Some ring theoretic equivalents to the Axiom of Choice

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ABSTRACT

This paper gives an historically oriented survey of some propositions concerning the existence of maximal one or two sided ideals, and their relation to the Axiom of Choice in Zermelo-Fraenkel set theory. It is observed that the statement "if a ring contains a nonzero idempotent element, then the ring has a maximal left and a maximal right ideal," is equivalent to the Axiom of Choice. More general conditions implying the existence of maximal one-sided ideals are given.

The first major result on the existence of maximal ideals in rings was obtained in 1929 by Wolfgang Krull and is now known as Krull's Theorem [8]: In a commutative ring with unity every proper ideal can be extended to a maximal ideal. Krull used the Well Ordering Principle to prove this result. It was known at the time that the Well Ordering Principle is equivalent to the Axiom of Choice (A.C.) in Zermelo-Fraenkel (Z.F.) set theory. Herein other conditions which guarantee the existence of maximal ideals (left, right, or two-sided) in general associative rings are considered, and their relations to A.C. in Z.F. set theory are considered. (A more detailed account assuming considerably more ring theoretic background will be given in [3]). Examples of rings without maximal ideals are discussed to show some of the limitations on the theory developed. Herein R will always denote a ring, associative, not necessarily commutative, with or without unity, and having more than one element. The term "ideal" without a modifier will mean a two-sided ideal.

Max Zorn presented a different proof of Krull's Theorem in 1934, which was published in 1935 [14]. Zorn was unaware of Krull's work at the time. Zorn's proof used a maximal principle he had formulated, which is now known as Zorn's Lemma: Let Ω be a nonempty family of subsets of a nonempty set S. If every chain Γ in $(\Omega, \subseteq)$ has its union, $\bigcup \Gamma$, in Ω, then $(\Omega, \subseteq)$ has a maximal term. Zorn's Lemma, which was popularized (in a more general form) by the Bourbaki in the early 1940's, has become a standard tool in transfinite algebra. In 1939, John Tukey [13] proved that Zorn's Lemma is equivalent to A.C. in Z.F. set theory. With only minor modification, Zorn's proof can be used to establish a more general result. By the early 1940's the following was ring theory "folklore."

Theorem 1 *If R has a left unity and I is a proper right (two-sided) ideal of R, then R contains a maximal right (two-sided) ideal which contains I.*
(Note: The obvious dual proposition using right unity and left ideals holds.)

Implicit in the seminal work of Nathan Jacobson in the mid-1940's, [7], is the following result:

Theorem 2 *If R has a nonzero idempotent, then R has a maximal left and a maximal right ideal.*

To obtain this from Jacobson's theory requires substantial knowledge of ring theory and of the Jacobson radical. To avoid this we next give an elementary proof, which uses Zorn's Lemma in a typical way. First, recall that if e is an idempotent in R, then $\mathbf{r}(e)=\{er - r: r\in R\}$ is a right ideal of R and $R = eR + \mathbf{r}(e)$, as a direct sum of right ideals.

Proof. If $\mathbf{r}(e)=0$, then e is a left unity and the desired result is immediate from Theorem 1. Let Ω be the set of all proper right ideals of R which contain $\mathbf{r}(e)$, and let Γ be a chain in $(\Omega, \subseteq)$. Then $T=\bigcup \Gamma$ is a right ideal of R and $\mathbf{r}(e)\subseteq T$. If $T=R$, then $e \in T$, and hence $e \in X$, for some $X\in \Gamma$. Then $R=eR + \mathbf{r}(e)\subseteq X$, a contradiction. Hence $T\in \Omega$ and by Zorn's Lemma $(\Omega, \subseteq)$ has a maximal term. Any such maximal term in $(\Omega, \subseteq)$ is a maximal right ideal of R. Similarly one obtains a maximal left ideal.

Since a left or a right unity in R is a nonzero idempotent, Theorem 2 implies the conclusion in Theorem 1 can be extended to include the existence of a maximal left ideal as well.

If the idempotent used in this proof is central (i.e., $ex=xe$, for each $x\in R$), then $\mathbf{r}(e)$ and eR will be ideals and the argument given above can be used to obtain the following:

Corollary 2A. *If R has a nonzero central idempotent, then R has a maximal ideal.*

In light of Theorem 2 and Corollary 2A, it is natural to ask if the existence of a nonzero idempotent implies the ring has a maximal two-sided ideal. The following example supplies a negative answer.

Example 1. Let V be a vector space over a field F with dim_FV = $\aleph_\omega$, where ω is the first infinite ordinal. Use End_FV for the full ring of linear transformations on V and define $R_\alpha = \{f \in End_F V : rank(f) < \aleph_\alpha\}$, for all ordinals $\alpha \leq \omega$ and let R = R_ω. It is well-known that the nonzero, proper ideals of R are the sets R_α, $\alpha < \omega$, [1]. The ring R has a plethora of idempotents, in fact it is von Neumann regular, but it has no maximal ideals. (The maximal one-sided ideals abound).

The standard example of a commutative ring without a maximal ideal is to take the trivial multiplication, xy=0 for all x,y, on the Prüfer quasi-cyclic group, C(p^∞).

In 1954, Dana Scott [11] asked if Krull's Theorem implies A.C. in Z.F. set theory. Some of the consequences in ring theory of working in Z.F. with the denial of A.C. were considered by Wilfrid Hodges in 1974, [4]. It is an understatement to say that the ring examples so obtained are "extremely pathological." In 1978, Hodges answered Scott's query, giving the even stronger result below.

Theorem 3 *(Hodges, [5]). In Z.F. set theory the statement "Every unique factorization domain has a maximal ideal" implies A.C.*

With this result one immediately gets that Theorem 1, Theorem 2, and A.C. are all equivalent in Z.F. set theory.

The line of investigation carried out by Hodges was extended by Y. Rav in 1988 to obtain the next result.

Theorem 4 *(Rav, [9]). In Z.F. set theory the statement " Every unique factorization domain can be mapped homomorphically onto a nonzero subdirectly irreducible ring" is equivalent to A.C.*

(Note: A ring R is subdirectly irreducible if the intersection of all the nonzero ideals of R is nonzero; see [2, p.64].)

Returning to the development of conditions that guarantee the existence of maximal one or two-sided ideals, the next theorem extends Theorems 1 and 2 and Corollary 2A.

Theorem 5 *Let $\bar{R}$ be a homomorphic image of R.*
(i)If $\bar{R}$ contains a nonzero element b such that $b^n = b^m \neq 0$, for some n > m, then R has a maximal left and a maximal right ideal. If this b is central in $\bar{R}$, then R contains a maximal two-sided ideal also.
(ii)If $\bar{R}$ contains a left or a right unity, then R has a maximal (left, right, two-sided) ideal.

Proof. Recall from the Correspondence Theorem for ring homomorphisms that a maximal (left, right, two-sided) ideal in $\bar{R}$ lifts to a maximal (respectively: left, right, two-sided) ideal in R. It is well-known that if $b^n = b^m \neq 0$, $n > m$, then some power of b is a nonzero idempotent. Also, if b is central, then so is any power of b. Then Theorem 2 and Corollary 2A give the desired maximal structures in $\bar{R}$, which lift to R. Similarly, use Theorems 1 and 2 to obtain part (ii).

The observation that a homomorphic image can have either of the desired elements hypothesized in (i) or (ii) without the pre-image having such an element shows that Theorem 5 is indeed a proper generalization of Theorem 2 and Corollary 2A. For example the ring $\mathbb{Z}_3$ is a homomorphic image of the ring of even integers, illustrating this observation.

The theories of the Jacobson radical and of the Brown-McCoy radical give general conditions for the existence of maximal one and two-sided ideals, respectively. (See [2] or [12] for a thorough treatment of these radicals and their relation to maximal one and two-sided ideals). However, the criteria in Theorems 2 and 5 for the existence of such ideals are both elementary and broad, and the proofs of these theorems require no sophisticated ring theory and nothing about radicals.

A worthwhile quest would be to find elementary necessary and sufficient conditions for a ring to have maximal (one or two-sided) ideals, as well as to investigate the relationships between having maximal one-sided ideals and maximal two-sided ideals. The relation of these results to the A.C. in Z.F. set theory would also be of interest. (For an encyclopedic summary of equivalences to A.C. in Z.F. set theory see [6],[10]).

A wide range of sufficient conditions, and some results on necessity, for existence of maximal one or two-sided ideals are given in [3].

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