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Identities for the Ternary Commutator

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This paper classifies all identities of degree 7 satisfied by the ternary commutator in an associative ternary algebra. (Seven is the lowest degree for which non-trivial identities exist.) These identities are ternary generalizations of the Jacobi identity for Lie algebras. © 1998 Academic Press

INTRODUCTION

If n is any positive integer, then an n-algebra is a vector space A over a field F together with a linear map ω: $A^{\otimes n} \to A$, where $A^{\otimes n}$ denotes the n-fold tensor power of A. In the case $n = 3$ we say that A is a ternary algebra (or triple system). To simplify notation, we write $a_1 a_2 \cdots a_n$ instead of $\omega(a_1 \otimes a_2 \otimes \cdots \otimes a_n)$ for $a_i \in A$ $(1 \leq i \leq n)$.

Given the n-algebra A, we can define a new n-ary operation ω^- on the same vector space by the formula

$$\omega^-(a_1 \otimes a_2 \otimes \cdots \otimes a_n) = \sum_{\pi \in S_n} \epsilon(\pi) a_{\pi.1} a_{\pi.2} \cdots a_{\pi.n}.$$

Here S_n denotes the symmetric group on n letters, and ϵ: $S_n \to \{\pm 1\}$ is the sign homomorphism. We call this operation the n-commutator, and usually write it as $[a_1 a_2 \cdots a_n]$. The operation ω^- is anticommutative in the sense that $[a_1 a_2 \cdots a_n] = 0$ whenever $a_i = a_j$ for some $i \neq j$. We write A^- for the n-algebra consisting of the vector space A with the operation ω^-. In the familiar case $n = 2$, we have an algebra A (not necessarily

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associative) with product ab, and the commutator $[ab] = ab - ba$ satisfying the anticommutative identity $[aa] = 0$.

We say that A is an associative n-algebra if the ordered product of $2n-1$ elements does not depend on the position of the parentheses; that is,

$$a_1 \cdots (a_i \cdots a_{i+n-1}) \cdots a_{2n-1} = a_1 \cdots (a_j \cdots a_{j+n-1}) \cdots a_{2n-1},$$

whenever $1 \le i < j \le n$. When $n = 2$ this reduces to the familiar identity $(ab)c = a(bc)$.

The commutator $[ab]$ in an associative 2-algebra satisfies the Jacobi identity $[[ab]c] + [[bc]a] + [[ca]b] = 0$. The anticommutative and Jacobi identities together define the variety of Lie 2-algebras. The Poincaré–Birkhoff–Witt theorem implies that any Lie 2-algebra is a subalgebra of A^- for some associative 2-algebra A. It follows that any identity satisfied by the commutator in every associative 2-algebra follows from the Jacobi identity.

Kurosh [K] (see also [BB, Sect. 15]) posed the question of determining all identities satisfied by the n-commutator in an associative n-algebra. For the case $n = 3$, it was shown in [B2] that there are no identities of degree 5. The purpose of this paper is (i) to show that in the case $n = 3$ the simplest non-trivial identities have degree 7, and (ii) to classify all the identities of that degree. These results are closely related to the problem of determining the correct definition of Lie n-algebra (equivalently, determining the correct generalization of the Jacobi identity to n-algebras). Some other papers which deal with this problem are [K], [BB], [F], [HW], [G1], [G2], [B2], [AP].

Most of the computations referred to in this paper were programmed in Maple V.4 and executed on a Sun Ultra 1 workstation. All the Maple procedures are available by e-mail from the author.

STATEMENT OF THE PROBLEM USING REPRESENTATION THEORY

In this section we express Kurosh's problem for $n = 3$ in terms of the representation theory of the symmetric group. We enumerate the permutations π_i of seven letters in lexicographical order:

$$\pi_1 = abcdefg, \qquad \pi_2 = abcdegf, \qquad \pi_3 = abcdfeg,$$
$$\pi_4 = abcdfge, \qquad \cdots, \qquad \pi_{5040} = gfedcba.$$

From now on we assume that the base field F is the field of complex numbers.

Let P_3^k denote the subspace of the free anticommutative ternary algebra spanned by the distinct multilinear monomials involving k pairs of brackets. Let $d = d_3^k$ denote the degree of these monomials, and D_3^k the dimension of P_3^k. From [B2] we have

$$d_3^k = 2k + 1, \qquad D_3^k = \frac{(3k)!}{k!6^k},$$

k	1	2	3	4	5
d_3^k	3	5	7	9	11
D_3^k	1	10	280	15400	1401400

The S_7-module P_3^3 has a basis consisting of

$$\binom{7}{3,2,2} = 210 \text{ monomials of type 1: } [[[\cdots]\cdot\cdot]\cdot\cdot],$$

$$\frac{1}{2}\binom{7}{3,3,1} = 70 \text{ monomials of type 2: } [[\cdots][\cdots]\cdot].$$

In Tables I and II these monomials are listed; the subscripts indicate the position of each monomial in the lexicographical ordering of the basis of P_3^3, and the superscripts give the sign and the index of the corresponding permutation of the seven letters.

Let Q_3^k denote the subspace of the free associative ternary algebra spanned by the $(2k+1)!$ permutations of $2k+1$ letters. We define a linear map E_3^k: $P_3^k \to Q_3^k$ by expanding each commutator in P_3^k as the alternating sum of the six permutations of its factors:

$$E_3^k \colon [xyz] \mapsto xyz - xzy - yxz + yzx + zxy - zyx.$$

Both P_3^k and Q_3^k are modules over the symmetric group S_d, where we define the action by permuting the symbols, not the positions. (All the identities we consider are multilinear, so this should not cause any confusion.) The commutator expansion map E_3^k is an S_d-module homomorphism, and the space I_3^k of all identities of degree $2k+1$ satisfied by the ternary commutator is the kernel of E_3^k (and hence also an S_d-module).

THE KERNEL OF THE COMMUTATOR EXPANSION MAP

It is shown in [B2] that $I_3^2 = \{0\}$, so there are no identities for the ternary commutator of degree 5 (that is, in which each term involves two pairs of brackets). The remainder of this paper is devoted to studying the space I_3^3. For convenience we omit the sub- and superscripts on P, Q, E, and I. Thus we are interested in the kernel I of the S_7-module homomorphism E: $P \to Q$. The matrix $[E]$ representing the linear map E has size 5040×280; the ij-entry is the coefficient of the ith associative monomial in the expansion of the jth anticommutative monomial.

TABLE I
Monomials of the Form [[[⋯] · ·] · ·]

$[[[abc]de]fg]_{1}^{+1}$	$[[[abc]df]eg]_{2}^{-3}$	$[[[abc]dg]cf]_{3}^{+5}$	$[[[abc]ef]dg]_{4}^{+9}$	$[[[abc]eg]df]_{5}^{-11}$
$[[[abc]fg]de]_{6}^{+17}$	$[[[abd]ce]fg]_{7}^{-25}$	$[[[abd]cf]eg]_{8}^{+27}$	$[[[abd]cg]ef]_{9}^{-29}$	$[[[abd]ef]cg]_{10}^{-29}$
$[[[abd]eg]cf]_{11}^{+35}$	$[[[abd]fg]ce]_{12}^{-41}$	$[[[abe]cd]fg]_{13}^{+49}$	$[[[abe]cf]dg]_{14}^{-51}$	$[[[abe]cg]df]_{15}^{+53}$
$[[[abe]df]cg]_{16}^{+57}$	$[[[abe]dg]cf]_{17}^{-59}$	$[[[abe]fg]cd]_{18}^{+65}$	$[[[abf]cd]eg]_{19}^{-73}$	$[[[abf]ce]dg]_{20}^{+75}$
$[[[abf]cg]de]_{21}^{-77}$	$[[[abf]de]cg]_{22}^{-81}$	$[[[abf]dg]ce]_{23}^{+83}$	$[[[abf]eg]cd]_{24}^{-89}$	$[[[abg]cd]ef]_{25}^{+97}$
$[[[abg]ce]df]_{26}^{-99}$	$[[[abg]cf]de]_{27}^{-99}$	$[[[abg]de]cf]_{28}^{+105}$	$[[[abg]df]ce]_{29}^{-107}$	$[[[abg]ef]cd]_{30}^{+113}$
$[[[acd]be]fg]_{31}^{+145}$	$[[[acd]bf]eg]_{32}^{-147}$	$[[[acd]bg]ef]_{33}^{+149}$	$[[[acd]ef]bg]_{34}^{+153}$	$[[[acd]eg]bf]_{35}^{-155}$
$[[[acd]fg]be]_{36}^{+161}$	$[[[ace]bd]fg]_{37}^{-169}$	$[[[ace]bf]dg]_{38}^{+171}$	$[[[ace]bg]df]_{39}^{-173}$	$[[[ace]df]bg]_{40}^{-177}$
$[[[ace]dg]bf]_{41}^{+179}$	$[[[ace]fg]bd]_{42}^{-185}$	$[[[acf]bd]eg]_{43}^{+193}$	$[[[acf]be]dg]_{44}^{-195}$	$[[[acf]bg]de]_{45}^{+197}$
$[[[acf]de]bg]_{46}^{+201}$	$[[[acf]dg]be]_{47}^{-203}$	$[[[acf]eg]bd]_{48}^{+209}$	$[[[acg]bd]ef]_{49}^{-217}$	$[[[acg]be]df]_{50}^{+219}$
$[[[acg]bf]de]_{51}^{-221}$	$[[[acg]de]bf]_{52}^{-225}$	$[[[acg]df]be]_{53}^{+227}$	$[[[acg]ef]bd]_{54}^{+227}$	$[[[ade]bc]fg]_{55}^{+289}$
$[[[ade]bf]cg]_{56}^{-291}$	$[[[ade]bg]cf]_{57}^{+293}$	$[[[ade]cf]bg]_{58}^{+297}$	$[[[ade]cg]bf]_{59}^{-299}$	$[[[ade]fg]bc]_{60}^{+305}$
$[[[adf]bc]eg]_{61}^{-313}$	$[[[adf]be]cg]_{62}^{+315}$	$[[[adf]bg]ce]_{63}^{-317}$	$[[[adf]ce]bg]_{64}^{-321}$	$[[[adf]cg]be]_{65}^{+323}$
$[[[adf]eg]bc]_{66}^{-329}$	$[[[adg]bc]ef]_{67}^{+337}$	$[[[adg]be]cf]_{68}^{-339}$	$[[[adg]bf]ce]_{69}^{+341}$	$[[[adg]ce]bf]_{70}^{+345}$
$[[[adg]cf]be]_{71}^{-347}$	$[[[adg]ef]bc]_{72}^{+353}$	$[[[aef]bc]dg]_{73}^{+433}$	$[[[aef]bd]cg]_{74}^{-435}$	$[[[aef]bg]cd]_{75}^{+437}$
$[[[aef]cd]bg]_{76}^{+441}$	$[[[aef]cg]bd]_{77}^{-443}$	$[[[aef]dg]bc]_{78}^{+449}$	$[[[aeg]bc]df]_{79}^{-457}$	$[[[aeg]bd]cf]_{80}^{+459}$
$[[[aeg]bf]cd]_{81}^{-461}$	$[[[aeg]cd]bf]_{82}^{-461}$	$[[[aeg]cf]bd]_{83}^{+467}$	$[[[aeg]df]bc]_{84}^{-473}$	$[[[afg]bc]de]_{85}^{+577}$
$[[[afg]bd]ce]_{86}^{-579}$	$[[[afg]be]cd]_{87}^{+581}$	$[[[afg]cd]be]_{88}^{+585}$	$[[[afg]ce]bd]_{89}^{-587}$	$[[[afg]de]bc]_{90}^{+593}$
$[[[bcd]ae]fg]_{91}^{-865}$	$[[[bcd]af]eg]_{92}^{-865}$	$[[[bcd]ag]ef]_{93}^{-869}$	$[[[bcd]ef]ag]_{94}^{-873}$	$[[[bcd]eg]af]_{95}^{+875}$
$[[[bcd]fg]ae]_{96}^{-881}$	$[[[bce]ad]fg]_{97}^{+889}$	$[[[bce]af]dg]_{98}^{-891}$	$[[[bce]ag]df]_{99}^{+893}$	$[[[bce]df]ag]_{100}^{+897}$
$[[[bce]dg]af]_{101}^{-899}$	$[[[bce]fg]ad]_{102}^{-899}$	$[[[bcf]ad]eg]_{103}^{-899}$	$[[[bcf]ae]dg]_{104}^{+915}$	$[[[bcf]ag]de]_{105}^{-917}$
$[[[bcf]de]ag]_{106}^{-921}$	$[[[bcf]dg]ae]_{107}^{+923}$	$[[[bcf]eg]ad]_{108}^{-929}$	$[[[bcg]ad]ef]_{109}^{+937}$	$[[[bcg]ae]df]_{110}^{-939}$
$[[[bcg]af]de]_{111}^{+941}$	$[[[bcg]de]af]_{112}^{+945}$	$[[[bcg]df]ae]_{113}^{-947}$	$[[[bcg]ef]ad]_{114}^{+953}$	$[[[bde]ac]fg]_{115}^{-1009}$
$[[[bde]af]cg]_{116}^{+1011}$	$[[[bde]ag]cf]_{117}^{-1013}$	$[[[bde]cf]ag]_{118}^{-1017}$	$[[[bde]cg]af]_{119}^{+1019}$	$[[[bde]fg]ac]_{120}^{-1025}$
$[[[bdf]ac]eg]_{121}^{+1033}$	$[[[bdf]ae]cg]_{122}^{-1035}$	$[[[bdf]ag]ce]_{123}^{+1037}$	$[[[bdf]ce]ag]_{124}^{+1041}$	$[[[bdf]cg]ae]_{125}^{-1043}$
$[[[bdf]eg]ac]_{126}^{+1049}$	$[[[bdg]ac]ef]_{127}^{-1057}$	$[[[bdg]ae]cf]_{128}^{+1059}$	$[[[bdg]af]ce]_{129}^{-1061}$	$[[[bdg]ce]af]_{130}^{-1065}$
$[[[bdg]cf]ae]_{131}^{+1067}$	$[[[bdg]ef]ac]_{132}^{-1073}$	$[[[bef]ac]dg]_{133}^{-1153}$	$[[[bef]ad]cg]_{134}^{+1155}$	$[[[bef]ag]cd]_{135}^{-1157}$
$[[[bef]cd]ag]_{136}^{-1161}$	$[[[bef]cg]ad]_{137}^{+1163}$	$[[[bef]dg]ac]_{138}^{-1169}$	$[[[beg]ac]df]_{139}^{+1177}$	$[[[beg]ad]cf]_{140}^{-1179}$
$[[[beg]af]cd]_{141}^{+1181}$	$[[[beg]cd]af]_{142}^{+1185}$	$[[[beg]cf]ad]_{143}^{-1187}$	$[[[beg]df]ac]_{144}^{+1193}$	$[[[bfg]ac]de]_{145}^{-1297}$
$[[[bfg]ad]ce]_{146}^{+1299}$	$[[[bfg]ae]cd]_{147}^{-1301}$	$[[[bfg]cd]ae]_{148}^{-1305}$	$[[[bfg]ce]ad]_{149}^{+1307}$	$[[[bfg]de]ac]_{150}^{-1313}$
$[[[cde]ab]fg]_{151}^{+1729}$	$[[[cde]af]bg]_{152}^{-1731}$	$[[[cde]ag]bf]_{153}^{+1733}$	$[[[cde]bf]ag]_{154}^{+1737}$	$[[[cde]bg]af]_{155}^{-1739}$
$[[[cde]fg]ab]_{156}^{+1745}$	$[[[cdf]ab]eg]_{157}^{-1753}$	$[[[cdf]ae]bg]_{158}^{+1755}$	$[[[cdf]ag]be]_{159}^{-1757}$	$[[[cdf]be]ag]_{160}^{-1761}$
$[[[cdf]bg]ae]_{161}^{+1763}$	$[[[cdf]eg]ab]_{162}^{-1769}$	$[[[cdg]ab]ef]_{163}^{+1777}$	$[[[cdg]ae]bf]_{164}^{-1779}$	$[[[cdg]af]be]_{165}^{+1781}$
$[[[cdg]be]af]_{166}^{+1785}$	$[[[cdg]bf]ae]_{167}^{-1787}$	$[[[cdg]ef]ab]_{168}^{+1793}$	$[[[cef]ab]dg]_{169}^{+1873}$	$[[[cef]ad]bg]_{170}^{-1875}$
$[[[cef]ag]bd]_{171}^{+1877}$	$[[[cef]bd]ag]_{172}^{+1881}$	$[[[cef]bg]ad]_{173}^{-1883}$	$[[[cef]dg]ab]_{174}^{+1889}$	$[[[ceg]ab]df]_{175}^{-1897}$
$[[[ceg]ad]bf]_{176}^{+1899}$	$[[[ceg]af]bd]_{177}^{-1901}$	$[[[ceg]bd]af]_{178}^{-1905}$	$[[[ceg]bf]ad]_{179}^{+1907}$	$[[[ceg]df]ab]_{180}^{-1913}$
$[[[cfg]ab]de]_{181}^{+2017}$	$[[[cfg]ad]be]_{182}^{-2019}$	$[[[cfg]ae]bd]_{183}^{+2021}$	$[[[cfg]bd]ae]_{184}^{+2025}$	$[[[cfg]be]ad]_{185}^{-2027}$
$[[[cfg]de]ab]_{186}^{+2033}$	$[[[def]ab]cg]_{187}^{-2593}$	$[[[def]ac]bg]_{188}^{+2595}$	$[[[def]ag]bc]_{189}^{-2597}$	$[[[def]bc]ag]_{190}^{-2601}$
$[[[def]bg]ac]_{191}^{+2603}$	$[[[def]cg]ab]_{192}^{-2609}$	$[[[deg]ab]cf]_{193}^{+2617}$	$[[[deg]ac]bf]_{194}^{-2619}$	$[[[deg]af]bc]_{195}^{+2621}$
$[[[deg]bc]af]_{196}^{+2625}$	$[[[deg]bf]ac]_{197}^{-2627}$	$[[[deg]cf]ab]_{198}^{+2633}$	$[[[dfg]ab]ce]_{199}^{-2737}$	$[[[dfg]ac]be]_{200}^{+2739}$
$[[[dfg]ae]bc]_{201}^{-2741}$	$[[[dfg]bc]ae]_{202}^{-2745}$	$[[[dfg]be]ac]_{203}^{+2747}$	$[[[dfg]ce]ab]_{204}^{-2753}$	$[[[efg]ab]cd]_{205}^{+3457}$
$[[[efg]ac]bd]_{206}^{-3459}$	$[[[efg]ad]bc]_{207}^{+3461}$	$[[[efg]bc]ad]_{208}^{+3465}$	$[[[efg]bd]ac]_{209}^{-3467}$	$[[[efg]cd]ab]_{210}^{+3473}$

TABLE II
Monomials of the Form $[[\cdots][\cdots]\cdot]$

$[[abc][def]g]_1^{+1}$	$[[abc][deg]f]_2^{-2}$	$[[abc][dfg]e]_3^{+4}$	$[[abd][cef]g]_4^{-10}$	$[[abd][ceg]f]_5^{-25}$
$[[abd][ceg]f]_6^{+26}$	$[[abd][cfg]e]_7^{-28}$	$[[abd][efg]c]_8^{+34}$	$[[abe][cdf]g]_9^{+49}$	$[[abe][cdg]f]_{10}^{-50}$
$[[abe][cfg]d]_{11}^{+52}$	$[[abe][dfg]c]_{12}^{-58}$	$[[abf][cde]g]_{13}^{-73}$	$[[abf][cdg]e]_{14}^{+74}$	$[[abf][ceg]d]_{15}^{-76}$
$[[abf][deg]c]_{16}^{+82}$	$[[abg][cde]f]_{17}^{+97}$	$[[abg][cdf]e]_{18}^{-98}$	$[[abg][cef]d]_{19}^{+100}$	$[[abg][def]c]_{20}^{-106}$
$[[acd][bef]g]_{21}^{+145}$	$[[acd][beg]f]_{22}^{-146}$	$[[acd][bfg]e]_{23}^{+148}$	$[[acd][efg]b]_{24}^{-154}$	$[[ace][bdf]g]_{25}^{-169}$
$[[ace][bdg]f]_{26}^{+170}$	$[[ace][bfg]d]_{27}^{-172}$	$[[ace][dfg]b]_{28}^{+178}$	$[[acf][bde]g]_{29}^{+193}$	$[[acf][bdg]e]_{30}^{-194}$
$[[acf][beg]d]_{31}^{+196}$	$[[acf][deg]b]_{32}^{-202}$	$[[acg][bde]f]_{33}^{-217}$	$[[acg][bdf]e]_{34}^{+218}$	$[[acg][bef]d]_{35}^{-220}$
$[[acg][def]b]_{36}^{+226}$	$[[ade][bcf]g]_{37}^{+289}$	$[[ade][bcg]f]_{38}^{-290}$	$[[ade][bfg]c]_{39}^{+292}$	$[[ade][cfg]b]_{40}^{-298}$
$[[adf][bce]g]_{41}^{-313}$	$[[adf][bcg]e]_{42}^{+314}$	$[[adf][beg]c]_{43}^{-316}$	$[[adf][ceg]b]_{44}^{+322}$	$[[adg][bce]f]_{45}^{+337}$
$[[adg][bcf]e]_{46}^{-338}$	$[[adg][bef]c]_{47}^{+340}$	$[[adg][cef]b]_{48}^{-346}$	$[[aef][bcd]g]_{49}^{+433}$	$[[aef][bcg]d]_{50}^{-434}$
$[[aef][bdg]c]_{51}^{+436}$	$[[aef][cdg]b]_{52}^{-442}$	$[[aeg][bcd]f]_{53}^{-457}$	$[[aeg][bcf]d]_{54}^{+458}$	$[[aeg][bdf]c]_{55}^{-460}$
$[[aeg][cdf]b]_{56}^{+466}$	$[[afg][bcd]e]_{57}^{+577}$	$[[afg][bce]d]_{58}^{-578}$	$[[afg][bde]c]_{59}^{+580}$	$[[afg][cde]b]_{60}^{-586}$
$[[bcd][efg]a]_{61}^{+874}$	$[[bce][dfg]a]_{62}^{-898}$	$[[bcf][deg]a]_{63}^{+922}$	$[[bcg][def]a]_{64}^{-946}$	$[[bde][cfg]a]_{65}^{+1018}$
$[[bdf][ceg]a]_{66}^{-1042}$	$[[bdg][cef]a]_{67}^{+1066}$	$[[bef][cdg]a]_{68}^{+1162}$	$[[beg][cdf]a]_{69}^{-1186}$	$[[bfg][cde]a]_{70}^{+1306}$

Simple S_7-modules will be denoted $[p]$, where p is a partition of 7. Thus the 15 distinct simple S_7-modules (with dimensions given in the second row) are

$[7]$	$[61]$	$[52]$	$[51^2]$	$[43]$	$[421]$	$[41^3]$	$[3^21]$	$[32^2]$	$[321^2]$	$[31^4]$	$[2^31]$	$[2^21^3]$	$[21^5]$	$[1^7]$
1	6	14	15	14	35	20	21	21	35	15	14	14	6	1

The space P decomposes as a direct sum $P = P' \oplus P''$ of S_7-modules, where P' is the span of the 210 monomials of type 1, and P'' is the span of the 70 monomials of type 2. (It is clear that these are submodules, since the action of S_7 does not affect the bracket arrangement.) We have the following result on the S_7-module structure of P' and P''. Similar results for the case $n = 2$ and $2 \leq k \leq 6$ can be found in [B1].

LEMMA. *The characters of the S_7-modules P', P'', and P are given in the following table. In the partitions labelling the conjugacy classes, parts equal to* 1 *are omitted.*

	id	2	22	222	3	32	322	33	4	42	43	5	52	6	7
P'	210	−50	14	−6	6	−2	2	0	0	0	0	0	0	0	0
P''	70	−20	6	−4	4	−2	0	1	0	2	0	0	0	−1	0
P	280	−70	20	−10	10	−4	2	1	0	2	0	0	0	−1	0

From this we obtain the multiplicities of the simple S_7-modules in P', P'', and P. The modules $[7], \ldots, [41^3]$ *do not occur in the decomposition*:

	$[3^21]$	$[32^2]$	$[321^2]$	$[31^4]$	$[2^31]$	$[2^21^3]$	$[21^5]$	$[1^7]$
P'	1	1	2	1	2	3	2	1
P''	0	0	1	0	1	1	1	1
P	1	1	3	1	3	4	3	2

Proof. First choose conjugacy class representatives in S_7. Apply each of these to each monomial of type 1, and determine the trace of each representative on P'. Then use the character table of S_7 to derive the decomposition of P'. The same method applies to the monomials of type 2, giving the decomposition of P''. (This computation took 44.46 sec.) ∎

The following theorem is the main result of this paper. We use the following notation:

X_i' denotes the monomials of type 1,

$X_i'(j)$ denotes the jth letter in X_i'.

$X_i'(j,k)$ denotes the set containing the jth and kth letters of X_i',

$X_i'(\hat{j})$ denotes the permutation obtained by omitting the jth letter of X_i',

$X_i'(\hat{j},\hat{k})$ denotes the permutation obtained by omitting the jth and kth letters of X_i'.

Similarly with X'' denoting the monomials of type 2.

THEOREM. *The kernel I of the commutator expansion map $E\colon P \to Q$ has dimension 7, and as S_7-modules $I \cong [21^5] \oplus [1^7]$. A generator of I is the alternating sum on b,c,d,e,f,g of $[[[bcd]ae]fg] + [[abc][def]g]$, which can also be expressed as*

$$\sum_{X_i'(4)=a} \epsilon\big(X_i'(\hat{4})\big)X_i' + \sum_{X_i''(1)=a} \epsilon\big(X_i''(\hat{1})\big)X_i''.$$

A basis for I is obtained by applying to this identity the permutations with indices $1+720i$ $(0 \le i \le 6)$; in cycle notation, these are $(), (ab), (acb), (adcb), (aedcb), (afedcb), (agfedcb)$.

A generator of the six-dimensional submodule of I isomorphic to $[21^5]$ is $(1/12$ of$)$ the linearization of the alternating sum on c,d,e,f,g of $3[[[acd]ae]fg] + 2[[[cde]af]ag] - [[acd][efg]a]$, which can also be expressed as

$$\sum_{X_i'(1,4)=\{a,b\}} \epsilon\big(X_i'(\hat{1},\hat{4})\big)X_i' + \sum_{X_i'(4,6)=\{a,b\}} \epsilon\big(X_i'(\hat{4},\hat{6})\big)X_i'$$
$$- \sum_{X_i''(1,7)=\{a,b\}} \epsilon\big(X_i''(\hat{1},\hat{7})\big)X_i''.$$

A basis for this submodule is obtained by applying to this identity the permutations with indices $1+120i$ $(0 \le i \le 6)$; these are $(\ ), (bc), (bdc), (bedc), (bfedc), (bgfedc)$.

A generator (and basis) of the one-dimensional submodule isomorphic to $[1^7]$ is $(1/24$ of$)$ the alternating sum on a,b,c,d,e,f,g of $[[[abc]de]fg] -$

$[[abc][def]g]$, *which can also be expressed as*

$$\sum_i \epsilon(X_i') X_i' - 3 \sum_i \epsilon(X_i'') X_i''.$$

Proof. These calculations, like those in the lemma, were done using Maple procedures on a Sun workstation.

The first procedure generated the 280 permutations corresponding to the monomials of types 1 and 2. The second procedure expanded the commutators in each of the monomials. Each expanded monomial is a list of $6^3 = 216$ permutations, which were stored together with their signs and lexicographical index. The third procedure used a matrix B of size 400×280 (initially set to zero); it then read the rows of the matrix $[E]$ in blocks of 120 into the last 120 rows of B and computed the reduced row−echelon form of B. After all 42 ($= 5040/120$) blocks were processed in this way, the matrix B was equal to the reduced row−echelon form of $[E]$. (This method was used to save memory, since the matrix $[E]$ was never stored all at once.) A basis for the nullspace of B was then computed; the nullspace of B is the same as the nullspace of $[E]$. (This entire calculation took 4520.30 s.) The results showed that the kernel I has dimension 7. The lemma shows that any seven-dimensional submodule of P is isomorphic to $[21^5] \oplus [1^7]$.

A second set of procedures took the basis vectors of the nullspace of $[E]$, computed the result of applying the commutator expansion map to each of these vectors, and verified that the result was always the zero vector. This shows that each basis vector of the nullspace of $[E]$ is indeed an identity satisfied by the ternary commutator. (This verification took 17.12 s.)

A third set of procedures determined a basis for the submodule generated by each of the basis vectors of the nullspace of $[E]$. Four of the basis vectors generated the whole kernel I, and the other three generated the six-dimensional submodule. The first two identities in the theorem (and the corresponding basis permutations) were obtained from these calculations. These first two identities have coefficients in $\{-1, 0, 1\}$. The list of 280 coefficients for the generator of I is

```
0000000000000000000000000000000000000000000000000000000000000000000000
00000000000000000000+-+000-+-000+-+000-+-000+-+000-+-000+-+000+-+000-+
-000+-+000-+-000+-+000-+-000-+-000+-+000-+-000+-+000-+-000+-+000-+-000
+-+--+-++-+--+-++-+-+-+--+-++-+--+-++-+--+-++-+-+-+--+-++-+-000000000
```

The list of coefficients for the generator of $[21^5]$ is

```
000000000000000000000000000000+-+000-+-000+-+000-+-000+-+000-+-000+-+0
00+-+000-+-000+-+000+-+000-+-000+-+000-+-000+-+000-+-000+-+000+-+000-+
-000+-+0000+-+-00-+-+00+-+-00+-+-00-+-+00+-+-00-+-+00+-+-00-+-+00+-+-0
00000000000000000000000-000+000-000+000-000+000-000-000+000--+-+-+--+-
```

(This computation took 35,060.70 s.)

A fourth set of procedures determined a basis for the one-dimensional submodule of I as follows. A general linear combination of the basis vectors of the nullspace of $[E]$ was computed. Each transposition in S_7 was applied to this general vector, and the result set equal to the negative of the general vector. The third identity in the theorem is a basis for the solution space of these equations. This identity was checked by applying the commutator expansion map and verifying that the result was the zero vector. (This calculation took 169.93 s.)

The above calculations were checked by a fifth set of procedures that read in the rows of $[E]$ one at a time; this allowed the computer to keep track of which rows resulted in an increase in the rank of B. The output consisted of a list of 273 row indices corresponding to a basis of the row space of $[E]$, thereby verifying that the kernel has dimension 7. These row

TABLE III
A Basis of the Row Space of the Commutator Expansion Matrix

1	2	3	4	5	6	7	8	9	10	11	12	13
14	15	16	17	18	19	20	21	22	23	25	26	27
28	29	31	32	33	34	35	36	37	38	39	40	41
42	43	44	45	46	47	49	50	51	52	53	55	57
58	59	61	62	63	64	65	67	68	69	70	71	73
74	75	76	77	79	81	83	89	91	92	93	94	95
97	98	99	100	101	103	105	107	113	121	122	123	124
125	127	129	130	131	133	137	139	145	146	147	148	149
151	153	154	155	157	159	160	161	163	165	166	167	169
170	171	172	173	175	177	179	181	183	185	187	189	191
193	194	195	196	197	199	201	203	209	211	213	215	217
218	219	220	221	223	225	227	233	241	243	245	247	249
250	251	257	273	275	281	289	290	291	292	293	295	297
299	305	311	313	314	315	316	317	319	321	323	329	335
337	338	339	340	341	343	345	347	353	361	363	377	401
425	433	434	435	436	437	439	441	443	449	457	458	459
460	461	463	465	467	473	481	577	578	579	580	581	583
585	587	593	721	722	723	724	725	727	729	730	731	733
737	739	745	746	748	751	753	755	757	761	763	769	770
781	785	787	793	811	841	842	844	850	865	889	913	937
970	1009	1033	1057	1153	1177	1297	1441	1447	1453	1459	1465	2161

indices are listed in Table III. (This calculation took 54,700.57 s to get to row 2161 and 131,015.17 s to get to row 5040.) ∎

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