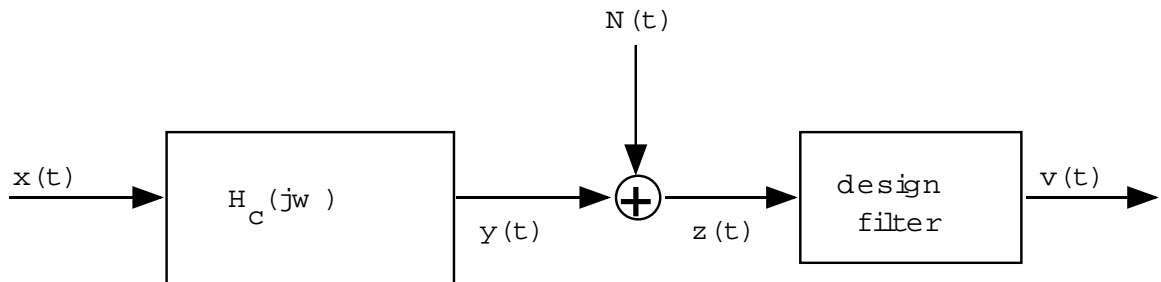Wiener Hopf Theory

1   Introduction

This theory arises when there are two related signals and it is desired to estimate the one signal from the other "in real time", that is, without delay. The following figure illustrates a typical application.

The tools used to design the filter involve the "spectrum" and "cross spectrum" of various signals. We begin with the spectrum of $x(t)$, $S_{xx}(j\omega)$, the channel frequency response, $H_C(j\omega)$ and the spectrum of the interference, $S_{NN}(j\omega)$. It is desired to find a stable, causal filter applied to $z(t)$ whose output, $v(t)$ matches $x(t)$ as closely as possible in the sense of minimizing

$\mathrm{AVE}\{(x(t) - v(t))^2\}.$

The various properties of spectra and cross-spectra are:

1 $S_{aa}(j\omega)$ is real and ≥ 0

2 If $a(t)$ is real for all t then $S_{aa}(-j\omega) \equiv S_{aa}(j\omega)$

3 $S_{ab}(j\omega) = S_{ba}^*(j\omega)$

4 if $b(t) = a(t) * h_C(t)$
   then

$$S_{ba}(j\omega) = S_{aa} H_C(j\omega)$$

$$S_{ab}(j\omega) = S_{xx} H_C^*(j\omega)$$

$$S_{bb}(j\omega) = S_{aa}(j\omega) |H_C(j\omega)|^2$$

5 if $c(t) = a(t) + b(t)$
   then

$$S_{cc}(j\omega) = S_{aa}(j\omega) + S_{ab}(j\omega) + S_{ba}(j\omega) + S_{bb}(j\omega)$$

For signals which are "independent" the cross spectrum is zero. In the above figure we assume the signal $x(t)$ and the interference, $N(t)$ are independent so that $S_{xN}(j\omega) \equiv 0.$

2 Wiener Hopf Theory

The explanation as to why the computation below gives the best estimate of the input at time t, given the output at all times $\{\tau < t\}$ depends upon a knowledge of stochastic processes (EE562a). Since EE562a is not a prerequisite for EE301a, we will have to accept it on faith.

3 Wiener-Hopf Calculation

This description applies only to the case when all spectra and cross-spectra are rational. That is, when the fromula for the spectra are ratios of polynomials in ω. Refering to the figure and the labels of the various signals the steps in the calulation:

Step 1: Factor

$$S_{zz}(\omega) = G^+(\omega) \cdot G^-(\omega)$$

where G^+ contains all of the zeros and poles with positive imaginary part (in upper half plain) and G^- contains all of the zeros and poles with negative imaginary part (in lower half plain).

Step 2:   Divide S_{xz} by G^- and do partial fraction expansion.
$$\left[\frac{S_{xz}(\omega)}{G^-(\omega)}\right] = \sum \frac{A_n}{\omega - a_n}$$

Step 3:   Construct a causal transfer function by taking only those terms whose poles have positive imaginary part.
$$\left[\frac{S_{xz}(\omega)}{G^-(\omega)}\right]_+ = \sum_{\mathcal{I}m(a_n)>0} \frac{A_n}{\omega - a_n}$$

Step 4:   Divide by G^+ to obtain the desired estimation filter:
$$H_{design}(\omega) = \frac{1}{G^+(\omega)} \cdot \left[\frac{S_{xz}(\omega)}{G^-(\omega)}\right]_+$$

Step 5:   Rewrite in rational form (ratio of two polynomials):
$$H_{design}(\omega) = \frac{P(\omega)}{Q(\omega)}$$