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On a New Segal Algebra

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Abstract. By means of a certain kind of 'atomic' representation a new Segal algebra $S_0(G)$ of continuous functions on an arbitrary locally compact abelian group G is defined. From various characterizations of $S_0(G)$, e. g. as smallest element within the family of all strongly character invariant Segal algebras, functorial properties of the symbol S_0 are derived, which are similar to those of the space $\mathscr{S}(G)$ of Schwartz—Bruhat functions, e. g. invariance under the Fourier transform, or compatibility with restrictions to closed subgroups. The corresponding properties of its Banach dual $S_0'(G)$ as well as some of their applications are to be given in a subsequent paper.

Introduction

Segal algebras, as introduced by Reiter ([30]), constitute a family of dense ideals of $L^1(G)$, for a locally compact group G. They have found much interest in the last decade, being very closely related to $L^1(G)$ in many instances, and showing a completely different behaviour in other respects. The Segal algebra $S_0(G)$ to be defined below is a good example for this ambiguity.

Let us begin by recalling some terminology concerning Segal algebras as well as harmonic analysis in general. Notations that are not explained explicitly are taken from Reiter's book ([30]). Throughout this paper G denotes a locally compact *abelian* group with Haar measure dx; the group operation is written as multiplication. Although some of the results below are true for arbitrary locally compact groups this seems to be the appropriate degree of generality for most of the applications to be given here, and the use of characters and the Fourier transform yields easier proofs in many situations.

For $y \in G$ the translation operator L_y is defined by

$$L_y f(x) := f(y^{-1} x), \ x \in G,$$

and for $t \in \hat{G}$ the multiplication operator M_t is given by

$$M_t f(x) := \langle x, t \rangle f(x), \ x \in G.$$

For any function f on G the functions $\check{f}$ and f^* are given by

$$\check{f}(x) := f(-x), \quad f^*(x) := \overline{f(-x)}, \quad x \in G.$$

$\mathcal{K}(G)$ denotes the (topological) vector space of all continuous functions on G with compact support (supp). It is endowed with its natural inductive limit topology. Its topological dual is $R(G)$, the space of all Radon measures. The space $L^1_{loc}(G)$ of all locally integrable functions on G is considered as a (closed) subspace of $R(G)$, i. e. two measurable functions which coincide locally almost everywhere (l. a. e.) are identified as usual. The topology on $L^1_{loc}(G)$ is thus generated by the family of seminorms (s_K), $s_K(f) := \int_K |f(x)|\,dx$, where K ranges over the family of all compact subsets of G. $(L^p(G), \|\ \|_p)$, $1 \leqslant p \leqslant \infty$, denotes the usual Lebesgue spaces on G. For any subspace B of $L^1_{loc}(G)$ and $K \subseteq G$, B_K denotes the subspace of all $f \in B$ satisfying supp $f \subseteq K$.

A Banach space $(B, \|\ \|_B)$ which is continuously embedded in $L^1_{loc}(G)$ is called (*strongly*) *translation invariant* if $L_y B \subseteq B$ for all $y \in G$ (and $\|L_y f\|_B = \|f\|_B$ for all $f \in B$, $y \in G$), and (*strongly*) *character invariant* if $M_t B \subseteq B$ for all $t \in \hat{G}$ (and $\|M_t f\|_B = \|f\|_B$ for all $f \in B$). A strongly translation invariant Banach space $(B, \|\ \|_B)$ is called a *homogeneous Banach space* on G if $\lim_{y \to e} \|L_y f - f\|_B = 0$ for all $f \in B$. If furthermore B is a dense subspace of $L^1(G)$ it is called a *Segal algebra* (cf. [30]).

The Fourier transform $\mathcal{F}_G$ defines an injective mapping from $L^1(G)$ into $C^0(\hat{G})$. Occasionally it is convenient to write $\hat{f}$ instead of $\mathcal{F}_G f$. $A(G) := \mathcal{F}_{\hat{G}}(L^1(\hat{G}))$ *, endowed with the norm inherited from $L^1(\hat{G})$ (i. e. $\|h\|_{A(G)} := \|f\|_1$ for $h = \hat{f}$) is a Banach algebra with respect to pointwise multiplication, called the *Fourier algebra*, as well as a homogeneous Banach space on G.

A Banach space $(B, \|\ \|_B)$ is called a (*left*) *Banach module over a Banach algebra* $(A, \|\ \|_A)$ if it is a module in the algebraic sense, satisfying in addition $\|ab\|_B \leqslant \|a\|_A \|b\|_B$ for all $a \in A$, $b \in B$ (cf. [32]). Any homogeneous Banach space B is a (left) Banach module over $L^1(G)$ with respect to convolution. Furthermore, one has $\lim_{\alpha \to \infty} \|e_\alpha * f - f\|_B = 0$ for any $f \in B$, if $(e_\alpha)_{\alpha \in I}$ is any *bounded approximate unit* for $L^1(G)$, i. e. if (e_α) is a bounded net in $L^1(G)$ satisfying $\lim_{\alpha \to \infty} \|e_\alpha * g - g\|_1 = 0$ for all $g \in L^1(G)$.

* Here G is identified with $(G^\wedge)^\wedge$ as usual.

Let B^1 and B^2 be two homogeneous Banach spaces on G, and T a bounded linear operator from B^1 into B^2. Then T is called a *multiplier* from B^1 into B^2 if it satisfies $TL_y = L_y T$ for all $y \in G$. This holds if any only if T is an *L^1-module homomorphism* (i. e. $T \in H_{L^1}(B^1, B^2)$: $T(g*f) = g*T(f)$ for all $g \in L^1(G)$, $f \in B^1$) (cf. [24] and [11]).

For any closed subgroup H of G the restriction mapping $f \mapsto f|H$ (defined for any continuous function f on G) is denoted by R_H. The canonical mapping T_H: $\mathscr{K}(G) \to \mathscr{K}(G/H)$ is given by the formula

$$T_H f(\dot{x}) = \int_H f(x\xi)\, d\xi, \quad \dot{x} = \pi_H(x),$$

where $\pi_H: G \to G/H$ denotes the canonical projection (cf. [30], III, 4). It extends to a contractive algebra homomorphism from $L^1(G)$ onto $L^1(G/H)$.

Given two functions f^1 and f^2 on G^1 and G^2 respectively, $f^1 \otimes f^2$ denotes the function on $G^1 \times G^2$ given by

$$f^1 \otimes f^2(x_1, x_2) = f^1(x_1) f^2(x_2), \quad x_i \in G^i, \ i = 1, 2.$$

Given two homogeneous Banach spaces B^1 and B^2 on G^1 and G^2 respectively, $B^1 \hat{\otimes} B^2$ denotes their *projective tensor product*, i. e. the space $\{f \mid f = \Sigma f_n^1 \otimes f_n^2, \ \sum \|f_n^1\|_{B^1} \|f_n^2\|_{B^2} < \infty\}$; this is a homogeneous Banach space on $G^1 \times G^2$ with the norm $\|f\|_{\hat{\otimes}} := \inf\{\sum \|f_n^1\|_{B^1} \|f_n^2\|_{B^2}, \ldots\}$.

The characteristic function of a subset $K \subseteq G$ is denoted by c_K. For convenience we shall often write only Σ instead of $\Sigma_{n=1}^{\infty}$. Numerical constants are denoted by C, $C_1, \ldots$, the same symbol possibly representing different values at its various occurrences.

Some 'historical' remarks are in order here. At the origin of this paper were two independant problems (about winter 1977). The first, posed by REITER in connection with the generalized Weil—Cartier theorem (cf. [31]), concerned the existence of a Segal algebra defined for arbitrary locally compact abelian groups and having certain functorial properties (known to hold for the Schwartz—Bruhat space $\mathscr{S}(G)$). The second one, posed by the author himself, concerned the existence of smallest elements in certain families of Segal algebras (cf. [10] and [14]). As the reader will see below, the solution to both problems turnes out to be the same space $S_0(G)$; this coincidence even considerably simplifies many arguments in proving further properties of $S_0(G)$.

Partial results concerning $S_0(G)$ and its dual $S_0'(G)$ have been given previously in two preliminary reports ([12]) and in several lectures by the author, first in Vienna in February 1979. A summary of results was given in [13]. This explains why two papers concerning the Segal algebra $S_0(G)$, by LOSERT ([26]) and POGUNTKE ([29]), could be published before this paper.

Since it turned out in the meantime that $S_0(G)$ as well as $S_0'(G)$ may be considered as (important) special cases of the so-called *spaces of Wiener's type* as treated by the author in [15] and [16], we restrict our attention here as far as possible to those aspects and properties which are special to $S_0(G)$.

The functorial properties (cf. Theorem 7 below) of the symbol S_0, in particular its invariance under the Fourier transform, imply corresponding properties of its Banach dual $S_0'(G)$. The characterization of $S_0'(G)$ as a space of quasimeasures on G suggests the term 'translation bounded quasimeasures' for its elements. Basic properties of $S_0'(G)$ and certain applications to harmonic analysis, in particular to the theory of Fourier transforms of unbounded measures and to the theory of multipliers (cf. e. g. [1], [20], [34], or [19], [24], [36]), are to be given in a subsequent paper.

The Smallest Strongly Character Invariant Segal Algebra

In this paper a new Segal algebra $S_0(G)$ of continuous functions is introduced. It is defined for arbitrary locally compact abelian groups G. The elements of this Segal algebra are characterized by the existence of certain "atomic" representations (cf. [5], where the usefulness of a similar concept in connection with the theory of real Hardy spaces is explained). It will be shown that the symbol S_0 has several interesting functorial properties. Most of these properties are derived from a characterization of $S_0(G)$ as the smallest strongly character invariant Segal algebra on G with respect to inclusion.

Definition 1. Let G be a locally compact abelian group, and let $Q \subseteq G$ be a (fixed) open subset of G with compact closure. Then $S_0(G)$ is defined as the set of those (continuous) $f \in L^1(G)$ which have a representation (as L^1-convergent sum) of the form

$$f = \sum_{n=1}^{\infty} L_{y_n} f_n, \tag{1}$$

where $y_n \in G, f_n \in A_Q(G), n \geqslant 1$, and $\sum \|f_n\|_{A(G)} < \infty$. Any representation of f of the form (1) satisfying this condition will be called an *admissible representation* of $f \in S_0(G)$.

It will be seen below that $S_0(G)$ is in fact independent of Q. The following characterization of $S_0(G)$ will be the basis for most of the results concerning this space.

Theorem 1. *Endowed with the norm*

$$\|f\|_{S_0} := \inf\{\sum \|f_n\|_{A(G)}, f = \sum L_{y_n} f_n \text{ admissible}\} \tag{2}$$

$S_0(G)$ is a strongly character invariant Segal algebra on G. It is the smallest such Segal algebra, i.e. it is continuously embedded in any other strongly character invariant Segal algebra. $S_0(G)$ is invariant under automorphisms of G.

Proof. Step I. First we show that $(S_0(G), \| \ \|_{S_0})$ is a strongly translation invariant Banach space, continuously embedded in $(L^1(G), \| \ \|_1)$. For any admissible representation (1) of f we have

$$\|f\|_1 \leqslant \sum \|L_{y_n} f_n\|_1 = \sum \|f_n\|_1 \leqslant \sum \|c_Q\|_1 \cdot \|f_n\|_{A(G)},$$

thus $\|f\|_1 \leqslant \|c_Q\|_1 \cdot \|f\|_{S_0}$ for any $f \in S_0(G)$. It is thus clear that $(S_0(G), \| \ \|_{S_0})$ is a normed space, continuously embedded in $L^1(G)$ and in $A(G)$; $S_0(G)$ is even continuously embedded in $W(G)$, cf. [10]. Furthermore, one clearly has $\|L_y f\|_{S_0} = \|f\|_{S_0}$ for $f \in S_0(G)$ and $y \in G$. Also $(S_0, \| \ \|_{S_0})$ is complete, since for any sequence $(f_k)_{k \geqslant 1}$ in $S_0(G)$ with $\sum \|f_k\|_{S_0} := C < \infty$ the continuous function $f := \sum_{k=1}^{\infty} f_k$ in $L^1(G)$ belongs to $S_0(G)$ and satisfies $\|f\|_{S_0} \leqslant C$.

Step II. Let us now verify that $S_0(G)$ is a strongly character invariant Segal algebra. To show that $\lim \|L_y f - f\|_{S_0} = 0$ $(y \to e)$ for any $f \in S_0(G)$, it is enough to verify this for those $f \in S_0(G)$ which have an admissible representation as a finite sum (these functions being dense in $S_0(G)$!), hence only for $f \in A_Q(G)$, i.e. for the "atoms". Since Q is an open set, there exists for every $f \in A_Q(G)$ some neighbourhood U of the identity such that for all $y \in U$ $\operatorname{supp}(L_y f) \subseteq Q$, i.e. $L_y f \in A_Q(G)$ hence $\|L_y f - f\|_{S_0} \leqslant \|L_y f - f\|_{A(G)}$, and thus $\|L_y f - f\|_{S_0} \to 0$ $(y \to e)$ for any $f \in A_Q(G)$. Thus $S_0(G)$ is a homogeneous Banach space, in particular a nontrivial ideal in $L^1(G)$.

That $(S_0(G), \| \ \|_{S_0})$ is strongly character invariant follows from its definition, since

$$M_t L_x f = L_x(\langle x, t\rangle M_t f), \ x \in G, \ t \in \hat{G}, \tag{3}$$

and

$$\| \langle x, t \rangle M_t f \|_{A(G)} = \| f \|_{A(G)} \text{ for all } f \in A(G). \tag{4}$$

By Wiener's theorem (cf. [30], VI, 1) $S_0(G)$ is a dense subspace of $L^1(G)$, as its L^1-closure is a closed ideal with empty cospectrum.

Step III. Let now any strongly character invariant Segal algebra $S(G)$ be given. We have to show that $S_0(G)$ is continuously embedded in $S(G)$. Define an auxiliary space on $\hat{G}$ by

$$S_1(\hat{G}) := \{h, h \in L^1(\hat{G}), \hat{h} \in S(G)\},$$

with the norm $\|h\|_{S_1} := \|h\|_1 + \|\hat{h}\|_S$. Now we show: $S_1(\hat{G})$ is a Segal algebra. The strong character invariance of $S(G)$ implies

$$\|L_t h\|_{S_1} = \|L_t h\|_1 + \|M_t \hat{h}\|_S = \|h\|_1 + \|\hat{h}\|_S = \|h\|_{S_1} \text{ for all } t \in \hat{G}.$$

In order to show $\lim \|L_t h - h\|_{S_1} = 0$ $(t \to \hat{e})$ for $h \in S_1(\hat{G})$ it will be sufficient to verify $\|M_t f - f\|_S \to 0$ for $f \in S(G)$, $t \to \hat{e}$. Since $S(G)$ is strongly character invariant it is sufficient to show this for $f \in S(G)$ with $\hat{f} \in \mathscr{K}(\hat{G})$ (these functions are dense in any Segal algebra, see [30], Chap. 6, § 2.2.iii). For such f it follows from the obvious relation $\|M_t f - f\|_1 \to 0$ for $f \in L^1(G)$, $t \to \hat{e}$, and the fact that the norms of $S(G)$ and $L^1(G)$ are equivalent on $\{f, f \in S(G), \operatorname{supp} \hat{f} \subseteq K\}$ for any compact subset $K \subseteq \hat{G}$ (cf. [30], Chap. 6, § 2.2.iv). Evidently, $S_1(\hat{G})$ is strongly character invariant and thus dense in $L^1(\hat{G})$ (see Step II).

Applying once more [30], Chap. 6, § 2.2.iii we obtain $\mathscr{K}(G) \cap A(G) \subseteq \hat{S}_1 \subseteq S(G)$, and with $K \supseteq Q$, the norms of $A(G)$ and $\hat{S}_1(G)$ are equivalent on $A_Q(G)$. In particular, there is a $C > 0$ such that

$$\|f\|_S \leqslant \|f\|_{\hat{S}_1} \leqslant C \|f\|_{A(G)} \text{ for all } f \in A_Q(G). \tag{5}$$

Thus for any $f \in S_0(G)$ having an admissible representation $f = \sum L_{y_n} f_n$, the series also converges in $S(G)$, and

$$\|f\|_S \leqslant \sum \|L_{y_n} f_n\|_S = \sum \|f_n\|_S \leqslant C \sum \|f_n\|_{A(G)} < \infty.$$

Hence $S_0(G) \subseteq S(G)$, and $\|f\|_S \leqslant C \|f\|_{S_0}$ for all $f \in S_0(G)$.

That the mapping $\alpha^* : f \mapsto \alpha^*(f)$, given by $\alpha^*(f)(x) := f(\alpha x)$ defines an automorphism of $S_0(G)$ for any $\alpha \in \operatorname{Aut} G$ follows from the fact that $\alpha^*(S_0(G))$ is a strongly character invariant Segal algebra, hence $S_0(G) \subseteq \alpha^*(S_0(G))$ by the minimality of $S_0(G)$. For reasons of symmetry equality must hold. This completes the proof of the Theorem.

We mention that it would have been possible to derive the main part of the above theorem from Corollary 4 of [14], by showing that the strongly character invariant Segal algebras are exactly those which are (essential) Banach modules over $A(G)$ with respect to pointwise multiplication. However, we have preferred to give an (altogether) slightly shorter direct proof, in order to make the paper more self contained.

Remarks: 1. It follows immediately from the theorem that $S_0(G)$ does not depend on the particular choice of Q, i.e. two different relatively compact subsets Q_1, Q_2 of G with nonvoid interior define the same space and equivalent norms. A direct proof using bounded uniform partitions of unity for $A(G)$ (cf. below) would give the same result. In particular, the norms of $S_0(G)$ and $A(G)$ are equivalent on $A_K(G)$ for any compact subset K of G.

2. It follows from the first remark that

$$\mathscr{K}(G) \cap A(G) = \{f \mid f \in \mathscr{K}(G), \hat{f} \in L^1(\hat{G})\} =: D(G)$$

is contained in $S_0(G)$ as a dense subspace. If $D(G) = \bigcup_K A_K(G)$, the union being taken over all compact subsets $K \subseteq G$, is endowed with its natural inductive limit topology, then the embedding of $D(G)$ into $S_0(G)$ is continuous (cf. [6], where it is shown that this space actually coincides with the space $D(G)$ introduced in [19], cf. [24], 5.1).

3. It is clear that $D(G) = S_0(G)$ $(= A(G))$ iff G is compact, in particular $S_0(\mathbb{T}) = A(\mathbb{T})$. It will be seen below that in many respects $S_0(G)$ behaves much more like $A(\mathbb{T})$ than $A(G)$ does in general. On the other hand one has $S_0(G) \subseteq L^1 \cap A(G)$ (again as a dense subspace), the sum (1) being absolutely convergent in $L^1(G)$ as well as in $A(G)$. If G is discrete both spaces coincide with $L^1(G)$. It can be shown, however, that the above inclusion is proper if G is non-compact and non-discrete. For the sake of shortness the proof is left to the interested reader. A much stronger strict inclusion has been shown by Losert ([26], cf. Remark 16 below).

4. It can be shown (cf. [12], I, Theorem 7) that there does not exist a minimal character invariant Segal algebra in general.

For the sake of completeness we mention two other characterizations of $S_0(G)$ that have been derived in a more general context elsewhere. We need one definition (see [14]).

Definition 2. A family $(\psi_i)_{i \in I} \subseteq A(G)$ is called a *bounded uniform partition of unity* in $A(G)$ if there exists a compact set $W \subseteq G$, and a (discrete) subset $Y = (y_i)_{i \in I} \subseteq G$ such that

A) $\sum_{i \in I} \psi_i(x) \equiv 1$;

B) $\sup_{i \in I} \|\psi_i\|_{A(G)} < \infty$;

C) $\operatorname{supp} \psi_i \subseteq y_i W$ for all $i \in I$;

D) $\sup_{y \in G} \# \{i \mid xK \cap y_i W \neq \emptyset\} < \infty$ for any compact set $K \subseteq G$.

For $G = \mathbb{R}^m$ there are particularly simple bounded uniform partitions of unity consisting of translates by elements of a lattice (e. g. $\mathbb{Z}^m$) of a single trapezoid-function in $A(G)$ (cf. [37], p. 91, for $m = 1$). Using structure theory the same can be done for arbitrary locally abelian groups (cf. [12], II, proof of Theorem 1.1, [29] or [34]).

Theorem 2. *For any $f \in A(G)$ the following is true:*

A) *For any given $(\psi_i)_{i \in I}$ as in Definition 2 one has*

$$f \in S_0(G) \Leftrightarrow \|f\|'_{S_0} := \sum_{i \in I} \|f \psi_i\|_{A(G)} < \infty. \tag{6}$$

Furthermore, $\| \ \|'_{S_0}$ defines an equivalent norm on $S_0(G)$.

B) *Let $h \in \mathcal{K}(G) \cap A(G)$, $h \neq 0$, be given. Then $f \in S_0(G)$ if and only if F_h: $F_h(z) := \|(L_z h) f\|_{A(G)}$ belongs to $L^1(G)$. Furthermore, $f \mapsto \|F_h\|_1$ defines another equivalent norm on $S_0(G)$.*

C) *Let $f_0 \in S_0(G)$, $f_0 \neq 0$ be given* (e. g. $f_0 \in A_Q(G)$). *Then*

$$S_0(G) = \{f \mid f = \sum f_n * M_{t_n} f_0,\ t_n \in \hat{G},\ f_n \in L^1(G),\ n \geq 1,\ \sum \|f_n\|_1 < \infty\} \tag{7}$$

and

$$\|f\|''_{S_0} := \inf \{\sum \|f_n\|_1, f = \sum \dots\} \tag{8}$$

defines an equivalent norm on $S_0(G)$ as well.

Proof. For a proof of A cf. [15], Theorem 2, but a special case is contained in the proof of Theorem 1.1 of [12], II (cf. also [26], Proposition 1). Assertion B is essentially a special case of the main result of [14], or of [29], and is equivalent to Proposition 2 of [24]. In order to prove C one shows first that the space defined there is a strongly character invariant Segal algebra with the norm $\| \ \|''_{S_0}$. Since $M_t(f * g) = M_t f * M_t g$, $t \in \hat{G}$, this can be done by standard arguments. Since the series used in (7) are obviously absolutely convergent in $S_0(G)$, it follows from the minimality of $S_0(G)$ that the two spaces coincide. The equivalence of both norms then follows.

Remarks: 5. Using the methods given in [15] it is even possible to show that $f \mapsto \| F_h \|_1$ defines an equivalent norm on $S_0(G)$ for any $h \in S_0(G)$, $h \neq 0$. In particular, one can say that $f \in A(G)$ belongs to $S_0(G)$ if any only if one has

$$\int \| (L_y f) f \|_{A(G)} \, dy < \infty .$$

This, of course, resembles the definition of the space $A^1(\mathbb{R})$ of good vectors given by EYMARD ([9]). The connections between $A^1(\mathbb{R})$ and $S_0(\mathbb{R})$ will be pointed out elsewhere (cf. [17]). More general spaces of a similar type have been considered by LEPTIN ([25]).

6. It is clear from above that $S_0(G)$ coincides with $W(A(G), L^1)$, the space of Wiener's type with *local* component $A(G)$ and *global* component $L^1(G)$, as introduced in [15].

7. Whereas the elements of $S_0(G)$ are originally defined by the existence of certain representations as sums, part A of Theorem 2 shows that $f \in S_0(G)$ may be characterized by conditions on a particular decomposition of f, namely $f = \sum_{i \in I} f \psi_i$ (observe that the series is norm convergent in $S_0(G)$!).

For the proof of the functorial properties of the symbol S_0 the characterization of $S_0(G)$ as a "convolution tensor product" of other Segal algebras (which are of Wiener's type as well) will be useful. For this purpose two definitions will be given.

Definition 3. Let V be an open, relatively compact subset of G. For $1 \leqslant p \leqslant \infty$ we set

$$W^p(G) := \{ f \mid f = \sum L_{y_n} f_n, \, y_n \in G, \, f_n \in L^p_V(G), \, n \geqslant 1, \, \sum \| f_n \|_p < \infty \}, \quad (9)$$

$$\| f \|_{W^p} := \inf \{ \sum \| f_n \|, \ldots \}, \quad (10)$$

the infimum again being taken over all admissible representations (as in (9)).

Proposition 3. *($W^p(G)$, $\| \; \|_{W^p}$) is a strongly translation invariant Banach space on G. The closure of $\mathcal{K}(G)$ in $W^p(G)$ (i.e. $W^p(G)$ for $1 \leqslant p < \infty$, and Wiener's algebra $W(G)$ for $p = \infty$) is a Segal algebra on G.*

Proof. For $1 \leqslant p < \infty$ the result follows from Theorem 5 of [14]. The case $p = \infty$, i.e. Wiener's algebra $W(G)$ requires only slight modifications (cf. [10]).

Remark 8. In [14] and [10] $W^p(G)$, $1 \leq p < \infty$, and $W(G)$ were characterized by a minimality property. Using methods of [15] it can be shown that $W^p(G)$ coincides with $W(L^1, L^p)$, or with W^p as defined in [22] for $G = \mathbb{R}$ and certain more general groups. In [2] this space is denoted by $l^1(L^p)$. Let us mention that a simple sufficient condition for $f \in L^1(\mathbb{R})$ to belong to $W(\mathbb{R})$ is that f be a continuous function of bounded variation.

Next we introduce the concept of a "convolution tensor product". We limit ourselves to a moderate degree of generality, sufficient for our purposes.

Definition 4. Let S be a Segal algebra; let $(B, \| \ \|_B)$ be a Banach space, continuously embedded in $L^1_{\mathrm{loc}}(G)$, which is a L^1-Banach module with respect to convolution (e. g. a homogeneous Banach space on G). The convolution tensor product $S \circledast B$ of these two spaces is given by

$$S \circledast B := \{f \mid f = \sum f_n * g_n,\ (f_n) \subseteq S,\ (g_n) \subseteq B,\ \sum \|f_n\|_S \|g_n\|_B < \infty\}. \tag{11}$$

We say that a Segal algebra admits *tensor product factorization* (or *feeble factorization*) if $S = S \circledast S$.

Lemma 4. *Let S and B be as in the definition. Then $S \circledast B$ is a homogeneous Banach space with respect to the norm*

$$\|f\|_{\circledast} := \inf\{\sum \|f_n\|_S \|g_n\|_B, \ldots\}, \tag{12}$$

the infimum being taken over all admissible representations of f. Furthermore, $S \circledast B$ is a (strongly) character invariant Banach space, if both S and B have this property.

Proof. The sums describing the elements of $S \circledast B$ being absolutely convergent in B it is clear that $(S \circledast B, \| \ \|_{\circledast})$ is a normed space continuously embedded into B. The completeness of $S \circledast B$ follows from the fact that absolutely convergent series in $S \circledast B$ are obviously convergent. Using the identity

$$L_y f = \sum L_y f_n * g_n, \quad y \in G, f \in S \circledast B \tag{13}$$

and the fact that translation is continuous in S it can be shown that $S \circledast B$ is a homogeneous Banach space. Finally, the character invariance follows from the formula

$$M_t f = \sum M_t f_n * M_t g, \quad t \in \hat{G}, f \in S \circledast B. \tag{14}$$

Remark 9. Using essentially the arguments given in ([33], Theorem 3.3) it can be shown that the convolution tensor product is isomorphic to the L^1-module tensor product $S \hat{\otimes}_{L^1} B$ of S and B as introduced by RIEFFEL ([32]). It is continuously embedded in B. It will be useful to keep this fact in mind, but we shall not make use of the theory of module tensor products in this paper.

Theorem 5. *Let* S_1, S_2 *be two Segal algebras on* G *satisfying* $S_0(G) \subseteq S_i \subseteq W^2(G)$, $i = 1, 2$. *Then*

$$S_1 \circledast S_2 = S_0 \tag{15}$$

and the corresponding norms are equivalent. In particular, $S_0(G)$ *has tensor product factorization.*

Proof. In view of the assumptions it will be sufficient to prove the first and the last inclusion of the following chain:

$$S_0 \subseteq S_0 \circledast S_0 \subseteq S_1 \circledast S_2 \subseteq W^2 \circledast W^2 \subseteq S_0 \tag{16}$$

(the corresponding norm inequalities then follow from the closed graph theorem). Since by Lemma 4 $S_0 \circledast S_0$ is a strongly character invariant Segal algebra the first inclusion follows from the minimality of S_0 (Theorem 1). The last inclusion follows essentially from the fact that one has $f * g \in A_Q(G)$ and

$$\|f * g\|_{A(G)} \leqslant \|f\|_2 \|g\|_2 \text{ for } f, g \in L^2_V(G), \text{ if } V^2 \subseteq Q,$$

and by means of a suitable rearrangement of an absolutely convergent double series.

Remarks: 10. It follows from the main result of [18] (Theorem 2.2) that $S_0(G)$ does not have weak factorization, i. e. that there exist elements $f \in S_0(G)$ which do *not* have a representation as

$$f = \textstyle\sum_{n=1}^{k} g_n * h_n, \; g_n, h_n \in S_0(G), \; 1 \leqslant n \leqslant k.$$

Even for $G = \mathbb{R}$ $S_0(\mathbb{R})$ seems to be the only proper Segal algebra S having tensor product factorization, at least under the restriction

$$\mathscr{F}_G S \subseteq \textstyle\bigcup_{p<\infty} L^p(\hat{G}).$$

That this is true for $G = \mathbb{T}$ ($S_0(\mathbb{T}) = A(\mathbb{T})$!) has been proved by C. C. GRAHAM (private communication) (cf. also [8], Theorem 3.1).

11. It is easily shown that one has

$$H_{L^1}(S, L^1) = H_{L^1}(S, S)$$

for any Segal algebra S having tensor product factorization. In particular,

$$H_{L^1}(S_1, S_2) \subseteq H_{L^1}(S_0, S_0)$$

for any pair (S_1, S_2) of strongly character invariant Segal algebras. One can even show that $H_{L^1}(W, W) \subseteq H_{L_1}(S_0, S_0)$ is a proper inclusion for $G = \mathbb{R}^m$ (cf. [12], II); there an explicit description of $H_{L^1}(S_0, S_0)$ is given and it is shown that $S_0(G)$ is a natural example of a Segal algebra whose multiplier algebra contains $M(G)$ as a proper subspace (cf. [2], and [36]).

As a consequence of Theorem 5 we obtain the following elementary characterization of $S_0(G)$:

Corollary 6. *Let V be a relatively compact subset of G with nonvoid interior. Then, given p, $2 \leqslant p \leqslant \infty$, one has*

$$S_0(G) = \{f \mid f = \sum L_{y_n}(f_n * g_n),\ f_n, g_n \in C_V(G),\ \sum \|f_n\|_p \|g_n\|_p < \infty\}, \quad (17)$$

and

$$\|f\| := \inf\{\sum \|f_n\|_p \|g_n\|_p, \ldots\} \quad (18)$$

defines an equivalent norm on $S_0(G)$.

Proof. The result follows from Theorem 5 and a verification that the right hand space coincides with $W^p \circledast W^p$, or from Theorem 1 and a direct verification that this space is a strongly character invariant Segal algebra contained in $S_0(G)$.

Remarks: 12. Corollary 6 shows that it is possible to restrict the set of "atoms" necessary for a representation of an arbitrary $f \in S_0(G)$ (cf. Definition 1) to those of a particularly simple form, i. e. to functions $f * g$, with $f, g \in C_V(G)$.

13. For any isomorphism $\alpha: G_2 \to G_1$ the bipositive algebra isomorphism $\alpha^*: L^1(G_1) \to L^1(G_2)$ given by

$$\alpha^*(f)(x) = f(\alpha x) \ \left(\alpha^*(f) \geqslant 0 \Leftrightarrow f \geqslant 0\right)$$

maps $(C_{\alpha Q}(G_1), \|\ \|_\infty)$ isometrically onto $(C_Q(G_2), \|\ \|_\infty)$. Corollary 6 therefore implies that α^* maps $S_0(G_1)$ onto $S_0(G_2)$. (This also gives another proof of the invariance of $S_0(G)$ under automorphisms.) Let us mention that the converse is also true, i. e. any bipositive algebra isomorphism is given in that way.

We are now ready to prove the main result of this paper which concerns the functorial properties of $S_0(G)$:

Theorem 7. *Let G, G_1, G_2 be locally compact Abelian groups, and let H be a closed subgroup of G. Then the following holds:*

A) $$\mathscr{F}_G[S_0(G)] = S_0(\hat{G});$$

B) $$T_H[S_0(G)] = S_0(G/H);$$

C) $$R_H[S_0(G)] = S_0(H);$$

D) $$S_0(G_1) \hat{\otimes} S_0(G_2) = S_0(G_1 \times G_2).$$

In A, B *and* C *the image spaces are endowed with the image (= quotient) norm inherited from $S_0(G)$.*

Proof. A) It follows from the arguments given at the beginning of step III in the proof of Theorem 1 that $\mathscr{F}_G[S_0(G)]$, endowed with the image norm, is a strongly character invariant homogeneous Banach space on $\hat{G}$. Since $S_0(G)$ is a Segal algebra contained in $A(G)$ one has $A(\hat{G}) \cap \mathscr{K}(\hat{G}) \subseteq \mathscr{F}_G[S_0(G)] \subseteq L^1(\hat{G})$. Consequently, $\mathscr{F}_G[S_0(G)]$ is also a dense subspace of $L^1(\hat{G})$, hence a strongly character invariant Segal algebra on $\hat{G}$. By Theorem 1 $S_0(\hat{G})$ is continuously embedded in $\mathscr{F}_G[S_0(G)]$. However, the same argument gives $\mathscr{F}_G^{-1}[S_0(\hat{G})] \supseteq S_0(G)$, hence equality.

B) Consider the space $T_H[S_0(G)] = S_0(G)/J_{S_0}(G, H)$ which is a Segal algebra on G/H with the quotient norm (cf. [30], VI, 2.7). Since for any $\dot{y} \in (G/H)^\wedge$ $y := \dot{y} \circ \pi_H$ defines a continuous character on G satisfying

$$T_H(M_y f) = M_{\dot{y}} T_H f, \tag{19}$$

it is evident that $T_H[S_0(G)]$ is a strongly character invariant Segal algebra on G/H. By Theorem 1 this implies $T_H S_0(G) \supseteq S_0(G/H)$. In order to show the reverse inclusion note first that for $K \subseteq G$ compact there is a constant $C > 0$ such that

$$\| T_H f \|_\infty \leqslant C \| f \|_\infty \text{ for all } f \in C_K(G), \tag{20}$$

in fact, one may take $C = \| T_H(c_K) \|_\infty$. Furthermore one has

$$\operatorname{supp}(T_H f) \subseteq \dot{K} = \pi_H(K) \text{ for } f \in L_K^\infty(G). \tag{21}$$

Now, by Corollary 6, there is a $C_1 > 0$ such that $f \in S_0(G)$ has a representation (13) with $\| f \| \leqslant C_1 \| f \|_{S_0}$. Since T_H is an algebra homomorphism of $L^1(G)$ onto $L^1(G/H)$, this implies

$$T_H f = \sum L_{\dot{y}_n}(T_H f_n * T_H g_n), \text{ with } \dot{y}_n = \pi_H(y_n). \tag{22}$$

The above arguments show that (22) is an admissible representation of $T_H f$ as an element of $S_0(G/H)$ in the sense of (17). Thus $T_H\colon S_0(G) \to S_0(G/H)$ is surjective; the continuity follows from

$$\sum \| T_H f_n \|_\infty \cdot \| T_H g_n \|_\infty \leqslant C^2 \sum \|f_n\|_\infty \cdot \|g_n\|_\infty \leqslant C^2 C^1 \|f\|_{S_0}. \quad (23)$$

C) results from B and A by the duality relation

$$R_{H^\perp} \mathscr{F}_G f = \mathscr{F}_{G/H}(T_H f). \quad (24)$$

It can also be proved directly from the definition, by means of restriction properties of the Fourier algebra $A(G)$.

D) Again, the inclusion $S_0(G_1 \times G_2) \subseteq S_0(G_1) \hat{\otimes} S_0(G_2)$ follows from the minimality of $S_0(G_1 \times G_2)$, as it is routine to verify that $S_0(G_1) \hat{\otimes} S_0(G_2)$ is a strongly character invariant Segal algebra (using the identity $(G_1 \times G_2)^\wedge = \hat{G}_1 \times \hat{G}_2$). On the other hand, for atoms $f_i \in A_{Q_i}(G_i)$, $i = 1, 2$, one has obviously

$$L_{y_1} f_1 \otimes L_{y_2} f_2 = L_{(y_1, y_2)}(f_1 \otimes f_2), f_1 \otimes f_2 \in A_{Q_1 \times Q_2}(G_1 \times G_2). \quad (25)$$

This yields $S_0(G_1) \hat{\otimes} S_0(G_2) \subseteq S_0(G_1 \times G_2)$, by a rearrangement of a double series.

Remarks: 14. Another proof of part B above has been derived by means of Theorem 5 from the relation $T_H[W(G)] \doteq W(G/H)$ by BÜRGER; the essential argument for a proof of the surjectivity is obtained by a modification of those given in [30], III.4.2 for $\mathscr{K}(G)$.

15. LOSERT ([26]) has shown that $S_0(G)$ is the only Segal algebra invariant under automorphisms which is defined for arbitrary locally compact abelian groups and satisfies all properties stated in Theorem 7. The Segal algebra $L^1 \cap A(G)$ which is the first candidate satisfying property A does not satisfy B or C, since $T_H(L^1 \cap A(G)) = L^1(G/H)$ if H is noncompact (this has been proved by KROGSTAD [23]). There is however another Segal algebra (also invariant under automorphisms of G) which satisfies properties A, B and C, namely $\mathfrak{B}_0(G) := \{f \mid f \in W(G), \hat{f} \in W(\hat{G})\}$, as has been shown by BÜRGER ([4]). It should be mentioned here that any space satisfying A – C (e. g. $S_0(G)$ or $\mathfrak{B}_0(G)$) represents a natural domain for Poisson's formula ([30], V, 5.1).

16. The other main result of [26] shows that $S_0(G)$ is a proper subspace of $\mathfrak{B}_0(G)$ if G is nondiscrete and noncompact. This also implies that it is impossible to describe $S_0(G)$ by conditions involving the "size" of f and $\hat{f}$ alone, in contrast to OSBORNE's results concerning

$\mathcal{S}(G)$ ([27]). More precisely, it is impossible to find homogeneous Banach spaces B_1 and B_2 on G and $\hat{G}$ respectively which are Banach modules over C^0 with respect to pointwise multiplication (e. g. such that $g \in B$, $f \in L^1_{loc}(G)$, $|f(x)| \leq |g(x)|$ l. a. e. implies $f \in B$ and $\|f\|_B \leq \|g\|_B$) so that $S_0(G)$ coincides with $\{f | f \in B_1, \hat{f} \in B_2\}$. In fact, such a representation would contradict the above result, since the minimality of Wiener's algebra ([10]) implies $W(G) \subseteq B_1$ and $W(\hat{G}) \subseteq B_2$.

17. A combination of assertions A and D yields: Given two locally compact abelian groups G_1 and G_2, the partial Fourier transform (i. e. the Fourier transform being only taken with respect to the second variable) maps $S_0(G_1 \times G_2)$ onto $S_0(G_1 \times \hat{G}_2)$.

The properties of the symbol S_0 proved in Theorem 7 are essentially the same as those of $\mathcal{S}$, where $\mathcal{S}(G)$ denotes the space of Schwartz—Bruhat functions (cf. [3]). For a comparison of these two spaces we need the following lemma.

Lemma 8. (i) *Let H be an open subgroup of G. Then $f \in L^1(H)$ belongs to $S_0(H)$ if and only if its trivial extension belongs to $S_0(G)$.*

(ii) *Let K be a compact subgroup of G, and let $f = \dot{f} \circ \pi_K$ be a K-periodic function on G. Then f belongs to $S_0(G)$ if and only if $\dot{f} \in S_0(G/K)$.*

Proof. (i) Let $f \in S_0(H)$ be given, having a representation (17). By replacing the f_n and g_n by their trivial extensions one obtains an admissible representation of the trivial extension of f as element of $S_0(G)$ (convolution now being taken in G). The converse follows immediately from Theorem 7.

(ii) Since one has $T_H f = \dot{f}$ one direction follows from Theorem 7, B. On the other hand, by Weil's formula ([30], III.4.5), any admissible representation of $\dot{f}$ according to Corollary 6

$$\dot{f} = \sum L_{\dot{y}_n} \dot{g}_n * \dot{h}_n, \dot{g}_n, \dot{h}_n \in C_{\dot{V}}(G/H),\ n \geq 1\,, \tag{26}$$

gives an admissible representation of f, by setting

$$f = \dot{f} \circ \pi_K = \sum L_{y_n} (\dot{g}_n \circ \pi_K) * (\dot{h}_n \circ \pi_K)\,, \tag{27}$$

if y_n satisfies $\pi_K(y_n) = \dot{y}_n$. Here we use the fact that

$$\dot{g}_n \circ \pi_K \in \mathcal{K}_{\pi_K^{-1}(\dot{V})}(G),\ \|\dot{g}_n * \pi_K\|_\infty = \|\dot{g}_n\|_\infty\,,$$

and that $\pi_K^{-1}(\dot{V})$ is a compact subset of G. This completes the proof. Another proof may be given using V.4.4.iii of [30].

We are now in the position to show that $S_0(G)$ contains the space $\mathscr{S}(G)$ of Schwartz—Bruhat functions for arbitrary locally compact abelian groups. For $G = \mathbb{R}^m$ this is Schwartz' space of rapidly decreasing functions. For the definition in the more general situation see the paper of BRUHAT [3]. A full discussion of the basic properties of $\mathscr{S}(G)$, in particular of the functorial properties for $\mathscr{S}$, formulated for S_0 in Theorem 7.A–C is given by REITER ([31], §§ VI and VII). We refer the reader to this paper for the definition and further details.

Theorem 9. *$S_0(G)$ contains $\mathscr{S}(G)$ as a dense subspace.*

Proof. For $G = \mathbb{R}^m$ this is Satz 3 of [29]. For arbitrary G we proceed as follows: For any $f \in \mathscr{S}(G)$ there exists an admissible pair (H, K) of subgroups of G, i.e. an open subgroup H of G and a compact subgroup K of H, such that H/K is an elementary locally compact abelian group, and such that f is K-periodic and vanishes outside H. In view of Lemma 8 and the corresponding assertions for $\mathscr{S}$ (cf. [31], VI, 8) it is sufficient to prove the inclusion $\mathscr{S}(G) \subseteq S_0(G)$ for an elementary group E. Such a group can be represented as a quotient of two closed subgroups H_1, H_2 of $\mathbb{R}^m$, i.e. $E \simeq H_1/H_2$ (cf. [31], VI, 1, x). The inclusion $\mathscr{S}(\mathbb{R}^m) \subseteq S_0(\mathbb{R}^m)$, together with the functorial properties of the symbols $\mathscr{S}$ and S_0 ([31], VI, 16 and VII, 1; resp. Theorem 7.B and C) imply

$$\mathscr{S}(E) = \mathscr{S}(H_1/H_2) = T_{H_1,H_2} R_{H_2} \mathscr{S}(\mathbb{R}^m) \subseteq T_{H_1,H_2} R_{H_2} S_0(\mathbb{R}^m) = S_0(E).$$

Remarks: 18. It is clear that the inclusion in Theorem 9 is in general a proper one, since $\mathscr{S}(G)$ is not a Banach space if G is not finite. Furthermore no decay at infinity is required for the elements of $S_0(G)$, G noncompact. More precisely, it is easy to construct for any given $g \in C^0(\mathbb{R}^m)$ some $f \in S_0(\mathbb{R}^m)$ such that $f(x_n) > |g(x_n)|$ for a suitable sequence $x_n \to \infty$. An explicit example for $f \in S_0(\mathbb{R}) \setminus \mathscr{S}(\mathbb{R})$ is provided by the density of the Cauchy distribution c_α. In fact, the two-sided monotony of $c_{\alpha/2}$ implies $c_{\alpha/2} \in W(\mathbb{R})$, hence $c_\alpha = c_{\alpha/2} * c_{\alpha/2} \subseteq$ $\subseteq W(\mathbb{R}) * W(\mathbb{R}) \subseteq S_0(\mathbb{R})$ for all $\alpha > 0$.

19. An argument using Theorem 5 instead of Theorem 2 in order to prove $\mathscr{S}(\mathbb{R}^m) \subseteq S_0(\mathbb{R}^m)$ is the following one. Since any $f \in \mathscr{S}(\mathbb{R}^m)$ is a continuous function satisfying $|f(x)| \leqslant C(1 + |x|)^{-m-1}$ it is evident that $\mathscr{S}(\mathbb{R}^m) \subseteq W(\mathbb{R}^m)$. Using now Theorem 5 and the fact that $\mathscr{S}(\mathbb{R}^m)$

has weak factorization, i. e. that any $f \in \mathscr{S}(\mathbb{R}^m)$ has a representation as a finite sum of convolution products of elements of $\mathscr{S}(\mathbb{R}^m)$ (cf. [7] for a general result, and [28] for an elementary proof) one obtains

$$\mathscr{S}(\mathbb{R}^m) \subseteq \operatorname{span}(\mathscr{S}(\mathbb{R}^m) * \mathscr{S}(\mathbb{R}^m)) \subseteq W(\mathbb{R}^m) \circledast W(\mathbb{R}^m) = S_0(\mathbb{R}^m).$$

We now give several applications of Theorems 5 and 7.A.

Theorem 10. *There is a $C > 0$ such that any $f \in S_0(G)$ can be written as a linear combination of four positive definite (positive) functions in $S_0(G)$; more precisely*

$$f = \sum_{k=0}^{3} i^k f_k, \ \|f_k\|_{S_0} \leqslant C \|f\|_{S_0}, \text{ and } f_k \gg 0 \ (f_k \geqslant 0). \tag{28}$$

Proof. By Theorem 5 $S_0(G)$ has tensor product factorization and the norm of $S_0 \circledast S_0$ is an equivalent norm on $S_0(G)$. On the other hand it follows directly from the definition that $S_0(G)$ is (isometrically) invariant under the involution * for $L^1(G): g^*(x) := \overline{g(-x)}$ (if $Q = Q^{-1}$). Thus there exists $C > 0$ such that any $f \in S_0(G)$ has a representation

$$f = \sum_{n=1}^{\infty} g_n * h_n^*, \text{ with } \sum \|g_n\|_{S_0} \|h_n\|_{S_0} \leqslant C \|f\|_{S_0}. \tag{29}$$

Without loss of generality we may suppose $\|g_n\|_{S_0} = \|f_n\|_{S_0}$. The polar decomposition gives

$$g * h^* = \tfrac{1}{4} \sum_{k=0}^{3} i^k (g + i^k h) * (g + i^k h)^*. \tag{30}$$

This implies the desired identity, if we set

$$f_k := \tfrac{1}{4} \sum_{n=1}^{\infty} (g_n + i^k h_n) * (g_n + i^k h_n)^*, \ k = 0, 1, 2, 3. \tag{31}$$

In view of the inequality

$$\|f_k\|_{S_0} \leqslant \tfrac{1}{4} \sum_{n=1}^{\infty} (\|g_n\|_{S_0} + \|h_n\|_{S_0})^2 \leqslant \sum_{n=1}^{\infty} \|g_n\|_{S_0} \|h_n\|_{S_0} \leqslant C \|f\|_{S_0} \tag{32}$$

it is clear that (31) is absolutely convergent in $S_0(G)$. This also implies $\mathscr{F}_G f_k \geqslant 0$, since it arises as a convergent sum (in $S_0(\hat{G})$) of the positive functions $\frac{1}{4} |\hat{g}_n + i^k \hat{h}_n|^2$ (by 7.A). By applying the above procedure to $\mathscr{F}_G f$ instead of f and the inverse Fourier transform, the decomposition into positive functions is obtained.

Remarks: 20. As in the case of $A(G)$ the decomposition into "positive" parts proved above cannot be the usual one which is applied to continuous functions (cf. [21], § 15).

21. It follows from the above representation (28) and (31) that any positive definite function in $S_0(G)$ can be obtained as a sum $\sum_{n=1}^{\infty} a_n f_n * f_n^*$ with $f_n \in S_0(G)$, $\|f_n\|_{S_0} \leqslant 1$ for $n \geqslant 1$, and *real* coefficients a_n, satisfying $\sum |a_n| < \infty$ $(f = f_1 - f_3)$. It is not clear whether it is sufficient to take positive coefficients in this case.

As another consequence of Theorem 5 we derive still another characterization of $S_0(G)$.

Theorem 11. *Let $S(G)$ be any strongly character invariant Segal algebra on G, and let $g \in S_0(G)$, $g \neq 0$, be given. Then $f \in L^1(G)$ belongs to $S_0(G)$ if and only if*

$$_S\|f\| := \int_{\hat{G}} \| M_t g * f \|_S \, dt < \infty , \tag{33}$$

and $_S\| \; \|$ defines an equivalent norm on $S_0(G)$.

Proof: (i) That $_{L^1}\| \; \|$ defines an equivalent norm on $S_0(G)$ follows from Theorem 2.B (cf. remark 5), by means of the identity

$$\| M_t g * f \|_1 = \| (L_t h) \hat{f} \|_{A(G)} \quad \text{for} \quad g = \hat{h}, h \in S_0(\hat{G}) . \tag{34}$$

(ii) In view of the minimality of $S_0(G)$ (as strongly character invariant Segal algebra on G) the general results follows if an equality of the form $_{S_0}\|f\| \leqslant C_3 \|f\|_{S_0}, f \in S_0(G)$, can be verified: By Theorem 5 there exists $C_1 > 0$ such that any $f \in S_0(G)$ has a representation

$$f = \sum f_n * g_n, \text{ with } \sum \|f_n\|_{S_0} \|g_n\|_{S_0} \leqslant C_1 \|f\|_{S_0} . \tag{35}$$

Without loss of generality we may suppose

$$\sup_{n \geqslant 1} \|g_n\|_{S_0} \leqslant C_1, \text{ and } \sum \|f_n\|_{S_0} \leqslant \|f\|_{S_0} . \tag{36}$$

Combining (35) and (26) one derives from i):

$$\int_{\hat{G}} \| M_t g * f \|_{S_0} dt \leqslant \int_{\hat{G}} \sum \| M_t g * f_n * g_n \|_{S_0} dt \leqslant$$

$$\leqslant \sum \int_{\hat{G}} \| M_t g * f_n \|_1 \|g_n\|_{S_0} dt \leqslant C_2 \sum {}_{L^1}\|f_n\| \leqslant C_3 \|f\|_{S_0},$$

and the proof is complete.

Remark: 22. It follows from Theorem 2.2 of [11] that one has $\| M_t g * f \|_1 \to 0$ for $t \to \infty$, for any $f \in L^1(G)$. $S_0(G)$ is thus described as a subspace of those $f \in L^1(G)$ for which this function satisfies an integrability condition.

We conclude this paper by showing that the minimality of $S_0(G)$ also implies invariance with respect to pointwise multiplication by characters of the second degree (cf. [31], Chap. II, also for the definition).

Lemma 12. *Let* $(S, \| \|_S)$ *be a strongly character invariant Segal algebra on* G, *and let* ψ *be a character of the second degree on* G. *Then*

$$S_\psi := \{\psi f \mid f \in S\}$$

is a strongly character invariant Segal algebra on G *with the norm* $\|\psi f\|_{S,\psi} := \|f\|_S$.

Proof. The map $f \mapsto \psi f$ defines an isometry from S onto S_ψ, and $(S_\psi, \| \|_{S,\psi})$ is a Banach space which is continuously and densely embedded in $L^1(G)$. By the functional equation for ψ there exists a continuous homomorphism $\varrho := G \to \hat{G}$ such that one has

$$[L_y(\psi f)](x) = \psi(x)\,\psi(-y)\,\langle x, \varrho(-y)\rangle\, L_y f(x) \quad \text{for } x, y \in G. \tag{37}$$

Using (37) we derive the strong translation invariance of S_ψ from the strong character invariance of S:

$$\| L_y(\psi f)\|_{S,\psi} = |\psi(-y)|\, \| M_{\varrho(-y)} L_y f\|_S = \|f\|_S = \|\psi f\|_{S,\psi}.$$

The continuity of the mapping $y \mapsto L_y(\psi f)$ from G into S_ψ follows from the estimate below, which is based on (37) as well.

$$\begin{aligned}\| L_y(\psi f) - \psi f\|_{S,\psi} &\leqslant \|\psi(-y)\, M_{\varrho(-y)}(L_y f - f)\|_S + \\ &\quad + \|\psi(-y)(M_{\varrho(-y)} f - f)\|_S + \|(\psi(-y) - 1) f\|_S \\ &= \|L_y f - f\|_S + \|M_{\varrho(-y)} f - f\|_S + \\ &\quad + |\psi(-y) - 1| \cdot \|f\|_S.\end{aligned}$$

Corollary 13. *Let* ψ *be a character of the second degree on* G. *Then* $A_\psi : f \to \psi f$ *defines an isomorphism of* $S_0(G)$ *onto itself.*

Proof. The above Lemma and the minimality of $S_0(G)$ imply $S_0(G) \subseteq \psi S_0(G)$. Since ψ^{-1} is also a character of the second degree, the converse inclusion holds as well.

It follows now that the operators $R_G(a)$, $a \in G$; $S_G(\hat{a})$, $\hat{a} \in \hat{G}$, $U_G(t)$; $W_G(\varrho)$ and $A_G(\psi)$ (cf. [31], p. 27) define isomorphisms of $S_0(G)$ onto itself. Therefore one could use $S_0(G)$ instead of $\mathcal{S}(G)$ as analytic tool for the proof of the generalized Weil—Cartier theorem (cf. [31], Chap. VI and VII).

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