Theoretical Computer Science 247 (2000) 229–255
Theoretical Computer Science
www.elsevier.com/locate/tcs

Domain representations of topological spaces

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Received October 1997; revised September 1998
Communicated by J.W. de Bakker

Abstract

A *domain representation* of a topological space X is a function, usually a quotient map, from a subset of a domain onto X. Several different classes of domain representations are introduced and studied. It is investigated when it is possible to build domain representations from existing ones. It is, for example, discussed whether there exists a natural way to build a domain representation of a product of topological spaces from given domain representations of the factors. It is shown that any T_0 topological space has a domain representation. These domain representations are very large. However, smaller domain representations are also constructed for large classes of spaces. For example, each second countable regular Hausdorff space has a domain representation with a countable base. Domain representations of functions and function spaces are also studied.

Keywords: Domain theory; Topological spaces; Representations; Computability

1. Introduction

In this paper we study domain representations of topological spaces and properties of such representations. The main reason for studying such representations is that they provide a uniform method to introduce computability on abstract spaces such as $\mathbb{R}$. Scott–Ershov domains carry a natural computability theory and the representing map from the domain onto the topological space imports the computability theory onto the topological space. We will in this paper not directly concern ourselves with computability but will instead study the notion of domain representability abstractly. The paper [9] is an extended abstract of this paper. Most results herein also appear in [8, Ch. 4].

The notion of domain representations is introduced in Section 3. The domain representations are classified depending on the properties of the representation. The primary classification is by the topological properties of the representing function. Several other useful properties that a representation may or may not have are also identified.

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PII: S0304-3975(99)00045-6

Our primary type of domains is Scott–Ershov domains, i.e., consistently complete algebraic cpos. We consider domain representations from continuous domains in Section 4. It is shown that the ordinary embedding of a continuous domain into an algebraic domain is a retract domain representation. This is further used to show that for large classes of representations from continuous domains it is possible to construct domain representations from Scott–Ershov domains with the same properties as the former.

Neighbourhood systems are introduced in Section 5. They are used to construct domain representations with many useful properties. In particular, it is shown that any regular Hausdorff space has an upwards-closed retract representation. The neighbourhood system chosen in this case consists of all the non-empty closed sets of the space. If the space is second countable, then a modification of the neighbourhood system gives a domain representation with a countable base. Furthermore, it is shown that spaces with upwards-closed retract representations are regular Hausdorff spaces. Hence, we have a complete characterisation of the spaces that have an upwards-closed retract representation.

In Section 6 we observe some limitations on spaces that have effective domain representations. These limitations are imposed by topological reasons since effective domains must be countably based.

Domain representations where the representing elements are a subset of the maximal elements of the domain are constructed in Section 7. It is possible to make such constructions for arbitrary T_0 spaces. However, these representations are very large and lack some of the properties that the representations constructed in Section 5 possess.

In Section 8 we study when it is possible to uniformly build domain representations for spaces obtained by a topological construction from domain representations of the old spaces. We have, for example, that retract representations are uniformly closed under retracts, subspaces, disjoint unions, and products.

Domain representations of functions are studied in Section 9.1. A function has a domain representation if there exists a domain function inducing it. Domain functions satisfying a natural condition always induce a continuous function on the represented spaces. Theorem 9.3 gives sufficient conditions so that representations of functions always exist.

Section 9.2 studies when the function space construction on domains can be used to give domain representations of function spaces. A representation of a function space induces a topology on the function space. Under some conditions this topology is proven (Theorem 9.7) to be exactly the compact-open topology.

2. Domain-theoretic background

In this subsection we will briefly review domain theory. We concentrate on giving the notions and hint at some results. The proofs are generally omitted and can be found in either [24] or [1]. Most of the material in this paper is based on what we call Scott–Ershov domains. Hence this section will lean towards that type of domains.

Let $D=(D,\sqsubseteq)$ be a partially ordered set. A subset $A\subseteq D$ is an *upper set* if $x\in A$ and $x\sqsubseteq y$ implies $y\in A$. Let $\uparrow A=\{y\in D\colon \exists x\in A(x\sqsubseteq y)\}$. We will abbreviate $\uparrow\{x\}$ by $\uparrow x$. The dual notions are *lower set* and $\downarrow A$. A subset $A\subseteq D$ is *directed* if $A\neq\emptyset$ and whenever $x,y\in A$ then there is $z\in A$ such that $x\sqsubseteq z$ and $y\sqsubseteq z$. The supremum, or least upper bound, of A (if it exists) is denoted by $\bigsqcup A$.

A *complete partial order*, abbreviated *cpo*, is a partial order, $D=(D;\sqsubseteq,\bot)$, such that $\bot$ is the least element in D and any directed set $A\subseteq D$ has a supremum, $\bigsqcup A$. This is also known as a pointed *dcpo* in the literature.

Note that our definition of cpo includes a bottom element. The existence of bottom elements is useful in, e.g., function space constructions. The more general form, without bottom, is not needed in our work. In addition, it is intuitively pleasing to have a bottom element since this will correspond to the trivial approximation of a point in a topological space. That is, the bottom element approximates the whole space.

Let D be a cpo. Then an element $a\in D$ is *compact* if whenever $A\subseteq D$ is a directed set and $a\sqsubseteq\bigsqcup A$, then $a\in\downarrow A$. The set of compact elements of D is denoted by D_{c}.

A cpo D is *algebraic* if for each $x\in D$, the set $\mathrm{approx}(x)=\downarrow x\cap D_{\mathrm{c}}$ is directed and $x=\bigsqcup\mathrm{approx}(x)$. A cpo D is *consistently complete* if $\bigsqcup A$ exists in D whenever $A\subseteq D$ is a consistent set, i.e., has an upper bound.

Definition 2.1. A *Scott–Ershov domain*, or simply *domain*, is a consistently complete algebraic cpo.

The topology normally used on domains is called the Scott topology. Let D be an algebraic cpo. A subset U of D is open if

(i) U is an upper set, and

(ii) $x\in U$ implies that there exists $a\in\mathrm{approx}(x)$ such that $a\in U$.

An easy observation is that the Scott topology on a domain is T_0. However the Scott topology fails to be T_1 on all domains except the trivial domain consisting of a single element.

The sets $\uparrow a$, for $a\in D_{\mathrm{c}}$, constitute a base for the Scott topology on a domain D.

Let D and E be domains. A function $f:D\to E$ is Scott continuous if f is monotone and

$$f\left(\bigsqcup A\right)=\bigsqcup f[A],$$

for any directed $A\subseteq D$. The notion of Scott continuity coincides with the notion of continuity induced from the Scott topology on the domains.

Any continuous function between domains is determined by its values on the compact elements.

Let D and E be domains and let $f:D_{\mathrm{c}}\to E$ be a monotone function. Then there exists a unique extension $g:D\to E$ of f such that $f=g|_{D_{\mathrm{c}}}$.

The function space $[D \to E]$ consists of all continuous functions from the domain D into the domain E. For $a \in D_c$ and $b \in E_c$ the step function $\langle a; b \rangle$ defined by

$$\langle a; b\rangle(x) = \begin{cases} b & \text{if } a \sqsubseteq x, \\ \bot & \text{otherwise} \end{cases}$$

is a continuous function. The compact elements of the function space are finite suprema of consistent sets of such step functions.

Domains are often constructed as the completion of some underlying structure. We will study the type of structure from which we can construct Scott–Ershov domains.

The compact elements D_c of a Scott–Ershov domain D form a conditional upper semilattice with least element, abbreviated *cusl*. That is, a cusl is a partially ordered set where a least upper bound exists for every pair of elements that have an upper bound.

An *ideal* is a directed lower set. The ideal completion over a cusl P is the set of all ideals over P, denoted $\mathrm{Idl}(P)$. When ordered by set inclusion the ideal completion of a cusl forms a Scott–Ershov domain. For a in a cusl P, $\downarrow a$ is an ideal, the *principal ideal* generated by a. The compact elements of $\mathrm{Idl}(P)$ are the principal ideals $\downarrow a$, for $a \in P$.

The representation theorem for Scott–Ershov domains tells us that any Scott–Ershov domain is the ideal completion of a cusl.

Theorem 2.2. *Let D be a Scott–Ershov domain. Then* $\mathrm{Idl}(D_c) \cong D$.

We clearly have the following equivalence, for $I \in \mathrm{Idl}(P)$:

$$\downarrow a \subseteq I \iff a \in I.$$

Thus the sets $B_a = \{I \in \mathrm{Idl}(P){:}\ a \in I\}$ for $a \in P$ form a base for the Scott topology on $\mathrm{Idl}(P)$.

Having introduced our main type of domains we will now briefly introduce a more general type of domains, namely the continuous domains.

Let x and y be elements of a cpo D. We say that x is *way below* y, denoted $x \ll y$, if for all directed subsets A of D, $y \sqsubseteq \bigsqcup A \Rightarrow x \in \downarrow A$. Let $\twoheaduparrow x$ denote $\{y \in D{:}\ x \ll y\}$ and $\twoheaddownarrow x$ denote $\{y \in D{:}\ y \ll x\}$.

An element x of a cpo D is compact if and only if it is way below itself. A subset B of D is a *basis* for D if for every $x \in D$ the set $\twoheaddownarrow x \cap B$ is directed and has supremum x.

Definition 2.3. A cpo D that has a basis is a *continuous cpo*. If D is in addition consistently complete then D is a *continuous domain*.

The definition of algebraic cpo is just a way of expressing that the set of compact elements is a basis for the cpo. Hence, any algebraic cpo is a continuous cpo.

If B is a basis for a continuous cpo D, then the sets $\twoheaduparrow b$, for $b \in B$, is a base for the Scott topology on D.

3. Domain representations

Representations of topological spaces by domains or embeddings of topological spaces into domains have been studied by several people. Weihrauch and Schreiber [30] considered embeddings of metric spaces into cpos with weight and distance. Stoltenberg-Hansen and Tucker [25, 27] introduced the notion of domain representability. Edalat [10–12] has used embeddings into continuous dcpos to study integration, measures and fractals. Edalat and Heckmann [13] and di Gianantonio [17] among others have also studied similar notions. Ershov's [14] representation of the Kleene–Kreisel continuous functionals is an early example of a domain representation. Scott [23] has proposed a category of equilogical spaces. The spaces in this category can also be used to represent topological spaces, see also [6, 3].

In a domain representation D of a space we isolate the set of representing elements D^{R} as those that contain *total* or *complete* information. This has led to the abstract study of domains with totality, i.e., domains with a distinguished subset of total elements. This sort of study has been pursued in connection with certain type structures by Berger [4, 5], Kristiansen and Normann [19], Normann [20, 22] and Waagbø [29].

The kind of representations or embeddings that are possible for a certain topological space are affected by the choice of domains. For example, any metric space can be embedded into the maximal elements of a continuous dcpo. For Scott–Ershov domains we know that the set of maximal elements is Hausdorff and has a clopen base, and hence, that any space embedded into the maximal elements of a Scott–Ershov domain is totally disconnected.

We mostly consider domain representations by Scott–Ershov domains here. This is due to Scott–Ershov domains having a simpler computability theory (not exploited here, however) and, in our experience, sufficiency in terms of representability. Sufficiency can to some extent be motivated by Theorems 4.3 and 4.4.

3.1. Classes of quotient maps

The primary classification of our domain representations will be the topological properties of the representing function. We introduce here the different classes of quotient maps that we will consider.

Definition 3.1. Let $f: X \to Y$ be a continuous function between the topological spaces X and Y. Then

(i) f is a *quotient map* if $V \subseteq Y$ is open if, and only if, $f^{-1}[V]$ is open,
(ii) f is *pseudo-open* if for any $y \in Y$ and any open set $U \subseteq X$ containing $f^{-1}[y]$, y is in the interior of $f[U]$,
(iii) f is *open* if $f[U]$ is open for any open subset U, and
(iv) f is a *retraction* if there exists $e: Y \to X$ such that $f \circ e = \mathrm{id}_Y$.

The notion of pseudo-open is due to Arhangelskij [2].

Remember that if $f : X \to Y$ is a quotient then $X/\sim$ and the image $f[X]$ are homeomorphic, where $\sim$ is the equivalence relation induced by f.

We observe some relationships between the introduced classes. Onto open maps are pseudo-open and onto pseudo-open maps are quotients. Retractions are pseudo-open. The notions of open and retraction are independent of each other. Moreover, the classes of quotient maps introduced above are all closed under composition.

3.2. *Classes of domain representations*

We now give the fundamental definition of domain representability. The notion is a stronger version of the one that appears in [25, 27].

Definition 3.2. Let D be a domain, $D^R \subseteq D$, and let X be a topological space. Suppose that φ is a continuous mapping from D^R onto X. We call the triple (D, D^R, φ)

(i) a *weak domain representation* of X;
(ii) a *domain representation* of X if φ is a quotient map;
(iii) a *pseudo-open domain representation* of X if φ is pseudo-open;
(iv) an *open domain representation* of X if φ is open;
(v) a *retract domain representation* of X if φ is a retraction;
(vi) a *homeomorphic domain representation* of X if φ is a homeomorphism.

We will sometimes drop the word domain from the notions above. We will also consider *continuous domain representations*, i.e., representations where the structure D is a continuous domain and not necessarily a Scott–Ershov domain. Classification of continuous domain representations is done by the same notions as otherwise introduced in this section.

The set D^R above will be called the set of representing elements. The representing domain D contains both proper approximations and total or complete representations of elements of X, the latter constituting the set D^R. Intuitively, D^R consists of those domain elements that contain sufficient information to completely determine an element in X via φ.

Each of the introduced classes of domain representations implies the earlier ones, with the exception that retract representations are not necessarily open representations. However, every retract representation (D, D^R, φ) of X with embedding η induces a homeomorphic (and hence open) representation $(D, \eta^{-1}[X], \varphi)$ of X.

If (D, D^R, φ) is a homeomorphic representation of X, then φ^{-1} is an embedding of X into D.

There are other criteria for suitability of a representation, besides the topological properties of the representing function φ, namely the kind of the domain D and the properties of the set D^R.

Definition 3.3. Let (D, D^R, φ) be a domain representation. The representation is *upwards-closed* if $d \in D^R$ and $d \sqsubseteq e$ implies $e \in D^R$ and $\varphi(e) = \varphi(d)$.

If the represented space X in the definition above is T_1, then it is redundant to require $\varphi(e)=\varphi(d)$.

In a natural representation we would like to consider all elements below a domain element as approximations to the point that element represents. In this setting, the representing elements will be *total* or *complete* in the sense that they contain total information about the point they represent. Since any element above a representing element contains more information we clearly see that any natural representation should be upwards-closed.

Let $(D, D^{\mathrm{R}}, \varphi)$ be a domain representation. If every element of D^{R} is a maximal element, then we say that it is a *representation by maximal elements* and note that it is upwards-closed. However, we have noted that only totally disconnected spaces can be given an upwards-closed homeomorphic domain representation by maximal elements. We can construct upwards-closed domain representations of a large class of spaces if we content ourselves with representing elements that are sufficiently high up in the domain so that no contradictory information can appear above them. More formally, we require only that each representing element is *total* in the sense that $\uparrow x$ is directed. We say that we have a *representation by total elements*.

A *dense representation* is a representation $(D, D^{\mathrm{R}}, \varphi)$ where D^{R} is dense in D. In Section 9.1 denseness is used to show that functions can be represented by (or lifted to) domain functions.

Many of our representations satisfy that for every $x \in X$ there exists a least representative d_x of x, or equivalently, $\varphi^{-1}[x]$ has a least element. Clearly, any homeomorphic representation has this property. A representation with this property is a *representation with least representatives.*

A domain representation is said to have the *closed image property* if $\varphi[\uparrow d \cap D^{\mathrm{R}}]$ is closed for all $d \in D$. This is equivalent to $\varphi[\uparrow a \cap D^{\mathrm{R}}]$ being closed for all $a \in D_{\mathrm{c}}$ since

$$\varphi[\uparrow d \cap D^{\mathrm{R}}] = \bigcap_{a \in \mathrm{approx}(d)} \varphi[\uparrow a \cap D^{\mathrm{R}}].$$

The following consistency requirement on representations intuitively has the consequence that all representations of a point are concentrated in a small part of the domain. A domain representation $(D, D^{\mathrm{R}}, \varphi)$ is *local* if for each $x \in X$, the set $\varphi^{-1}[x]$ is consistent. Clearly, any homeomorphic representation is local.

4. Algebraic and continuous representability

In this section we show that any continuous domain has a retract representation by an algebraic domain. This implies that any space that has a continuous domain representation also has a domain representation via the representations of the continuous domain.

The following lemma is well-known.

Lemma 4.1. *Let D be a continuous cpo. Then D can be embedded into an algebraic cpo E via a continuous embedding projection pair.*

Proof. Let B be a basis for D and let $E = \mathrm{Idl}(B, \sqsubseteq)$. Then E is an algebraic cpo since $\sqsubseteq$ is a preorder. Define $e : D \to E$ by $x \mapsto \mathord{\downarrow\!\!\downarrow} x \cap B$, and $p : E \to D$ by $I \mapsto \bigsqcup_D I$. It is clear that (e, p) constitutes a continuous embedding projection pair. □

Let D be a continuous cpo and let E be constructed as in Lemma 4.1. Let (e, p) be the continuous embedding projection pair.

Corollary 4.2. (i) *If D is a continuous ω-cpo then E can be chosen to be an algebraic ω-cpo.*

(ii) *If D is consistently complete then E is also consistently complete.*

Proof. For (i) choose a countable basis B in the proof of the lemma. The other is trivial. □

Summarising so far we have the following result.

Theorem 4.3. *Every continuous domain D has a dense retract domain representation.*

Proof. By Lemma 4.1 and the corollary, (E, E, p, e) is a retract domain representation of D. The representation is clearly dense. □

The representation above will be called the canonical representation of a continuous domain.

Suppose $(D, D^{\mathrm{R}}, \varphi)$ is a continuous domain representation of X. Clearly, $(E, p^{-1}[D^{\mathrm{R}}], \varphi \circ p)$ is a domain representation of X. Will the properties of the continuous domain representation lift to the new domain representation of X? Since the representation of the continuous domain is a retract representation, we cannot assert that the domain representation is open even if the continuous domain representation is open. Furthermore, the representation is not by maximal elements even if the continuous domain representation is by maximal elements. However, the other properties considered in this paper are preserved. The possible loss of openness should be compared with the simplicity and concreteness gained by using algebraic domains.

Theorem 4.4. *Let $(D, D^{\mathrm{R}}, \varphi)$ be a continuous domain representation of X and let (E, E, p, e) be the canonical domain representation of D constructed from the base B.*

(i) *If $(D, D^{\mathrm{R}}, \varphi)$ is by total elements, then $(E, p^{-1}[D^{\mathrm{R}}], \varphi \circ p)$ is a domain representation by total elements of X.*

(ii) *If $(D, D^{\mathrm{R}}, \varphi)$ is upwards-closed, then $(E, p^{-1}[D^{\mathrm{R}}], \varphi \circ p)$ is an upwards-closed domain representation of X.*

(iii) *If $(D, D^{\mathrm{R}}, \varphi)$ has least representatives, then $(E, p^{-1}[D^{\mathrm{R}}], \varphi \circ p)$ is a domain representation with least representatives of X.*

(iv) *If* (D, D^R, φ) *has the closed image property, then* $(E, p^{-1}[D^R], \varphi \circ p)$ *is a domain representation with the closed image property of* X.

(v) *If* (D, D^R, φ) *is local, then* $(E, p^{-1}[D^R], \varphi \circ p)$ *is a local domain representation of* X.

Proof. (i) Suppose that $I \in p^{-1}[x]$, where $x \in D^R$. Let J and K be ideals extending I. We want to show that $J \cup K$ generates an ideal. For this we have to show that any pair of elements from J and K is consistent, so let $a \in J$ and $b \in K$. Since J is an ideal containing I, we have that $a \sqcup x$ exists in D, likewise $b \sqcup x$ exists. By the totality of x, $a \sqcup x$ and $b \sqcup x$ are consistent in D and hence also a and b.

(ii) Let $I \in p^{-1}[D^R]$. Suppose that $\varphi \circ p(I) = x$ and that $I \sqsubseteq J$. By monotonicity $p(I) \sqsubseteq p(J)$ and by D^R being upwards-closed we have $\varphi \circ p(J) = x$.

(iii) If d_x is the least representative of x in D, then $e(d_x)$ is the least representative of x in E.

(iv) Consider the image of a basic open set:

$$\varphi[p[\uparrow(\downarrow a \cap B) \cap p^{-1}[D^R]]] = \varphi[\uparrow a \cap D^R].$$

The latter is closed by the closed image property of (D, D^R, φ).

(v) Suppose $x, y \in p^{-1}[D^R]$ and $\varphi \circ p(x) = \varphi \circ p(y)$. Then $p(x)$ and $p(y)$ are consistent since (D, D^R, φ) is local. Clearly, the ideal $\downarrow(p(x) \sqcup p(y)) \cap B$ is an upper bound of x and y. $\square$

In order to show a similar result for dense representations some caution is required in the choice of base from which E is constructed.

Theorem 4.5. *Let* (D, D^R, φ) *be a dense continuous domain representation of* X *and let* (E, E, p, e) *be the canonical domain representation of* D *constructed from the base* $B' = \{a \in B : \Uparrow a \neq \emptyset\}$, *where* B *is a base for* D. *Then* $(E, p^{-1}[D^R], \varphi \circ p)$ *is a dense domain representation of* X.

Proof. A basic open set in E is of the form $\uparrow(\downarrow a \cap B)$, where $a \in B$. Since D^R is dense and $\Uparrow a \neq \emptyset$, there exists $x \in D^R$ such that $x \in \Uparrow a$. Clearly, $\downarrow x \in \uparrow(\downarrow a \cap B) \cap p^{-1}[D^R]$. $\square$

5. Standard representations by domains of filter bases

5.1. Neighbourhood systems

This subsection introduces the notion of neighbourhood systems. These structures will be used in the subsequent subsections to construct domain representations.

The interior and closure of a subset $A \subseteq X$ are denoted by A° and $\overline{A}$, respectively.

Definition 5.1. Let X be a topological space and let P be a family of non-empty subsets of X such that $X \in P$. Then $P=(P;\supseteq,X)$ is a *neighbourhood system* if the following are satisfied:
(i) if $A,A' \in P$ and $A \cap A' \neq \emptyset$ then $A \cap A' \in P$, and
(ii) if $x \in U$, where U is open, then $(\exists A \in P)(x \in A^\circ \subseteq A \subseteq U)$.

Examples of neighbourhood systems are: the non-empty closed sets of a regular space; the non-empty compact sets of a locally compact regular space; and all non-empty sets of a base for the topology together with the set X. The former two may be called *closed* neighbourhood systems and the latter an *open* neighbourhood system.

Condition (i) makes P ordered with reverse inclusion into a cusl. Hence, the ideal completion $D=\mathrm{Idl}(P)$ is a domain. The elements of D are ideals in $(P,\supseteq)$, i.e., they are *filter bases* in the topological sense. The Scott topology on D is generated by the basic open sets $\uparrow\downarrow A = B_A = \{I \in D\colon A \in I\}$ for $A \in P$.

The elements of P may be seen as approximations of elements of X. These approximations are consistent if they have a non-empty intersection. P is an approximation for X in the sense of [27].

For each element of the space X we define two ideals of special interest.

Definition 5.2. Let P be a neighbourhood system for X and let $x \in X$.
(i) $I_x = \{A \in P\colon x \in A^\circ\}$.
(ii) $J_x = \{A \in P\colon x \in A\}$.

Clearly, $I_x \subseteq J_x$, and if P is an open neighbourhood system then $I_x = J_x$. For any $A \in P$ there exists $x \in A$. Clearly, $J_x \in B_A$. Thus, the set $\{J_x\colon x \in X\}$ is dense in D.

Define $\eta : X \to D$ by $\eta(x) = I_x$.

Lemma 5.3. (i) *The function η is continuous.*
(ii) *If η is injective, then η is an embedding of X into D.*

Proof. (i) $\eta(x) \in B_A \Leftrightarrow A \in I_x \Leftrightarrow x \in A^\circ$.
(ii) Since $\{A^\circ\colon a \in P\}$ is a base for the topology on X by Definition 5.1(ii), and $A^\circ = \eta^{-1}[B_A]$ by part (i), η is an embedding. □

An ideal I *converges* to a point $x \in X$, denoted $I \to x$, if for every open set U containing x, there is an $A \in I$ such that $x \in A \subseteq U$, or equivalently, if the filter base corresponding to I converges to x. We note that $I \to x$ if, and only if, $I_x \subseteq I$.

5.2. Homeomorphic representations for T_0-spaces

Theorem 5.4. *Any T_0-space X has a dense homeomorphic representation.*

Proof. Let P be a neighbourhood system consisting of all the non-empty sets of a base for the topology together with the set X.

Due to the T_0-property of X, the function $\eta : X \to D$ defined as above is injective. Hence, by Lemma 5.3, η is an embedding of X into D. Thus, $(D, \eta[X], \eta^{-1})$ is a homeomorphic representation of X.

Since P is an open neighbourhood system we have $I_x = J_x$, and hence, $D^R = \eta[X]$ is dense in D. □

In general, the representation above is not upwards-closed and not by maximal or total elements as the following example shows.

Example 5.5. The ideals I_x need not be maximal, in fact they need not even be total. Suppose $X = \mathbb{R}$ and let P consist of all non-empty open intervals. Then I_x consists of all open intervals containing x. Let I_0^+ and I_0^- be the ideals generated by $I_0 \cup \{(0,1)\}$ and $I_0 \cup \{(-1,0)\}$ respectively. The ideals I_0^+ and I_0^- are not consistent, i.e., I_0 is not total.

Theorem 5.6. *A space X with a retract representation (D, D^R, φ, η) is a T_0-space.*

Proof. If x and x' are inseparable by open sets, then the same holds true for $\eta(x)$ and $\eta(x')$ since η is an embedding. Hence, $\eta(x) = \eta(x')$ since domains are T_0. Applying φ we have $\varphi(\eta(x)) = \varphi(\eta(x'))$, i.e., $x = x'$. □

The theorem above also holds when D is a continuous domain.

5.3. Upwards-closed retract representations for regular Hausdorff spaces

Let P be a neighbourhood system for a Hausdorff space X and let $D = \mathrm{Idl}(P)$. Let D^R be the set of converging ideals. The Hausdorff property implies that every converging ideal has a unique limit point. Define $\varphi : D^R \to X$ by mapping a converging ideal to its limit point.

Let $x \in X$ and $A \in P$. By the properties of a neighbourhood system, $x \in \overline{A}$ if, and only if, there exists an ideal I containing A and converging to x. Thus, φ will have the closed image property since $\varphi[B_A \cap D^R] = \overline{A}$.

It is clear that φ is onto and that the representation will be upwards-closed. However, in order to show continuity of φ, we need to strengthen (ii) in Definition 5.1 to:

(ii)′ if $x \in U$, where U is open, then $(\exists A \in P)(x \in A^\circ \subseteq \overline{A} \subseteq U)$.

Lemma 5.7. *If the neighbourhood system P satisfies (ii)′, then φ is continuous.*

Proof. Let $U \subseteq X$ be an open set and let $I \in D^R$ converge to some point $x \in U$. By (ii)′ there exists a set $A \in P$ satisfying that $x \in A^\circ \subseteq \overline{A} \subseteq U$. It is clear that any converging ideal containing A converges to some point in $\overline{A}$. Hence, B_A is an open neighbourhood of I contained in $\varphi^{-1}[U]$, i.e., φ is continuous. □

There are two immediate choices for a neighbourhood system P satisfying (ii)′ for a regular Hausdorff space X.

(i) For any base, all the non-empty basic open sets together with the set X.
(ii) All non-empty closed sets of X.

The former is the one chosen in Section 5.2. However, the latter choice will give a local representation by total elements. By Example 5.5, the former choice will not have these properties. Thus, the latter seems to be superior.

Summarising we have the following theorem.

Theorem 5.8. *Any regular Hausdorff space X has a local dense upwards-closed retract representation by total elements with least representatives and the closed image property.*

Proof. Choose P to consist of all non-empty closed sets of the space. Let $D = \mathrm{Idl}(P)$ and let D^{R} be the set of all converging ideals in D.

The representation is dense since $\{J_x : x \in X\} \subseteq D^{\mathrm{R}}$ is dense in D.

Clearly, the ideal J_x is the greatest ideal converging to x. Hence, the representation is local and by total elements.

The ideal I_x is the least ideal representing x. $\square$

By restricting D^{R} to $\eta[X]$ we get a homeomorphic representation by total elements of X. However, this representation is neither dense nor upwards-closed in general, and it may also lack the closed image property.

Let X be a second countable regular Hausdorff space and let B be a countable base for the topology on X. Let P consist of all finite non-empty intersections of sets in $\{\overline{U} : U \in B\}$. Then P satisfies the stronger version of neighbourhood systems. Note that P is countable, i.e., the constructed domain D has a countable base.

An *effective domain* is a domain where consistency is decidable and the partial supremum function is computable. For approximations with irregular shapes it is not clear how to effectively decide whether two approximations are consistent. However, the following example shows that there exists an effective domain representation of the reals.

Example 5.9. Let P consist of the non-empty rational intervals. Then P is a neighbourhood system for $\mathbb{R}$. To determine whether two rational intervals intersect it is sufficient to make a few comparisons of rational numbers. The supremum is again a rational interval and comparisons of rational numbers again suffice to compute this interval. The operations are clearly effective. Thus, the constructed domain representation of $\mathbb{R}$ is effective.

Clearly, Theorem 5.8 does not yield an effective domain in general. In [7] a general construction of effective domain representations of metric spaces is given.

Theorem 5.10. *Let $(D, D^{\mathrm{R}}, \varphi, \eta)$ be an upwards-closed retract representation of X. Then X is a regular Hausdorff space.*

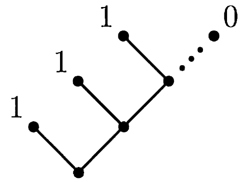

Fig. 1. Domain D representing the Sierpinski space.

Proof. Let x_1 and x_2 be distinct points in X. Then $\eta(x_1)$ and $\eta(x_2)$ are inconsistent by upwards-closed. Hence, there exist disjoint open sets U_i such that $\eta(x_i)\in U_i$. Thus, $\eta^{-1}[U_i]$ are disjoint open sets containing x_i.

Let x be in the open set U. Then $\eta(x)\in\varphi^{-1}[U]$. There exists $a\in D_c$ such that $\eta(x)\in\uparrow a\cap D^R\subseteq\varphi^{-1}[U]$.

$$\begin{aligned}x &\in \eta^{-1}[\uparrow a\cap D^R] \quad \text{(open)}\\ &\subseteq \eta^{-1}[\downarrow\uparrow a\cap D^R] \quad \text{(closed)}\\ &\subseteq \varphi[\downarrow\uparrow a\cap D^R]\\ &= \varphi[\uparrow a\cap D^R] \quad \text{(by upwards-closed)}\\ &\subseteq \varphi[\varphi^{-1}[U]]\\ &= U. \qquad \square\end{aligned}$$

Corollary 5.11. *The spaces with upwards-closed retract representations are exactly the regular Hausdorff spaces.*

It is not possible to drop the requirement of retract in the theorem above as the following example shows.

Example 5.12. Let X be the Sierpinski space, i.e., $X=\{0,1\}$ and the topology on X is $\{\emptyset,\{1\},X\}$. The Sierpinski space is T_1, but not Hausdorff. However, we can give an upwards-closed open representation of the Sierpinski space.

Build a domain D as in Fig. 1. Let D^R be the set of maximal elements of D. Define φ as indicated in the figure, i.e., the only non-compact element is mapped to 0, the rest of the maximal elements are mapped to 1.

Clearly, (D,D^R,φ) represents X. It is upwards-closed since D^R consists of maximal elements. Any basic open set in D contains a maximal element that represents 1. Hence, the forward image of any basic open set is open. Thus, the representation is open.

5.4. Representations for spaces with clopen bases

A *clopen base* is a base where each basic open set is also closed. A topological space X is *totally disconnected* if every pair of distinct points can be separated by a *disconnection*. This means that if $x\neq y$, then there exist disjoint open sets U and V containing x and y respectively, such that $U\cup Y=X$. Any totally disconnected space is Hausdorff. A T_0-space that has a clopen base is totally disconnected. A space that is compact and totally disconnected has a clopen base.

Theorem 5.13. *A space X with a clopen base has a dense homeomorphic representation by maximal elements with the closed image property.*

Proof. Let P consist of the non-empty sets in a clopen base for X together with the set X. Then P is a neighbourhood system for X satisfying the requirement (ii)$'$. Let $D = \mathrm{Idl}(P)$ and let D^{R} be the converging ideals. By the proof of Theorem 5.8 this is a dense retract representation with the closed image property.

Clearly, $I_x = J_x$. Hence, φ is injective, i.e., φ is a homeomorphism. The ideals J_x are maximal in D. □

That the representation is by maximal elements implies that it is local and that it is upwards-closed.

In the other direction we can show that already spaces that have local upwards-closed open representations must have clopen bases. We start by recording the following topological fact.

Lemma 5.14. *Let $f\colon X \to Y$ be a continuous open onto mapping and let B be an open base for the topology on X. Then $\{f[U]\colon U \in B\}$ is an open base for the topology on Y.*

Proof. Any set of the form $f[U]$ is open since f is an open mapping. Suppose that y is a point in some set V open in Y. Since f is onto we can choose an element $x \in f^{-1}[y]$. The set $f^{-1}[V]$ is open in X and contains x, hence there exists an $U \in B$ such that $x \in U \subseteq f^{-1}[V]$. Clearly $y = f(x)$ is an element of the open set $f[U]$. Moreover, $f[U] \subseteq V$ since $U \subseteq f^{-1}[V]$. Thus the set $\{f[U]\colon U \in B\}$ is an open base. □

Now, we show that local upwards-closed open representations have the closed image property. This is used together with the lemma above to find a clopen base for the represented space.

Lemma 5.15. *A local upwards-closed open representation has the closed image property.*

Proof. Let $a \in D_{\mathrm{c}}$. The set $U = D \setminus {\downarrow}{\uparrow} a$ is open in D. The image $\varphi[U \cap D^{\mathrm{R}}]$ is open since φ is open. The sets $\varphi[{\uparrow} a \cap D^{\mathrm{R}}]$ and $\varphi[U \cap D^{\mathrm{R}}]$ cover X since, by upwards-closed, $\varphi[{\uparrow} a \cap D^{\mathrm{R}}] = \varphi[{\downarrow}{\uparrow} a \cap D^{\mathrm{R}}]$. Assume that $x, y \in D^{\mathrm{R}}$ satisfy $x \in {\uparrow} a, y \in U$ and $\varphi(x) = \varphi(y)$. By the representation being local, x and y are consistent, contradicting that ${\uparrow} a \cap = \emptyset$. Thus, $\varphi[{\uparrow} a \cap D^{\mathrm{R}}]$ and $\varphi[U \cap D^{\mathrm{R}}]$ are disjoint, and so, $\varphi[U \cap D^{\mathrm{R}}]$ is the complement of $\varphi[{\uparrow} a \cap D^{\mathrm{R}}]$. Thus, $\varphi[{\uparrow} a \cap D^{\mathrm{R}}]$ is closed since the complement is open. □

Theorem 5.16. *The following are equivalent*:

(i) *X is T_0 and has a clopen base.*

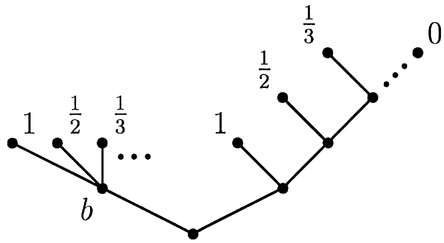

Fig. 2. Domain D representing the metric space $\{1/n\colon n\in\mathbb{N}\}\cap\{0\}$.

(ii) *X has a homeomorphic representation by maximal elements.*
(iii) *X has a local upwards-closed open representation.*
(iv) *X is T_0 and has an open representation with the closed image property.*

Proof. (i) $\Rightarrow$ (ii): By Theorem 5.13.

(ii) $\Rightarrow$ (iii): Trivial.

(iii) $\Rightarrow$ (iv): Let $x,y\in D^{\mathrm{R}}$ be such that $\varphi(x)\neq\varphi(y)$. By upwards-closed, x and y cannot be consistent, since otherwise $\varphi(x)=\varphi(x\sqcup y)=\varphi(y)$. Hence, there exists $a\in\mathrm{approx}(x)$ such that a and y are inconsistent. By the representation being open, $\varphi[\uparrow a\cap D^{\mathrm{R}}]$ is open, and clearly, $\varphi(x)\in\varphi[\uparrow a\cap D^{\mathrm{R}}]$. Assume that there exists $z\in\uparrow a\cap D^{\mathrm{R}}$ such that $\varphi(z)=\varphi(y)$. Then, by the representation being local, y and z are consistent, contradicting that a and y are inconsistent. Thus, $\varphi(y)$ is not in the open set $\varphi[\uparrow a\cap D^{\mathrm{R}}]$, i.e., X is T_0.

The representation has the closed image property by Lemma 5.15.

(iv) $\Rightarrow$ (i): By Lemma 5.14 and the closed image property, $\{\varphi[\uparrow a\cap D^{\mathrm{R}}]\colon a\in D_{\mathrm{c}}\}$ is a clopen base for X. $\square$

The use of the local property in the theorem above is an easy but crude way to establish the result. One gets the feeling that upwards-closed open representations represent totally disconnected spaces. However, this is false, in general, and an exact characterisation of the spaces that have upwards-closed open representations has not been established. Example 5.12, for instance, gave an upwards-closed open representation of a T_0-space which is not totally disconnected.

The following example shows that upwards-closed open representations need not have the closed image property. Hence, the naïve choice of candidate for a clopen base does not work. Yet, the space has a clopen base and is totally disconnected.

Example 5.17. Let $X=\{0\}\cup\{1/n\colon n\in\mathbb{N}\}$ with the subspace topology from $\mathbb{R}$. Construct a domain D as in Fig. 2 and let D^{R} be the set of maximal elements. Define φ as indicated in the figure. Then $(D,D^{\mathrm{R}},\varphi)$ is an upwards-closed open representation of X.

The base $\{\uparrow a\colon a\in D_{\mathrm{c}}\}$ in D induces a base $B=\{\varphi[\uparrow a\cap D^{\mathrm{R}}]\colon a\in D_{\mathrm{c}}\}$ in X by Lemma 5.14. The base B is not clopen since the forward image of the basic open set $\uparrow b$, where b is the element depicted in Fig. 2, is not closed. However, the set $\{\varphi[\uparrow a \cap D^{\mathrm{R}}]\colon a \in D_{\mathrm{c}} \setminus \{b\}\}$ is a clopen base for the topology on X.

We have not been able to construct an upwards-closed open representation of, for example, the reals. In fact, we know of no Hausdorff space which has an upwards-closed open representation and which is not totally disconnected.

The results in this subsection do not hold for continuous domain representations as is apparent from the following proposition.

Proposition 5.18. *Any normal space has an upwards-closed open retract continuous domain representation with the closed image property.*

Proof. Let X be a normal space and let P consist of all the non-empty open subsets of X. Define $\prec$ on P by $U \prec V \Leftrightarrow \overline{V} \subseteq U$. By normality, $(P, \prec)$ is an abstract basis. The ideal completion $D = \mathrm{Idl}(P, \prec)$ is a continuous domain.

Let D^{R} be the set of ideals that contain a neighbourhood base for some point and let $\varphi : D \to X$ be defined by $\varphi(I) = x$ if I contains a neighbourhood base of x. Then $(D, D^{\mathrm{R}}, \varphi)$ is an upwards-closed open continuous domain representation of X. It is open since an ideal I converging to $x \in X$ belongs to the basic open set $\Uparrow(\downarrow U)$ if, and only if, $x \in U$.

Define $\eta : X \to D^{\mathrm{R}}$ by $\eta(x) = \{U \in P\colon x \in U\}$. Clearly, $\varphi \circ \eta = \mathrm{id}$.

We leave to the reader to verify that $\varphi[\uparrow(\downarrow U) \cap D^{\mathrm{R}}] = \overline{U}$. □

6. Representations from first or second countable domains

The results in this section limit the class of topological spaces that can be given a computability theory by a domain representation since any effective domain must be second countable (or countably based), and hence, first countable. They are consequences of standard topological facts.

Definition 6.1. (i) A topological space X is *sequential* if a set $A \subseteq X$ is open if and only if every sequence $(x_n)_n$ converging to a point in A is eventually in A.

(ii) A topological space X is a *Fréchet* space if a point lies in the closure of a set if and only if there is a sequence in the set converging to the point.

Lemma 6.2. *A first countable space is Fréchet and any Fréchet space is sequential.*

Proof. Standard. □

Proposition 6.3. (i) *Every space represented by a first countable domain is sequential.*

(ii) *Every space pseudo-openly represented by a first countable domain is a Fréchet space.*

(iii) *Every space with a retract representation from a second countable domain is second countable.*

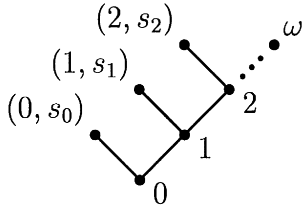

Fig. 3. The domain $D_{s,x}$.

Proof. (i) The domain is first countable, and hence, sequential. Sequential spaces are closed under quotient images. Thus, X is sequential.

(ii) The domain is first countable, and hence, Fréchet. Fréchet spaces are closed under pseudo-open images. Thus, X is a Fréchet space.

(iii) Let η be the embedding of X into a second countable domain D. Then $\{\eta^{-1}[\uparrow a \cap \eta[X]]: a \in D_c\}$ is a countable base for X. □

7. Representations by maximal elements

Any T_0 space can be represented by maximal elements. However, the representation map of the representations constructed here have weaker topological properties than the ones constructed in Section 5. We will start by giving the construction for a simpler case.

7.1. Representations by maximal elements for sequential spaces

We give a direct construction of a domain representation of a sequential space. In the proof below let s denote a sequence whose elements are s_n for $n \in \mathbb{N}$.

Theorem 7.1. *Any sequential space X has a representation by maximal elements.*

Proof. Let $\mathscr{S} = \{(s, x): s \in X^{\mathbb{N}}, x \in X$, and s converges to $x\}$. For each $(s, x) \in \mathscr{S}$ construct a domain $D_{s,x}$ (see Fig. 3) whose compact elements are

$$\{(n, s_n): n \in \mathbb{N}\} \cup \mathbb{N}.$$

The ordering on $D_{s,x}$ is

$$n \sqsubseteq m \Leftrightarrow n \leqslant m,$$

$$n \sqsubseteq (m, s_m) \Leftrightarrow n \leqslant m.$$

The pairs (n, s_n) are maximal.

Let D be the separated sum of $\{D_{s,x}: (s, x) \in \mathscr{S}\}$. The set of maximal elements of D is $D_m = \{(n, s_n)_{D_{s,x}}: n \in \mathbb{N}, (s, x) \in \mathscr{S}\} \cup \{\omega_{D_{s,x}}: (s, x) \in \mathscr{S}\}$. Define a function $\varphi : D_m \to X$ by

$$\varphi(y) = \begin{cases} s_n & \text{if } y = (n, s_n)_{D_{s,x}}, \\ x & \text{if } y = \omega_{D_{s,x}}. \end{cases}$$

To show that φ is continuous it is sufficient to show that φ is continuous on each $D_{s,x} \cap D_m$. Let $U \subseteq X$ be open and let $V = \varphi^{-1}[U] \cap D_{s,x}$. Clearly any point of the form $(n, s_n) \in V$ is interior since (n, s_n) is compact in $D_{s,x}$. If $\omega \in V$ then there must exist an n such that $s_m \in U$ for all $m \geqslant n$ since s converges to x in the open set U. Hence, $\{y \in D_{s,x}: n \sqsubseteq y\} \cap D_m$ is an open set of D_m included in V, i.e., x is an interior point of V. Thus φ is continuous on every $D_{s,x} \cap D_m$, and hence, on D_m.

Suppose $A \subseteq X$ and $\varphi^{-1}[A]$ is open in D_m. If $s \to x \in A$ then $\omega_{D_{s,x}} \in \varphi^{-1}[A]$ which is open in D_m, hence there exists an open set $\{y: n_{D_{s,x}} \sqsubseteq y\} \cap D_m \subseteq \varphi^{-1}[A]$ such that $\omega_{D_{s,x}} \in \{y: n_{D_{s,x}} \sqsubseteq y\} \cap D_m$. Thus, the sequence s is eventually in A. Since any sequence converging to a point of A eventually is in A and X is a sequential space we have that A is open. We have shown that φ is a quotient map. Thus, (D, D_m, φ) is a representation of X. $\square$

7.2. Representations by maximal elements for arbitrary spaces

We now generalise the above construction to show that any topological space has a representation by maximal elements. The construction will use nets instead of sequences. For each net a domain is constructed and we take the separated sum of these to be the representing domain. However, the nets over a space constitute a proper class. We solve this problem by limiting the cardinality of the nets. Hence, we are left with a set of nets.

Definition 7.2. A space X is said to have a *locally κ-based* topology if every point of X has a neighbourhood base of cardinality less than κ.

Note that saying that a space is locally ω_1-based is simply saying that it is first countable.

Lemma 7.3. *Let X be a locally κ-based space for some $\kappa > \omega$ and let $N: A \to X$ be a net converging to some $x \in X$. Then there exists a net $N': A' \to X$ converging to x, such that $A' \subseteq \kappa$, $N'[A'] \subseteq N[A]$ and such that A' does not have a greatest element.*

Proof. Let B be a local base at x of cardinality less than κ and let $f: B \to \kappa$ be an injective function. Let $A' = f[B]$ and order A' by $a \leqslant a' \Leftrightarrow f^{-1}(a) \supseteq f^{-1}(a')$. Clearly, A' is a directed set. For any neighbourhood U of x we have that N is eventually in U, hence we can choose a point x_U in $U \cap N[A]$. Let $N'(a) = x_{f^{-1}(a)}$. Then N' is a net satisfying $N'[A'] \subseteq N[A]$ and $A' \subseteq \kappa$. Clearly, N' converges to x.

If A' has a greatest element then we can modify the net by adding an ω-chain on top of A' and letting the net be constant on the ω-chain. We leave the formal details to the reader. $\square$

The proof of the following theorem is very similar to the proof of Theorem 7.1. In fact the previous proof is the special case when the only nets considered are the ω-sequences.

Theorem 7.4. *Any topological space X has a representation by maximal elements.*

Proof. Choose κ such that the topological space X is locally κ-based; any $\kappa > 2^{|X|}$ is sufficient. Let

$$\mathcal{N} = \{ (N,x)\colon N \text{ is a net converging to } x \in X, \\ \text{and } A_{N,x} \subseteq \kappa \text{ does not have a greatest element}\},$$

where $A_{N,x} = \text{dom } N$.

For each pair $(N,x) \in \mathcal{N}$ we construct a domain $D_{N,x}$. Let E be the domain of all lower sets in $A_{N,x}$. Clearly, $A_{N,x}$ is the greatest element of E. Let $D_{N,x}$ be the domain obtained by augmenting E with new compact maximal elements

$$(a, N(a)), \quad \text{for } a \in A_{N,x},$$

where $e \in E$ is below $(a, N(a))$ if $e \subseteq {\downarrow} a$. The element $A_{N,x} \in E$ is not of the form ${\downarrow} a$ since $A_{N,x}$ does not have a greatest element. Hence, $A_{N,x}$ is maximal in $D_{N,x}$.

Let D be the separated sum of $\{D_{N,x}\colon (N,x) \in \mathcal{N}\}$. Clearly,

$$D_m = \{A_{N,x}\colon (N,x) \in \mathcal{N}\} \cup \bigcup_{(N,x)\in\mathcal{N}} \{(a, N(a))\colon a \in A_{N,x}\}.$$

Define a function $\varphi : D_m \to X$ by

$$\varphi(y) = \begin{cases} N(a) & \text{if } y = (a, N(a))_{D_{N,x}}, \\ x & \text{if } y = A_{N,x}. \end{cases}$$

We show that φ is continuous. Let $U \subseteq X$ be open and let $V = \varphi^{-1}[U] \cap D_{N,x}$. Clearly any point of the form $(a, N(a)) \in V$ is interior since $(a, N(a))$ is compact in $D_{N,x}$. If $A_{N,x} \in V$ then there exists an $a \in A_{N,x}$ such that $N(b) \in U$ for all $b \geqslant a$ since N converges to x in the open set U. Hence, ${\uparrow}{\downarrow} a \cap D_m$ is an open set included in V, i.e., $A_{N,x}$ is an interior point of V. Thus, φ is continuous on every $D_{N,x}$, and hence, on D.

Now, we show that φ is a quotient. Suppose $S \subseteq X$ and $\varphi^{-1}[S]$ is open in D_m. We have to show that S is open, which we do by showing that any net converging to some point in S must eventually be in S. By Lemma 7.3, we only need to consider nets whose domains are subsets of κ and lack a greatest element. For a net N converging to $x \in S$ we have that $A_{N,x} \in \varphi^{-1}[S]$. Since $\varphi^{-1}[S]$ is open in D_m, there exists $d \in (D_{N,x})_c$ such that $A_{N,x} \in {\uparrow} d \cap D_m \subseteq \varphi^{-1}[S]$. The compact element d is on the form ${\downarrow} a_1 \cup \cdots \cup {\downarrow} a_n$. Since $A_{N,x}$ is directed, there exists an $a \in A_{N,x}$ such that $a_i \leqslant a$, for $i = 1, \ldots, n$. Clearly, $\varphi[{\uparrow}{\downarrow} a \cap D_m] \subseteq S$, and so, for each $b \geqslant a$ in $A_{N,x}$, we have $N(b) \in S$, i.e., the net N is eventually in S. □

Moreover, since the image of a converging net under a continuous function is a converging net, it is easy to lift a continuous function to representations of this kind even though it is not covered by Theorem 9.3.

8. Uniform closure properties

Now we study uniform closure properties of representations under topological constructions of spaces. Thus we are interested in when we can construct a new representation of the newly created space in a canonical way from the representation(s) of the old space(s). All constructions made in this section preserve effectivity.

8.1. Quotients, pseudo-open and open images, and retracts

Given a representation of a space we can represent certain images of that space with the same domain and with the same set of representing elements by composition of the representing function and the image map.

Proposition 8.1. *Representability, pseudo-open representability, retract representability and open representability is uniformly closed under quotients, pseudo-open images, retracts and open images, respectively.*

Proof. The classes of quotient maps introduced in Section 3 are all closed under composition. □

8.2. Subspaces

Given a representation $(D, D^{\mathrm{R}}, \varphi)$ of a space X, we are interested in when D with φ restricted to the inverse image of a subset $Y \subseteq X$ is a representation of Y. The following result is easily obtained.

Proposition 8.2. *Let $(D, D^{\mathrm{R}}_X, \varphi)$ be a weak representation of X and let Y be a subset of X, and let $D^{\mathrm{R}}_Y = \varphi^{-1}[Y]$.*

(i) *If $(D, D^{\mathrm{R}}_X, \varphi)$ is a weak, pseudo-open, open, retract or homeomorphic representation, then $(D, D^{\mathrm{R}}_Y, \varphi|_{D^{\mathrm{R}}_Y})$ is a representation of the same kind of Y.*

(ii) *If $(D, D^{\mathrm{R}}_X, \varphi)$ is a representation and Y is either an open or a closed subset of X, then $(D, D^{\mathrm{R}}_Y, \varphi|_{D^{\mathrm{R}}_Y})$ is a representation of Y.*

The following example shows that (ii) in the proposition above cannot be strengthened to an arbitrary subset Y of X, in fact, it does not hold for G_δ subsets.

Example 8.3. Let Z be the following subset of the euclidean plane; $Z = \{(x, 0)\colon 0 \neq x \in \mathbb{R}\} \cup \{(\frac{1}{n}, 1)\colon 0 < n \in \mathbb{N}\} \cup \{(0, 1)\}$. Let $(D, D^{\mathrm{R}}, \varphi)$ be a retract representation of $\mathbb{R}^2$ constructed as in Theorem 5.8. By Proposition 8.2 (i) $(D, D^{\mathrm{R}}_Z, \varphi)$, where $D^{\mathrm{R}}_Z = \varphi^{-1}[Z]$, is a retract representation of Z. Let X be the projection of Z onto its first coordinate. If p is the projection then $(D, D^{\mathrm{R}}_Z, p \circ \varphi)$ is a representation of X since p is a quotient. Let $Y = X \setminus \{\frac{1}{n}\colon 0 < n \in \mathbb{N}\}$ and let $D^{\mathrm{R}}_Y = D^{\mathrm{R}}_Z \cap \varphi^{-1}[p^{-1}[Y]]$. Then $(D, D^{\mathrm{R}}_Y, p \circ \varphi|_{D^{\mathrm{R}}_Y})$ is a weak domain representation of Y.

Let U be the open ball centred in $(0,1)$ with radius $\frac{1}{2}$. By the construction of D there is $\downarrow A \in D_c$ such that $(0,1) \in A^\circ \subseteq A \subseteq U$. Clearly, if $I \in B_A \cap D_Y^R$ then $\varphi(I) = (0,1)$, i.e., $p(\varphi(I)) = 0$. Thus $B_A \cap D_Y^R \subseteq \varphi^{-1}[p^{-1}[0]]$. On the other hand, if $I \in \varphi^{-1}[p^{-1}[0]]$ then $\varphi(I) = (0,1)$, and hence, $I \in B_A$. We have shown $B_A \cap D_Y^R = \varphi^{-1}[p^{-1}[0]]$. It follows that $p \circ \varphi|_{D_Y^R}$ is not a quotient since $\{0\}$ is not an open set in Y.

8.3. Disjoint sums and direct limits

We now briefly consider disjoint sums and direct limits of topological spaces.

Proposition 8.4. *Weak representation, representability, pseudo-open representability, retract representability and open representability are uniformly closed under disjoint topological sum.*

Proof. Let X be the disjoint topological sum of $\{X_i\colon i \in I\}$ and suppose (D_i, D_i^R, φ_i) are weak representations of X_i. Let D be the separated sum of $\{D_i\colon i \in I\}$. Clearly $(D, \biguplus_i D_i^R, \biguplus_i \varphi_i)$ is a weak representation of X.

Observe that $\biguplus_i \varphi_i$ has the required property if each φ_i has that property. □

Proposition 8.5. *Weak representability and representability are uniformly closed under direct limits.*

Proof. By Propositions 8.1 and 8.4 using the standard construction of a direct limit as a quotient of a disjoint sum. □

8.4. Products

In this subsection we consider uniform representations of cartesian products. The situation here is, perhaps surprisingly, somewhat problematic. For weak representations, however, it is straightforward.

Proposition 8.6. *For every $i \in I$ let (D_i, D_i^R, φ_i) be a weak representation of X_i. Then $(\prod D_i, \prod D_i^R, \prod \varphi_i)$, where $\prod \varphi_i(\prod d_i) = \prod \varphi_i(d_i)$, is a weak representation of $\prod X_i$.*

Proof. Clearly, $\prod \varphi_i$ is onto and continuous. □

The following example shows that we cannot uniformly construct a representation of the product space from a pair of pseudo-open representations. This provides a counterexample to uniform constructions of representations and also of pseudo-open representations.

Example 8.7. Define q on $\mathbb{Q}$ by

$$q(x) = \begin{cases} 0 & \text{if } x \in \mathbb{Z}, \\ x & \text{otherwise.} \end{cases}$$

Let $\mathbb{Q}' = q[\mathbb{Q}]$ have the quotient topology induced by q.

Let $(D, D^{\mathrm{R}}, \varphi)$ be a retract representation of $\mathbb{Q}$ constructed as in Theorem 5.8. Let $\varphi' = q \circ \varphi$. Then $(D, D^{\mathrm{R}}, \varphi')$ is a pseudo-open representation of $\mathbb{Q}'$ since φ' is the composition of pseudo-open maps.

If $\varphi \times \varphi' = (\mathrm{id} \times q) \circ (\varphi \times \varphi)$ is a quotient, then $\mathrm{id} \times q$ would be a quotient. However, $\mathrm{id} \times q$ is not a quotient, see, e.g., [16]. Thus, $(D \times D, D^{\mathrm{R}} \times D^{\mathrm{R}}, \varphi \times \varphi')$ is not a representation of $\mathbb{Q} \times \mathbb{Q}'$.

Proposition 8.8. *Let $(D_i, D_i^{\mathrm{R}}, \varphi_i)$, for $i \in I$, be an open representation of X_i. Then $(\prod D_i, \prod D_i^{\mathrm{R}}, \prod \varphi_i)$ is an open representation of $\prod X_i$.*

Proof. By Lemma 8.6, it is a weak representation. A subbase for $\prod D_i$ are the sets $U_i \times \prod_{j \neq i} D_j$, where U_i is open in D_i, for $i \in I$. The images $\varphi_i[U_i \cap D_i^{\mathrm{R}}] \times \prod_{j \neq i} \varphi_j[D_j^{\mathrm{R}}]$ constitute a subbase of $\prod X_i$, hence the result. □

Proposition 8.9. *Let $(D_i, D_i^{\mathrm{R}}, \varphi_i, \eta_i)$, for $i \in I$, be a retract representation of X_i. Then $(\prod D_i, \prod D_i^{\mathrm{R}}, \prod \varphi_i, \prod \eta_i)$ is a retract representation of $\prod X_i$.*

Proof. By Proposition 8.6, $\prod \varphi_i$ is continuous. Clearly, $\prod \varphi_i \circ \prod \eta_i = \mathrm{id}$.

The set $U_i \times \prod_{j \neq i} D_i$, where U_i is open in D_i, is a subbasic open set in $\prod D_i$.

$$\left(\prod \eta_i\right)^{-1} \left[(U_i \cap D_i^{\mathrm{R}}) \times \prod_{j \neq i} D_j^{\mathrm{R}} \right] = \eta_i^{-1}[U_i \cap D_i^{\mathrm{R}}] \times \prod_{j \neq i} X_i.$$

The right-hand side is a subbasic open set in $\prod X_i$ since η_i is continuous. Thus, $\prod \eta_i$ is continuous. □

9. Functions and function spaces

9.1. Representing continuous functions

We will in this section study when functions between represented spaces can be represented. We start with the definition of the notion.

Definition 9.1. Let $(D, D^{\mathrm{R}}, \varphi)$ and $(E, E^{\mathrm{R}}, \psi)$ be domain representations of X and Y respectively. A continuous function $f : X \to Y$ is *represented* by a continuous function $\bar{f} : D \to E$ if $\psi(\bar{f}(x)) = f(\varphi(x))$, for all $x \in D^{\mathrm{R}}$. See Fig. 4.

We now give sufficient conditions for a continuous function between representing domains to induce a continuous function. We merely state the following easy but important result.

Proposition 9.2. . *Let $(D, D^{\mathrm{R}}, \varphi)$ be a representation of X and $(E, E^{\mathrm{R}}, \psi)$ be a weak representation of Y. Let $\bar{f} : D \to E$ be continuous such that $\bar{f}[D^{\mathrm{R}}] \subseteq E^{\mathrm{R}}$ and*

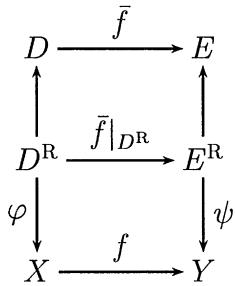

Fig. 4. Representing a function f.

assume $\bar{f}$ respects the equivalence relations induced by φ and ψ. Then $\bar{f}$ induces a unique continuous function $f:X \to Y$.

The following theorem tells us that for a large class of domain representations it is possible to represent any continuous function. In particular, representations of functions exist if the spaces are represented by domains constructed as in Sections 5.2 and 5.3.

Theorem 9.3. *Let (D, D^R, φ) be a dense weak representation of X, and let (E, E^R, ψ, η) be a retract representation of Y. Then every continuous function $f:X \to Y$ is represented by some continuous function $\bar{f}:D \to E$.*

Proof. The construction of $\bar{f}$ is done in two steps. First, let $f' = \eta \circ f \circ \varphi$. By hypothesis, this is a continuous function from D^R to E^R, which induces f since $\psi \circ f' = \psi \circ \eta \circ f \circ \varphi = f \circ \varphi$. The function f' may be considered as a continuous function from D^R to E as well since E^R has the subspace topology induced from E.

Secondly, the function f' is extended to a function $\bar{f}:D_c \to E$ by $\bar{f}(a) = \sqcap f'[\uparrow a \cap D^R]$. The infimum is well-defined since $\uparrow a \cap D^R$ is non-empty by density of D^R, and non-empty infima exist in consistently complete cpos. Clearly, $\bar{f}$ is monotone, and hence, it has a unique extension to D.

We now show that $\bar{f}$ is indeed an extension of f', i.e., that $\bar{f}(d) = f'(d)$ for $d \in D^R$. Let $a \in \text{approx}(d)$, then $\bar{f}(a) \sqsubseteq f'(d)$ since $d \in \uparrow a \cap D^R$. Conversely, if $b \in \text{approx}(f'(d))$, then $d \in f'^{-1}[\uparrow b]$. Hence, there exists $a \in \text{approx}(d)$ such that $f'[\uparrow a \cap D^R] \subseteq \uparrow b$, and so, $b \sqsubseteq \bar{f}(a) \sqsubseteq f'(d)$. Thus, $f'(d) = \bigsqcup_{a \in \text{approx}(d)} \bar{f}(a) = \bar{f}(d)$. $\square$

Constructions, such as the one in the theorem above, have been studied earlier, see for example [15].

9.2. Function spaces

In this section we consider representations of function spaces built by the function space construction on domains. Compare the work done by di Gianantonio [17] on representations of functions and functionals over the reals.

Let $(D, D^{\mathrm{R}}, \varphi)$ and $(E, E^{\mathrm{R}}, \psi)$ be representations of the topological spaces X and Y, respectively. Let us further assume that every continuous function $f: X \to Y$ is represented by a continuous function $\bar{f}: D \to E$. Then let

$$[D \to E]^{\mathrm{R}} = \{ f \in [D \to E] \colon f[D^{\mathrm{R}}] \subseteq E^{\mathrm{R}} \text{ and } (\forall x, y \in D^{\mathrm{R}})(\varphi(x) = \varphi(y) \Rightarrow \psi(f(x)) = \psi(f(y))) \}.$$

That is, $[D \to E]^{\mathrm{R}}$ consists of the continuous functions from D to E inducing continuous functions from X to Y. Thus there is an epimorphism $\vartheta : [D \to E]^{\mathrm{R}} \to (X \to Y)$ so that $([D \to E], [D \to E]^{\mathrm{R}}, \vartheta)$ is a representation of $X \to Y$, the continuous functions from X to Y. This representation induces a topology τ on $X \to Y$, the quotient topology obtained from the Scott topology on $[D \to E]$. The question is now how this topology is related to other topologies on $X \to Y$ and what properties it has.

Definition 9.4. Let X and Y be topological spaces.
(i) The sets $W(x, U) = \{ f \colon f(x) \in U \}$, for $x \in X$ and U an open subset of Y, form a subbase for the *pointwise* topology on the function space $X \to Y$.
(ii) The sets $W(K, U) = \{ f \colon f[K] \subseteq U \}$, for K a compact set in X and U an open set in Y, form a subbase for the *compact-open* topology on the function space $X \to Y$.
(iii) A topology on the function space $X \to Y$ is *jointly continuous* if the evaluation function eval: $(X \to Y) \times X \to Y$ defined by $\mathrm{eval}(f, x) = f(x)$ is continuous.

From general topology (see, e.g., [18]) we know that if a topology is jointly continuous then it is *finer* (has more open sets) than the compact-open topology and that the compact-open topology is finer than the pointwise topology.

From domain theory we know that the topology on function spaces of domains is exactly the pointwise topology and that the topology on function spaces is jointly continuous, hence the pointwise and the compact-open topology coincide for the function space construction on domains.

Since the Scott topology on function spaces of domains is jointly continuous a natural question is how close the induced topology is to being jointly continuous. It can be proved that under natural conditions, the induced topology is finer than the compact-open topology. The next two lemmas show this for slightly different conditions on the representations.

Lemma 9.5. *Let $(D, D^{\mathrm{R}}, \varphi)$ be a pseudo-open representation of X with least representatives, and let $(E, E^{\mathrm{R}}, \psi)$ be a representation of Y. Suppose that every continuous function $f: X \to Y$ can be lifted to a continuous function $\bar{f}: D \to E$. Then the topology τ on $X \to Y$ induced by the representation $([D \to E], [D \to E]^{\mathrm{R}}, \vartheta)$ is finer than the compact-open topology.*

Proof. It is sufficient to show that each $W(K, U)$ belongs to τ. Let $S = W(K, U)$ for some K and U. We show that $S \in \tau$ by showing that $\vartheta^{-1}[S]$ is open in $[D \to E]^{\mathrm{R}}$. Let

$f \in \vartheta^{-1}[S]$. We will show that we can find an open neighbourhood of f which also is a subset of $\vartheta^{-1}[S]$, thereby showing that $\vartheta^{-1}[S]$ is open.

Let $x \in K$ and let $d_x \in D^R$ be the least representative of x. Then $\vartheta(f)(x) \in U$ so $f(d_x) \in \psi^{-1}[U]$ and hence there exists $b_x \in \mathrm{approx}(f(d_x))$ such that $\psi[\uparrow b_x \cap E^R] \subseteq U$. By continuity of f there exists $a_x \in \mathrm{approx}(d_x)$ such that $b_x \sqsubseteq f(a_x)$. Note that the step function $\langle a_x;\, b_x \rangle$ is an approximation of f. We have $\varphi^{-1}[x] \subseteq \uparrow a_x$ since d_x is the least representation of x. Since φ is pseudo-open we have that $x \in (\varphi[\uparrow a_x \cap D^R])^\circ$. Clearly the open sets $(\varphi[\uparrow a_x \cap D^R])^\circ$, for $x \in K$, cover K. Hence there exists a finite subset of $\{a_x\colon x \in K\}$, say $\{a_1, \dots, a_n\}$, such that $K \subseteq \bigcup_{i=1}^n (\varphi[\uparrow a_i \cap D^R])^\circ$. We have that $\langle a_i; b_i \rangle \sqsubseteq f$, for $i \in \{1, \dots, n\}$. Hence $\bigsqcup_{i=1}^n \langle a_i; b_i \rangle$ exists and is below f.

We will now show that any function represented by a function in the basic open set $\uparrow \bigsqcup_{i=1}^n \langle a_i; b_i \rangle$ belongs to S. Let $g \in \uparrow \bigsqcup_{i=1}^n \langle a_i; b_i \rangle \cap [D \to E]^R$. For $x \in K$ there exists a representation $\bar{x}$ of x such that $\bar{x} \sqsupseteq a_i$, for some i, since $K \subseteq \bigcup_{i=1}^n (\varphi[\uparrow a_i \cap D^R])^\circ$. But $g(a_i) \sqsupseteq b_i$ since $\langle a_i; b_i \rangle \sqsubseteq g$. Hence $g(\bar{x}) \in \psi^{-1}[U]$, i.e., $\vartheta(g)(x) \in U$. Thus $\vartheta(g)[K] \subseteq U$, i.e., $\vartheta(g) \in S$. □

Lemma 9.6. *Let (D, D^R, φ) be a retract representation of X, and let (E, E^R, ψ) be a representation of Y. Suppose that every continuous function $f: X \to Y$ can be lifted to a continuous function $\bar{f}: D \to E$. Then the topology τ on $X \to Y$ induced by the representation $([D \to E], [D \to E]^R, \vartheta)$ is finer than the compact-open topology.*

Proof. Let $K \subseteq X$ be a compact set and let $U \subseteq Y$ be an open set. Choose an open set $V \subseteq E$ such that $V \cap E^R = \psi^{-1}[U]$. We will now show $\vartheta^{-1}[W(K, U)] = W(\eta[K], V) \cap [D \to E]^R$. Let $f \in \vartheta^{-1}[W(K, U)]$. Then $f[\varphi^{-1}[K]] \subseteq V$, and so, in particular, $f[\eta[K]] \subseteq V$. For the other direction let $f \in W(\eta[K], V) \cap [D \to E]^R$. Then, since f induces a function mapping K to U, $\varphi^{-1}[K]$ is mapped into V by f.

The set $W(\eta[K], V)$ is compact-open, and hence, Scott-open, as observed above. □

We will now show that if X is a locally compact Hausdorff space and if the representations are of a certain kind, then the topology τ induced on $X \to Y$ by the representation of the function space will be the compact-open topology. Moreover, τ will be jointly continuous.

Theorem 9.7. *Let $(D, D^R, \varphi, \eta_X)$ be a dense retract representation with the closed image property of a locally compact Hausdorff space X and let (E, E^R, ψ, η_Y) be a retract representation of a space Y. Then $([D \to E], [D \to E]^R, \vartheta, \varepsilon)$ is a retract representation of $X \to Y$ with the compact-open topology.*

Proof. The embedding ε is obtained by a slight modification of the construction in Theorem 9.3. Let $f' = \eta_Y \circ f \circ \varphi$ and define $f'': D'_c \to E$ by $f''(a) = \sqcap f'[\uparrow a \cap D^R]$, where D'_c is the set of all $a \in D_c$ such that $\varphi[\uparrow a \cap D^R]$ is not only closed, but compact. Finally, let $\varepsilon(f)(d) = \bigsqcup\{f''(a)\colon a \in \downarrow d \cap D'_c\}$.

The closed image property implies that if $b \in D_c$ is above some $a \in D'_c$, then $\varphi[\uparrow b \cap D^R]$ is compact, i.e., $b \in D'_c$, since closed subsets of compact sets are compact. Thus, $\varepsilon(f)(d)$ is the supremum of either an empty set or a directed set, i.e., ε is well-defined.

An argument similar to the one in Theorem 9.3 shows that $\varepsilon(f)$ induces f, i.e., $\vartheta \circ \varepsilon = \mathrm{id}$. The argument requires that $\downarrow d \cap D'_{\mathrm{c}}$ is non-empty, for $d \in D^{\mathrm{R}}$. This follows from local compactness of X.

The induced topology τ is finer than the compact-open topology by Lemma 9.6. This implies that ϑ is continuous.

For continuity of ε, let $\varepsilon(f) \in \uparrow \langle a; b\rangle$ with $a \in D_{\mathrm{c}}$ and $b \in E_c$. By definition of ε we may restrict to $a \in D'_{\mathrm{c}}$. Now,

$$\begin{aligned} \varepsilon(f) \in \uparrow\langle a;\, b\rangle &\Leftrightarrow b \sqsubseteq f''(a) \\ &\Leftrightarrow f'[\uparrow a \cap D^{\mathrm{R}}] \subseteq \uparrow b \\ &\Leftrightarrow f[\varphi[\uparrow a \cap D^{\mathrm{R}}]] \subseteq \eta_Y^{-1}[\uparrow b] \\ &\Leftrightarrow f \in W(\varphi[\uparrow a \cap D^{\mathrm{R}}], \eta_Y^{-1}[\uparrow b]). \qquad \square \end{aligned}$$

The theorem is a generalisation of a result by di Gianantonio [17]. Note that this result cannot be lifted to functionals since function spaces fail to be locally compact in general.

The embedding η_X is only used in showing continuity of ϑ by use of Lemma 9.6. Hence, in view of Lemma 9.5, the theorem can also be formulated with the condition that the representation $(D, D^{\mathrm{R}}, \varphi)$ of X should be a dense pseudo-open representation with least representatives and the closed image property.

Acknowledgements

I would like to thank Viggo Stoltenberg-Hansen for numerous ideas and suggestions. The anonymous referee is thanked for generalising and sharpening some of the results.

References

[1] S. Abramsky, A. Jung, Domain theory, in: S. Abramsky, D.M. Gabbay, T.S.E. Maibaum (Eds.), Handbook of Logic in Computer Science, vol. 3, Oxford University Press, Oxford, 1994, pp. 1–168.

[2] A. Arhangelskij, Some types of factor mappings, and the relations between classes of topological spaces, Sov. Math. Dokl. 4 (1963) 1726–1729.

[3] A. Bauer, L. Birkedal, D.S. Scott, Equilogical spaces, preprint, 1998.

[4] U. Berger, Total sets and objects in domain theory, Ann. Pure Appl. Logic 60 (1993) 91–117.

[5] U. Berger, Density theorems for the domain-with-totality semantics of dependent types, Applied Categorical Structures, to appear.

[6] L. Birkedal, A. Carboni, G. Rosilini, D.S. Scott, Type theory via exact categories, Proc. 13th Ann. IEEE Symp. on Logic in Computer Science, 1998, pp. 188–198.

[7] J. Blanck, Domain representability of metric spaces, Ann. Pure Appl. Logic 83 (1997) 225–247.

[8] J. Blanck, Computability on topological spaces by effective domain representations, Uppsala Dissertations in Mathematics, vol. 7, 1997.

[9] J. Blanck, Domain representations of topological spaces, Electron. Notes Theoret. Comput. Sci. 13 (1998).

[10] A. Edalat, Dynamical systems, measures, and fractals via domain theory, Inform. Comput. 120 (1995) 32–48.
[11] A. Edalat, Domain theory and integration, Theoret. Comput. Sci. 151 (1995) 163–193.
[12] A. Edalat, Power domains and iterated function systems, Inform. Comput. 124 (1996) 182–197.
[13] A. Edalat, R. Heckmann, A computational model for metric spaces, Theoret. Comput. Sci. 193 (1998) 53–73.
[14] Y.L. Ershov, Model C of partial continuous functionals, in: R.O. Gandy, J.M.E. Hyland (Eds.), Logic Colloquium 76, Studies in Logic and Foundations in Mathematics, vol. 87, North-Holland, Amsterdam, 1977, pp. 455–467.
[15] M.H. Escardo, Injective spaces and function spaces, Topology Appl. 89 (1998) 75–120.
[16] S.P. Franklin, Spaces in which sequences suffice, Fund. Math. 57 (1965) 107–115.
[17] P. di Gianantonio, Real number computability and domain theory, Inform. and Comput. 127 (1996) 11–25.
[18] J.L. Kelley, General Topology, Van Nostrand, Princeton, NJ, 1955.
[19] L. Kristiansen, D. Normann, Semantics for some constructors of type theory, in: Behara, Fritsch, Lintz (Eds.), Symposia Gaussiana, Walter de Gruyter, Berlin, 1995, pp. 201–224.
[20] D. Normann, A hierarchy of domains with totality but without density, in: S.B. Cooper, T.A. Slaman, S.S. Wainer (Eds.), Computability, Enumerability, Unsolvability, Cambridge University Press, Cambridge, 1996, pp. 233–257.
[21] D. Normann, The continuous functionals, in: E.R. Griffor (Ed.), Handbook of Computability Theory, Elsevier, Amsterdam, 1998.
[22] D. Normann, Closing the gap between continuous functionals and recursion in ${}^3E$, Arch. Math. Logic 36 (1997) 269–287.
[23] D.S. Scott, A new category? unpublished manuscript, 1996.
[24] V. Stoltenberg-Hansen, I. Lindström, E.R. Griffor, Mathematical Theory of Domains, Cambridge University Press, Cambridge, 1994.
[25] V. Stoltenberg-Hansen, J.V. Tucker, Complete local rings as domains, J. Symbolic Logic 53 (1988) 603–624.
[26] V. Stoltenberg-Hansen, J.V. Tucker, Algebraic and fixed point equations over the inverse limits of algebras, Theoret. Comput. Sci. 87 (1991) 1–24.
[27] V. Stoltenberg-Hansen, J.V. Tucker, Effective algebra, in: S. Abramsky, D.M. Gabbay, T.S.E. Maibaum (Eds.), Handbook of Logic in Computer Science, vol. 4, Oxford University Press, Oxford, 1995, pp. 357–526.
[28] V. Stoltenberg-Hansen, J.V. Tucker, Computable rings and fields, in: E.R. Griffor (Ed.), Handbook of Computability Theory, Elsevier, Amsterdam, 1998.
[29] G. Waagbø, Domains-with-totality semantics for intuitionistic type theory, Ph.D. Thesis, Oslo, 1997.
[30] K. Weihrauch, U. Schreiber, Embedding metric spaces into cpo's, Theoret. Comput. Sci. 16 (1981) 5–24.