

1 Conjunctive Normal Form

We already defined a literal to be either an atomic proposition or the negation of an atomic proposition. A conjunction is an expression of the form $p_1 \wedge p_2 \wedge \dots \wedge p_n$. A disjunction is an expression of the form $q_1 \vee q_2 \vee \dots \vee q_n$. A formula is in CNF if it is a conjunction of disjunctions of literals. Some examples of CNF expressions

$$1. \neg((p \wedge q) \wedge r) \Leftrightarrow (\neg p \vee \neg q) \wedge \neg r$$

Follow the following steps to get a formula into CNF.

1. Remove all \rightarrow and \leftrightarrow operators by replacing them with equivalent formulas.

- $p \rightarrow m \Leftrightarrow \neg p \vee m$
- $p \leftrightarrow m \Leftrightarrow (p \wedge m) \vee (\neg p \wedge \neg m)$

2. Move the \neg operators as far into the formula as possible using *double complement* and *De Morgan's laws*.

- $\neg \neg m \Leftrightarrow m$.
- $\neg(m \wedge p) \Leftrightarrow (\neg m \vee \neg p)$
- $\neg(m \vee p) \Leftrightarrow (\neg m \wedge \neg p)$

3. Move all conjunctions (\wedge) outside the disjunctions (\vee) using *distributive laws*.

- (a) $(m \wedge p) \vee r \Leftrightarrow (m \vee r) \wedge (p \vee r)$
- (b) $m \vee (p \wedge r) \Leftrightarrow (m \vee p) \wedge (m \vee r)$

Here is an example of such a conversion. Suppose that we are trying to convert the expression $(p \rightarrow q) \rightarrow (\neg r \rightarrow (s \wedge t))$.

$$\begin{aligned} (p \rightarrow q) \rightarrow (\neg r \rightarrow (s \wedge t)) &\Leftrightarrow \neg(p \rightarrow q) \vee (\neg r \rightarrow (s \wedge t)) \\ &\Leftrightarrow \neg(\neg p \vee q) \vee (\neg r \rightarrow (s \wedge t)) \\ &\Leftrightarrow \neg(\neg p \vee q) \vee (r \vee (s \wedge t)) \\ &\Leftrightarrow (p \wedge \neg q) \vee ((r \vee s) \wedge (r \vee t)) \\ &\Leftrightarrow ((p \wedge \neg q) \vee (r \vee s)) \wedge ((p \wedge \neg q) \vee (r \vee t)) \\ &\Leftrightarrow (p \vee r \vee s) \wedge (\neg q \vee r \vee s) (p \vee r \vee t) \wedge (\neg q \vee r \vee t) \end{aligned}$$

Notice that the CNF expression can become very long.