Odd Perfect Numbers

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1 Introduction

Mathematicians have long been interested in knowing which types of numbers are perfect numbers. I quote Oystein Ore: “Perfect numbers are essential elements in numerological speculations. God created the world in 6 days, which is a perfect number. The moon circles the earth in 28 days, again a symbol of perfection in the best of all possible worlds.” As the last proposition in the ninth book of his *Elements*, Euclid proved that for a prime p, if $2^p - 1$ is prime, then $2^{p-1}(2^p - 1)$ is perfect. Euler further showed that every even perfect number has the form given by Euclid. So we have a complete characterization of even perfect numbers. What about odd perfect numbers? There are certain results on the minimum size of an odd perfect number, what types of prime factors such a number could have, and Euler's characterization of odd perfect numbers, which says that any odd perfect number must have the form $n = p^{\alpha} m^2$, where p is prime and $p \equiv \alpha \equiv 1 \pmod 4$, which implies that $n \equiv 1 \pmod 4$. Yet, we do not know of any specific odd perfect number; worse still, it remains an open question whether odd perfect numbers exist in the first place or not.

The aim of this article is to bring to light an elementary proof of Touchard's Theorem on odd perfect numbers. In 1953, Jacques Touchard proved that any odd perfect number must have the form $n = 12m + 1$ or $n = 36q + 9$. His proof used a recursion relation derived by Balth. van der Pol in 1951 using a nonlinear partial differential equation. The proof presented here, discovered recently by Judy A. Holdener, is much more elementary and shorter than the original proof. This proof appeared in the American Mathematical Monthly, Volume 109, Number 7.

2 Terminology

Definition 1 *The sigma function $\sigma(n)$ denotes the sum of the positive divisors of a positive integer n:*

$$\sigma(n) = \sum_{d|n} d.$$

Note that $\sigma(n)$ is a multiplicative function, that is, given m, n relatively prime, $\sigma(mn) = \sigma(m)\,\sigma(n)$.

Definition 2 *A number n is called perfect if the sum of its divisors is equal to twice itself, that is, $\sigma(n) = 2n$.*

It is of interest to note that Holdener's definition of a perfect number in words is wrong: “A natural number is said to be *perfect* if it is equal to twice the sum of its divisors.”

3 Touchard's Theorem

Recall that Touchard's Theorem states that any odd perfect number must have the form $n = 12m+1$ or $n = 36q+9$. To prove the result, we shall need the following lemma.

Lemma 3 *If n is an odd number of the form $n = 6k-1$, then n is not perfect.*

Proof: Observe that $n = 6k-1 = 3(2k)-1$, so $n \equiv -1 \pmod 3$. Any number k is of the form $n \equiv 0, 1, -1 \pmod 3$, so $k^2 \equiv 0, 1 \pmod 3$. This implies that n is not a perfect square. Consider any pair $(d, n/d)$, where d is a divisor of n. Because $n = d \cdot n/d \equiv -1 \pmod 3$, we have two cases:

$$\begin{aligned} d &\equiv 1 \pmod 3 \text{ and } n/d \equiv -1 \pmod 3, \text{ or} \\ d &\equiv -1 \pmod 3 \text{ and } n/d \equiv 1 \pmod 3. \end{aligned}$$

Hence we see that for any divisor d of n, we have $d + n/d \equiv 0 \pmod 3$. We shall employ this observation to pair the divisors of n to establish a congruence relation for $\sigma(n)$. We have

$$\sigma(n) = \sum_{d|n,\, d<\sqrt{n}} \left(d + \frac{n}{d}\right) \equiv 0 \pmod 3.$$

But why is this possible? That is to say, why are there an even number of divisors of n? Holdener does not mention this in her proof, so she probably assumed readers take this for granted or are aware of it. This relies on our observation that n is not a perfect square. Consider the canonical prime factorization of n, $n = p_1^{a_1} p_2^{a_2} \cdots p_k^{a_k}$, where each $a_i \geq 1$. Because n is not a perfect square, at least one of the a_i's is odd. Suppose a_j is odd. Recall that the number of divisors of n is $(a_1+1)(a_2+1)\cdots(a_k+1)$, which is even because a_j+1 is. So we see that $\sigma(n) \equiv 0 \pmod 3$. But $2n = 12k-2 \equiv 1 \pmod 3$. Thus $n = 6k-1$ cannot be perfect.

Using this lemma, we now proceed to prove Touchard's Theorem.

Theorem 4 *Any odd perfect number must have the form $n = 12m+1$ or $n = 36q+9$.*

Proof: By the lemma, n cannot have the form $n = 6k-1$, so $n = 6k+1$ or $n = 6k+3$. We have two cases.

Case 5 *Suppose $n = 6k+1$. By Euler's characterization of odd perfect numbers, $n = 4l+1$. So $4l+1 = 6k+1 \Rightarrow 2l = 3k$. Since $2l$ is even, we see that k is even. Letting $k = 2m$, we conclude that $n = 6(2m)+1 = 12m+1$.*

Case 6 *Now suppose $n = 6k+3$. Again, we also have $n = 4l+1$. We get $4l+1 = 6k+3 \Rightarrow 2l = 3k+1$. Since $2l$ is even, $3k$ must be odd, so k must be odd. Letting $k = 2m+1$, we infer that $n = 6(2m+1)+3 = 12m+9$. Now either $3 \mid m$ or $3 \nmid m$. Write $n = 12m+9 = 3(4m+3)$. If $3 \nmid m$, then $3 \nmid 4m+3$, so $\gcd(3, 4m+3) = 1$. Using the fact that $\sigma(n)$ is a multiplicative function, we see that*

$$\sigma(n) = \sigma(3(4m+3)) = \sigma(3)\,\sigma(4m+3) = 4\sigma(4m+3),$$

because $\sigma(3) = 1+3 = 4$. So in this case, $\sigma(n) \equiv 0 \pmod 4$. However, $2n = 12k+6 \equiv 2 \pmod 4$. Hence n cannot be perfect. Thus $3 \mid m$ and we deduce that $n = 12(3q)+9 = 36q+9$.