalization of Green's formulas, namely, for any two functions u and v continuous with their third derivatives in S, the following relation holds

$$\int_S(u\Delta\Delta v-v\Delta\Delta u)\mathrm{d}x\mathrm{d}y=\int_L\left(v\frac{d\Delta u}{dn}-u\frac{d\Delta v}{dn}\right)\mathrm{d}l+\int_L\left(\Delta v\frac{du}{dn}-\Delta u\frac{dv}{dn}\right)\mathrm{d}l \tag{19}$$

where d/dn denotes the derivative in the direction of the inner normal to the contour L.

Next, Mathieu introduced two biharmonic functions

$$v(x,y)=\int_L\rho'(a,b)\ln\frac{1}{r}\mathrm{d}l,\quad r^2=(x-a)^2+(y-b)^2$$

$$w(x,y)=\int_L\rho(a,b)\left(r^2\ln\frac{1}{r}+\frac{1}{2}r^2\right)\mathrm{d}l, \tag{20}$$

called the first (logarithmic) and second potentials, respectively, for some smooth functions $\rho'(a,b)$ and $\rho(a,b)$ on the contour L.

Mathieu developed a theory of this second potential. Based upon relation (19), he proved that inside the simply connected domain S there exists a unique, continuous in its third derivatives, function u with prescribed values of u and du/dn on L, and this function can be expressed as a sum of first and second potentials,

$$u(x,y)=v(x,y)+w(x,y) \tag{21}$$

By using in (19) the biharmonic function $\Pi=r^2\ln r$, Mathieu established the relation

$$u=\frac{1}{8\pi}\int_L\left[\Pi\frac{d\Delta u}{dn}-u\frac{d\Delta\Pi}{dn}+\Delta\Pi\frac{du}{dn}-\Delta u\frac{d\Pi}{dn}\right]\mathrm{d}l \tag{22}$$

This equation provides the value of the biharmonic function at any point inside the domain S by means of the values u, du/dn, Δu and $d(\Delta u)/dn$ given on the contour L.

These results were proven in another way in the doctoral thesis by Koialovich [229] defended on February 2, 1903 at St Petersburg University as a consequence of his more general relations for linear partial differential equations with constant coefficients. (These results were first announced [230,231] on December 28, 1901 at the XIth Congress of Russian Natural Scientists and Physicians which by tradition took place at the very end of the year.) It should be noted, however, that in the committee report [232] on this thesis,

[4]Biographical data and a short survey of his scientific works can be found in [226–228].

signed among others by prominent Russian mathematicians Korkin and Markov, these results were mentioned only scarcely in one short paragraph.

Further discussion of Green's function for clamped elastic plates can be found in [233–240].

For the Stokes flow, Green's function was imployed by great Dutch physicist Hendrik Antoon Lorentz (1853–1928) who considered [241] the action of a force in an interior of a viscous incompressible fluid with negligible inertia forces. By using the well-known device of surrounding the point where the force acts by a small sphere and then allowing its radius to vanish, he derived the now famous integral equation for slow viscous flows which relates the velocity vector at any point inside the fluid to a boundary integral which involves the stresses and the velocities on its boundary. This formula has been used extensively in the past two decades or so in the so-called boundary-element method.

Lorentz [241], and later independently Hancock [242], interpreted the well-known Stokes [135] solution for the slow flow induced by a sphere moving through a highly viscous fluid as the sum of two solutions which are singular at the center of the sphere. One of these is a doublet which is also present in an inviscid flow. The second one, according to [242] ''is a singularity peculiar to viscous motion, which will here (for want of a better word) be called a *stokeslet*.'' For the 2D case this term (a more appropriate name according to [243] could be *lorentzlet*) coincides with the second Green's function of the biharmonic problem.

3.3 Method of complex variables

The idea of application of complex variable theory to solve the biharmonic equation looks very natural in view of the great success attained by such an approach for harmonic functions. Goursat [202] established that arbitrary biharmonic function U can be represented via two analytic functions $\phi(z)$ and $\chi(z)$ of the complex variable $z=x+\mathrm{i}y$ as

$$2U=\bar{z}\phi(z)+z\overline{\phi}(z)+\chi(z)+\overline{\chi}(z), \tag{23}$$

where the bar sign indicates a complex conjugate. Another version of the derivation of this important formula is given by Muskhelishvili [244].

In the theory of elasticity for 2D plane stress or plane strain problems the idea of application of complex variable traces back to Clebsch [218] and Love [14]; see, also, Tedone and Timpe [95] (p 163) for details. Clebsch [218] (Section 31) derived the representation of the functions X_x+Y_y and $X_x-Y_y+2\mathrm{i}X_y$ via one function of z and the same function of $\bar{z}$. These expressions, see also Kolosov [245], were rather cumbersome and contained some combinations of Lamé's constants λ and μ. Clebsch did not use them for solving any specific problem for plane stress.

Love [14], following Lamé [1], wrote the 2D equations for the Cartesian components u and v of the displacement vector in a form

$$(\lambda+2\mu)\frac{\partial\Theta}{\partial x}-2\mu\frac{\partial\omega}{\partial y}=0, \quad (\lambda+2\mu)\frac{\partial\Theta}{\partial y}+2\mu\frac{\partial\omega}{\partial x}=0 \tag{24}$$

with $\Theta=\partial u/\partial x+\partial v/\partial y$, $2\omega=\partial v/\partial x-\partial u/\partial y$, and established that the expression $(\lambda+2\mu)\Theta+2\mu\mathrm{i}\omega$ is an analytic function of the complex variable z.

The same approach was developed by Chaplygin [246] around 1900, but he did not pursue further this avenue. Later Filon [132] established rather complicated complex representations and used them for construction of various real expressions for displacements and stress components in a finite rectangle in form of Fourier series and in finite polynomial terms.

Kolosov [247,248] was the first author who developed and systematically applied the complex variables method[5]. Based upon Maurice Lévy's Eqs. (17), with the first two rewritten in the form,

$$\frac{\partial(2T)}{\partial y}+\frac{\partial(N_1-N_2)}{\partial x}=-\frac{\partial(N_1+N_2)}{\partial x}$$
$$\frac{\partial(2T)}{\partial x}-\frac{\partial(N_1-N_2)}{\partial y}=-\frac{\partial(N_1+N_2)}{\partial y} \tag{25}$$

Kolosov derived the following relations

$$N_1+N_2=\frac{1}{2}\{\Phi(z)+\Phi(\bar{z})\}$$
$$2T+\mathrm{i}(N_1-N_2)=\mathrm{i}(\alpha+\mathrm{i}\beta)\frac{d\Phi(z)}{dz}+F(z) \tag{26}$$

where Φ and F are arbitrary complex functions of their arguments, $\alpha(x,y)$ and $\beta(x,y)$ are real functions representing any solutions of the system

$$\frac{\partial\alpha}{\partial x}-\frac{\partial\beta}{\partial y}=-1\,, \quad \frac{\partial\alpha}{\partial y}+\frac{\partial\beta}{\partial x}=0 \tag{27}$$

Therefore, the solution of the 2D problem is completely defined if two analytic functions $\Phi(z)$ and $F(z)$ can be found based upon the prescribed boundary conditions at the contour.

Kolosov [248] also provided the analog of (26) for any curvilinear orthogonal isothermic coordinate system. By chosing some concrete expressions for $\Phi(z)$ and $F(z)$ he obtained anew the already known results [209,219] for circular domains. In addition, he solved some typical boundary problems for a halfplane, a circle, and a plane with circular and elliptical openings. The results were presented [255] at the IV International Mathematical Congress in Rome (Italy), section of Mechanics and Mathematics, on April 11, 1908. Hadamard was the chairman of the session, and Runge, Boggio, and Volterra participated in discussion.

More then one and a half years later, Kolosov presented two talks [256,257] at the XII Congress of Russian Natural Scientists and Physicians, held in Moscow from December

[5]Gurii Vasili'evich Kolosov (1867–1936) (or Kolossoff in the French spelling of his name) graduated from St Petersburg University where he did his master's dissertation on a solid body rotation with one fixed point. From 1902 till 1913 he worked at Yur'ev [Tartu] University. After 1913, he worked at the Electrotechnical Institute and the University in St Petersburg (Leningrad) where he spent the rest of his career. In 1931 upon a suggestion by Academicians Krylov and Chaplygin he was elected a corresponding member of the Academy of Sciences of the USSR. A more detailed biography of Kolosov and discussion of his scientific works can be found in [249–254].

28, 1909–January 6, 1910 (according to the Julian calendar which was in usage in Russia that time). Renowned Russian mathematicians and mechanicians Steklov, Joukowsky, Chaplygin, and Timoshenko participated in the discussion and made some comments. In the first talk [256], Kolosov suggested an interesting method of solution of the biharmonic problem (1), (2) based upon searching for the second derivatives U_{xx}, U_{yy}, and U_{xy}, instead of the function U itself. After some transformations this reduces the problem to that of finding two analytic functions $\Phi(z)$ and $F(z)$ under the condition that along the boundary the function $(\alpha+\mathrm{i}\beta)d\Phi(z)/dz-\mathrm{i}F(z)+\frac{1}{2}\exp(-2\mathrm{i}\theta)(\Phi(z)+\Phi(\overline{z}))$ is given. Here θ denotes the angle between the normal at the boundary at the point (x,y) and the positive x-axis. Finally, the problem is reduced to the known Riemann-Hilbert problem. The method was generalized to arbitrary orthogonal curvilinear coordinates and then applied to some specific contours: a line, a circle, an ellipse, and even a rectangle. It seems that this method remains unnoticed and it deserves further elaboration.

In the second talk [257], Kolosov repeated the derivation of his main formulas (26) and provided an expression for components of the displacement vector. He also derived an integral Fredholm equation, but he did not investigate its properties. Later, a similar approach was essentially developed by Fok [258,259].

All these results entered in Kolosov's doctoral dissertation [206][6]. The defense took place at St Petersburg University on 21 November 1910, and Academician Vladimir Andreevich Steklov (1863–1926) and Professor Dmitrii Konstantinovich Bobylev (1842–19170) were the official opponents. According to [250,251] Kolosov had some troubles during the defense. The matter was that Academician Steklov, who immediately understood and appreciated the main idea of usage of two analytical functions, had noticed some fault in formulas (26) when in Section 12 Kolosov applied them to an arbitrary isothermic coordinate system. His concern was that the right hand side does not generally represent an analytic function. From February–April 1910, Steklov exchanged several letters with Bobylev and Kolosov, see [254] for full texts. Kolosov had to accept these comments and he included an Appendix into his dissertation with a long quotation from Steklov's letter. That is why the date on the title page of the printed dissertation does not correspond to reality—the entire work had been bound after April 1910. Later Kolosov [260,261] referred to the year 1910 as the date of publication of his dissertation.

Kolosov[7] [68] summarized his studies on the solution of the biharmonic problem by means of the complex variables method of finding two analytic functions $\Phi(z)$ and $F(z)$ based upon prescribed boundary conditions at the contour. In addition to already existing approaches, he developed [262] the method of "complex compensation." This method was based upon application of the Schwarz integral (a representation that gives the analytic complex function under prescribed value of its real part at some closed contour). This method, being rather powerful and straightforward, remains rather unnoticed and it deserves further elaboration.

The method of complex variables in 2D elasticity problems developed by Kolosov was successfully followed [263,264] by Muskhelishvili,[8] a pupil of Kolosov at St Petersburg Electrotechnical Institute. Later Muskhelishvili [244,265–270] considerably extended the method by adding the idea of the Cauchy integral and conformal mapping, and solved a large number of specific problems summarized in his remarkable treatise [37–39]. Additional references and detailed exposition of the complex variables method can be found in [33,271].

It is worthwhile to note that a similar complex variables method has been also suggested by Stevenson [272,273] and Poritsky [274], with Kolosov's formulas being derived anew without any references to his works. This circumstance received severe critique from Muskhelishvili in the third edition of his treatise [38] (Section 32). Radok in a translator's note in [39] (p 115) mentioned that he received some explanations from both authors. Stevenson wrote that in the years 1939–1940 when he worked on his paper he was admittedly ignorant of prior works in that area. However, later Stevenson acknowledged the priority of Kolosov and Muskhelishvili by referring to six papers by Kolosov, dating as far back as 1909, of four papers by Muskhelishvili, the first of which appeared in 1919, and to the joint paper by both authors, published in 1915. Poritsky indicated that he deduced his formulas in 1931, although his paper was not published until 1945. By that time, the Russian works had been given a fair amount of publicity in the USA and therefore he quoted only one paper [392], merely for the purpose of acknowledging that he had been anticipated; see [275,276] for further details.

Another usage of complex variables was suggested by Nikolai Mikhailovich Gersevanov (1879–1950) in his master's dissertation [207]. Considering the inhomogeneous biharmonic Eq. (3) for bending a plate with linearly distributed loading $p(x,y)=Px+Qy+R$ (for example, for a sluice gate), he presented the general solution in the form

[6] The dissertation has also been printed in parts in several issues of the *Scientific Notes of Yur'ev [Tartu] University* in 1911 with two additional pages with main statements.

[7] It is interesting to note that the book had slightly different titles on the cover and title pages; this leads to somewhat confusing references in the literature.

[8] There are different spellings, Muschelišvili, Muskhelov of this Georgian name in various publications in French and German (and even in Russian!) journals. Nikolai Ivanovich Muskhelishvili (1891–1975) graduated in 1915 from the Physico-Mathematical Faculty of St Petersburg University, and on presentation of a diploma thesis he was retained by the Department of Theoretical Mechanics for preparation for an academic career. From 1917–1920 he taught at Petrograd University and also at other higher educational institutions of Petrograd. In 1920, Muskhelishvili moved to Tiflis [Tbilisi], where he worked at Tbilisi University and Tbilisi Politechnic Institute. In 1939, he was elected as a Academician of the Academy of Sciences of the USSR, and in 1941 he was elected as a President of the Georgian Academy of Sciences, and the Director of the Georgian Mathematical Institute. Muskhelishvili's fundamental monograph [37] was honored in 1941 with the Stalin Prize of the first order; it has been translated into English, Chinese, and Roumanian and is widely known among specialists. In 1945, Muskhelishvili was awarded the title of Hero of Socialist Labor. He was a deputy of the Supreme Soviet of the USSR of all councils.

$$w=\frac{1}{2}[\phi_1(x-\mathrm{i}y)+\phi_2(x+\mathrm{i}y)]+\mathrm{i}y[\phi_1'(x-\mathrm{i}y)-\phi_2'(x+\mathrm{i}y)]+[\phi_3(x-\mathrm{i}y)+\phi_4(x+\mathrm{i}y)]+\frac{Py^4x}{24}+\frac{Qy^5}{120}+\frac{Ry^4}{24} \quad (28)$$

where four continuous functions $\phi_1(x-\mathrm{i}y)$, $\phi_2(x+\mathrm{i}y)$, $\phi_3(x-\mathrm{i}y)$, $\phi_4(x+\mathrm{i}y)$ are defined from four functional equations corresponding to the boundary conditions (4) for the clamped plate. The author derived that these functions $\phi_q(z)$ can be presented in the form of Taylor series expansions

$$\phi_q(z)=\phi_q^{IV}(0)\frac{z^4}{4!}+\phi_q^{V}(0)\frac{z^5}{5!}+\ \dots$$

and provided an algorithm for finding the values of the coefficients $\phi_q^{IV}(0)$, $\phi_q^{V}(0)$, ...

Gersevanov [207] himself did not consider any numerical example of application of the developed scheme. For some reasons (not clear now) Bubnov, who in 1910 was one of the chief designers of the Russian Imperial Navy, suggested to Fridman, (the future renowned expert in the dynamical metereology and general cosmology, who had just graduated from St Petersburg University) to make some practical calculations based upon that method. In June 1910, Fridman wrote a letter [277] to Steklov (probably, his scientific adviser at the University)

> Some days ago after sending a letter to AN Krylov I received from Mr Bubnov some information about the project. It appears neccesary to solve by the method of N Gersevanov the equation $\Delta\Delta v=a$ for boundary contours consisting of two parabolas of the n-th order. This approach theoretically looks highly cumbersome; I don't know yet how it will work numerically. Anyway that project is very timely and I thank you very much for your help.

and some time later in the second letter

> The matter with calculations based upon Gersevanov's method is very bad; Gersevanov did not prove either the convergence of the Taylor's series or the possibility of finding the coefficients at all. I have tried to apply this method to a simpler Dirichlet problem for a halfplane; it does not work. I am going to discuss this issue with Bubnov.

Later Gersevanov [278] also presented the general integral for the components N_1, N_2, T of the stress tensor in the Maurice Lévy Eqs. (17)

$$N_1=\mathrm{i}y\phi_1'(z)-\mathrm{i}\phi_2(z),$$
$$N_2=2\phi_1(z)-\mathrm{i}y\phi_1'(z)+\mathrm{i}\phi_2(z),$$
$$T=-\mathrm{i}\phi_1(z)-y\phi_1'(z)+\phi_2(z) \quad (29)$$

where $z=x+\mathrm{i}y$ and ϕ_1 and ϕ_2 are two arbitrary functions. Previously Gersevanov [207,279] used more complicated expressions that contain four arbitrary functions (and even the Lamé constants λ and μ, which however can be easily excluded). Gersevanov [278–280] considered by this approach some problems for an elastic halfplane and found simple closed form expressions for various given boundary conditions at the surface. He pointed out a mistake in the solution by Puzyrevskii [281] connected with an (apparent) nouniqueness of the solution. In my opinion, this method for the case of the halfplane (as an alternative approach to Fourier transforms) is rather useful and it deserves further elaboration.

Sobrero [282] suggested the method based upon usage of the so-called *hypercomplex functions* which was developed in [283,284] for the representation of the stress function. However, this approach appeared to be much less effective for solution of specific problems.

4 THE *PRIX VAILLANT* COMPETITION

The engineering problem of bending of a clamped, rectangular, thin plate by normal pressure constantly attracts the attention of mathematicians. As the famous Russian scientist and naval architect Academician Alexei Nikolaevich Krylov (1863–1945) (or Kriloff in French spelling of his name) recollected, [285]:

> In the summer of 1892 I worked in Paris on the project of the Drzewiecki's submarine. Before leaving for Paris, I received from Professor Korkin several of his articles and a letter for Hermite. Upon arrival in Paris, I went to Hermite and was received very warmly. Hermite asked me about Korkin, the Naval Academy, *etc*. Then I said to Hermite that it would be very important for shipbuilding to obtain a solution of the differential equation with the boundary conditions being that the contour of the plate is fixed. Hermite called his son-in-law Picard and said to him: "Look, Captain Kriloff suggests an excellent topic, which can be used for the *Grand Prix des Mathématiques*. Think about this." Approximately a year later this topic was suggested by the Paris Academy of Sciences.

This recollection does not seem to be completely correct. In 1894, the journal *l'Intermédiare des Mathématiciens* was founded with an original idea of providing room for the exchange of opinions among professional mathematicians and interested people by stating questions and (possibly) getting answers. In the first issue of this journal, Picard [286] put the question No 58 in the following words:

> Le problème de l'équilibre d'une plaque rectangulaire encastrée revient à l'intégration de l'équation
>
> $$\Delta\Delta u=a$$
>
> (a étant une constante, et Δf représentant $\partial^2 f/\partial x^2+\partial^2 f/\partial y^2$), u s'annulant sur le périmètre du rectangle, ainsi que la dérivée du/dn prise

> dans le sens de la normale. La solution de ce problème peut-elle être obtenue par des séries ou des intégrales définies?

As the index to the first 20 volumes for the years 1894–1913 shows, this question remained without answer. A similar question (not related to the rectangle only) was repeated ten years later; it received short replies by Boggio with a few Italian references and Maillet with reference to Flamant's textbook [287].

Only in 1904 did the French Academy of Sciences suggest that topic for the competition of the *Prix Vaillant* (and not the *Grand Prix Mathématique*) for the year 1907, with a prize of 4000 francs. The condition for the competition was first announced in *Comptes rendus des séances de l'Académie des Sciences* 1904, **139**, 1135:

> PRIX VAILLANT (4000^{fr}).
>
> L'Académie met au concours, pour l'année 1907, la question suivante:
>
> *Perfectionner en un point important le problème d'Analyse relatif à l'équilibre des plaques élastiques encastrées, c'est-à-dire le problème de l'intégration de l'équation*
>
> $$\frac{\partial^4 u}{\partial x^4}+2\frac{\partial^4 u}{\partial x^2\partial y^2}+\frac{\partial^4 u}{\partial y^4}=f(x,y)$$
>
> *avec les conditions que la fonction u et sa dérivée suivant la normale au contour de la plaque soient nulles. Examiner plus spécialement le cas d'un contour rectangulaire.*
>
> Les Mémoires devront être envoyés au Secrétariat *avant le* 1^{er} *janvier* 1907.

Poincaré (1854–1912), Picard (1856–1941), and Painlevé (1863–1933) were named *les rapporteurs*, that is, it was their task to judge twelve memoirs submitted for consideration. They presented extended reports [288–290], and in December 1907, the authoritative commission consisting of Jordan, Appell, Humbert, Maurice Lévy, Darboux, and Boussinesq decided to share the prize (asking for additional money for that purpose) between Jacques Hadamard (1865–1963) (three quarters of the value), Arthur Korn (1870–1945), Giuseppe Lauricella (1868–1913), and Tommaso Boggio (1877–1963),[9] and gave a special notice to the work by Stanislaw Zaremba (1863–1942).

In the memoirs awarded the *Prix Vaillant*, Hadamard [291], Korn [292], Lauricella [293–295], Boggio [189],[10] and Zaremba [296,297] considered mainly the biharmonic problem for a singly connected interior with a smooth boundary contour. In all cases, some integral equations either for the original biharmonic function or for some auxiliary harmonic functions were written down. By means of the just established Fredholm integral equation theory, it was proven that under rather general conditions a unique solution to the problem under consideration exists.

The main goal of Hadamard's [291] rather voluminous memoir (of 128 pages) is to study "fonction de Green d'ordre deux," Γ_A^B, of the biharmonic problem (3), (4) for the clamped elastic plate, that is, to investigate the general properties of the deflection $w(B;A)$ at an arbitrary point B inside the domain S with a smooth boundary L under the action of a unit normal force applied at a point A. Hadamard showed that the value of Γ_A^A is finite and positive and the inequality $(\Gamma_A^B)^2\leqslant\Gamma_A^A\Gamma_B^B$ holds well. He considered the interesting problem of how the biharmonic Green's function changes under a small deformation of the domain, and he derived the nonlinear integro-differential equation for the variation of the Green's function mentioning that "it is in no way an exception in mathematical physics." He also studied the variational properties of the Green's function and put forward the isoperimetric conjecture that the maximum value of the functional $G(P,P)$ considered on the set of domains with a prescribed perimeter, is attained for a circular domain with a center at P. This has an important connection with the solution of extreme problems and problems related to conformal mapping; see [298] for further mathematical details.

Another question addressed in [291] was the so-called "Boggio's conjecture." Boggio [188] had put forward a conjecture that the biharmonic Green's function is always *positive* inside a convex domain. In other words, the deflection of any point of a clamped plate coincides with the direction of an applied concentrated force. Boggio proved this conjecture for a circular domain by means of some obvious inequalities applied to the explicit expression of the Green's function. Hadamard [291] suggested another "physically evident" conjecture that the value of the Green's function increases with decreasing domain. In the talk presented at IVth International Congress of Mathematicians in Rome in September 1908, Hadamard [299] (p 14) stated that

> M Boggio qui a, le premier, noté la signification physique de Γ_B^A, en a déduit l'hypothése que Γ_B^A était toujours positit. Malgré l'absence de démonstration rigoureuse, l'exactitude de cette proposition ne parait pas douteuse pour les aires convexes.

Hadamard, however, mentioned the necessity to put some additional assumptions on the domain, for the Green's function G has an alternating sign for an annulus with a large ratio between external and internal radii. (Enliš and Peetre [300] proved that G is not positive for an arbitrary ratio of the radii.)

After Hadamard, the Boggio-Hadamard conjecture received considerable attention, mainly among applied mathematicians. It finally appeared that it is *wrong*! Garabedyan [301] showed that G changes sign inside the elliptical domain $x^2+(\frac{5}{3}y^2)\leqslant 1$. Shapiro and Tegmark [302] showed that non-positivity of G for the elongated ellipse $x^2+25y^2\leqslant 1$ can be easily obtained by considering the polynomial $P(x,y)=(x^2+25y^2-1)(1-x)^2(4-3x)$ that satisfies both

[9]Boggio was extremely fortunate to escape with his life as 78,000 people were killed by an earthquake that on December 28, 1908 struck Messina, northeastern Sicily, where he held the position of Professor of Rational Mechanics.

[10]There were no special publications later on, but Poincaré in his report [290] presented a detailed survey of the entrant essay.

boundary conditions (4). Since $\Delta\Delta P > 0$ everywhere inside the ellipse, then assuming $G \leqslant 0$ one arrives at $P \leqslant 0$, which is obviously incorrect. Other (exotic) examples of domains bounded by analytical curves, for which Green's function changes sign, were provided in [303,304].

Duffin [305] suggested that Green's function for a half-strip $x \geqslant 0$, $|y| \leqslant 1$ may change sign because of asymptotic behavior $C\exp(-\sigma x)\cos(\tau x - \phi)$ for large x on the line $y = 0$. Following Boggio [188] and Hadamard [299], he also formulated two conjectures. First, he supposed that the change of sign occurs for rectangles with a ratio of sides greater than four. Secondly, he supposed that for a square plate the Boggio-Hadamard conjecture holds well. (This statement is incorrect.)

Another approach to the Boggio-Hadamard conjecture was developed by Hedenmalm [306]. He was interested in additional conditions for the positivity of the biharmonic Green's function, and he applied the idea of Hadamard of changing G with changing form of a domain. By introducing into consideration the function $H(P,P_0) = \Delta_P G(P,P_0) - g(P,P_0)$ with $g(P,P_0)$ being the harmonic Green's function for the Dirichlet problem for the Laplace equation, he proved that for a star-shaped domain with boundary given by analytic curve, $G(P,P_0) \geqslant 0$ in the domain if and only if $H(P,P_0) \geqslant 0$ in the whole domain including the boundary.

Lauricella [295] in his winning memoir developed another approach to solve the biharmonic problem (1) and (2). He introduced two unknown functions

$$u = \frac{\partial U}{\partial x}, \quad v = \frac{\partial U}{\partial y} \tag{30}$$

with an auxiliary function

$$\theta = \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = \Delta U \tag{31}$$

Lauricella considered the following equations

$$\frac{\partial u}{\partial y} = \frac{\partial v}{\partial x}, \quad \Delta\theta = 0 \tag{32}$$

inside the domain S with boundary conditions

$$u = g(l)\frac{\mathrm{d}x}{\mathrm{d}n} + \frac{\mathrm{d}f}{\mathrm{d}l}\frac{\mathrm{d}x}{\mathrm{d}n} = f_1(l)$$

$$v = g(l)\frac{\mathrm{d}y}{\mathrm{d}n} - \frac{\mathrm{d}f}{\mathrm{d}l}\frac{\mathrm{d}x}{\mathrm{d}n} = g_1(l) \tag{33}$$

at the contour L.

After extensive transformations Lauricella reduced the boundary value problem (32), (33) to the system of two Fredholm integral equations with respect to the unknown functions u and v. He proved uniqueness of the solution for the case of a finite domain bounded by a smooth contour.

Later Sherman [307] independently established a similar integral equation for a complex function $\omega(z)$ conditions, which is usually called the Lauricella-Sherman equation. A detailed review of Sherman's numerous articles on this topic as well as other possible types of integral equations can be found in [38,39,106].

General approaches based on the integral equations appeared not effective when dealing with the rectangular domain—the case that was specially mentioned in the condition for the *Prix Vaillant*. Lauricella [295] wrote down at length the representation for the deflection of the clamped rectangular plate in terms of functions u and v. (The general case of the inhomogeneous biharmonic equation can be easily reduced to the homogeneous one by choosing any particular solution.) In fact, the infinite system of linear algebraic equations for the coefficients in the Fourier series for these functions completely concides with one obtained by Mathieu [200,201] for an elastic rectangle. Lauricella did not provide any numerical results, only referring to Koialovich's [229] (or Coialowitch, as he wrote) doctoral dissertation. This approach was further developed by Schröder [308,309] in extensive papers which, however, remained not known because of World War II. Later Schröder [310] considered the case of a rectangular domain, where he also used Lauricella's method.

The results of Korn [292] and Zaremba [296,297], being interesting at the time, did not have much impact on the further development in the biharmonic problem: references to these memoirs are scarce today, both in mathematical and engineering studies. References to these studies in general context of the *Prix Vaillant* competition are given in [37–39,298].

I do not know what other eight memoirs submitted for the *Prix Vaillant* competition were. It is highly possible that the manuscript by Haar (1885–1933) [311] (based upon his Göttingen dissertation guided by Hilbert) was among them. In several publications [11,12,312–314] it was mentioned that among the twelve memoirs submitted to the *Prix Vaillant* competition there was one authored by Walter Ritz. This epoch-making study was not to be crowned, and it was not even discussed in the commission reports. The reasons for that are not very clear. According to Forman [314] (p 481), Ritz's manuscript, 38 pages in folio, together with a referee's summary, is in the archives of the French Academy of Sciences (it still would be interesting to find these sheets!), while the obituary note by Fueter [312] (p 102) stated that Ritz presented the memoir in time, but it had been simply lost. In any case, in April 1908 Poincaré visited Göttingen (where Ritz then resided) and expressed his deep regrets that this very original investigation had not been honored. Poincaré mentioned that the Academy would award Ritz another prize. Finally, in 1909 Ritz was awarded (unfortunately, posthumously) the *Prix Leconte* of the Academy of Sciences for his works in mathematical physics and mechanics as it was stated in *Comptes rendus des séances de l'Académie des Sciences* 1909, **149**, 1291. For further discussion of Ritz's work, see Section 5.6.3 of the present article.

5 METHODS AND RESULTS FOR SOME CANONICAL DOMAINS

The term "canonical domain" usually refers to the domain whose boundary (or boundaries) is formed by a coordinate line (or lines) of some typical 2D coordinate systems, eg, rectangular, polar, elliptical, or bipolar ones. The usage of these systems can often provide a considerable simplification and permit one to obtain an analytical solution of the biharmonic problem. In what follows we restrict our consideration to the most commonly used canonical domains of finite dimensions: a circle (or ring), an elliptic region, an eccentric circular ring, and a rectangle. Out of consideration remain, however, a sector and an annular sector (a curved thick beam) that also obtained a great deal of attention in the literature. The biharmonic problem for an infinite wedge represents one of the benchmark problems that is extremely important for understanding the peculiarities of local behavior of a biharmonic function in a vicinity of a non-smooth boundary. Besides, we briefly consider the solutions of the biharmonic problems in some outer infinite regions, namely, a plane, a halfplane, and a layer with circular and elliptical openings, the problems that traditionally have a strong technological importance in civil engineering and shipbuilding. (The equally important problems of several nearby openings remain, however, out of the scope of this review.) The most typical infinite domains: a plane, a halfplane, and a layer remain out of consideration, too—there exists a great number of textbooks and monographs already mentioned in the Introduction that contain detailed expositions of these problems.

5.1 Circle and circular ring

The circular domain is obviously the most common one for solving explicitly the biharmonic problem. This has been done by many authors in almost innumerable publications. In what follows, we restrict our consideration only to the most important steps; for sake of uniformity of description of the results of many authors we will consider the circular domain $0 \leqslant r \leqslant a,\ 0 \leqslant \theta \leqslant 2\pi$ in the cylindrical coordinates (r,θ).

5.1.1 General homogeneous biharmonic problem

The explicit solution of the general biharmonic problem (1), (2) was obtained a long time ago in several ways. One way, employed by Venske [180], consists in using the fact that any biharmonic function U can be written in the form

$$U(r,\theta)=u(r,\theta)+r^2\, v(r,\theta) \tag{34}$$

with u and v being the harmonic functions in the interior of the circle. By searching for these functions in the form of Fourier series on the complete trigonometric system $\cos n\theta$, $\sin n\theta$ with $n=0,1,\ \ldots$ and expanding the boundary conditions (2) in Fourier series one obtains an independent system of equations to determine the Fourier coefficients for every number n. The main question, however, is the convergence of the Fourier series, and their ability to present the biharmonic function U in a form suitable for numerical evaluation.

Another approach belongs to the Italian mathematicians Almansi [185] and Lauricella [193]. By means of representation (34) they reduced the problem to two Dirichlet problems in the circle and they represented the solution of the boundary value problem (1), (2) in two different forms of definite integral over the contour $r=a$ that provide the finite expression for the above-mentioned Fourier series. In a short comment Volterra [194] demonstrated that both these expressions can be transformed one into another; see also [195].

5.1.2 Bending of a clamped circular plate

In his epoch-making memoir Poisson [315] was the first to consider the bending of a circular thin isotropic elastic plate of thickness h and radius a clamped at the boundary $r=a$ under an action of axisymmetric pressure $p(r)$ at its top surface. He got the general solution to the boundary value problem (3), (4) in closed form; see Todhunter and Pearson [21] for this long expression.

For particular cases of uniform loading p_0 and concentrated force P applied at the center of the plate (here, in fact, Poisson used the notion of δ-function while mentioning that the load has sensible values only when the values of r are insensitive and some integrals then have to be suppressed) he obtained

$$w(r)=\frac{p_0}{64\pi a^2 D}(a^2-r^2)^2 \tag{35}$$

and

$$w(r)=\frac{P}{8\pi D}\left[-r^2\ln\frac{a}{r}+\frac{1}{2}(a^2-r^2)\right] \tag{36}$$

respectively.

In 1862, Alfred Clebsch (1833–1872), then age 29, being a Professor at the Polytechnic school at Karlsruhe published a book [218], based upon his lectures on the theory of elasticity. Notwithstanding his position at the technical school this book certainly was not suited for the technician—it was highly mathematical, with a wealth and ingenuity of analysis of the more theoretical parts of elasticity. The chief value of the book lies in the novelty of the analytical methods and solutions of several new elasticity problems. In the French translation [3], performed by Saint-Venant (age 86!) and Flamant, there are a lot of amendments which increase the volume to more than twice the length (as well as the correction of many of the innumerable errata of the original). In Sections 75 and 76 of [218] (or pp 763–778 of the French translation [3]) the general problem of small deflection of a thin isotropic clamped plate of thickness h is dealt with. Clebsch wrote down the general equation for the bending of the plate also subjected to the stretching T in the middle plate. Supposing the normal load $p(r,\theta)$ to be known in sines and cosines of multiple angles of θ, and then expressing w in like form, and assuming for simplicity $T=0$, Clebsch obtained a set of equations in the form:

$$\left(\frac{d^2}{dr^2}+\frac{1}{r}\frac{d}{dr}-\frac{n^2}{r^2}\right)^2 w_n=\frac{p_n}{D},\quad n=0,1,\ \ldots, \tag{37}$$

where $w_n(r)$ is the coefficient of $\cos n\theta$ or $\sin n\theta$ in $w(r,\theta)$, and $p_n(r)$ the coefficient of the like terms in $p(r,\theta)$.

Clebsch presented the explicit expression for the particular case of a clamped edge $r=a$ under concentrated normal load P applied at the point (r_0,θ_0), that is the Green's function for the biharmonic problem in the circular domain. This solution had mainly mathematical interest, for it was rather difficult to employ it for practical calculations. However, Clebsch managed to sum the Fourier series and to find a simple engineering formula for the deflection f_{r_0} of the plate at the point of applied load. This value appeared finite:

$$f_{r_0}=\frac{P}{8\pi D}\left(\frac{a^2-r_0^2}{2}+r_0^2\ln\frac{r_0}{a}\right) \tag{38}$$

and for the most interesting case of central load, $r_0=0$

$$f_0=\frac{Pa^2}{16\pi D} \tag{39}$$

Michell [179] developed an elegant method of inversion and presented the deflection w at some point A under the concentrated force of value P applied at some point C as:

$$w=\frac{P}{8\pi D}\left[-R^2\ln\frac{R'}{R}+\frac{1}{2}(R'^2-R^2)\right], \tag{40}$$

where R and R' are the distances from point A to points C and C', the inverse to C with respect to the circle of radius a; see, [18] (Section 314b) for further details. This solution was used in [316] to consider more general cases of local loading of a clamped circular plate.

Apparently, being unaware of Michell's solution Föppl [317] considered this problem anew and he obtained the representation for deflection in the complicated form of a Fourier series. By using bipolar coordinates Melan [318], Flügge [319], and Müller [320] constructed another (more simple) expression for the deflection in Michell's solution.

5.1.3 *Stresses in a circular plate and a circular ring*

Clebsch [218] (Section 74) also addressed a general solution for a circular plate (under conditions of plane stress) subjected to a given system of forces acting parallel to the plane of the plate, but himself did not provide a discussion of any specific problem.

Based upon general representation (18) Timpe [181,219], Timoshenko [321–323], Wieghardt [324], Filon [325], and Köhl [326] considered several practical cases of concentrated and distributed loads acting at the surface(s) of a circular disc (or a ring). There were some delicate questions concerning a choice of constants B_1' and F_1' in order to provide the single values for not only stresses, but also for the radial and circumferential components of a displacement vector in a complete ring. It appeared that the relations

$$B_1'(1-2\nu)+H_1(2-2\nu)=0,$$

$$F_1'(1-2\nu)+D_1(2-2\nu)=0$$

must be fulfilled for the case of plane deformation (and with corresponding change of ν for plane stress). Therefore, the stresses in the circular ring will generally depend on Poisson's ration ν. However, if loadings applied to the inner and outer surfaces of the ring provide separately zero total force, than the constants B_1', H_1, F_1', D_1 turn to zero. In particular, Papkovich [80] (p 506) pointed out the mistakes made in the textbooks by Timoshenko [43] and Filonenko-Borodich [58] while considering the benchmark problem of concentrated force acting in an infinite elastic plate as a limiting case of a plate with a small circular hole under prescribed nonaxisymmetric load. This problem, in fact, had been correctly solved by Michell [178] and cited already by Love [16–18].

It should be noted that the problem for the circular disc under action of concentrated forces acting at its surface has been first solved in a closed form by the great German physicist Heinrich Hertz (1857–1894) [327] by the method of images; see [43,80,91] for detailed explanation. The same problem by the method of complex variables was solved by Kolosov and Muskhelov [263] and later reproduced by both Kolosov [68] and Muskhelishvili [37–39].

5.2 Ellipse and elliptical ring

Although the biharmonic problem in an elliptic region has received relatively little attention so far, it provides, however, a wonderful example of a simple closed-form analytical solution of an important engineering problem.

In his talk communicated to the Summer Meeting of the 34th Session of the Institution of Naval Architects on July 13, 1893 Bryan [328] discussed how the general mathematical Kirchhoff theory of thin elastic plates could be applied to calculate the stresses in a thin elastic plate that is bent under pressure. Giving a talk before practical naval engineers, the applied mathematician, as we could call him now, Bryan did not attempt to go through the long and complicated analysis and mentioned that "at a future I would be prepared to apply the results to calculate the stresses in a circular, elliptic, or rectangular area exposed to fluid pressure, in the hope that such calculations may serve as a basis for future experimental or other investigations on the subject." At the very end of his talk, Bryan [328] said:

> I find that the solution assumes a very simple form when the boundary of the plate is elliptical (or other form of any conic section), and is built in, provided that the pressure is either uniform over the plate, or is hydrostatic pressure proportional to depth. ⟨ . . . ⟩ I only regret that it has been found too late to incorporate into the present paper the results which I have arrived at so far; but I trust the delay may allow of this work being put into a more complete form before it is published.

Although in 1901 George Hartley Bryan (1864–1928), the Fellow of the Royal Society since 1895, was awarded the gold medal of the Institution of Naval Architects for a paper on the effect of bilge keels on the oscillations of a ship, his scientific interest gradually moved to aviation, to a class of problems known now as "flutter." (For his book *Stability in Aviation* published in 1911, Bryan was presented with the gold medal of the Royal Aeronautical Society.) Bryan had never published the promised results. Instead, he communicated to Love the elegant solution on bending of an elliptic plate $x^2/a^2+y^2/b^2=1$ by uniform normal loading p. This solution was immmediately presented in Love [15] (p 199) with reference to Bryan:

$$w = \frac{p}{8D\left(\frac{3}{a^4}+\frac{3}{b^4}+\frac{2}{a^2b^2}\right)}\left(1-\frac{x^2}{a^2}-\frac{y^2}{b^2}\right)^2 \qquad (41)$$

It is easy to check that expression (41) satisfies both the governing equation (3) and boundary conditions (4) because $\partial w/\partial x = 0$ and $\partial w/\partial y = 0$ at the contour.

Love briefly referred to (41) in subsequent editions of his treatise, [16,18] (Section 310), without much use of it. In contrast, this solution was thoroughly discussed in textbooks by Timoshenko [92] (Section 52), Timoshenko [44] (Section 56), and Timoshenko and Woinowsky-Krieger [47] (Section 71). In particular, it appeared that the maximum stresses are at the ends of the short axis. The expressions for distribution of shear forces and normal pressure on the contour were also provided.

Bubnov [4] (p 21) used this remarkable solution to assume the following postulate:

''If we have four plates of the same thickness, having the following form of boundaries:

(1) A rectangle with one side $2a$, the other being very long;
(2) A rectangle with one side $2a$, the other $2b$ $(b>a)$;
(3) An ellipse with the axis $2a$ and $2b$ $(b>a)$;
(4) A circle with the diameter $2a$;

all subjected to the same pressure, the corresponding stresses and strains in the first plate are greater than in second, in second greater than in the third, and in the fourth they the least.''

Therefore, if we denote p_1, p_2, p_3, and p_4 the pressures causing the same maximum bending stresses, we have for plates clamped on the boundaries

$$p_1 : p_2 : p_4 = \frac{3}{8} : \frac{3}{8}\left(1+\frac{2}{3}\left(\frac{a}{b}\right)^2+\left(\frac{a}{b}\right)^4\right) : 1$$

Boggio [188] and Leibenzon [329] considered the problem of construction of the Green's function (a concentrated force acting at the center of an ellipse) for a clamped elliptical plate, and Bremekamp [330] considered the classical biharmonic problem (1), (2) in an elliptical region with semi-axes a and b along x- and y-axes. The solutions were searched for in the elliptical coordinates (ξ, η), where $x = c\cosh\xi\sin\eta$, $y = c\sinh\xi\cos\eta$, c being half the focus distance, by expansion of two auxiliary harmonic functions into Fourier series in $\sin n\eta$ and $\cos n\eta$. The general representation for the biharmonic function consists of four Fourier series, the two pairs of them corresponding to even and odd parts on the coordinate η. Finally, two independent recurrent infinite systems were derived. Each equation in them (besides the first two) contains Fourier coefficients with indices $n-2$, n, and $n+2$. The algorithm of solution based upon a specially constructed Taylor expansion was employed that permitted one to express the coefficients explicitly. Additional references on the general problem of bending of elliptical elastic plates by various loadings can be found in [331,332].

The elastic 2D problem for an elliptical region was first considered by Tedone [333], but his solution is very difficult to understand. Muskhelishvili [268,334] gave a rather simple solution, which was later reproduced in [38,39] by the method of conformal mapping together with his method of complex variables for two complex functions. Instead of an obvious mapping of the ellipse onto a circle (which led to complications in the solution), the special mapping of an elliptic ''ring'' with an empty region between foci was used. Again, the recurrent infinite system was obtained that contains coefficients with indices $k+2$ and k and their conjugates. This system could be solved recurrently, starting from two known first coefficients. Sherman [335] employed his integral equation method to solve the same problem.

The elastic problems for the domain enclosed by two confocal ellipses (or an elliptic arc clamped at the horizontal plane) were considered by Belzeckii [336], Timpe [337],[11] and Sheremet'ev [338] by means of the Fourier series expansions.

It should be noted that all these studies contain no (or very little) numerical data for the stress field, which might represent a possible engineering interest.

5.3 Stress concentration around openings

For many years engineers have been in doubt as to the effect, on distribution of stress, of punching a hole in the center of a tie-bar or other simple tension member. Common sense made it evident that the resulting distribution of stress in the immediate neighborhood of the hole must be far from uniform, but it was not an easy matter to estimate the relative importance of the local increases in stress intensity. That this increase of stress might well be very considerable was evident from the fact that Grübler [339] had shown that the piercing of a small hole in the center of a rotating disc had the effect of doubling the maximum stress as compared with the stress in an unpierced disc subjected to the same centrifugal forces; see also Stodola [340] (or English translation [341], p 383) for an important note of danger of boring a hole for the shaft.

The stress concentration problems provide a vast area of application of the solutions of the biharmonic problem in the theory of elasticity; see, for example, fundamental books by Neuber [342] and Savin [81–84], and review papers by Biezeno [343], Timoshenko [344], Sternberg [345], and Neuber and Hahn [346] for detailed lists of publications. Below we present a few typical examples in the history of these problems.

5.3.1 *Stress concentration around a circular opening*

It is a common statement that (almost) every textbook on the theory of elasticity and structural mechanics published in the twentieth century, in a chapter (or chapters) devoted to 2D problems, contains a section about stress concentration around a circular opening in an infinitely large elastic plate subjected to a uniform tension in a certain direction at infinity. This problem is traditionally attributed to German scien-

[11] According to [39] (Section 64) that solution is wrong, because Timpe did not use the complete system of functions to represent the biharmonic function.

tist and engineer G Kirsch[12] and is usually considered as the starting point of the vast area of the stress concentration problems.

But, in fact, the first problem of stress concentration had been considered by Love [14], who studied the displacement field in an infinite elastic plate with a circular cavity subjected to a shear displacement $U=sy$, $V=0$ at an infinite distance $y\to\infty$. In terms of stress these conditions meant an application of uniform shearing stress at infinity. By use of a general representation for the displacements u, v in the cylindrical coordinates r, θ, established by him earlier, along with the condition that there is no traction across the surface $r=a$, Love obtained (“the work may be left to the reader,” as he wrote)

$$u=\left(\frac{\lambda+2\mu}{\lambda+\mu}\frac{a^2}{r}+\frac{1}{2}r-\frac{1}{2}\frac{a^4}{r^3}\right)s\sin 2\theta$$

$$v=\left(\frac{\mu}{\lambda+\mu}\frac{a^2}{r}+\frac{1}{2}r+\frac{1}{2}\frac{a^4}{r^3}\right)s\cos 2\theta-\frac{1}{2}sr \tag{42}$$

Apparently, this solution has been overlooked by all followers except Suyehiro [348], and later Föppl [349], who obtained it independently. For unknown reasons Love omitted this solution from the subsequent editions [16–18].

In an extensive talk read before the 39th general meeting of *des Vereines deutscher Ingenieure* in Chemnitz on 8 June 1898, Kirsch [350] stated that the tangential stresses (τ_{ru} in his notation) at the end points of a diameter of the hole drawn at right angles to the direction of tension are three times greater than the applied uniform tension p. (At the end points of the diameter parallel to the direction of tension the tangential stress is equal to the applied tensile stress.) Traditionally, it is mentioned that Kirsch himself did not provide any derivation of the final correct analytical expressions for the stress tensor in the polar coordinates (r,θ)

$$\frac{\sigma_r}{p}=\frac{1}{2}\left(1-\frac{a^2}{r^2}\right)+\frac{1}{2}\left(1-\frac{a^2}{r^2}\right)\left(1-\frac{3a^2}{r^2}\right)\cos 2\theta$$

$$\frac{\sigma_u}{p}=\frac{1}{2}\left(1+\frac{a^2}{r^2}\right)-\frac{3}{2}\left(1-\frac{3a^4}{r^4}\right)\cos 2\theta$$

$$\frac{\tau_{ru}}{p}=-\frac{1}{2}\left(1-\frac{a^2}{r^2}\right)\left(1+\frac{3a^2}{r^2}\right)\sin 2\theta \tag{43}$$

and virtually each author suggested his own (rather simple) derivation of the Kirsch solution. (In some cases the name of Kirsch was not mentioned at all; for example, Howland [351] ascribed this solution to Southwell, which, in turn, has been published in [352]. Love [16,18] in a rather detailed reference list of authors did not mention Kirsch's name, either.)

In fact, Kirsch's paper [350] contains a few Appendices and in the second one an analytical representation for the displacements in the Cartesian coordinates of the infinite elastic body (with a reference to the system of equations in displacements given in the textbook of Föppl [353]) were provided without any deviation. How Kirsch managed to find these rather complicated expressions for displacements and stress (some parts of them contain terms like $(x^2+y^2)^{-3}$ and $(x^2+y^2)^{-4}$) really remains unclear. But in the new edition of his very popular that time textbook, August Föppl (1854-1924) gave an expression [127] (Section 55)

$$F=\frac{1}{4}p\left\{r^2-2a^2\ln r-\frac{(r^2-a^2)^2}{r^2}\cos 2\theta\right\} \tag{44}$$

for the stress function which provides the components (43) of the stresses. Föppl also wrote down without derivation the expressions for displacements in the Cartesian coordinates addressing now the paper of Kirsch [350]!

Apparently independently of Föppl [127], Velikhov [354] and Timoshenko [321] in Russia addressed the same Kirsch's problem. While the first of these studies again contained initially some empirical expressions for the stress components in rectangular coordinates, and then by not very clear procedure of fitting it, provided the accurate expressions (with an extensive experimental verification of results and practical recommendations for distribution of rivets in an elongated plate), the second study provided a strict derivation of the stress function. The method is based upon considering a rather extensive circular ring plate with nonuniform normal and tangential loadings over its large circle $r=R$. These distributions correspond to simple expressions for stresses of a uniformly loaded plate, written in polar coordinates. The solution for the stress function of this auxiliary problem has been obtained in a closed form by means of two terms of the Fourier series. Finally, by letting $R\to\infty$ in the final expressions, Timoshenko [321] obtained the results of Kirsch [350] and Föppl [127] for the stresses and the stress function; see also studies by Timoshenko [43,91,355–357] for further details.

It might seem strange, but all these solutions remained unnoticed by naval engineers at the beginning of the twentieth century. In the September 1, 1911, issue of *Engineering* (p 291), one of the leading applied journals of that time, an editorial note “The distribution of stress round deck-openings” was published with a brief discussion of the solution presented by Dr Suyehiro, of the Department of Naval Architecture, Tokyo Imperial University, in which that author had been “congratulated on a distinct addition to the number of known solutions in the mathematical theory of elasticity.” In fact, Suyehiro's [358] solution was nothing more than a detailed repetition of Kirsch's solution[13].

In Austria a lot of theoretical and experimental studies devoted to the Kirsch problem were done by Leon [359–362], and Leon and Willheim [363]; see also Leon and Zidlicky [364] and Preuss [365] for further references.

[12] In [347] there was published a short obituary note for Gustav Kirsch (1841–1901) who, after graduation from the Gewerbeinsitut in Berlin, studied further at Faculté des Sciences (Sorbonne) and the Eidgenössischen Polytechnikum in Zürich. After defending his Doktor promovirt degree in 1869 at the philosophical Fakultät of the Universität Leipzig, he was almost 30 years in Chemnitz as a Professor of the Gewerbeakademie.

[13] Suyehiro [348] did really make a new contribution to the problem by considering the distribution of stresses near a plugged circular hole.

There were a lot of experimental studies [366–377] by the English school of photoelasticity founded and headed by Ernest George Coker (1869–1946), delivered mainly at the meetings of the Institution of Naval Architects, (see also Coker and Filon [27] for a detailed review), that contained experimental testing data and a confirmation of the theoretical results for that important problem.

Having left Kiev in 1920 to go first to Zagreb (Yugoslavia), and then to the USA, Timoshenko[14] published two papers [355,356] which contain an approximate solution of the stress concentration problem in a circular ring of inner and outer diameter d and D, respectively, loaded at the outer side by specially distributed loads corresponding to the uniaxial tension in an infinite plate with a hole of diameter d. Provided that D is large compared with d, he used the elementary theory of the bending of curved bars and came to the conclusion that for $5<D/d<8$ the results obtained agree closely with the exact Kirsch's solution. When $D/d<5$, the hole has an essential effect on the distribution of the forces acting at the external boundary of the ring. When $D/d>8$, the elementary theory of curved bars when the inner radius is very small in comparison with the outer one provides insufficient accuracy. These results partly entered into an extensive talk Timoshenko and Dietz [385] delivered at the Spring Meeting of the American Society of Mechanical Engineers, Milwaukee, May 16–21, 1925[15].

Both this talk and the analytical Kirsch solution met a severe reaction from Swain [386] who was at that time a Professor of Civil Engineering at Harvard University and one of the leading figures in bridge design. He had just published a textbook [387] in which on pp 121–123 he pointed out that if the result of threefold increase of the stress on the edge of the hole is independent of the size then it will (assuming the material to be perfectly homogeneous and elastic) be the same if the diameter of the hole be diminished to an infinitesimal size. Based upon ''common sense,'' the author took the illigitimate step of equating this infinitesimal to zero, thus abolishing the hole, with the ''result'' $3=1$, which he advanced as a proof of some error in the Föppl [-Kirsch] solution (he reproduced, however, the main formulas (43) for stress) and concluded that ''it is unnecessary to give their derivation.'' Further the author provided additional arguments based upon elementary strength of materials reasons to support his conclusion, and he noted on p 122:

> Perhaps in this may be found the fallacy in the theoretical demonstration, but the writer has not gone through with it. He has no time for such illusory mathematical recreations.

[14] A detailed account of the life and scientific results of that outstanding scholar who produced a considerable input on the development of the many fields of mechanics of solids in many countries can be found in a fascinating autobiography by Stephen Prokopovich Timoshenko (1878–1972) [378,379] and in the introductory article in [380]; see also [381–383]. The book [384] contains not only a list of Timoshenko's numerous books and articles, but a list of references about Timoshenko in numerous ''Who's whos'' and reviews on some of his books and articles published in various archival and review journals.

[15] It seems strange, but this remarkable talk has not been reproduced in Timoshenko [380].

In the discussion, Swain [386] had doubted the Kirsch solution in the following (rather peculiar) words:

> It may be worthwhile to examine first this so-called exact solution. It is based on two assumptions. The first, of course, is that there is a hole. The second is that the width of the plate is infinite. The last assumption means that there is no hole at all, because a hole of finite diameter d in a plate of infinite width is the same as a hole of no diameter in a plate of finite width w, since
>
> $$\frac{d}{infinity}=\frac{0}{w}$$
>
> here we have an instance of the character of some of the demonstrations that are now being put forward as founded on the theory of elasticity and as being ''exact.'' This one, as above stated, is founded on the assumptions that there is a hole and that there is no hole; in other words, that a thing is and is not at the same time. It gives results for the stress at the edge of the hole which are independent of the diameter of the hole. Of course, it is easy for any practical man to see that such results are absolutely worthless as applied to any practical case.

and he concluded

> In the judgement of the present author, engineering today is being and has been demoralized by the abuse of mathematics and of testing. Mathematics is an invaluable tool, a necessary tool, but it is a dangerous tool, because the tool itself is so interesting that those who are expert in its use but do not understand the meaning or the physical limitations of the problems to which it is applied will misuse the tool.

Timoshenko [388] in the discussion of the talk retorted

> In his discussion Professor Swain makes a reference to his book on the Strength of Materials in which the problem on stress concentration is discussed in an elementary way; but by using a simple beam formula (see page 123, Eq. 13, of Professor Swain's book), it is impossible to disprove the exact solution. The errors in his reasoning have been indicated also by another author (see *Engineering*, July 31, 1925, page 144), and the writer hopes that in the next edition of his book Professor Swain will give a more satisfactory discussion of such an important question as stress concentration produced by notches and holes.

and he concluded with general comments

> In conclusion, the writer desires to make some remarks in general about analytical and experimental methods in modern technical literature.

> The trend of modern industrial development is more and more toward the free acceptance and application of the teaching of pure science. This general tendency can be seen also in the increased use of the mathematical theory of elasticity for solving technical problems. In many cases of modern design the elementary solutions obtained by the application of the theory of strength of materials are insufficient, and recourse has to be made to the general equations of the theory of elasticity in order to obtain satisfactory results. All problems on stress concentration are of this kind. They involve highly localized stresses and elementary methods like those given in Professor Swain's book, pp. 122 and 123, cannot give a satisfactory solution. Only a complete analysis of stress distribution, together with experiments such as discussed in the authors' paper, can be expected to yield sufficient data for a practical design.

Later, in his William Murray Lecture presented at the Annual meeting of the Society for Experimental Stress Analysis in New York, December 1953, Timoshenko [344] gave more mild reminiscences about this discussion. Swain [389] did not follow Timoshenko's suggestion about correction in the new edition of his book. Anyway, the problem of scaling stated by Swain is really important; see recent paper [390] for further discussion.

5.3.2 *Stress concentration around an elliptical opening*

Kolosov [206] (Section 5) [260,261], by using his method based upon the theory of complex variables solved the 2D problem of stress distribution produced in an in infinite plate with an elliptic opening caused by uniform uniaxial tension at infinity[16]. He presented explicit expressions and showed that the maximum stress is especially large if the major axis of the ellipse is perpendicular to the direction of tension in the plate. The maximum stress occurs at the opening boundary along this axis and increases with an increase of the major axis to the minor axis ratio of the ellipse. Although one of these studies has been published in German in one of the leading mathematical journals of that time, the results remained unnoticed by a wide circle of practical engineers. This solution has been reproduced in Kolosov [68]. Kolosov's solution in terms of complex variables has been simplified by Muskhelishvili [244,391,392] and later has been reproduced in [37–39], and here the solution occupies only two pages of large print. The same problem has been considered by Föppl [393] as an example of his very complicated method of conformal mapping. The solution occupied five large pages of small print that corresponds to about twenty pages of normal academic typesetting; Muskhelishvili [39] (p 344) has even admitted that he has not succeeded in understanding this method.

[16]In fact, this problem was considered by a similar approach in a draft note by Chaplygin [246] written around 1900. Chaplygin, however, did not publish these results during his life.

Independently of Kolosov, Inglis[17] addressed [395] the same problem in a talk read at the Spring Meeting of the 54th Session of the Institution of Naval Architects, March 14, 1913. In the first part of his talk, he presented a summary of the more important results and conclusions, with a lot of instructive figures, while in the second part the mathematical treatment of the problem in the elliptical coordinates is briefly outlined. This was one of the comparatively few attempts that have been made at that time to adapt the mathematical theory of elasticity to the practical problems encountered in naval architecture. Inglis mentioned that his paper is an endeavor to answer questions concerning the stresses around a crack stated in a lecture of Professor Hopkinson read before the Sheffield Society of Engineers and Metallurgists in January 1910.

Inglis [395] established that for an elliptical hole in a plate with the major OA and minor OB semi-axes being of length a and b, respectively, subjected to a tensile stress R at the direction perpendicular to the major semi-axis OA, if the material is nowhere strained beyond its elastic limit, a tensile stress occurs at the point A with the value $R(1+2a/b)$, and a compression stress at the point B of magnitude R. On exploring the plate along the major axis, the tensile stress rapidly decreases, and at a short distance attains approximately its average value R. Advancing along the minor axis the compression stress soon changes to a small tensile stress, and this gradually tends to zero.

If the major axis of the ellipse makes an angle α with the direction of the pull, the tensile stress at the ends of this axis is $R[a/b-(1+a/b)\cos 2\alpha]$. For such a case, however, the greatest tension does not occur exactly at these ends, and the value given may be considerably exceeded. The general expression for tangential tensile stress Q along the edge of the hole (with an angle θ from a positive direction of the major axis) reads as

$$Q=R\,\frac{1-m^2+2m\cos 2\alpha-2\cos 2(\theta-\alpha)}{1-2m\cos 2\theta+m^2} \tag{45}$$

with $m=(a-b)/(a+b)$.

Inglis [395] extended these results to a few cases important for shipbuilding, namely the case of a square hole with rounded corners, the case of a crack starting from the edge of a plate, and the case of a notch which is not necessarily elliptic in form. Viewing a crack as the limiting case of the

[17]Sir Charles Edward Inglis (1875–1952) was educated at King's College, Cambridge, where he gained a first-class mechanical science Tripos in 1898. In 1901, he joined the teaching staff of the Engineering Department of Cambridge University. During World War I, Inglis was able to make an immediate and valuable contribution to military engineering and had devised a light portable tubular bridge, which was accepted as standard equipment. For this work he received the OBE. On being demobilized, with the rank of Major, he returned to Cambridge University and in 1919 he was appointed Professor of Engineering—or Professor of Mechanical Sciences, as he was later known—a position in which he served until his retirement twenty-five years later. His great services to the cause of engineering education (Inglis was for a long time Head of the Department of Engineering at Cambridge University) were recognized by the Knighthood which he received in the Birthday Honours of 1945. Education at its best, Inglis said, should aim at something much deeper than the memorization of a number of facts and formulas and be more lasting. The good of education was the power of reasoning, and the habit of mind which remained when all efforts of memorization had faded into oblivon. A short account of Inglis' life with the references to some of his papers devoted to mathematics in relation to mechanical engineering and university training of engineers can be found in [394] and in obituary notices published in *Engineering*, 1952, **173**, 528, and *The Engineer*, 1952, **193**, 570 where a nice pencil portrait was supplied.

elliptical hole in which the minor axis is vanishingly small, $a \gg b$, he stated that the stress at the end of the crack of arbitrary form is proportional to the square root of the length of the crack, and inversely proportional to its radius of curvature. (This result was highly appreciated by Hopkinson during subsequent discussion.) In answering the questions in the discussion which followed, Inglis admitted that "concerning the direction in which a crack will spread, theory, I think, tells us little or nothing."

Inglis's solution in elliptical coordinates was obtained anew by Pöschl [396] and repeated with full details in [27,43,45,46]. Experimental measurements [375,397] based upon the photoelasticity method provided good agreement with Inglis's theoretical expressions for various types of openings and cracks.

5.3.3 *Halfplane and layer with a circular opening*

The solution for a semi-infinite plate with one circular hole subjected to the presence of traction either at the edge of the hole or at infinity was obtained by Jeffery [2] using bipolar coordinates. Gutman [398] applied this solution to calculate stress distribution around a tunnel. Mindlin [399,400] found a small mistake in the expression for stresses and provided the corrected solution. The same elasticity problem and the mathematically similar problem of a slow creeping flow of a viscous fluid over a halfplane with a circular rigid cylinder (either stationary or uniformly rotating) were independently considered in bipolar coordinates and thoroughly discussed in the dissertation by Krettner [401] and papers of his adviser Müller [402,403]. Apparently, due to the conditions of war, these studies which contained a lot of numerical data concerning distributions of stresses, velocity field and forces, and torque acting at the rigid cylinder and some other interesting results went almost unnoticed.

Howland [351] considered the more complicated problem of an elastic infinite layer bounded by two parallel edges $y = \pm b$ that contains a circular hole with $r = a$ midway between the edges and subjected to tensions at both ends at infinity. A solution of the problem was sought by the method of successive approximations that is analogous to the alternating process of Schwarz. The biharmonic stress function χ was sought as

$$\chi = \chi_0' + \chi_0 + \chi_1 + \chi_2 + \dots \tag{46}$$

where the terms of the series are each, separately, solutions of the biharmonic equation and have, in addition, the following properties: χ_0' gives the stresses at infinity and none on the edges $y = \pm b$; $\chi_0' + \chi_0$ satisfies the conditions on the rim $r = a$ of the hole and at infinity, but not on the edges, ie, it is the solution for an infinite plane; the term χ_1 cancels the stresses due to χ_0 on the edges $y = \pm b$, but introduces stresses on the rim of the hole; χ_2 cancels these, but again produces stresses on the edges, and so on.

If the series is truncated after χ_{2r} it will give a value of χ satisfying all the conditions exactly except those on the edges $y = \pm b$. If the residual tractions due to χ_{2r} are small enough, this value of χ is adequate for practical purposes. Similarly, if the series is truncated after χ_{2r+1} the resulting value of χ satisfies all the conditions except those at the rim of the hole $r = a$. If the additional tractions due to χ_{2r+1} are small enough, the solution is again sufficient in practice. Howland [351] established that if $\lambda < 0.5$, with $\lambda = a/b$, it is never necessary to proceed beyond χ_8, while if $\lambda < 0.25$ it is possible to stop at χ_2. Values $\lambda > 0.5$ would lead to very laborious computations[18].

Howland and Knight [404] modified this solution to find the stream function corresponding to the slow rotation of a rigid cylinder placed symmetrically between parallel unmoved walls in a very viscous flow. In comparison to an infinite plane the influence of walls produces a considerable increase in the torque couple to maintain the same angular velocity: for $\lambda = 0.5$ it increases by 25%.

5.4 Eccentric cylinders

A solution of the 2D biharmonic problem in the domain inside two eccentric cylinders with coincident axes traditionally attracts great interest in engineering. The question of slow motion of an incompressible viscous fluid between two uniformly rotating cylinders is a key question in the field of tribology when considering the hydrodynamic theory of fluid-film lubrication. The question of what happens in a thin layer between a journal and bearing has a longstanding history. Considerable input for this problem was made in the period 1883–1886 when independently Nikolai Pavlovich Petrov (1836–1920) [405] and Osborn Reynolds (1842–1912) [406] suggested the hydrodynamical theory of lubrication. Petrov was mainly interested in experimental verification of the hypothesis of application of the Navier-Stokes equations and especially the non-slip conditions at the rigid boundaries for such type of flow. Therefore, he used an assumption of coincidence of the axis of journal and bearing considering in fact an axisymmetric problem. He provided an engineering formula for the dependence of the friction force upon viscosity of fluid (and also the external friction) and angular velocity. Joukovskii [407] pointed out the necessity of an eccentricity between axes of the journal and bearing in order to get a supporting force. This problem was thoroughly considered by Reynolds [406] and was first presented in two unpublished talks before the 44th meeting of the British Association for the Advancement of Science, Montreal, Canada, on August 28 and September 2, 1884 with Stokes, Rayleigh, and W Thomson among the listeners; see [408] for full details. Reynolds established the main equilibrium equations for pressure and torque distribution along the circle and performed a huge approximate integration in terms of trigonometric expansion. He expressed the Fourier coefficients of the sine and cosine terms in the form of Taylor series of dimensionless eccentricity c up to c^{11}. (Later Petrov [409] extended these expressions up to the terms c^{29}, and he also pointed out some small mistakes in the previous ones.)

Arnold Sommerfeld (1868–1951) [137] mentioned this

[18] It is interesting to read now an acknowledgment in the article, "In making the calculations we have had the use of two calculating machines. One of these was obtained with a grant from the Government Grant Committee of the Royal Society, to whom our thanks are due. We also gratefully acknowledge the assistance of the Research Committee of University College, Southampton, who have made possible the hire of a second machine."

mammoth and quite unnecessary approximate integration and developed an accurate theory based upon the Stokes flow approximation. He established the biharmonic Eq. (9) for the stream function (V in his notation). He also pointed out the analogy with an elastic problem of bending of a clamped elastic eccentric circular plate (in fact, at the inner circle a constant angle of inclination should be given) and mentioned that the solution of this problem had not been obtained yet and it could lead to very complicated expressions. Based upon physical reasonings for a velocity field, in a thin layer between a journal and bearing, Sommerfeld neglected some terms in the governing biharmonic Eq. (9) (it is worth noting that such an approach of neglection of terms in the governing linear Stokes flow equation has recently been employed by Hills and Moffatt [410] for the much more complicated case of the 3D flow in a wedge) and obtained a simple closed form expression

$$V=\frac{a}{2}\rho^2\ln\rho+(b-2a)\frac{\rho^2}{4}+c\ln\rho+d \tag{47}$$

with ρ being the radial coordinate, and a, b, c, d some (later defined) functions of a circumferential angle ϕ. Based upon this approximate solution Sommerfeld [137] discussed some examples and defined all necessary mechanical quantities important in practical applications of fluid-film lubrication. Anthony GH Michell (1870–1959) [411] extended Sommerfeld's solution for two inclined planes to the case when one plane has a finite width. Michell got a patent on this practically important case that appeared to be very successful.

The complete solution of the slow journal bearing flow 2D biharmonic problem for arbitrary thickness of layer and radii of cylinders was constructed by Nikolai Egorovich Joukovskii (1847–1921) and Sergei Alexeevich Chaplygin (1869–1942) in their joint (a rather rare case for scientists of that time) paper [412][19]. Since then this benchmark paper has been reprinted 10 times—probably, a record for any scientific publication! (Mercalov [414] provided the detailed exposition of this article in a review paper for Russian technical encyclopedia[20].) In this paper which was based upon previous studies, Joukovskii [416] and Chaplygin [417], the authors made use of Neumann's bipolar coordinates, in which one family of coordinate lines gives two eccentric circles—the boundaries of the cylinders. After a rather ingenious transformation, the stream function W was obtained explicitly (here the uniform distribution of velocities at the boundaries is essential). The authors obtained the analytical expressions for the force and momentum of interaction between the journal and the bearing. In conclusion there was shown the derivation from these formulas the approximate Sommerfeld expression for a thin layer.

In spite of an extensive German summary published in *Jahrbuch über die Fortschritte der Mathematik*, this study went almost unnoticed in countries other than Russia (or USSR). For example, Müller [402,403] constructed anew the solution in the bipolar coordinates and added a lot of figures showing the distribution of streamlines. Independently, Wannier [418] and Ballal and Rivlin [419] solved the same problem. This journal bearing flow served as one of the first examples, Aref and Balchandar [420], Chaiken, Chevray, Tabor, and Tan [421], of the chaotic advection paradigm in Lagrangian turbulence.

A similar elastic problem about stress distribution in a region enclosed by eccentric cylinders (and a limiting case of a halfplane with a circular hole) was addressed in several studies, including Jeffery [2], Chaplygin and Arzhannikov [422], Gutman [398], Mindlin [399,400], Müller [320], and Ufliand [423]. All these authors used the bipolar coordinates and constructed the explicit solution. In particular, it was analytically proven that under uniform normal pressure applied at either outer or inner cylinder boundaries the maximum stresses will occur at the boundary of the inner cylinder at the thinnest part, if the eccentricity is not too high (otherwise, the maximum appears at the outer boundary).

5.5 Infinite wedge

Venske [180] was the first who considered the solution of the biharmonic problem in a sector wedge domain of angle $\alpha\pi$ defined in the polar coordinates (r,ϕ) for $0\leqslant r<\infty$, $-\frac{1}{2}\alpha\pi\leqslant\phi\leqslant\frac{1}{2}\alpha\pi$. Representing the biharmonic function in the form $v=U+r^2V$, with U and V being harmonic functions, and seeking the solution for U and V as

$$U=\int_0^\infty\{(a_\mu e^{\mu\phi}+a'_\mu e^{-\mu\phi})\cos(\mu\ln r)+(b_\mu e^{\mu\phi}+b'_\mu e^{-\mu\phi})\sin(\mu\ln r)\}d\mu \tag{48}$$

and a similar expression for V with unknown coefficients $c_\mu,\ldots,d'_\mu$, Venske, in fact, employed the Mellin transformation. He did not, however, present any further details concerning determination of the unknown coefficients. Venske only wrote a final explicit expression for the "second" Green's function $v(r,\phi,r_0,\phi_0)$ in such a domain with *integer* $\alpha=n=1,2$ (a half-plane or a plane cut along a semi-infinite straight line) when a concentrated force is applied at some inner point.

Maurice Lévy (1838–1910) [424] considered the problem of elastic stress distribution in a wedge $0\leqslant r<\infty$, $0\leqslant\theta\leqslant\beta$ loaded by uniform or linear normal forces at the side $\theta=0$ (or y-axis in the Cartesian coordinates). The representation for the (not mentioned explicitly) biharmonic Airy function was chosen in the form of polynomials of the second or third degrees in x and y. The expressions for components of the stress tensor in rectangular coordinates look rather simple, and Lévy suggested to use this solution in the analysis of stresses in masonry dams. (Galerkin [425] used such a solu-

[19] Amazingly, this article was first published in 1904 as a separate issue for the XIIth volume of the journal *Trudy Otdeleniya Fizicheskikh Nauk Imperatorskogo Obshchestva Lyubitelei Estestvoznaniya, Antropologii i Etnographii* (see a photocopy of the title page in [413], p 45), and it was really published in the XIIIth volume of that journal which appeared only in 1906.

[20] Krylov [415] pointed out: "Many of Joukovskii's works had a practical importance; if he, like Lord Kelvin, had developed them up to practical applications and had taken out patents, he would also have had his own yacht, villas, and castles. It's enough to mention his theory of lubrication—it contains all of Michell's journal bearing theory, which had brought millions to Michell. Joukovskii never patented anything and he provided all his discoveries for common usage, seeing science not as a mean of personal enrichment, but of increasing the knowledge of mankind."

tion for a more general case of a truncated wedge when studying the problem of stresses in dams and retaining walls with trapezoidal profiles.) Later, these solutions were obtained independently by Fillunger [426]. This author noticed, however, that for the case of uniform loading the stress components contain in the denominator the term $\tan\beta-\beta$ which may turn into zero for some value $\beta_0>\pi$.

Michell [179] considered two particular cases of compression and flexure of a wedge $0\leqslant r<\infty$, $-\alpha\leqslant\theta\leqslant\alpha$ by concentrated forces P or Q applied at the apex of the wedge in the direction of the axis $\theta=0$ and in a perpendicular direction, respectively. By usage of his general solution [176] of the biharmonic Eq. (7) in polar coordinates, Michell chose the particular expressions for the stress functions

$$\chi=Ar\theta\sin\theta, \quad \text{or} \quad \chi=Br\theta\cos\theta \tag{49}$$

for the two cases, respectively. For these solutions, only the radial stresses σ_r are nonzero, and they increase indefinitely as r^{-1} when $r\to 0$. The constants A and B were defined from the conditions of equilibrium

$$\int_{-\alpha}^{\alpha}\sigma_r\cos\theta r\,\mathrm{d}\theta=P, \quad \text{or} \quad \int_{-\alpha}^{\alpha}\sigma_r\sin\theta r\,\mathrm{d}\theta=Q \tag{50}$$

for any finite portion $0\leqslant r\leqslant a$, $-\alpha\leqslant\theta\leqslant\alpha$ of the wedge. For the particular case $\alpha=\frac{1}{2}\pi$, the normal force P provides the solution for a halfplane already obtained by Flamant [427,428]. It is worth noting that same solutions were independently obtained by Mesnager [204], and since then these solutions have been traditionally included in many textbooks on the theory of elasticity, see, *eg*, Love [16–18], Timoshenko [43,91], Papkovich [80], and Lur'e [35], to name only a few.

Action of concentrated forces at some points of the sides of the wedge were considered in detail by Wieghardt [429] in a far less known paper. This paper was published in German in a journal which later ceased publication and, therefore, it was forgotten and did not exert any long-living impact in the theory of elasticity and fracture mechanics. (Its recent English translation of 1995 deserves, in our opinion, special attention.) Wieghardt [429] used the so-called *Sommerfeld transformation* in order to use the Flamant [427] solution. Several expressions are presented to solve the plane stress problem in elastic wedge shaped bodies under concentrated forces applied at its sides. Wieghardt also considered acute angles of the wedge in order to apply his theory to Bach's problem of roller bearing case fracture, for which he derived the first mixed-mode fracture criterion. He described the structure of the stress field for any wedge-type notch, including the crack as a special case of a plane with a semi-infinite straight cut. The solutions presented are associated with the splitting and cracking of elastic bodies. Wieghardt [429] correctly stated that "knowledge of the theoretical stress distribution does not allow one to evaluate crack initiation upon exceeding of the loading with certainty; and it is not at all possible to determine the path of further cracking." Finally, the differences between the developments presented in this study and the partially incorrect approaches by Venske [180] regarding wedge domains are emphasized.

Carothers [430] considered some typical cases of wedge loading bearing in mind the discussion of the technically important problem of determining the stress in a masonry dam (see, [18], Section 151 or [91], Section 36, for additional references). Among others he briefly mentioned an elementary solution for an infinite wedge loaded by a concentrated couple M at the apex with the stress function χ

$$\chi=-\frac{M}{2}\,\frac{\sin 2\theta-2\theta\cos 2\alpha}{\sin 2\alpha-2\alpha\cos 2\alpha} \tag{51}$$

and stresses

$$\sigma_r=\frac{2M}{r^2}\,\frac{\sin 2\theta}{\sin 2\alpha-2\alpha\cos 2\alpha}, \quad \sigma_\theta=0$$

$$\tau_{r\theta}=\frac{M}{r^2}\,\frac{\cos 2\theta-\cos 2\alpha}{\sin 2\alpha-2\alpha\cos 2\alpha} \tag{52}$$

These stresses satisfy the conditions of equilibrium

$$\int_{-\alpha}^{\alpha}(\sigma_r\cos\theta-\tau_{r\theta}\sin\theta)\,r\,\mathrm{d}\theta=0$$

$$\int_{-\alpha}^{\alpha}(\sigma_r\sin\theta+\tau_{r\theta}\cos\theta)\,r\,\mathrm{d}\theta=0$$

$$\int_{-\alpha}^{\alpha}\tau_{r\theta}r^2\mathrm{d}\theta=M \tag{53}$$

for any finite portion $0\leqslant r\leqslant a$, $-\alpha\leqslant\theta\leqslant\alpha$ of the wedge.

However, Fillunger [431] in a completely forgotten paper constructed exactly the same solution! He observed that for the specific acute angle of the wedge 2α (or 2Φ in his notation) equal to $2\Phi=257°27'13''$, the root of the equation $2\Phi=\tan 2\Phi$, the denominator in (51) becomes zero and the solution in the form (52) does not exist. Having pointed out this, Fillunger did not discuss in detail this paradoxical result.

Later, all these elementary solutions for concentrated forces and couples at the appex were repeated by Inglis [432] who wrote that "the object of this paper is to popularize certain stress distributions which, in the opinion of the author, ought to be better known than they are at present." Remarkably, neither Carothers [430] nor Inglis [432] (and later Miura [433], Coker and Filon [27], Bay [434], and Papkovich [80] who also described in detail these solutions) have noticed the specific acute wedge angle 2Φ when formally the solution for the concentrated couple does not exist.

Sternberg and Koiter [435] called attention to the paradoxical result in the Carothers solution for a specific angle $2\alpha^*$. (They did not mention [431] at all.) Their remarkable paper on the so-called "Carothers paradox" (the name "Fillunger paradox" seems far more appropriate) started an interesting discussion concerning the physical meaning of a concentrated couple applied at the apex of a wedge. This couple can be considered either (Sternberg and Koiter [435], Barenblatt [436], Harrington and Ting [437]) as a limit as $r_0\to 0$ of loading by antisymmetric normal forces $p(r)$ on a small lateral part $0\leqslant r\leqslant r_0$ provided that

$$\int_0^{r_0} p(r)\,\mathrm{d}r=0, \quad 2\int_0^{r_0} p(r)r\,\mathrm{d}r=M \tag{54}$$

or (Neuber [438]) as a limit with $r_0 \to 0$ of a truncated wedge $r_0 \leqslant r < \infty$, $-\alpha \leqslant \theta \leqslant \alpha$ with free lateral sides and applied tangential force (or displacement) producing the couple M. In the first case, it appears that the solution for the wedge angle $2\alpha < 2\alpha^*$ tends with $r_0 \to 0$ to the Carothers-Inglis solution (51), while for the wedge angle $2\alpha \geqslant 2\alpha^*$ the solution crucially depends upon the distribution of $p(r)$ even when $r_0 \to 0$ and does not turn into the elementary solution. Here we have an example of the self-similar solution of the ''second'' kind according to Barenblatt [436]. Later, this paradox has been addressed anew [439–445].

Apparently Brahtz [446–448] being engaged in the mathematical analysys of stresses in the Grand Coulee Dam was the first author who considered by means of the Mellin transform the general case of loading of a wedge at its sides. He presented the explicit solution for the stress function and provided concrete results for the wedge with the angle $3\pi/2$ loaded by a concentrated force. He used the residues method to calculate the integrals in the Mellin transforms. Independently, Shepherd [449], Abramov [450], and Figurnov [451] considered by the same approach the general case of loading of a wedge at its sides. The latter author presented the explicit solution for the stress function, but he did not provide any concrete results. All these papers went almost unnoticed by the successors, besides [452,453], see also [40,454]. (While the third author published his results in a conference proceedings, in Russian, this situation seems more strange for the first two authors who published their studies in well-established American and British journals.) Only Papkovich [80] repeated briefly the main lines of the Abramov's solution and suggested as problems (!) in his course to consider ten typical loads, mentioning a possible connection with a practical problem of breaking of ice by an ice-breaker. Lur'e and Brachkovskii [455][21] developed a similar solution and considered the case of a concentrated normal force applied at one side; they applied the residue method to calculate the integrals. The same approach based upon the Mellin transform has been developed independently by Sakharov [456] and Tranter [457]; later it was repeated with full detail in the books by Sneddon [458], Ufliand [94], Lur'e [35], and Tranter [459]. The detailed experimental study of a concentrated force acting at the apex of the wedge was performed in [460,461]; see also Coker and Filon [27] for further details.

Woinowsky-Krieger [462,463] constructed an analytical solution in the form of integrals for a clamped plate $0 \leqslant r < \infty$, $0 \leqslant \theta \leqslant 2\alpha$ loaded by a concentrated force in some inner point lying on the diagonal $\theta = \alpha$. He provided results for bending moments and shearing forces (reactions) along clamped sides for typical values of $\alpha = \frac{1}{4}\pi, \frac{1}{2}\pi, \pi$. He presented a figure showing that the reaction (which is proportional to the third derivative of the deflection) for a quarter plane ($\alpha = \frac{1}{4}\pi$) has oscillatory behavior near the apex, but he did not discuss this result at all.

The creeping steady flow of a very viscous fluid in a wedge domain bounded by the walls $\theta = 0$, $\theta = \alpha$ with a uniform velocity V sliding motion of the wall $\theta = 0$, was considered independently by Goodier [145,147] and Taylor [464,465]. The solution for the stream function ψ reads, [466]

$$\psi = Urf(\theta) \tag{55}$$

with

$$f(\theta) = \frac{(\alpha - \sin\alpha\cos\alpha)\theta\sin\theta + \sin^2\alpha\,\theta\cos\theta - \alpha^2\sin\theta}{\alpha^2 - \sin^2\alpha}$$

Taylor [465] (p 314), in particular, noticed:

> The palette knives used by artists for removing paint from their palettes are very flexible scrapers. They can therefore only be used when α is nearly 180°. In fact artists instinctively hold their palette knives in this position.

Taylor also pointed out the fact of a logarithmic singularity of the shear stress along the wall needed to support the prescribed uniform finite velocity of the wall; this prediction is clearly unrealistic. Presumably, one of the asumptions of the creeping flow breaks down near the vicinity of a sharp andgle. More general cases of the nonuniform tangential velocity applied at side walls were considered by Moffatt [467], Jeffrey and Sherwood [468], and Krasnopolskaya [469].

There is another interesting aspect of the biharmonic problem in a wedge domain which deserves special attention from both mathematical and engineering points of view. It concerns the nature of the homogeneous biharmonic function in a wedge domain, say, a deflection w around a plate corner having two clamped edges. Ritz [470] made the remark that it may not be possible to develop a solution of the governing biharmonic equation into a Taylor series at the corner point. Rayleigh [471] argued that all partial derivatives of the plate deflection must vanish at the corner point. He erroneously concluded that the deflection at a distance r from the corner diminishes more rapidly than any power of r. In spite of a short note by Nádai [472], who pointed out the possibility to investigate the question by contructing a solution for a sickle form clamped plate in bipolar coordinates, (such a solution has been constructed by Woinowsky-Krieger [473]) only Dean and Montagnon [474] pointed out the possibility that the biharmonic function may vary as a fractional power of r had appeared to be overlooked, and in such a case partial derivatives beyond a certain order will be infinite at $r = 0$. These authors discovered that in an infinite wedge domain filled with a very viscous fluid with fixed walls $\theta = 0$ and $\theta = \alpha$, there can exist a non-zero stream function $\psi(r,\theta) = r^{n+1} f_n(\theta)$, with the values of n satisfying the equation

$$\sin n\alpha = \pm n \sin\alpha \tag{56}$$

where the plus or minus sign corresponds to two different types of symmetry with respect to the diagonal line $\theta = \frac{1}{2}\alpha$.

[21] In fact, this paper had been published in 1946 because of World War II, but the second author was killed in 1941 in the battle for Leningrad.

The authors found that for values of α less than 146.3° the values of n satisfying Eq. (56) must be complex, but they did not discuss the structure of the velocity field near the origin. Moffatt [475] discovered that these complex roots lead to an infinite eddy structure (later named ''Moffatt eddies'') near the apex of the wedge. These eddies were visualized by Taneda [476]. Subsequent developments which were summarized in [468,477,478] provide an understanding of the complicated structure of the streamline patterns for various infinite domains with corners.

However, it should be pointed out that exactly the same results concerning the eigenfunctions in an infinite elastic wedge $0 \leqslant r < \infty$, $\alpha \leqslant \theta \leqslant \alpha$, with its sides $\theta = \pm\alpha$ free of stresses were first obtained by Brahtz [448]. He used the expression for a stress function $F(r,\theta) = r^{\beta+1}\psi_\beta(\theta)$, where the ''corner function'' $\psi_\beta(\theta)$ satisfies the differential equation

$$\psi_\beta'''' + [(\beta+1)^2 + (\beta-1)^2]\psi_\beta'' + (\beta^2-1)^2\psi_\beta = 0 \qquad (57)$$

and boundary conditions $\psi_\beta(\pm\alpha) = \psi_\beta'(\pm\alpha) = 0$. For the eigenvalue β one obtains two equations (56) with changes n to β and α to 2α. It was proven that if $\alpha \neq 2/\pi$ and $\alpha \neq \pi$ the roots β of that equation are complex; approximate expressions for the roots were given. Brahtz [448] used these corner functions to calculate the stress distribution in the Grand Coulee Dam.

The same eigenfunctions were also obtained by Tölke [479] who provided extensive tables with the complex eigenvalues. Considering the problem for a finite wedge he used the method of least squares to define the coefficients of eigenfunctions expansion. Apparently, due to the political situation of that time, this remarkable study, in spite of its detailed German review, has been completely overlooked. Sobrero [480] also obtained the same Eq. (56) considering the problem of the elastic stress distribution in a wedge with angle α. He stated, however, that if $270° \leqslant \alpha < 360°$, the stresses vary as $r^{-0.5}$ without a sensible error (the author claimed that these theoretical results were fully confirmed by photoelastic experiments). Being published in a rather unknown journal this paper also went completely unnoticed. And only Williams was lucky enough, for his short talk [481] has been recognized and widely cited.

5.6 Rectangle

The overriding importance of a clamped rectangular elastic plate and a very long elastic rectangular prism or thin sheet subjected to surface normal loadings only at their sides as crucial elements in structural mechanics and shipbuilding has given rise to a large number of works where the question was treated by different approaches and, in fact, these problems are connected with some important findings in mathematics as well as in engineering. The history of the biharmonic problem for the bending of a clamped rectangular plate and for the stretching of an elastic rectangle is rather fascinating. Love [6] addressed this problem as ''one of the classical problems in the Theory of Elasticity.''

For a thin elastic plate the normal deflection w satisfies the nonhomogeneous biharmonic Eq. (3). In various engineering structures (bulkheads of a ship, for example) the edges of the plate are firmly clamped, or attached to angle irons which allow no side motions. The deflection w must vanish at the edge; and, in addition, the tangent plane at every point of the edge must remain fixed when the plate is bent. As a matter of practice it is extremely difficult to clamp a plate efficiently. There is nearly always a small inclination at the edges of the tangent plate to the original xy-plane. In careful experiments [9,10,482] this may be of the order of magnitude $1'$. Moreover, the attachment structure at the edges may be stiff, but cannot be completely rigid. But for the theoretical reasoning it is typical to disregard this and think of the plate as perfectly clamped.

On the other hand, in the theory of elasticity the determination of stresses in an infinite rectangular prism with the surface loads being the same along the generating line of the prism (the state of plane strain) or thin sheet or plate under thrust in its own plane (the state of plane stress) reduces to the solution of the 2D biharmonic Eq. (7) for the Airy stress function. The boundary conditions corresponding to the system of self-equilibrating normal and shear forces applied at the rectangular boundary can also be written in terms of the prescribed values of the stress function χ and its normal derivative at the contour. Discontinuous and concentrated forces are also admissible, and the problem of a rectangular beam supported at two places and bent by a weight W applied between them is the benchmark one [6,132].

An important consideration in the formulation of the boundary conditions consists of the satisfaction or violation of the conditions of symmetry of the shear stresses at the corner points. In the framework of continuum mechanics the boundary is considered to be a surface that is different from the rest of body, and, therefore, it is possible to prescribe any values of forces on it. Some misunderstanding of this circumstance may lead to both the paradoxial conclusion of Winslow [483] that ''stress solutions satisfying all boundary conditions will be in general impossible'' and to additional relations [484] between stresses at the corner points.

The 2D boundary problem for a rectangle represents the particular case of the famous Lamé problem of the equilibrium of an elastic parallelepiped under any system of normal forces on its sides. Lamé [129] considered the 3D problem to be as complicated as the famous problem of three-bodies in celestial mechanics: ''C'est une sorte d'énigme aussi digne d'exercer la sagacité des analystes que le fameux problème des trois corps de la Mécanique céleste.'' (It seems now that he has underestimated the difficulty of the second one.) Apparently under Lamé's influence the competition for the *Grand Prix de Mathématiques* of the French Academy of Science of Paris for a solution of this problem was announced in 1846 for the year 1848. According to the announcement published in *Comptes rendus des séances de l'Académie des Sciences* 1846 **22** 768–769, the condition for the award was:

> *Trouver les intégrales des équations de l'équilibre intérieur d'un corps solide élastique et homogène dont toutes les dimensions sont finies, par exemple d'un parallélépipède ou d'un cylin-*

> *dre droit, en supposant connues les pressions ou tractions inégales exercées aux différents points de sa surface*.
>
> Le prix consistera en une médaille d'or de la valeur de *troix mille francs*.
>
> Le Comission chargée de proposer le sujet du prix était composée de MM Arago, Cauchy, Lamé, Sturm, Liouville rapporteur.

There were no entries and this topic was suggested (along with the last Fermat theorem!) then two times for the years 1853 and 1857, and had been initially prolonged for the year 1861, but already in 1858 it was changed into another question, see *Comptes rendus des séances de l'Académie des Sciences*, 1858, **46**, 301. The only entrant for this long competition was a memoir *"De l'Equilibre intérieur d'un corps solide, élastique, et homogène"* marked with motto "Obvia conspicimus, nubem pellente Mathesi" submitted by William John Macquorn Rankine (1820–1872) for the year 1853 (its main results were published in [485,486]) but it did not receive an award; see Todhunter and Pearson [22] (Section 454).

An excellent example of an engineering approach to the problem of bending of a narrow rectangle resting on two supports under a concentrated force applied at the middle of the upper side was given by Stokes in 1891. He took so much interest in Carus Wilson's [487] photoelastic experimental result of two dark spots existing in the glass beam at which there is no double reflection (indicating the so-called neutral points or, equivalently, places of equal normal stresses) that he developed an approximate theory published as a letter supplementary to [487] to account for it. By means of this theory Stokes provided a formula and found the correct positions of the neutral points which agreed completely with Wilson's observations.

The clamped rectangular plate was not only an important test problem for any new method, but, in many cases, new engineering methods were invented to solve exactly that problem. For example, Nielsen [488], Marcus [489], Bortsch [490], Bay [491–495], Varvak [496], Conway, Chow and Morgan [497], and Beyer [498] applied the finite-difference method specially for the problems of the clamped rectangular plate and the finite elastic rectangle. (In fact, Richardson [499] was the first author who developed the finite-difference method to solve the biharmonic problem in domains consisting of several rectangles with application to a masonry dam.) Pan and Acrivos [500] applied this method to the steady Stokes flow in a rectangular cavity. Similarly, the paper by Biezeno and Koch [501] contains an approach when the clamped rectangular plate is divided into parts, with the corresponding approximation of the surface loading. The relaxation method developed by Southwell [502,503] for various problems of the theory of elasticity was also applied in [504,505] to study the biharmonic problem for a rectangle.

Below we describe several major approaches to solve the biharmonic problem in a rectangle and to obtain reliable results concerning important mechanical characteristics of structural elements. An excellent survey of several approximate methods for the solution of rectangular plate bending problems is given by Leissa, Clausen, Hulbert, and Hopper [506].

5.6.1 Grashof's empirical formula for a uniformly loaded plate

Apparently Franz Grashof (1826–1893) [507] (Section 234) was the first author who obtained an approximate solution for a practically important case of a rectangular plate $|x| \leqslant a$, $|y| \leqslant b$ bent by a uniform normal pressure p_0 applied to its surface. He considered the plate as a collection of elementary clamped beams parallel to both axes; at any given point the intersecting beams must deflect the same amount. By using the elementary solution for a clamped beam Grashof suggested an approximate solution

$$w_G(x,y) = \frac{p_0}{2Eh^3}\,\frac{(a^2-x^2)^2(b^2-y^2)^2}{f(a,b)} \tag{58}$$

with some function $f(a,b)$ yet to be determined. He choose the expression $f(a,b)=(a^n+b^n)^{4/n}$ with integer n that provided the correct asymptotic behavior of deflection while $a \to \infty$ or $b \to \infty$. By comparison with the solution for a circular plate, and not very rigorous reasons, he suggested taking the value $n=4$. Due to the assumption of the clamped beam analogy the expression w is independent of ν. (Love [18] in Section 314 gives, however, an expression which differs from (58) by factor $1-\nu^2$ in the nominator.) This empirical, or rather hypothetical, solution satisfies the boundary conditions exactly, but does not satisfy Eq. (3). Grashof's solution was constantly addressed to in the old textbooks on applied mechanics and the theory of elasticity, [19,508–515].

Formula (58) gives relatively good results, considering its empirical nature. According to the experimental data [9,10] Grashof's rule for the deflection of a rectangular plate with sides 4 inches and 2 inches gave a deflection at the center of 0.0437 inch, which approximates reasonably closely to that found, viz, 0.0410 inch. This rule for the stress at the ends of a short diameter of the plate gave, however, a stress of 50% in excess of that found from the experiments of Laws and Allen [516]. The error in center deflection $w_{\max}$ for a square plate was about 13%, while the error in the maximum edge moment was about 23%. The errors are less for rectangular plates; see [517] for further details.

The expressions for the maximum deflection (which occurs at the center of the plate) and stress (which occurs at the middle of a long side) were often used by practical naval architects Read [518], Yates [8], and Elgar [519] at the end of nineteenth century. Bryan [328], probably, was the first who stressed the necessity to use more accurate mathematics when solving a specific engineering problem. He mentioned briefly that the case of a rectangular plate with clamped edge seems to be unsolvable (except with the help of elliptical functions, which are quite complicated for all practical purposes), but he did not enter into any further explanations.

Already in the beginning of the 20th century Grashof's formula appeared in doubt in comparison with experiments of Bach [520]. Based upon these and his own experiments, a German shipbuilder Felix Pietzker suggested [521] a new formula for maximum deflection and stresses with some empirical coefficients given by figures and tables. Pietzker's short book was rather popular among naval architects, for it had a second edition as well as Russian [522] and English [523] translations. Pietzker pointed out that in the case of rectangles with the side ratio a/b below 0.33, the maximum stress remains almost the same as though the short sides do not exist. (A review paper by Lamble and Shing [12] contains figures showing comparative data of various approaches for these coefficients depending on the ratio a/b.) Pietzker [523] (p 42) also made a characteristic remark that "the stress in the plate corners is uncertain for the time being."

5.6.2 *Polynomial and Fourier series solutions*

In structural mechanics there is a constant interest in analyzing the stress and displacements in a (long) rectangular strip $0 \leqslant x \leqslant l$, $|y| \leqslant c$ in order to compare results with an elementary beam theory.

Mesnager [203] suggested using the biharmonic polynomials of various integer orders. (Later Zweiling [214] gave an extensive listing of biharmonic polynomials.) These expressions satisfy identically the homogeneous biharmonic Eq. (7) and have some arbitrary constants. By choosing these constants appropriately, it appeared possible to satisfy exactly some simple boundary conditions over long sides $y = \pm c$, and by means of the Saint-Venant principle to satisfy integrally the boundary conditions over the short sides. A special attention has been paid to the benchmark problem of the so-called "simply supported" finite strip loaded either by uniform normal pressure over its top side $y = c$ or by its own weight. (Already Airy [166–168] had considered this problem but his polynomials were not biharmonic ones.) The solution in terms of the biharmonic polynomials of the fifth order was given [132,181,219]; it has been since then repeated in the beginning of the twentieth century in the textbooks on the theory of elasticity by Föppl [127], Timoshenko [91], Föppl and Föppl [28]. It appeared that an additional term in the stress X_x connected with the two-dimensionality of the problem is small in comparison with the main term according to the elementary solution of the strength of materials provided that $2c/l$ is much less than one. A similar conclusion is true for the displacement of the center line $v(x,0)$ that corresponds to a deflection of the beam in the elementary theory.

A shortcoming of the polynomial solutions consists in the impossibility to consider some practically important loadings (eg, concentrated forces), see Belzeckii [524]. This shortcoming can be partially overcome by considering the solutions for the stress function in the form of a Fourier series on some complete systems of trigonometric functions. Already Ribière [525] in his dissertation used Fourier series represenations for the stress function on the complete system $\cos n\pi x/l$. In this way, it appeared possible to exactly satisfy the boundary conditions over the sides $y = \pm c$. However, it was impossible to satisfy fully the conditions over the two short sides, $x = 0$ and $x = l$. Here one inevitably has $u = 0$, $v \neq 0$, $X_x \neq 0$, $Y_x = 0$, and mechanically it corresponds to an infinite periodically loaded strip with simple supports. If the ratio of the rectangle's sides is large, it was believed (according to the Saint-Venant principle) that, at a long distance from the short ends, the effect of any self-equilibrated system of loads may be neglected, and the boundary conditions are fulfilled only for total tension, total shear, and total bending moment.

Filon [132][22], and independently Belzeckii [527], used a similar approach, but with the complete system $\sin n\pi x/l$. in the Fourier series of the stress function. Here one has $u \neq 0$, $v = 0$, $X_x = 0$, $Y_x \neq 0$ that corresponds to the conditions of "free support." Here, also, the accuracy of satisfaction of the boundary conditions at the short ends was not checked. By means of this solution these authors considered a few interesting problems for beams lying on two supports. In particular, Filon [132] solved the problem of compressing of a finite elastic rectangle by two normal forces symmetrically placed at points $(0,c)$ and $(0,-c)$. For a sufficiently long rectangle the normal stress $Y_y(x,0)$ equals to zero at $|x| = 1.35c$ (independently on the ratio l/c for $l > 4c$), and the pressure is replaced by a tension. This result permits one to understand a simple experiment when an elastic block, acted upon by a concentrated load on its upper surface, cannot lie having full contact with a smooth rigid plane, and at a certain distance away from the force the ends lift off the plane. An accurate analysis of this remarkable phenomenon has to rely upon the solution of a complicated mixed problem with an unknown boundary, but a rough estimate of the dimensions of the area in contact can be made considering the area where the normal stresses Y_y are positive.

Papkovich [80] presented a complete comparative analysis of the Ribière and Filon–Belzeckii solutions for several most common loadings of a rectangular plate. This served as a basis for a detailed study of some practical cases of bending of box-shaped rectangular empty beams which are widely used in shipbuilding.

By combining a solution in the form of a Fourier series for a halfplane Bleich [528] considered an interesting case when normal concentrated forces are applied at the centers of the short sides of a rectangle; see also [28], Section 56, and [43], Section 20. This solution can be used for a quantitative estimate of the Saint-Venant principle: even for this extreme case the distribution of the stress X_x over the cross section is almost uniform for the distance c from the short ends; see Meleshko [529] for further details.

[22] Louis Napoleon George Filon (1875–1937) was the son of Augustin Filon, the French littérateur who was tutor to the Prince Imperial. He began Latin and Greek before he was six. Filon's ambition was to be a sailor. He was always drawing pictures of boats at sea and some good models of ships he made at this time are still in existence. In later life, this old ambition showed itself in his keen interest in the theory of navigation and in his one form of relaxation, yachting. Filon graduated from University College, London, and he took his BA degree in 1896 with a gold medal for Greek. He was a student of Karl Pearson and Micaiah JM Hill, two teachers for whom he had an affection and reverance. In 1898, Filon was elected to an 1851 Studentship and went to King's College, Cambridge. Here he published his benchmark studies on the theory of elasticity in which he developed the theory of "generalized plane stress." In 1910, he was elected to the Fellowship of the Royal Society, of which he later became Vice-President. After World War I, Filon served as Vice-Chancellor of the University of London; he was a Vice-President of the London Mathematical Society for the two years 1923–1925. Towards the end of 1937, Filon fell a victim to the typhoid epidemic in Croydon, and he died on December 29. A more detailed biography of Filon with a complete list of his scientific works can be found in [526].

5.6.3 Ritz method

Swiss born physicist Walter Ritz (1887–1909) lived a short but brilliant life in scientific aspects. He studied in Göttingen under David Hilbert, did his PhD thesis in Leiden under Hendrik Lorenz and had one joint paper with Albert Einstein on the theory of relativity[23], see [312,314] for a detailed account of the life of this extraordinary scientist. Conscious of his imminent death from consumption, Ritz [530] published a short account of his study presented for the *Prix Vaillant* competition. This work had been submitted in May 1908 to the Göttingen Academy of Sciences by well-known German applied mathematician Carl Runge (1856–1927). Already in September 1908 the extended version of this benchmark study (the ''habilitationsschrift'' dissertation) was published [470] in the first issue of the 108th volume of the famous Crelle *Journal für die reine und angewandte Mathematik*. (The whole volume was dated by the year 1909, and this sometimes leads to the incorrect dating of Ritz's memoir.) This paper still deserves attentive study for its richness with ideas and unsurpassable clearness of presentation. Ritz proved that the problem of integrating Eq. (3) with boundary conditions (4) can be reduced to the following *variational* problem: from the set of functions satisfying the boundary conditions (4) it is required to find that one which gives the *minimum* value of the potential energy W of the deformed plate,

$$W = D\int_S \left[\frac{1}{2}(\Delta w)^2 - fw\right] dx\, dy, \quad \text{with} \quad f = \frac{p}{D} \tag{59}$$

Among several choices of the trial functions for a rectangular plate $0 \leqslant x \leqslant a$, $0 \leqslant y \leqslant b$ Ritz [470] in Section 11 employed an expression

$$w_{MN} = \sum_{m=1}^{M}\sum_{n=1}^{N} a_{mn}\xi_m(x)\eta_n(y) \tag{60}$$

Here $\xi_m(x)$ and $\eta_n(y)$ are the eigenfunctions of transverse vibration of the elastic beams $0 \leqslant x \leqslant a$ and $0 \leqslant y \leqslant b$, respectively, satisfying differential equations

$$\frac{d^4\xi_m}{dx^4} = \frac{\kappa_m^4}{a^4}\xi_m, \quad \frac{d^4\eta_n}{dy^4} = \frac{\kappa_n^4}{b^4}\eta_m \tag{61}$$

with zero boundary conditions on the functions and their first derivatives at the ends of their intervals. The values of κ_m and κ_n are the roots of the equation $\cos\kappa\cosh\kappa = 1$. In Section 9 of his paper, Ritz provided the explicit expressions for these well-known functions which are too long to be reproduced here; see [531] (Section 172). Ritz pointed out that sometimes the functions from the exact solution of one problem may be used in the approximate solution of another; the functions giving the deflection of a clamped beam were in fact used in the form of products to represent the approximate deflection of a clamped rectangular plate.

It should be noted that series (60), when differentiated term by term, does not satisfy the differential Eq. (3) for w, but the proof was given that it must tend, as the numbers M and N increase indefinitely, to an expression which does satisfy this equation. (Another version of the proof was given in a short note [532].)

The minimizing procedure for the functional

$$J_{MN} = \int_0^b\int_0^a \left[\frac{1}{2}(\Delta w_{MN})^2 - fw_{MN}\right] dx\, dy \tag{62}$$

that is, $\partial J_{MN}/\partial a_{mn} = 0$, leads to the system of linear algebraic equations

$$\int_0^b\int_0^a [(\Delta w_{MN})\Delta(\xi_m\eta_n) - f\xi_m\eta_n]dx\, dy = 0,$$

$$(m = 1, \ldots, M, \quad n = 1, \ldots, N) \tag{63}$$

Integrating twice by parts and taking into account the zero boundary conditions for functions ξ_m, η_n and their first derivatives, Ritz arrived at the following system of $M\times N$ linear algebraic equation for the coefficients a_{mn}

$$\int_0^b\int_0^a (\Delta\Delta w_{MN} - f)\xi_m\eta_n dx\, dy = 0,$$

$$(m = 1, \ldots, M, \quad n = 1, \ldots, N) \tag{64}$$

Dealing with this (and *not* with the original one (63) as it usually assumed) system Ritz performed extensive calculations for a square plate ($b = a$) under uniform load p_0 and presented three approximations (with $M = N = 1$, $M = N = 3$, and $M = N = 5$; only odd values of m and n were involved) for the deflection. The first approximation reads as $w = 0.6620 l\xi_1(x)\eta_1(y)$, where $l = 8\cdot 10^{-4}p_0a^4/D$. It is important that the additional terms with coefficients $a_{13}, a_{31}, \ldots, a_{55}$, in the second and the third approximations for deflection were found to be 1/20 of the first coefficient a_{11} or less, and that the main coefficient was only slightly changed. (Later these simulations were completely reproduced in [533–535].)

Ritz's contributions to the most difficult problems of equilibrium of a clamped rectangular plate and steady vibrations of a rectangular plate with free edges were greatly appreciated by famous mathematician Jules Henri Poincaré (1854–1912) and physicist John William Strutt (Lord Rayleigh) (1842–1919). Poincaré wrote a special letter [536] (p XVI) in the foreword of Ritz's *Oeuvres* volume, where he emphasized the superiority of Ritz's ''une méthode d'ingénieur'' over the purely mathematical Fredholm integral equations approach when concrete numerical results are needed. Poincaré also presented this volume to the French Academy on behalf of the Swiss physical society (see *Comptes rendus des séances de l'Académie des Sciences*, 1911, **153**, 924).

Rayleigh [537] also called attention to the remarkable study of Ritz [470] and noted that ''the early death of the talented author must be accounted a severe loss to Mathematical Physics.'' At the same time Rayleigh remarked

> But I am surprised that Ritz should have regarded the method itself as new. An integral involving an unknown arbitrary function is to be made a minimum. The unknown function can be represented by a series of known functions with

[23]It was, in fact, joint expressions of contradictory views on the subject.

arbitrary coefficients—accurately if the series be continued to infinity, and approximately by a few terms. When the number of coefficients, also called generalized coordinates, is finite, they are of course to be determined by ordinary methods so as to make the integral a minimum.

In this respect, Rayleigh referred to several sections of his treatise [531], where a similar approach had been successfully used.

The Ritz variational method immediately received a great deal of attention. Timoshenko [379] (p 114) told how he in 1909 had found in the library of Kiev Polytechnic Institute the journal with Ritz's paper and he suggested this topic as a diploma work to one of his students. Performing this task Pistriakoff [538] repeated Ritz's calculations and extended them to several other values of a/b ratio.

Later, the Ritz method was applied for a rectangular clamped plate in the thesis by Paschoud [539], Salvati [540] and papers [541–552]. Stresses in an elastic rectangle under discontinuous loading at its opposite sides are calculated by Hajdin [553]. The Stokes flow in a rectangular cavity with one moving wall was studied by Weiss and Florsheim [554].

There are many pure mathematical papers summarized in Kryloff [555]; see also [313,556,557] which were connected with general aspects of the convergence of the Ritz method. The mathematical question of convergence of the solution of the biharmonic problem for a clamped plate was addressed by Kryloff [558], Trefftz [99,559,560], Courant [561], Friedrichs [562], Wegner [548], and Rafal'son [563]. A weak point in the Ritz method is that it does not contain an algorithm to estimate the accuracy of the approximation. More importantly, a suitable selection of the basic functions is often difficult to make and laborious computations are sometimes necessary. A detailed treatment of the variational methods is given in the textbooks by Leibenzon [69] and Sokolnikoff [41].

Among various interesting fields of application of the Ritz method one can mentioned the studies of the eigenfrequencies and modes of vibrations of a clamped rectangular thin elastic plate by Rayleigh [537], Stodola [340,341,564], Young [565], and many others referred to by Bateman [566], Courant [567], and Leissa [568]. Mathematician Davydov [569] mentioned that by the year 1932 he already collected 291 references for the Ritz method in various problems of statics and dynamics of elastic rods and plates.

5.6.4 *Energy methods*

Timoshenko [570] suggested a slight modification of the Ritz method—the "energy method" or the "principle of least work," as he usually called it [379] (p 114). The method consists of applying the Lagrange variational equation

$$\delta V=\int_S p\,\delta w\,\mathrm{d}x\,\mathrm{d}y \tag{65}$$

that is, by equating the change of potential energy of bending to the work done by the external loading under prescribed plate deflection compatible with boundary conditions (4). Here the potential energy of bending is given by Kirchhoff [126] as

$$V=\frac{D}{2}\int_S\left\{(\Delta w)^2-2(1-\nu)\left[\frac{\partial^2 w}{\partial x^2}\frac{\partial^2 w}{\partial y^2}-\left(\frac{\partial^2 w}{\partial x\partial y}\right)^2\right]\right\}\mathrm{d}x\,\mathrm{d}y \tag{66}$$

(It can be shown that the integral of the term in square brackets vanishes for the clamped plate.)

For the uniformly loaded clamped rectangular plate occupying the region $0\leqslant x\leqslant a$, $0\leqslant y\leqslant b$, Timoshenko [570] obtained

$$w=\frac{p_0a^4b^4\left(1-\cos\frac{2\pi x}{a}\right)\left(1-\cos\frac{2\pi y}{b}\right)}{\pi^4 D(3a^4+3b^4+2a^2b^2)} \tag{67}$$

that satisfies boundary conditions (4). The same expression was independently obtained by Lorenz [571] and it has been repeated in his (formerly well-known) treatise [19].

In spite of the French summary and the extended German abstract, Timoshenko's (and Pistriakoff's) studies apparently remained completely unknown. As Krylov later observed [11] (p 160): "Their investigations are published in the Transactions of the Polytechnic Institute, in Russian, of course, which means for Western Europe almost the same as Chinese!"

Similar calculations based upon the energy method were performed later by many authors, the results not always agreeing, being highly dependent on the choice of the approximation functions. Another choice of approximation functions (which has also been briefly mentioned by Ritz [470]) used by father and son Föppl [28] and later Leibenzon [69] provided the expression:

$$w=\frac{7p_0}{128D\left(a^4+b^4+\frac{4}{7}a^2b^2\right)}(x^2-a^2)^2(y^2-b^2)^2 \tag{68}$$

Similar calculations based upon the variational method had been done in the book by Hager [572]. This study was highly estimated in the review paper by Föppl [317]. However, according to Galerkin [573] and Mesnager [544], there was an essential error in the expression for inner work due to omission of the shearing forces.

By using the same energy approach Timoshenko [574] considered the plane stress or strain equilibrium of an elastic rectangle $|x|\leqslant a$, $|y|\leqslant b$ subjected to normal symmetric load $S(1-y^2/b^2)$, parabolically distributed at its sides $x=\pm a$. Due to the Michell theorem about independence of the stresses in a simply connected domain upon the Poisson ratio ν, the expression for the potential energy U might be simplified by setting $\nu=0$, and in terms of the stress function ϕ it then reads

$$U=\frac{1}{2E}\int_{-a}^{a}\int_{-b}^{b}\left[\left(\frac{\partial^2\phi}{\partial y^2}\right)^2+\left(\frac{\partial^2\phi}{\partial x^2}\right)^2+2\left(\frac{\partial^2\phi}{\partial x\partial y}\right)^2\right]\mathrm{d}x\,\mathrm{d}y \tag{69}$$

Choosing the doubly symmetric stress function in the form

$$\phi=\frac{1}{2}Sy^2\left(1-\frac{y^2}{6b^2}\right)+(x^2-a^2)^2(y^2-b^2)^2(\alpha_1+\alpha_2x^2+\alpha_3y^2) \qquad (70)$$

that satisfies all boundary conditions (while the governing biharmonic Eq. (7) is not satisfied at all; the mathematical question of convergence of this representation still deserves attention) and substituting it in (70) Timoshenko minimized the potential energy U,

$$\frac{\partial U}{\partial \alpha_1}=0, \quad \frac{\partial U}{\partial \alpha_2}=0, \quad \frac{\partial U}{\partial \alpha_3}=0 \qquad (71)$$

In such a way he obtained a rather cumbersome system of linear equations for $\alpha_1,\ \alpha_2,\ \alpha_3$. Timoshenko displayed the distribution of the normal stress σ_x in the middle cross-section $x=0$ for the square $(a=b)$ and rectangular $(a=2b)$ domains. For the latter case this distribution appeared to be almost uniform with a mean value $\frac{2}{3}S$. Similar results were also obtained by Inglis [575] for a shear loading at the surface. By the same approach James Norman Goodier (1905–1969), a PhD student of Timoshenko at the University of Michigan, considered [576] the bending of a finite rectangle. All these examples have been reproduced in the textbooks by Timoshenko [43] and Papkovich [80]. Later Leibenzon [70,69] considered the same benchmark examples by means of the Castigliano theorem.

It should be noted that in all these studies Ritz's name has not explicitly been mentioned. This omission of reference to the Ritz name caused a violent attack by von Krzywoblocki, at that time a Professor of Gasdynamics and Theoretical Aerodynamics at the University of Illinois, Urbana. He wrote two papers [577,578], and participated in discussions on papers [497,579] stating that Timoshenko ''did a great injustice to the late Swiss physicist,'' not mentioning the name of Ritz. More attentive reading of Timoshenko's textbooks shows that this scientist (who usually was very careful with references in his books) always paid a tribute to the fundamental method of Ritz, see [91] (Introduction), [43] (Section 80), and [45] (p 158) in Chapter 6 ''Strain-energy methods'' which, however, has been completely omitted in the third edition [46]. Ritz did not consider 2D elasticity problems, and it was Timoshenko [574] who first applied the energy method (and pointed out the possibility of simplification of the expression for the potential energy by setting the Poisson ratio equal to zero) to solve the important problem of the elastic equilibrium of a rectangle.

Filonenko-Borodich[24] suggested [582,583] to use a special complete system of functions that permit one to satisfy identically the static equations of equilibrium and boundary conditions at all sides of a rectangular domain. The representation for stresses in the rectangle has enough arbitrariness to satisfy the conditions of compatibility by means of the Castigliano theorem. By usage of this approach Filonenko-Borodich [582] and Danilovskaya [584] considered in full detail the equilibrium of an elastic rectangle.

Another complete set of polynomials was employed by Horvay [585–588] and Horvay and Born [589] to study various problems of equilibrium of a semi-infinite elastic strip under a self-equilibrating load at its end. This approach caused an extensive discussion, see [579].

5.6.5 *Bubnov—Galerkin method*

It has already been mentioned that Ritz, starting from the problem of minimization of potential energy of bending of a rectangular clamped plate, arrived at the Eq. (64) (number (41) on page 38 of the original paper [470]) more suitable for practical calculations. In his seminal paper [590], Galerkin[25] explaining the essence of Ritz's method, considered bending by uniform load of a clamped plate $|x|\leqslant a/2$, $|y|\leqslant b/2$ and chose the deflection in the form

$$w=\sum_{k=2}^{3}\sum_{n=2}^{3}A_{kn}(a^2-4x^2)^k(b^2-4y^2)^n \qquad (72)$$

(That expression was also briefly mentioned by Ritz [470] as a possible choice of trial functions.)

Galerkin directly substituted that expression into the governing Eq. (3), multiplied it subsequently by $(a^2-4x^2)^k(b^2-4y^2)^n\mathrm{d}x\mathrm{d}y$, with $k,n=2,3$ and integrated over the plate area. In such a way he obtained a linear system of four algebraic equations for defining the values of A_{kn}. Galerkin solved that system for three values of ratio $b/a=1,1.5,2$.

The approach provided reasonable values of deflection, bending moments, and shearing forces for the plate, which were compared with results of previous studies by Hencky [593], Bubnov [594], and Galerkin [573]. Galerkin [590] explained the essence of his approach for the first time with the example of a simply supported rectangular plate, which admits the exact analytical expression for deflection either in Navier's or Lévy's form of double or single Fourier series, respectively. He did not, however, mention that his approach for a simply supported plate had already been described in the textbook by Bubnov [594] (Section 21). (This example was considered by the *same* approach by Simič [595] but, this study was, probably, not well-known in Russia at that time.)

Heinrich Hencky (1885–1951) studied the paper of Galerkin [590] (he was a Russian prisoner during World War I and

[24]Mikhail Mitrofanovich Filonenko-Borodich (1885–1962) graduated in mathematics from Kiev University in 1909 and got a railway engineer degree from Moscow Institute of Engineers of Ways of Communications. He got his Doctor of Science degree in 1935 without submission of a dissertation, due to his important and practical studies in the railway design. From 1931 till the end of his life he worked at Moscow University and at the Military Engineering Academy. He was a Major General of the Engineering Corps since 1943. A more detailed biography of Filonenko-Borodich and discussion of his scientific works, including an original approach to the famous Lamé's problem of an elastic parallelepiped, can be found in [580,581].

[25]After graduating from St Petersburg Technological Institute in 1899 Boris Grigor'evich Galerkin (1871–1945) got a job at the Kharkov locomotive plant. In 1906, he participated in the revolutionary movement and was imprisoned. From 1909 Galerkin began teaching at St Petersburg Technological Institute, and in 1920 he was promoted to head of the structural mechanics Chair. By this time he also held two Chairs, one in elasticity at the Leningrad Institute of Engineers of Ways of Communications and one in structural mechanics at Leningrad University. Galerkin was a consultant in the planning and building of many of the Soviet Union's largest hydrostations. In 1929, in connection with the building of the Dnepr dam and hydroelectric station, Galerkin investigated stress in dams and breast walls with a trapezoidal profile. From 1940 until his death, Academician Galerkin was head of the Institute of Mechanics of the Academy of Sciences in Moscow. A more detailed biography of Galerkin and discussion of his scientific works, including his numerious papers on thin elastic plates, can be found in [591,592].

during this internment in Saratov he learned the Russian language) and later used [596] the same method for the determination of the stress field in an elastic rectangle. Working at that time at the Technische Hoogschule Delft and being personally acquainted with Galerkin, he drew Biezeno's attention to that approach. Cornelis Benjamin Biezeno (1888–1971) [5] in his opening lecture at the First International Congress on Applied Mechanics, referred to the paper [590] and called this approach the "Galerkin method." It is probably since then that this name has been widely used. On the other side, in their recommendation letter for Galerkin for election as a corresponding member of the USSR Academy of Sciences, Academicians Ioffe, Krylov, and Lazareff [556] did not explicitly mention this method among the achievements of the candidate; they only wrote that "Galerkin has to develop new methods of calculus when studying many rather complicated and difficult problems of theory of elasticity." Only several years later when Galerkin had been elected an Academician and occupied an important place in Soviet mechanics (see, eg, a special issue of leading Soviet journal *Prikladnaya Matematika i Mekhanika* 1941, **5**, No 3, devoted to the 70th anniversary of Galerkin's birth) the developed method was completely connected with his name. This method received wide appreciation in applied mathematics and mechanics; see [597–601] for detailed reviews of the vast literature on the subject.

On the other hand, already Biezeno [602] and [5] (p 14) stated that "the GALERKIN and RITZ methods are *identical*," according to his previous study [603]. However, Biezeno [602] also mentioned that the Galerkin method requires much less ciphering than that of Ritz (in its traditional formulation), and is therefore preferable. Later Hencky [604] published a short note with the similar result; this equivalence already established by Ritz (see [536] p 228) has been mentioned by Timoshenko in his textbooks [45] (p 159) and [47] (Section 81). Davydov, a professor of mathematics at the Zhukovskii Air Force Academy in Moscow, in a letter [569] to Papkovich, expressed an opinion, that "Galerkin's 'method' does not exist at all, but there exists the Galerkin 'scheme' to calculate the coefficients in the Ritz method." Papkovich [605] replying to that letter wrote:

> I share your opinion about Galerkin's algorithm. To tell more, I am in doubt whether one can consider "Galerkin's scheme" as a new one. Probably, this scheme has been used by great mathematicians of the last century and the century before last. I do not know. We do not read their original works, and it seems likely that such an approach has been used by someone. This approach looks very similar to the main algorithm for developing the prescribed function into the Fourier series. Maybe it was used a long time ago for other developments of the prescribed function into series, and for a representation of a seeking function in form of the series. Further investigation of this issue seems very attractive.

Sommerfeld [606] pointed out that Kirchhoff and Ohm were predecessors of Rayleigh and Ritz.

Later on, by direct calculations of variation of the potential energy in (66) Leibenzon [69] proved that Eq. (65) for a clamped plate of any shape can be written in the form:

$$\int_S (D\Delta\Delta w - p)\,\delta w\,\mathrm{d}x\,\mathrm{d}y = 0 \tag{73}$$

As it was mentioned above, the same equation was obtained by Ritz [470] for the calculations in the rectangular domain, in Eq. (64).

A detailed study of this subject by Grigolyuk [599,600] shows that the main idea of the "Galerkin method" was suggested by naval architect, Professor Bubnov[26] as early as May 1911 in a referee report [613] on Timoshenko's memoir, submitted for the competition for the Zhuravskii prize. (This prize was established in 1902 and valued at one year's professor salary, named after the famous Russian railway engineer Dmitrii Ivanovich Zhuravskii (1820–1891). Timoshenko in 1911 was the only single recipient to have ever received the prize.) Bubnov explained (on four printed pages only!) the essence of a method other than Ritz's (or "energy," as Timoshenko preferred to call it) with examples of the Euler stability of a rod and a simply supported rectangular plate compressed in its plane by opposite normal loads at the contour sides. Moreover, Bubnov [594] in Section 22 already applied this method to the more complicated stability problem of a uniformly loaded rectangular plate under additional normal and shearing loads along its contour. It now seems that the name "Bubnov-Galerkin method" as it was widely used in Russian literature [557,601,611,614–616] should be more appropriate; see Meleshko [617], Grigolyuk [600,612] for further historical details.

An interesting modification of the Bubnov–Galerkin procedure has been suggested by Biezeno and Koch [501]; see also [5,25]. They chose the representation of deflection in the clamped plate $|x|\leqslant a$, $|y|\leqslant b$ as

$$w=\left(\frac{x^2}{a^2}-1\right)^2\left(\frac{y^2}{b^2}-1\right)^2\sum_{k,n=1}^{2} f_{kn}\left(\frac{x^2}{a^2}\right)^{n-1}\left(\frac{y^2}{b^2}\right)^{k-1} \tag{74}$$

which satisfies the boundary conditions (4) and substituted it into governing Eq. (3). In this way, some fictitious load $\overline{p}$ instead of p has appeared in the right-hand-side of Eq. (3). Then, the coefficients f_{kn} are determined by the conditions that the integrals $\int\int p\,\mathrm{d}x\,\mathrm{d}y$ and $\int\int \overline{p}\,\mathrm{d}x\,\mathrm{d}y$ taken over well-chosen regions of the plate surface, are equal. In the particular case of a uniform load p_0, a quadrant of the plate limited by the two axes $x=0$ and $y=0$ and by the lines $x=a$ and $y=b$, was subdivided into four parts by the lines $x=a/2$ and

[26] Ivan Grigor'evich Bubnov (or Boobnoff, according to the French spelling of his name) (1872–1919) graduated *cum laude* in 1896 from the Naval Academy in St Petersburg. He worked as a naval architect, and was a head (1908–1914) of the model basin of the Imperial Russian Navy. At his final years Bubnov was Professor at St Petersburg Naval Academy and Major General of the Corps of Naval Architects. He died in March 1919 from typhoid during the civil war in Russia. The detailed story of his life, a general overview of his scientific advances, including an input into the development of the nonlinear theory of bending of plates and his role in Russian naval architecture as the Chief Designer of battleships and first submarines can be found in [23,600,607–612].

$y = b/2$. After laborious but straightforward calculations of integrals, the system of four equations which determines the coefficients f_{kn} was written down. In the original paper [501] the aspect ratio $a/b = 1.5$ was considered in full detail.

Another modification of the variational approach was suggested [618] by Leonid Vital'evich Kantorovich (1912–1983) (the Nobel prize winner in economics); see also [619] and [533–535] for further details. The main idea of this method is to reduce a search for the of minimum of a functional (the total potential energy of the system) depending upon two variables to the problem of minimizing of the functional that depends on several functions of one variable. Applying the well-known Euler equation of the calculus of variations to the problem gives a set of ordinary differential equations in these functions to solve.

For the benchmark case of the clamped elastic plate $|x| \leqslant a$, $|y| \leqslant b$ under a uniform load p_0, one seeks the solution as follows [534,535,620]

$$w_N(x,y) = \sum_{n=1}^{N} \phi_n(y) f_n(x), \quad \phi_n(y) = (y^2 - b^2)^2 y^{2n-2} \tag{75}$$

where the known functions $\phi_n(y)$ satisfy the conditions $\phi_n(\pm b) = 0$, $\phi_n'(\pm b) = 0$. Restricting to $N = 1$ and applying the [Ritz]-Bubnov-Galerkin procedure

$$\int_{-b}^{b} \left(\Delta\Delta w_1 - \frac{p_0}{D} \right) \phi_1(y) \mathrm{d}y = 0 \tag{76}$$

one obtains the solution of the fourth-order ordinary differential equation

$$f_1(x) = A \cosh\frac{\alpha x}{b} \cos\frac{\beta x}{b} + B \sinh\frac{\alpha x}{b} \sin\frac{\beta x}{b} + \frac{p_0}{24D} \tag{77}$$

where $\alpha = 2.075$, $\beta = 1.143$ are the roots of the charactersitic equation, and A, B are explicitly expressed via α and β.

5.6.6 Complex variable approach

Dixon[27] in the series of papers [622,623] summarized in his Presidential Address [7] read before the London Mathematical Society showed that the solution to the problem of the rectangular clamped plate (which was originally proposed to him by his colleague, a professor of engineering in Belfast) depends on the discovery of a function $f(z)$ of a complex variable z satisfying the functional equation

$$f(z+a) - f(z-a) = 2caf'(z)$$

with c some known constant. This functional differential equation can be written as an integral equation and further analysis leads to both Poisson's theory of mixed difference equations and the theory of certain linear integral equations and, finally, to an infinite system of linear algebraic equations. Dixon [7] noted that it is not clear whether this system could be truncated by the theory of infinite determinants, and said very little about the possible application of the whole theory to numerical calculation.

Love[28] suggested [626] an approach wherein the solution involves the conformal mapping of the rectangle onto the circular region of unit radius. This, finally, leads to the solution of an infinite system of linear equations of rather complicated, but triangular, structure. The solution of that system can be found one by one (or a general determinant formula can be written down), but the asymptotic behavior cannot be seen easily from this solution. Love restricted himself to the first three terms of the series. He obtained a value of the center's deflection only 2.5% greater than Hencky's [593] and did no more arithmetic. The same approach was used later in [627–630]. It is worth noting that the general idea of application of conformal mapping (but for smooth contours only) was suggested by Levi-Cevita [192].

The approach by Muskhelishvili [37] leads finally to the solution of the singular integral equation with a kernel of Cauchy's type and provides an effective method for treating the biharmonic problem in a domain with a smooth contour. Magnaradze [631,632] gave the general proof of applicability of that method to contours with sharp angles, while Deverall [633] obtained concrete results for a clamped rectangular plate. Further developments in this direction are summarized by Belonosov [634,635].

5.6.7 Method of superposition

This analytical approach was suggested by Lamé [129] in the twelfth of his famous lectures on the mathematical theory of elasticity when considering the equilibrium of a 3D elastic parallelepiped under any system of normal loads acting on its sides. Briefly mentioned by Lamé [1] in Section 102 and Thomson and Tait [636] in Section 707 as a possible method of solution of the Laplace equation in a rectangle, the method of superposition for the 2D biharmonic equation was developed by Mathieu [199–201] to solve the problem of the equilibrium of an infinite rectangular prism with the surface loads being uniform along the generating line of the prism. He considered, however, the equation of equilibrium written in traditional form of the Lamé equations for two components of displacement. The main idea of the method, concisely expressed in [199], consists of using the sum of two ordinary Fourier series of the complete systems of trigonometric functions in x and y coordinates in order to represent

[27]Alfred Cardew Dixon (1865–1936) in 1883 entered Trinity College, Cambridge and he graduated in 1886 as Senior Wrangler (placed first). He had been taught by a number of famous mathematicians at Cambridge, including Glaisher, Rouse Ball, and Forsyth, and he attended lectures by Cayley. Dixon was appointed a Fellow of Trinity College in 1888 and was awarded a Smith's prize. Dixon was appointed to the Chair of mathematics at Queen's College, Galway, Ireland in 1893, and in 1901 he was appointed to the Chair of mathematics at Queen's College, Belfast. After he retired from his chair in Belfast in 1930 he served as president of the London Mathematical Society from 1931 until 1933. See [621] for a detailed account of his life and the scientific works of this mathematician, who as a devout Methodist, was active in the philharmonic orchestra.

[28]Augustus Edward Hough Love (1863–1940) graduated from Cambridge and held the Sedleian Chair of Natural Philosophy at Oxford from 1899. He was elected under the old Statutes, before the retiring age had been invented, and he continued to lecture and examine until shortly before his death. An expert on spherical harmonics, Love discovered the existence of waves of short wavelength in the Earth's crust. The ideas in this work are still much used in geophysical research and the short wavelength earthquake waves he discovered are called the "Love waves." He was a Fellow of the Royal Society and a corresponding member of the Institute of France. He received many honors, the Royal Society awarded him its Royal Medal in 1909 and its Sylvester Medal in 1937, while the London Mathematical Society awarded him its De Morgan Medal in 1926. For more details of his life, a nice pencil sketch of him with moustache charmingly reminiscent of a frozen waterfall made during a lecture *ca* 1938, and a complete list of publications see [23,624,625].

an arbitrary biharmonic function in the 2D domain $|x| \leqslant a$, $|y| \leqslant b$. Each of these series satisfies identically the biharmonic equation inside the domain and has a sufficient functional arbitrariness for fulfilling the two boundary conditions on sides $|x| = a$ or $|y| = b$. Because of the interdependency, the expression for a coefficient of a term in one series will depend on all the coefficients of the other series and vice versa. Therefore, the final solution involves solving an infinite system of linear algebraic equations providing finally the relations between the coefficients and loading forces.

Mathieu in an elaborate memoir [200] suggested the method of successive approximations for solving that system, and proved its convergence for a square plate. This memoir has been completely reproduced in the second part of his lectures on the theory of elasticity [201]; traditionally, all references to the Mathieu's studies are only restricted by these lectures. Mathieu did not, however, provide any concrete numerical results for stresses in such a domain based upon his solution. As Filon [132] (p 153) noted later, "the solution is, however, so complex in form, and the determination of the constants, by means of long and exceedingly troublesome series, so laborious, that the results defy all attempts at interpretation." Similar opinions were expressed in the textbooks by Papkovich [80] and Timoshenko and Goodier [46]. These views, as it was shown in [529] are too critical: after a proper treatment, Mathieu's method appears rather simple for numerical exploration.

Because of permanent internal tensions of that time between Parisian and non-Parisian mathematicians (see [227]) these remarkable results went completely unnoticed. As we already mentioned, in 1894 Picard suggested this topic as a question worth thinking about on the pages of *l'Intermédiare des mathématiciens*, and it remained practically unanswered as the Index of contents of this journal for years 1894–1913 shows. (With the beginning of WWI this journal ceased publication.)

Meanwhile, in Russia the biharmonic problem (1), (2) was addressed by a mathematician Koialovich[29] in his doctoral dissertation [229]. The defense took place in February 1903 at the Faculty of Physics and Mathematics of St Petersburg University.

Koialovich constructed the analytical solution of the biharmonic problem, and even provided some numerical results. He considered separately the two problems of either finding the biharmonic function which has the prescribed value at the boundary with the value of its normal derivative being zero, or finding the biharmonic function equal to zero at the contour and having the prescribed value of its normal derivative. Each of these problems was then subdivided into three simpler problems, depending on whether the biharmonic functions are even in both variables, even in x and odd in y (or vice versa), or, finally, odd in both variables.[30]

Koialovich [229] employed the Mathieu method of superposition with a particular choice of the complete trigonometric systems in the Fourier series on the intervals $|x| \leqslant a$ and $|y| \leqslant b$ for each of the six problems and considered in great detail the solution of each of them. First, he used *finite* numbers of terms in both Fourier series (N and K, respectively). Therefore, the boundary conditions can be satisfied only approximately, within the accuracy of representation of the functions by the finite number of terms in the Fourier series. The finite system for the unknown coefficients X_n and Y_k was written in general form as

$$X_n = \sum_{k=1}^{K} a_k^{(n)} Y_k + b_n, \quad 1 \leqslant n \leqslant N$$

$$Y_k = \sum_{n=1}^{N} c_n^{(k)} X_n + d_k, \quad 1 \leqslant k \leqslant K \tag{78}$$

with some positive elements $a_k^{(n)}$ and $c_n^{(k)}$. The algorithm of successive approximations was suggested to solve this system, for it has been pointed out that direct numerical solution of the linear system cannot provide all the necessary information about how these Fourier coefficients may change when increasing N and K. It was strictly proven that the algorithm of successive approximations is convergent when the number of iterations tends to infinity, and the coefficients approach some values, depending on N and K. The next step was to increase these values of N and K, while conserving their fixed ratio. It was also proven that this second limiting process is convergent. Later Sobolev [639] gave a general proof of convergence of that approach which is equivalent to the Schwarz alternating algorithm; see [533–535,566] for details. Thus, it was stated that the final representation of the biharmonic function in terms of *infinite* Fourier series is convergent and satisfies both boundary conditions at all sides of the rectangle.

The way of investigation of infinite systems used by Koialovich [229] traces back to the memoir by Fourier written in 1807 (see [640] for the full original text) and published in the year 1822 in his famous book [641]. It is equivalent to the solution of the *infinite* system of linear algebraic equations

$$X_n = \sum_{k=1}^{\infty} a_k^{(n)} Y_k + b_n, \quad 1 \leqslant n \leqslant \infty$$

$$Y_k = \sum_{n=1}^{\infty} c_n^{(k)} X_n + d_k, \quad 1 \leqslant k \leqslant \infty \tag{79}$$

[29]In the literature, there are different spellings of his name: Coialowitch, Kojalovič, Kojalovitch, Kojalowicz, Koyalovich, Koyalovicz. Boris Mikhailovich Koialovich (1867–1941) was a son of the well-known Russian historian Mikhail I Koialovich; he knew seven foreign languages (including old Greek and Latin) and was a good chess player. He constantly played chess with the great Russian mathematician Andrei A Markov, and he even managed to win against great grand masters Lasker and Alekhin in 1912 and 1924, respectively. During his whole life, Koialovich was interested in pure mathematics: in the period 1895–1924 he wrote many textbooks and lecture notes on calculus, geometry, differential equations, and probability, and in the period 1903–1916 he published about 70 reviews of various mathematical books. He wrote several important articles on different mathematical topics for the new Russian edition of the Brockhaus–Efron Allgemeine Enzyklopädie. His biography (not touching some obscure events of his last years of life in Perm and Leningrad) is presented in [637].

[30]Already in his master's dissertation Koialovich ([638], p 7) wrote: "We are deeply convinced that only research in integrating of differential equations may be fruitful, that is always based upon practical applications, *ie*, upon specific examples. Nothing is easier than writing general discussions of the theory of integrating differential equations, but such discussions, for the most part, remain fruitless if they do not follow from researching specific types of theories."

by a traditional way, named by Riesz [642] the "method of reduction," assuming that coefficients with subscripts higher than a chosen value may be neglected. Then for given numbers N, K the unknowns X_n, $(1 \leqslant n \leqslant N)$ and Y_k, $(1 \leqslant k \leqslant K)$ in the finite system (78) have some specific values which can be found by any technique (for example, Fourier [641] even obtained closed analytical expressions for his specific system). It is obvious that the values of these unknowns vary as we increase their number and the number of the equations which ought to determine them. It is desired to find the limits towards which the values of the unknowns converge as the number of equations $N+K$ increases indefinitely. These limits are called the solution of the infinite system (79). The main tasks are to establish whether the system has a (unique) solution, to indicate the way of finding (by means of a finite number of operations) the approximate values for the unknowns X_n, Y_k, and to estimate the error of these approximations.

Koialovich [229] considered a particular example of his general formulas, namely, the uniformly loaded clamped rectangular plate with sides ratio $a/b=2$ (this ratio was in the neighborhood of that most commonly occurring in shipbuilding practice). He chose the deflection in the form

$$w=\frac{p_0}{24D}(b^2-y^2)^2+U(x,y) \tag{80}$$

and for the function U he obtained the biharmonic problem (1), (2) with a zero value of normal derivative at the contour. The function U was represented as a sum of two finite Fourier series with the trigonometric functions $\cos n\pi x/a$ and $\cos k\pi y/b$ with $n,k=1,2,\ldots$ Koialovich found the numerical values of the Fourier coefficients, restricting himself to only five terms in each of the two series. The values of these coefficients appeared to decrease rather rapidly, but the rate of that decrease was not investigated. The deflection of the plate at some typical points was also calculated (the error in satisfying the boundary conditions was about 4% of the maximum value at the center), and a figure with a form of the bent plate was presented. It was more than enough for the mathematical dissertation!

However, Timoshenko who was doing his one year compulsory military service at that time in a St Petersburg sapper regiment and attended the defense, later remarked, [379] (p 72):

> No one in the debate stressed the technological importance of the work. The main item which was discussed by the official opponents (the then famous Russian mathematicians Korkin and Markov) debated mainly the conditions for convergence of the series in terms of which the solution was presented.

Timoshenko permitted a small inexactness: besides Korkin and Markov (an inventor of the "Markov chains"), who were Koialovich's supervisors, the official report [232] was signed by two other members of the scientific Council. This report really contained a detailed survey of the mathematical part of the dissertation, but it did not provide any discussion of the important results for the physical problem.

Koialovich was a professor at the Technological Institute in St Petersburg, and he was also interested in a practical applications of his doctoral dissertation. Academician Krylov recalled (see [11], p 159) that at the beginning of the 20th century he was the Director of the Model Basin of the Imperial Russian Navy in St Petersburg, and how he organized an experimental proof of Koialovich's [229] theoretical results. The tank was constructed, in which plates up to 0.25 inch in thickness, and nearly 3 feet by 6 feet could be investigated. The clamping was effected by tightening the edges of the plate between very rigid angle irons and steel bars $1\frac{1}{4}$ inch in thickness and nearly 10 inch wide. Several measurements were carried out by Bubnov, who at that time was Krylov's assistant. Instead of the deflection, the change of inclination under pressure was measured by means of Poggendorf's scale and mirror method. The experimental data for the deflection of a plate corresponded quite accurately to the calculated results.

Koialovich's solution has been referred to in subsequent articles by Lauricella [295], Galerkin [573,590], and Leibenzon [329], in the dissertation by Kolosov [206], and in the textbooks by Timoshenko [44,92] and Timoshenko and Woinowsky-Krieger [47]. But the very first reference to that solution was made by the Naval Architect of the Imperial Russian Navy Bubnov in the talk [4] read at the Spring Meeting of the 43rd Session of the Institution of Naval Architects in London. In this extensive paper, which contained a lot of new scientific and engineering ideas, Bubnov noted that the convergence of the Fourier series left much to be desired for practical calculations. Regarding the maximum value of stresses in the finite plate, he postulated that the maximum value of stresses in the finite clamped plate with sides $2a$ and $2b$ is reached at the boundary in the middle of the longer side $2b$. The absolute value is sandwiched between the values of a infinite clamped panel with the side $2a$ and that of an elliptical clamped plate with the minor and major axes $2a$ and $2b$, respectively. These two problems allow simple analytical solutions, providing a rather accurate estimate for the stresses in the finite rectangular plate. In the interesting discussion that followed Bubnov's talk, Bryan (see [4], p 48) doubted this postulate. He pointed out that the slowness in the convergence of the Fourier series suggests that there may be a very great tendency to break at the corners of the rectangle. In the written answer Bubnov did not contest this opinion, but he did not fully agree with it.

Later the same year Bubnov published the extended Russian version of his talk in a series of papers, and in May 1904 the whole study [643] was defended in an adjunct dissertation at the shipbuilding department of the St Petersburg Polytechnic Institute. In one of these papers, Bubnov [644] repeated briefly Koialovich's solution and presented numerical values for the Fourier coefficients. Bubnov mentioned that a considerable amount of numerical work still needed to be performed: in order to get an accuracy of 1% in the deflection in the center of the plate it appeared necessary to go to as many as to 15 terms in the Fourier series, doing all the calculations with five digit accuracy. He solved the finite system of $9+9$ equations obtained by the method of reduction from the infinite system for the specific problem of bending of a clamped rectangular plate with sides ratio $b/a=2$. Based upon numerical data Bubnov guessed without any proof that *all* coefficients A_n and B_k (in his notation) in the Fourier series behave as $A_n=(-1)^n\,C\,n^{-3}$ and $4B_k=(-1)^{k+1}\,C\,k^{-3}$ for $n,k>9$ with the *same* constant of value $C=0.2886$. How Bubnov arrived at this remarkable result—we can only guess now. But, the fact remains: already in 1902 he knew the right behavior of the coefficients, the asymptotic law which was strictly proven by Koialovich [645] more than a quarter century later! Unfortunately, neither Bubnov in 1902 nor Koialovich in 1930 took full advantage of the knowledge of the asymptotic law, and the principal question of the convergence of the Fourier series has not, however, been investigated.

In the second volume of his textbook [594], Bubnov addressed the biharmonic problem (3), (4) in much more detail. This analysis was presented in the extensive lecture course in the Naval Academy in St Petersburg, and that volume was written already in 1912 and had been printed out in April 1914 in only 400 copies. (Almost 40 years later, this part of the lectures has been reprinted in [646].) Not stated explicitly, Bubnov used the asymptotic law for the Fourier coefficients in order to calculate the important mechanical characteristics such as bending moments and shear forces along the edge of the clamped plate. Without using that law the Fourier series appeared to be divergent—that circumstance had been especially emphasized. Unfortunately, after Bubnov's premature death during the civil war in Russia, see [607], these facts went practically unnoticed by further investigators, in spite of references made in [44,47].

Lauricella [295] also considered the biharmonic problem in a rectangle. He used his general approach of the decomposition of the problem by means of two auxiliary harmonic functions u and v, and constructed the representations for these functions by superposition of two ordinary Fourier series on the same system of complete trigonometric functions as Koialovich [229] and Bubnov [644]. Many years later Schröder [310] used the same approach in a large memoir in which he considered in detail all four types of symmetry of the biharmonic function (or eight combinations of symmetry of functions u and v). This excellent paper, however, did not receive proper attention at that time. One reason could be that this study had been done during WWII and after the war was published in German. Anyway, neither Lauricella [295] nor Schröder [310] provided any numerical data for solving the infinite systems for specific domains.

The method of superposition also got a new impulse when Hencky in his dissertation [593], submitted in October 1913 to the Technische Hochschule Darmstadt, used other complete systems of the trigonometric functions $\cos(2m-1)\pi x/2a$ and $\cos(2l-1)\pi y/2b$ with $m,l=1,2,\ldots$ Numerical results showed the fast rate of decrease of the Fourier coefficients when the finite system had been solved, but the rate of convergence for the Fourier series, especially for bending moments and shear forces at the boundary, was not discussed properly. This solution was used (or, sometimes, simply repeated) by many authors, [18,30,44,47,66,76,92,533–535,573,647–673] to obtain the numerical data for deflections, bending moments, and shear forces for a wide range of clamped rectangular plates under uniform and concentrated loadings.

Being unaware of some previous studies based upon Hencky's solution Inglis [11] used the same representation for the deflection. This analyst used only two terms in each of the Fourier series and demonstrated reasonable satisfaction of the boundary conditions for deflection, but details of his solution had been omitted. As famous Russian scientist and naval architect Academician Krylov, who was at that time in England for talks about the former Russian Navy fleet and attended the meeting, has noted ([11], p 160) in the discussion:

> In treating mathematical subjects two of the greatest masters give us quite different models. Thus Euler enters into every detail of his reasoning and calculations, illustrating them profusely by examples, and explains exactly how his actual work was performed; on the other hand, the "princeps mathematicorum" Gauss presented results of his investigations in the most concise and elegant manner: "After you have erected a building you do not leave the scaffolding," he used to say. Professor Inglis' paper presented in these thirteen pages is developed in an "ultra-Gaussian" manner. Owing to its importance, this paper must be studied from the beginning to the end by every student of naval architecture, pencil in hand, without omitting a single letter or figure. But before the student succeeds in mastering it, he will have used a great many pencils.

Anyway, Inglis's solution really looked like a good engineering solution and deserved the admiration which was shown in the discussion.

Being already back in the USSR, Academician Krylov submitted in October 1928 to the Academy of Sciences an extensive study performed by Koialovich [645]. In that remarkable memoir Koialovich turned to the general mathematical theory of infinite systems of linear algebraic equations, keeping in mind, however, its possible application to the problem of bending of the clamped rectangular plate. For the infinite system (79) Koialovich developed a powerful method of the so-called *limitants*, the special quantities that can be defined after solving the finite system (78), and which define completely some bounds for *all* other unknowns. This method is similar to Cauchy's method of majorant functions in the theory of differential equations. The principal difference of the proposed method from the traditional method of reduction consists in the following. The traditional method can provide only numerical values of the first unknowns—all others are simply put equal to zero. Koialovich's approach gives the underestimated and overestimated values of *all* unknowns by using a successive approximation algorithm when solving some specially constructed systems.

In spite of the short (and not precise!) reference in Kantorovich and Krylov [533,535] and the German abstract in one of the leading abstract journals of that time, the main positive results of that study seem to have been almost ignored by Koialovich's contemporaries. The "Renaissance" of this outstanding memoir (an introduction to it was written in the best traditions of old mathematicians) was started by a short note by Grinchenko and Ulitko [674] and the book by Grinchenko [64]; see also [675,676] for further details.

Koialovich [645] studied the example of the system corresponding to a clamped rectangular plate with $a=1$, $b=2$, the example that he had already considered in his dissertation [229]. By using a proper choice of limitants Koialovich managed to find numerically (without any computer!) the bounds for all unknowns. All calculations were presented with intermediate data of ten successive approximations and occupy 16 printed pages. Koialovich [645] (p 43) wrote:

> The numerical aspect seems at first very narrow, dry, and low. But, developing it, I realized that it opens up an interesting area, rich in results important not only for applications, but for theory as well. Almost never have I regretted the time I spent on repeated numerical solutions of a system: each time, I learned something new.

His numerical data supported the (empirical) Bubnov law: in terms of X_n and Y_k all unknowns seem to tend to the same single constant.

In the last section of his paper Koialovich, using the notions of limitants and supposing two *additional* properties of the coefficients of matrices (these conditions are not essential as it had later appeared), proved that

$$\lim_{n\to\infty} X_n = \lim_{k\to\infty} Y_k = G \tag{81}$$

with some constant value G. At the very end of his memoir he mentioned that the law (81) could be used (as Bubnov had supposed) as a base for a new, more powerful algorithm of solving infinite systems. Namely, by putting

$$X_n = G + x_n, \quad Y_k = G + y_k \tag{82}$$

substituting it into (79), one obtains a new system with the unknowns x_n and y_k, which can effectively be solved by the method of reduction (now, $x_n \to 0$ and $y_k \to 0$ as $n\to\infty$ and $k\to\infty$, respectively).

Koialovich [645] also stated an important question about the possibility of finding the value of *G a priori*, without solving the infinite system, but he himself did not provide any further development. Probably, his age (he was 63 already, and he died in December 1941 during the siege of Leningrad) and subsequent long discussions between Koialovich [677–679] and Kuz'min [680,681] prevented him, unfortunately, from the analysis of that interesting question. (The discussion, "the seven years war" according to the Academician Krylov [251], concerned the general conditions which must be imposed on the matrix and free terms of an *arbitrary* infinite system in order to have a unique solution and how this solution can be obtained. These difficulties are connected, of course, with double infinities (the number of terms and the number of equations) which are involved in the system (79). The arguments of these papers, being very interesting by themselves, represent only partial interest for the physical problem in question: already Mathieu [123] proved the uniqueness of the solution of the biharmonic problem when the values of the function and its normal derivative are prescribed at the contour of any finite domain.)

For the particular case of a square plate, by using the Koialovich [229]-Bubnov [594,644] representation for the deflection, Grinchenko and Ulitko [674] found an *explicit* value of the constant G which appears not equal to zero. It is interesting to note that for Hencky's [593] representation the value $G=0$, as it has been found by Meleshko and Gomilko [682]; see also Meleshko and Gomilko [676] for a detailed analysis of the general mathematical problem. This completely justified the method of simple reduction employed by many authors to solve the infinite system and to accurately estimate the values of deflections and bending moments. However, the asymptotic behavior of the terms x_n and y_k when $n\to\infty$ and $k\to\infty$, respectively, are important to establish the local distribution of deflection and shearing forces near the corner point; see [683].

For 2D plane elasticity problems of equilibrium of an elastic rectangle after Mathieu studies [200,201], the method of superposition has been employed in several articles [484,497,575,684–697] and books [30,89,698,699]. All these studies provide an immense amount of numerical data concering distribution of stresses at various inner cross sections of the rectangle, but none of them took into account the asymptotic law (81) for the unknown coefficients in the infinite systems. As it has been first shown by Grinchenko [64] the simple reduction method of solving of the infinite systems cannot provide an accurate determination of stresses near corner points: a finite extra value Gab/π in the values of stresses σ_x and σ_y at the corner point cannot be removed by any increase of the number of terms in the finite Fourier series; see also Meleshko [529] for further explanations. It it interesting to note that for an example considered by Pickett [690]—a square loaded by a parabolically distributed normal load at the sides $|x|=a$, the benchmark example of studies by his teacher Timoshenko [43,574], the value of G can be found explicitly, as it was proven by Meleshko and Gomilko [676].

By usage of the method of superposition the steady Stokes flows in a rectangular cavity was analyzed only in a few studies. Takematsu [700] on one (!) journal page presented the general scheme of the method. Meleshko [617,701] obtained the solutions for arbitrary (including discontinuous) distribution of velocities at the cavity's boundary. The algebraic work involved is rather cumbersome, but the final formulas are very simple for numerical evaluation.

Looking back upon the method of superposition one can state that this approach enables one to deal, after a proper treatment, with all important physical problems connected with the biharmonic equation in the rectangular domain. The developed numerical algorithm seems to overcome the difficulty [4] (p 21) connected with the use of "the Fourier's series, whose convergency leaves much to be desired for practical calculation" and to obtain very accurate data by means of only a few terms in the Fourier series. The proposed way of consideration of the infinite systems suggested by physical problems provides a direct and powerful algorithm for solving rather complicated 2D biharmonic problems for the rectangle.

5.6.8 *Method of eigenfunctions expansion*

An elegant analytical approach for considering the biharmonic problem in a rectangle utilizes a natural generalization of the expansion in eigenfunctions for the Laplacian boundary value problem. Apparently, this method was initiated, in the fundamental memoir by Dougall [702] who considered the general problem of the equilibrium of thick elastic infinite layer $|z|\leqslant h$ under given forces. Dealing mainly with problems in cylindrical coordinates, he briefly mentioned that for plane strain in the layer the flexural (antisymmetric with respect of z) system of stresses (or ''modes'' as he called them)

$$\sigma_x = C\mathrm{i}\,e^{\mathrm{i}\kappa x}[(3-\cosh 2\kappa h)\sinh \kappa z + 2\kappa z \cosh \kappa z]$$

$$\sigma_z = C\mathrm{i}\,e^{\mathrm{i}\kappa x}[(1+\cosh 2\kappa h)\sinh \kappa z - 2\kappa z \cosh \kappa z]$$

$$\tau_{xz} = Ce^{\mathrm{i}\kappa x}[(1-\cosh 2\kappa h)\cosh \kappa z + 2\kappa z \sinh \kappa z] \qquad (83)$$

with an arbitrary constant C keeps the sides $z=\pm h$ free of tractions, provided that κ is a root of the equation

$$\sinh 2\kappa h - 2\kappa h = 0 \qquad (84)$$

Dougall established that besides an obvious triple root $\kappa=0$, Eq. (84) has complex roots falling into groups of four symmetrically placed with respect to the axes of the complex plane. He also found that the asymptotic behavior of the four members of each group is given by $\kappa_r h = \pm 1/2 \ln(4r\pi+\pi) \pm \mathrm{i}(r\pi + 1/4\,\pi)$ for large r.

Next, if $P(z)$ and $Z(z)$ are continuous functions odd and even on z, respectively, with conditions

$$\int_{-h}^{h} P(z)z\,\mathrm{d}z = 0, \quad \int_{-h}^{h} Z(z)\,\mathrm{d}z = 0$$

represent the normal and tangential loads at the end $x=0$ of the semiinfinite strip $x\geqslant 0, |z|\leqslant h$, then complex coeffiicents C_r should exist, such that simultaneously

$$\left.\begin{aligned} &\mathrm{i}\sum_r C_r[(3-\cosh 2\kappa_r h)\sinh \kappa_r z + 2\kappa_r z \cosh \kappa_r z] = P(z)\\ &\sum_r C_r[(1-\cosh 2\kappa_r h)\cosh \kappa_r z + 2\kappa_r z \sinh \kappa_r z] = Z(z) \end{aligned}\right\} \qquad (85)$$

Dougall [702] did not suggest any algorithm for determining these coefficients from two series expansions (85). Probably, he did not even notice the unusual situation of the necessity of defining the coefficients C_r from *two* functional equations in contrast to an ordinary Fourier series expansion.

Filon [703] was the first author who made an attempt to define these coefficients. Firstly, he considered the general problem of expanding a given function $f(x)$ in a series of functions $\phi(\kappa_r, x)$, where κ_r is the (real or complex) root of equation $\psi(\kappa)=0$. Based upon Cauchy's theory of residues, Filon established a general theorem for expanding a polynomial into a series of functions of the form $\phi(\kappa_r x)$. Next, he addressed the possibility of applying the method to a series of functions $\phi(\kappa_r, x)$ where κ_r and x do not appear exclusively as a product $\kappa_r x$. Referring to Dougall [702] and considering the ''flexural'' solution of the biharmonic equation in a semi-infinite strip $|x|\leqslant b, y\geqslant 0$, Filon arrived at a system of equations similar to (85). He managed to express explicitly (and *uniquely*, as he believed) the coefficients C_r by means of only *one* (first) equation in (85), provided $P(x)$ was a polynomial. He gave an example of such an expansion,

$$x^3 = \frac{3}{5}xb^2 - \sum_r \frac{b}{\kappa_r^2}\left[\kappa_r x \frac{\cosh \kappa_r x}{\sinh \kappa_r b} + (2-\kappa_r b)\frac{\sinh \kappa_r x}{\sinh \kappa_r b}\right] \qquad (86)$$

where the summation extends to $\mathrm{Re}\,\kappa_r > 0$. (The equivalence of the eigenfunctions in the expansions (85) and (86) can be readily established by means of Eq. (84).)

This paradoxical mathematical result of the necessity of only *one* boundary condition for normal loading leaving the shear end stresses arbitrary, probably appeared so unusual to Filon (and, probably, to many others) that almost no further papers were published on the subject for the next 33 years! The single exception was the paper by Andrade [704] who used this approach to solve the problem of shear in an elastic rectangle with sides $x=\pm a$ free of loading and prescribed tangential displacements at the sides $y=\pm b$. Corresponding to the stress field (83), displacements had to satisfy the prescribed boundary conditions. It was found feasible to work with more than three roots of Eq. (84). (Andrade had also performed the accurate study of locations of these roots and defined their accurate values.) Six real constants were determined by the collacation approach, that is by setting the displacements at the points $x=0$, $x=a/2$, and $x=a$ equal to their prescribed values. The laborious arithmetic was done on a ''Brunsviga'' machine, the personal computer of that time. The results for shear stress distributions in several cross sections were found to be in reasonable correspondence with experimental measurements. Andrade [704] also noticed that Filon [703] used only one equation, but he did not pursue this avenue.

The reference to the Filon's paper [703] was made by Lur'e [705] who introduced the name ''homogeneous'' solutions for the eigenfunctions (83), and Prokopov [706], who suggested to satisfy the prescribed shear stresses only at one middle point, referred to the application of the Saint-Venant principle.

The next step in developing of the eigenfunctions approach was taken by Papkovich[31]. In the list of problems to Chapter X of his fundamental treatise [80] he suggested the following

Problem 31. Obtain the solution of the plane problem for a rectangular strip with two sides $y=\pm b$ free of loading while the sides $x=0$ and $x=l$ are loaded by arbitrary forces, by means of the representation

$$\phi=\sum_i X_i(x)Y_i(y)$$

where each function $Y_i(y)$ satisfies equations

$$Y_i''''(y)+\left(\frac{u_i}{b}\right)^2 Y_i''(y)=0$$

(with u_i are some constants), and the boundary conditions

$$Y_i(\pm b)=Y_i'(\pm b)=0$$

In this formulation the functions $X_i(x)$ are to be determined, and not be chosen proportional to $\exp(\kappa x)$ as in Dougall's [702] representation. The functions $Y_i(x)$ correspond to the eigenfunctions of the stability problem of a clamped elastic beam.

In an extensive hint to that problem occupying almost two pages printed in small letters, after some interesting transformations based upon application of the Bubnov's method and solving the infinite system of differential equations for the functions $X_i(x)$, Papkovich finally arrived at the representation for the symmetric (extensional) stress distribution with even functions with respect to the x-axis loading

$$\phi=\sum_{k=2,4,\ldots} A_k e^{s_k x}F_k(y) \tag{87}$$

where the complex functions $F_k(y)$ are

$$F_k(y)=\sum_{i=2,4,\ldots}\left(\frac{\cos\dfrac{i\pi y}{2b}}{\cos\dfrac{i\pi}{2}}-1\right)\left[\left(\frac{s_k b}{\pi}\right)^2-\left(\frac{i}{2}\right)^2\right]^{-2}$$

and s_k being the roots of equation

$$\sum_{i=2,4,\ldots}\left(\frac{s_k b}{\pi}\right)^4\left[\left(\frac{s_k b}{\pi}\right)^2-\left(\frac{i}{2}\right)^2\right]^{-2}=-\frac{1}{2}$$

(there is a misprint in the original text on p 485) or, equivalently,

$$\sin(2s_k b)=-2s_k b \tag{88}$$

Papkovich established that this equation, besides the obvious one, $s_0=0$, has only complex roots

$$2s_k b=\pm\alpha_k\pm i\beta_k$$

with $\pi<\alpha_2<1.5\pi$, $3\pi<\alpha_4<3.5\pi$, $5\pi<\alpha_4<5.5\pi$, *etc*,

The representation (87) along with inequalities for α_k shows that self-equilibrated loads applied at short sides $x=0$ and $x=l$ decrease at the distance of $2b$ to a factor of approximately 1/25. This provides some quantitative estimates for the correctness of the Saint-Venant principle even without finding values A_k. The way of the construction of the solution of the biharmonic equation in a rectangular domain was discussed in much more detail in the textbook by Papkovich [707]. In Sections 32–35, he provided similar discussion of a problem important for shipbuilding of bending of rectangular plates firmly clamped at the opposite sides $|y|=b$ under any aritrary loading at the edges $x=0,1$. These studies first remained hardly known except in the USSR, and only in a naval architects' community [709]. These estimates were later confirmed in [710]; see further extensive discussions in [119–121]. See also Horgan [711,712] for application of such results to the classic problem of estimating the entrance length for Stokes flows in a parallel plate channel.

This eigenfunctions expansion approach got much more popularity after independently publishing in 1940 by Papkovich [713] and Fadle [714]. (The first paper had been also published in German, according to the Soviet tradition of that time, but later, after Papkovich's untimely death in 1946, was not even mentioned in the detailed list of his publications in the obituary. Politics, sometimes, has an unusual influence on science.) The second paper was based upon the dissertation [715], in which Fadle attributed this approach to Tölke [479]; note also the misprint in the year on the first page of [714]. Papkovich [713] mentioned the possibility of applying the Gram-Schmidt orthonormalization process for constructing an orthonormal set of functions from a linearly independent set in order to define the complex unknown coefficients in series expansion (this paragraph for some reason was absent in the German version of the paper), but he did not provide any numerics. On the contrary, Fadle [714,715] performed extensive calculations for several practical distributions of normal and tangential loads at the sides of a square plate. The boundary conditions were satisfied by means of the least squares method, that is, by considering instead of two functional equations (85) the procedure of minimization of a joint sum consisting of the squares of the differences between prescribed loads and finite sums of eigenfunctions.

In his next paper on the subject, Papkovich [716] studied the problem of bending a rectangular plate with clamped opposite sides $y=\pm b$ by a system of loads either symmetric or antisymmetric on y, by prescribing any of two physical quantities: deflection w, angle of inclination $\partial w/\partial x$, bending

[31]Petr Fedorovich Papkovich (1867–1946) graduated with a gold medal from the Shipbuilding Department of St Petersburg Politechnic Institute, where he attended the lectures on the theory of vibration by Krylov and on structural mechanics of a ship by Bubnov. In 1912–1916 he served at the design bureau of the Admiralty shipbuilding factory. In 1919 he became a professor at Petrograd [Leningrad] Politechnic Institute succeeding Bubnov and Timoshenko, and in 1925 he was appointed as a head of the Department of the Structural Mechanics of a ship. During the period of more than twenty years Papkovich held positions as a lecturer in mechanics, theory of vibration, structural mechanics of a ship at Leningrad Shipbuilding Insitute, Leningrad University, and the Naval Academy. In 1933 Papkovich was elected a Corresponding Member of the Academy of Sciences of the USSR. In 1940 he was promoted to the rank of Rear-Admiral of the Corps of Naval Architects. He was twice decorated with the high distinction of the USSR, the Order of Lenin. He wrote a textbook [80] which contains a most interesting exposition of solutions of most problems in the theory of elasticity by means of a single unified approach. Another textbook by Papkovich [707] in two volumes (almost 1800 printed pages) was awarded the Stalin (or State, as it was bashfully named in the USSR some time ago) Prize in 1946. This treatise is remarkable for its completeness, see Timoshenko [23], Section 90. Further details of Papkovich's life and scientific works can be found in [708] and in the obituary notice published in *Prikladnaya Matematika i Mekhanika*, 1946, **10**, 305–312.

moment M_x or shear force Q_x at the ends $x = \pm a$. For the symmetric case of loading the biharmonic function w was represented in the form of a series

$$w = \sum_{k=1}^{\infty} a_k Y_k(y) \frac{\cosh \lambda_k x}{\cosh \lambda_k a},$$

$$Y_k(y) = \frac{\cos \lambda_k y}{\cos \lambda_k b} - \frac{y}{b} \frac{\sin \lambda_k y}{\sin \lambda_k b} \tag{89}$$

The complex eigenfunction $Y_k(y)$ identically satisfies the both boundary conditions at the sides $|y| = b$, provided that the eigenvalue $\lambda_k = s_k$ is the complex root of Eq. (88).

The most significant input in the eigenfunctions method was made by Papkovich [713,716] who established the special biorthogonality property

$$\int_{-b}^{b} \{Y_k''(y) Y_n''(y) - \lambda_k^2 \lambda_n^2 Y_k(y) Y_n(y)\} \mathrm{d}y = 0, \quad k \neq n \tag{90}$$

(This important condition has been independently rediscovered by Smith [717].) It readily permits *two* boundary conditions, either w, M_x or $\partial w/\partial x$, Q_x, to explicitly determine the coefficients a_k. In particular, by chosing $w = 0$, one arrives at the nontrivial expansion of zero in a series of nonorthogonal eigenfunctions $Y_k(y)$. That expansion is *nonunique* depending on the choice of the value M_x. This nonuniqueness explains Filon's paradox; a detailed account is given by Gomilko and Meleshko [718]. For the classical biharmonic problem with prescribed values of w and $\partial w/\partial x$, Papkovich [716] established the integral equation for the auxiliary function $\partial^2 w/\partial x^2$ at $x = a$, and suggested an algorithm for succesive approximations to its solution. In the textbook Papkovich [707] considered some particular engineering problems of ship plating. From the engineering point of view the problem is only partially solved when a mathematical expression for the deflection has been determined. The calculation of the moments and shear forces at different points is likely to be just as difficult and tedious.

In spite of a simple idea at the heart of the method of eigenfunctions expansion, it can hardly be recommended for engineering applications, requiring too many additional calculations; see, for example, the numerical data of Fadle [714,715], Koepcke [719], Gurevich [720], and Gaydon [721,722] for an elastic rectangle. Besides this, the main question of the convergence of these nonorthogonal complex series at the sides $|x| = a$, already anticipated by Dougall [702], is not at all trivial. It was investigated in [723–739], to name the most significant papers.

Due to these and some other studies referred to in books by Timoshenko and Goodier [46] (Section 26), Grinchenko [64], and Lourie and Vasil'ev [740] and review articles by Dzhanelidze and Prokopov [741], Vorovich [742], and Prokopov [743,744], the mathematical problem of developing two arbitrary functions into series of eigenfunctions of non-self-adjoint operators is now clarified in great detail.

An original method of solving the biharmonic problem in a rectangular region was suggested by Grinberg and Ufliand [745] and developed in [746,747]. The method is based upon a construction in the rectangular domain by means of a recurrent sequence of operations on a special set of complete orthogonal harmonic functions. By using Green's formulas the expansions of $\nabla^2 w$ and w were then constructed. The method provides reliable results for the deflection at the center, even with only a few basic functions. A similar engineering approach was independently suggested by Morley [748].

It may seem strange, but the eigenfunctions expansion method has only begun to be applied to Stokes flow problems in a rectangular cavity recently. Papers by Sturges [749], Shankar [750], and Srinivasan [751] provide interesting data concerning the structure of streamline patterns in the cavity, including the Moffatt eddies near quiet corners.

6 CONCLUSION

History, to paraphrase Leibniz, is a useful thing, for its study not only gives to men of the past their just due but also provides those of the present with a guide for the orientation of their own endeavors. (Recall Abel's statement, quoted in the remarkable talk by Truesdell [753] p 39, that he had reached the front rank quickly "by studying the masters, not their pupils.")

At the end of the 19th century, Karl Pearson (1857–1936) wrote in the preface to the monumental treatise [21] (pp X-XI):

> The use of a work of this kind is twofold. It forms on the one hand the history of a peculiar phase of intellectual development, worth studying for the many side lights it throws on general human progress. On the other hand it serves as a guide to the investigator in what has been done, and what ought to be done. In this latter respect the individualism of modern science has not infrequently led to a great waste of power; the same bit of work has been repeated in different countries at different times, owing to the absence of such histories as Dr Todhunter set himself to write. It is true that the various *Jahrbücher* and *Fortschritte* now reduce the possibility of this repetition, but besides their frequent insufficiency they are at best but indices to the work of the last few years; an enormous amount of matter is practically stored out of sight in the *Transactions* and *Journals* of the last century and of the first half of the present century. It would be a great aid to science, if, at any rate, the innumerable mathematical journals could be to a great extent specialised, so that we might look to any of them for a special class of memoir. Perhaps this is too great a collectivist reform to expect in the near future from even the cosmopolitan spirit of modern science. As it is, the would-be researcher either wastes much time in learning the history of his subject, or else works away regardless of earlier investigators. The latter course has been singularly prevalent with even some first-class British and French mathematicians.

On the other hand, almost one hundred years later Albert Edward Green (1912–1999), one of the prominent figures in the theory of elasticity in the twentieth century, said while receiving the Timoshenko medal of the American Society of Mechanical Engineers (cited according to *Journal of Applied Mechanics*, 1999, **66**(4), p 837)

> On looking back over the history of science one realizes that most of us can only hope to place one small brick—if that—in the edifice—and even that may get knocked out by following generations.

As it may be seen from the preceding pages, the long fascinating history of the biharmonic problem in the period of the last 125 years or so completely confirms both of these quotes. The history of the biharmonic problem that has been represented in a great number of diverse works reveals a great variety of mathematical methods specially developed for its solution. Most of the methods can provide acceptable results for engineering purposes, even though the rigorous mathematical requirements regarding convergence cannot be completely answered in all cases[32]. Of course it should be kept in mind that any engineering formula giving a relatively simple solution and connecting some physical quantities, is the consequence of some assumptions, and it is necessary to see in it not only pure numbers. But we might definitely state that the fruitful interaction between mathematical and engineering approaches provides the solution of the 2D biharmonic problem with both mathematical and engineering accuracy.

I have started this review with words of Lamé[33], one of the great mathematicians and engineers (French mathematicians, however, considered him too practical, and French engineers too theoretical), and I want to end it with words from his last lecture course in the Sorbonne [758]:

> Écartez à tout jamais la division de la science en Mathématiques pures en Mathématiques appliquées.

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[32]Inglis [752] (p 272) in the lecture at the centenary anniversary of the Insitution of Mechanical Engineers, said: "In contrast to real mathematicians, engineers are more interested in the contents of the tin than in the elegance of the tin-opener employed."

[33]Gabriel Lamé (1795–1870) was a student at the École Polytechnique and later a professor there. Between these times (1820–1830) he was in Russia, see [754,755]. He worked on a wide variety of different topics. Lamé's work on differential geometry and contributions to Fermat's Last Theorem are important. Here he proved the theorem for n=7 in 1839. However, Lamé himself once stated that he considered his development of curvilinear coordinates [1] as his greatest contribution to mathematical physics. On the applied side he worked on engineering mathematics and elasticity where two elastic constants are named after him. Gauss considered Lamé the foremost French mathematician of his generation, while in the opinion of Bertrand, Lamé had a great capacity as an engineer. Further details can be found in [756,757].

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