Available online at www.sciencedirect.com
SCIENCE DIRECT®
LINEAR ALGEBRA AND ITS APPLICATIONS
Linear Algebra and its Applications 414 (2006) 1–18
www.elsevier.com/locate/laa

Ternary analogues of Lie and Malcev algebras

Murray R. Bremner [a,*], Luiz A. Peresi [b]

[a] *Research Unit in Algebra and Logic, Department of Mathematics and Statistics, University of Saskatchewan, 106 Wiggins Road, Saskatoon, SK, Canada S7N 5E6*
[b] *Instituto de Matemática e Estatística, Universidade de São Paulo, Rua do Matão, 1010-Cidade Universitária, CEP 05508-090 São Paulo, Brazil*

Received 30 March 2005; accepted 11 September 2005
Available online 27 October 2005
Submitted by H. Schneider

Abstract

We consider two analogues of associativity for ternary algebras: total and partial associativity. Using the corresponding ternary associators, we define ternary analogues of alternative and assosymmetric algebras. On any ternary algebra the alternating sum $[a, b, c] = abc - acb - bac + bca + cab - cba$ (the ternary analogue of the Lie bracket) defines a structure of an anticommutative ternary algebra. We determine the polynomial identities of degree $\leqslant 7$ satisfied by this operation in totally and partially associative, alternative, and assosymmetric ternary algebras. These identities define varieties of ternary algebras which can be regarded as ternary analogues of Lie and Malcev algebras. Our methods involve computational linear algebra based on the representation theory of the symmetric group.

Keywords: Ternary algebras; Polynomial identities; Computational linear algebra; Representations of the symmetric group

* Corresponding author.
E-mail addresses: bremner@math.usask.ca (M.R. Bremner), peresi@ime.usp.br (L.A. Peresi).

0024-3795/$ - see front matter
doi:10.1016/j.laa.2005.09.004

1. Preliminaries

1.1. Binary algebras

An *algebra* (not necessarily associative) consists of a vector space A over a field $\mathbb{F}$ together with a bilinear product

$$A \times A \to A$$

denoted by juxtaposition ab. The *associator* is the trilinear function

$$(a, b, c) = (ab)c - a(bc).$$

We call A *associative* if it satisfies the polynomial identity

$$(a, b, c) = 0$$

for all $a, b, c \in A$. We call A *alternative* if it satisfies the polynomial identities

$$(a, a, b) = 0 \quad \text{and} \quad (b, a, a) = 0$$

for all $a, b \in A$. If char $\mathbb{F} \neq 2$ then these identities are equivalent to the identities

$$(a_{\sigma(1)}, a_{\sigma(2)}, a_{\sigma(3)}) = \epsilon(\sigma)(a_1, a_2, a_3)$$

for all $a_1, a_2, a_3 \in A$, where $\sigma \in S_3$ (symmetric group) and $\epsilon : S_3 \to \{\pm 1\}$ is the sign homomorphism. We call A *assosymmetric* if it satisfies the polynomial identities

$$(a_{\sigma(1)}, a_{\sigma(2)}, a_{\sigma(3)}) = (a_1, a_2, a_3)$$

for all $a_1, a_2, a_3 \in A$. General references on the theory of nonassociative algebras are Schafer [19] and Zhevlakov et al. [23].

The *Lie bracket* (or *commutator*) is the bilinear function

$$[a, b] = ab - ba.$$

In any associative algebra the Lie bracket satisfies anticommutativity and the Jacobi identity

$$[[a, b], c] + [[b, c], a] + [[c, a], b] = 0$$

for all $a, b, c \in A$. This is the minimal identity (non-trivial identity of lowest degree) in the sense that every identity of degree $\leqslant 3$ satisfied by the commutator in every associative algebra is a consequence of anticommutativity and the Jacobi identity. These two identities define the variety of *Lie algebras*. (By the theorem of Poincaré, Birkhoff and Witt, it is known that every identity satisfied by the commutator in every associative algebra is a consequence of anticommutativity and the Jacobi identity.) In any alternative algebra the Lie bracket satisfies anticommutativity and the Malcev identity

$$\begin{aligned}[][a, c], [b, d]] = {} & [[[a, b], c], d] + [[[b, c], d], a] \\ & + [[[c, d], a], b] + [[[d, a], b], c]\end{aligned}$$

for all $a, b, c, d \in A$. This is the minimal identity in the sense that every identity of degree $\leqslant 4$ satisfied by the commutator in every alternative algebra is a consequence of anticommutativity and the Malcev identity. These two identities define the variety of *Malcev algebras*. (It is still an open problem whether every identity satisfied by the commutator in every alternative algebra is a consequence of anticommutativity and the Malcev identity.) Every assosymmetric algebra is Lie-admissible: the Lie bracket satisfies anticommutativity and the Jacobi identity. Every Lie algebra is a Malcev algebra. Malcev algebras were introduced by Malcev [14] (as "Moufang–Lie algebras") and were given their present name by Sagle [18]. For recent developments in the theory of Malcev algebras see Pérez-Izquierdo and Shestakov [15] and Shestakov [20].

1.2. *n-Ary algebras*

An *n-ary algebra* consists of a vector space A over a field $\mathbb{F}$ together with a multilinear map

$$\overbrace{A \times \cdots \times A}^{n \text{ factors}} \to A$$

denoted by concatenation $a_1 a_2 \cdots a_n$.

On A we define $n-1$ multilinear $(2n-1)$-ary operations, called the *total associators*, by

$$\begin{aligned} &t_i(a_1, a_2, \ldots, a_{2n-1}) \\ &\quad = a_1 \cdots (a_i \cdots a_{i+n-1}) \cdots a_{2n-1} - a_1 \cdots (a_{i+1} \cdots a_{i+n}) \cdots a_{2n-1} \end{aligned}$$

for $1 \leqslant i \leqslant n-1$. We call A *totally associative* if it satisfies the polynomial identities

$$t_i(a_1, a_2, \ldots, a_{2n-1}) = 0$$

for $1 \leqslant i \leqslant n-1$. We define another multilinear $(2n-1)$-ary operation, called the *partial associator*, by

$$\begin{aligned} &p(a_1, a_2, \ldots, a_{2n-1}) \\ &\quad = \sum_{i=1}^{n} (-1)^{(n-1)i} a_1 \cdots a_{i-1}(a_i \cdots a_{i+n-1}) a_{i+n} \cdots a_{2n-1}. \end{aligned}$$

(These definitions come from the theory of duality for quadratic operads; see Gnedbaye [6].) We call A *partially associative* if it satisfies the polynomial identity

$$p(a_1, a_2, \ldots, a_{2n-1}) = 0.$$

(In the case $n = 2$ both total and partial associativity reduce to binary associativity.) We use these associators to define n-ary analogues of certain varieties of nonassociative algebras.

We call A *totally alternative* if it satisfies the polynomial identities

$$t_i(a_1, a_2, \ldots, a_{2n-1}) = 0 \quad \text{whenever } a_j = a_k \text{ for some } j \neq k$$

for $1 \leqslant i \leqslant n$. Equivalently (char $\mathbb{F} \neq 2$) each total associator t_i is an alternating function of its $2n-1$ arguments. We call A *partially alternative* if it satisfies the polynomial identities

$$p(a_1, a_2, \ldots, a_{2n-1}) = 0 \quad \text{whenever } a_j = a_k \text{ for some } j \neq k.$$

Equivalently (char $\mathbb{F} \neq 2$) the partial associator p is an alternating function of its $2n-1$ arguments.

We call A *totally assosymmetric* if each total associator t_i is invariant under all permutations of its $2n-1$ arguments. We call A *partially assosymmetric* if the partial associator p is invariant under all permutations of its $2n-1$ arguments.

The *alternating sum* (or *n-ary commutator*) is the multilinear operation

$$[a_1, a_2, \ldots, a_n] = \sum_{\sigma \in S_n} \epsilon(\sigma) a_{\sigma(1)} a_{\sigma(2)} \cdots a_{\sigma(n)}.$$

(In the case $n = 2$ this reduces to the familiar Lie bracket.) We write A^- for the n-ary algebra which has the same underlying vector space as A but where the original operation is replaced by the alternating sum. Then A^- is an *anticommutative* n-ary algebra in the sense that

$$[a_{\sigma(1)}, a_{\sigma(2)}, \ldots, a_{\sigma(n)}] = \epsilon(\sigma)[a_1, a_2, \ldots, a_n]$$

for every permutation $\sigma \in S_n$.

One way to generalize the concept of Lie algebra to the n-ary case is to consider the minimal identities satisfied by the alternating sum in every totally or partially associative n-ary algebra. Other n-ary analogues of Lie algebras have been introduced by Filippov [5], Gnedbaye [6] and Hanlon and Wachs [7].

Similarly, we can generalize the concept of Malcev algebra to the n-ary case by considering the minimal identities satisfied by the alternating sum in every totally or partially alternative n-ary algebra. Another n-ary analogue of Malcev algebras has been introduced by Pozhidaev [16].

1.3. Ternary algebras

In this paper we study the case $n = 3$. The minimal identity in the totally associative case, which has degree 7, was found by Bremner [1]. The minimal identity in the partially associative case, which has degree 5, was found by Gnedbaye [6].

We use computational linear algebra based on the representation theory of the symmetric group to classify all identities of degree $\leqslant 7$ satisfied by the alternating sum in every totally or partially associative, alternative, or assosymmetric algebra. With respect to the total associators we have the following results:

1. In the totally associative case, we recover the identity of Bremner [1] in degree 7; this identity is equivalent to two irreducible identities.

2. In the totally alternative case, we show that in degree 7 the alternating sum satisfies only one of the two irreducible identities obtained in the totally associative case.
3. In the totally assosymmetric case, we show that the identities are the same as in the totally associative case.

With respect to the partial associator we have the following results:

1. In the partially associative case, we recover the identity of Gnedbaye [6] in degree 5, and show that it implies all the identities in degree 7.
2. In the partially alternative case, we show that there are no identities in degree $\leqslant 7$: the minimal identity for the alternating sum has degree $\geqslant 9$.
3. In the partially assosymmetric case, we show that the identities are the same as in the partially associative case.

We now specialize the definitions in the n-ary case to $n = 3$. We have a ternary algebra A with product denoted abc. The alternating sum is

$$[a, b, c] = abc - acb - bac + bca + cab - cba.$$

The first and second total associators are

$$(abc)de - a(bcd)e, \quad a(bcd)e - ab(cde).$$

Since the symmetric group S_5 is generated by the two permutations (12) and (12345), we see that total alternativity is equivalent (char $\mathbb{F} \neq 2$) to the multilinear identities

$$\begin{aligned}
&(abc)de - a(bcd)e + (bac)de - b(acd)e,\\
&a(bcd)e - ab(cde) + b(acd)e - ba(cde),\\
&(abc)de - a(bcd)e - (bcd)ea + b(cde)a,\\
&a(bcd)e - ab(cde) - b(cde)a + bc(dea).
\end{aligned}$$

Total assosymmetry is equivalent to the multilinear identities

$$\begin{aligned}
&(abc)de - a(bcd)e - (bac)de + b(acd)e,\\
&a(bcd)e - ab(cde) - b(acd)e + ba(cde),\\
&(abc)de - a(bcd)e - (bcd)ea + b(cde)a,\\
&a(bcd)e - ab(cde) - b(cde)a + bc(dea).
\end{aligned}$$

The partial associator is

$$(abc)de + a(bcd)e + ab(cde).$$

Partial alternativity is equivalent (char $\mathbb{F} \neq 2$) to the multilinear identities

$$\begin{aligned}
&(abc)de + a(bcd)e + ab(cde) + (bac)de + b(acd)e + ba(cde),\\
&(abc)de + a(bcd)e + ab(cde) - (bcd)ea - b(cde)a - bc(dea).
\end{aligned}$$

Partial assosymmetry is equivalent to the multilinear identities

$$(abc)de + a(bcd)e + ab(cde) - (bac)de - b(acd)e - ba(cde),$$
$$(abc)de + a(bcd)e + ab(cde) - (bcd)ea - b(cde)a - bc(dea).$$

2. Representations of the symmetric group

The first papers which used the representation theory of the symmetric group to classify polynomial identities of algebras were Malcev [13] and Specht [21]. In the present paper, our methods are based on Hentzel's idea to use a computer program to implement the structure of the group ring of the symmetric group in order to decompose polynomial identities into components which are irreducible (in the sense of group representations), thereby reducing the identities to the smallest possible pieces and permitting the study of identities of higher degree than would otherwise be possible. The original references for these computational methods are Hentzel [8,9]. Some more recent papers which discuss computational methods in the application of representation theory to the study of polynomial identities are Bremner and Hentzel [2,3] and Hentzel and Peresi [10,11].

2.1. *The group ring* $\mathbb{F}S_n$

We recall in outline Young's structure theory for $\mathbb{F}S_n$ where $\mathbb{F}$ has characteristic zero. General references on the representation theory of the symmetric group are Rutherford [17] and James and Kerber [12]. Young's original papers are collected in [22].

Let λ be a partition of n, and let d_λ be the number of standard tableaux associated to λ. Then $\mathbb{F}S_n$ is isomorphic to the direct sum over all partitions λ of full matrix subalgebras of size $d_\lambda \times d_\lambda$:

$$\mathbb{F}S_n \cong \bigoplus_\lambda M_{d_\lambda}(\mathbb{F}).$$

In the natural representation of S_n a basis E_{ij} $(1 \leqslant i, j \leqslant d_\lambda)$ of the full matrix subalgebra corresponding to λ is given by the following construction. Enumerate the standard tableaux corresponding to λ as $T_1, T_2, \ldots, T_d$ where $d = d_\lambda$. For tableau T_i let R_i be the sum over the row permutations and let C_i be the alternating sum over the column permutations. Let s_{ij} be the permutation which interchanges tableaux T_i and T_j. Define elements E_{ij} in $\mathbb{F}S_n$ by

$$E_{ii} = \frac{d}{n!} C_i R_i, \quad E_{ij} = E_{ii} s_{ij} \quad (i \neq j).$$

Then we have the multiplication formula

$$E_{ij} E_{k\ell} = \epsilon_{jk} E_{i\ell},$$

where the scalars ϵ_{ij} take the values $\{-1, 0, 1\}$. Clifton [4] gives a rule for computing the ϵ_{ij} and a simple modification of the E_{ij} which provides a basis satisfying the usual matrix unit relations.

2.2. Polynomial identities

Let $I = I(a_1, \ldots, a_n)$ be a multilinear nonassociative polynomial of degree n. We think of I as an identity satisfied by some nonassociative algebra A. Suppose that there are t distinct inequivalent association types in degree n. Then we can write

$$I = \sum_{i=1}^{t} I^{(i)},$$

where $I^{(i)}$ consists of the terms of I having association type i. Since all the terms in $I^{(i)}$ have the same association type, we can regard $I^{(i)}$ as a linear combination of permutations of the n variables $a_1, \ldots, a_n$; that is, $I^{(i)}$ is an element of the group ring $\mathbb{F}S_n$. Since there are t association types, we can regard the complete identity I as an element of the direct sum of t copies of the group ring:

$$I \in \bigoplus_{i=1}^{t} \mathbb{F}S_n{}^{(i)}.$$

Each of the t copies of the group ring is the direct sum of full matrix rings corresponding to the partitions of n:

$$I \in \bigoplus_{i=1}^{t} \bigoplus_{\lambda} M_{d_\lambda}^{(i)}(\mathbb{F}).$$

For each partition λ we extract the t components of I in the matrix rings corresponding to λ, and regard I as a sum of components indexed by λ:

$$I = \sum_{\lambda} I^{(\lambda)}, \quad I^{(\lambda)} \in \bigoplus_{i=1}^{t} M_{d_\lambda}^{(i)}(\mathbb{F}).$$

Since the matrix rings are orthogonal summands of the group ring, we can process the components $I^{(\lambda)}$ one partition λ at a time, thereby reducing the size of the computations very substantially.

Each component $I^{(\lambda)}$ is represented by a matrix $R(I, \lambda)$ of size $d_\lambda \times td_\lambda$. Left multiplications in the group ring are equivalent to row operations in the matrix rings. Therefore, the reduced row-echelon form of $R(I, \lambda)$ gives a canonical form for the component of I in the irreducible representation of S_n labelled by λ. The nonzero rows of this row canonical form are generators for the irreducible components of the representation generated by I. The rank of this matrix (the number of independent generators) is called the *rank of the identity I in representation λ*.

This construction extends in a straightforward manner to any finite set of identities $S = \{I_1, \ldots, I_k\}$. For each identity I_j we obtain a matrix of size $d_\lambda \times td_\lambda$. We stack the k matrices and obtain a matrix $R(S, \lambda)$ of size $kd_\lambda \times td_\lambda$. In $R(S, \lambda)$ the vertical blocks of size $kd_\lambda \times d_\lambda$ represent the component in each association type of the representation labelled by λ generated by the k identities. As before, we consider the row canonical form of the matrix $R(S, \lambda)$ and call the rank of this matrix the *rank of the set S of identities in representation λ*.

2.3. Ternary operations

2.3.1. Degree 5

We have a total of four association types: three for a nonassociative ternary operation

$$1: (abc)de, \quad 2: a(bcd)e, \quad 3: ab(cde),$$

and one more for an anticommutative ternary operation

$$4: [[abc]de].$$

(From now on we omit the commas in this symbol.) We consider a partition λ of 5 with corresponding irreducible representation of dimension d. For the totally associative case we have $k = 2$ identities represented by a matrix of size $2d \times 4d$ with a zero submatrix of size $2d \times d$ for association type 4. For the totally alternative and assosymmetric cases we have $k = 4$ identities represented by a matrix of size $4d \times 4d$ with a zero submatrix of size $4d \times d$ for association type 4. For the partially associative case we have $k = 1$ identity represented by a matrix of size $d \times 4d$ with a zero submatrix of size $d \times d$ for association type 4. For the partially alternative and assosymmetric cases we have $k = 2$ identities represented by a matrix of size $2d \times 4d$ with a zero submatrix of size $2d \times d$ for association type 4.

There are $k = 3$ trivial identities for an anticommutative ternary operation:

$$[[abc]de] + [[bac]de], \quad [[abc]de] + [[acb]de], \quad [[abc]de] + [[abc]ed].$$

These identities are represented by a matrix of size $3d \times 4d$ with a zero submatrix of size $3d \times 3d$ for association types 1, 2, 3.

There is $k = 1$ identity expressing the expansion of the anticommutative type $[[abc]de]$ in terms of the nonassociative types. This expansion has $6^2 = 36$ terms in the first three association types, and one term in the fourth type, namely $-[[abc]de]$. It is represented by a matrix of size $d \times 4d$.

2.3.2. Degree 7

We have a total of 14 association types: 12 for a nonassociative ternary operation

$$\begin{array}{rrr} 1: ((abc)de)fg, & 2: (a(bcd)e)fg, & 3: (ab(cde))fg, \\ 4: a((bcd)ef)g, & 5: a(b(cde)f)g, & 6: a(bc(def))g, \\ 7: ab((cde)fg), & 8: ab(c(def)g), & 9: ab(cd(efg)), \\ 10: (abc)(def)g, & 11: (abc)d(efg), & 12: a(bcd)(efg) \end{array}$$

and two more for an anticommutative ternary operation:

$$13: [[[abc]de]fg], \;\; 14: [[abc][def]g].$$

We consider a partition λ of 7 with corresponding irreducible representation of dimension d. For a multilinear identity $I = I(a, b, c, d, e)$ of degree 5 in a ternary operation, we have eight distinct ways to lift I to degree 7:

$$
\begin{aligned}
&I(afg, b, c, d, e), \quad I(a, bfg, c, d, e), \quad I(a, b, cfg, d, e),\\
&I(a, b, c, dfg, e), \quad I(a, b, c, d, efg), \quad fgI(a, b, c, d, e),\\
&fI(a, b, c, d, e)g, \quad I(a, b, c, d, e)fg.
\end{aligned}
$$

For the totally associative case this gives $k = 16$ identities in degree 7 represented by a matrix of size $16d \times 14d$ with a zero submatrix of size $16d \times 2d$ for association types 13 and 14. For the totally alternative and assosymmetric cases we have $k = 32$ identities represented by a matrix of size $32d \times 14d$ with a zero submatrix of size $32d \times 2d$ for association types 13 and 14. For the partially associative case we have $k = 8$ identities represented by a matrix of size $8d \times 14d$ with a zero submatrix of size $8d \times 2d$ for association types 13 and 14. For the partially alternative and assosymmetric cases we have $k = 16$ identities represented by a matrix of size $16d \times 14d$ with a zero submatrix of size $16d \times 2d$ for association types 13 and 14.

There are $k = 7$ trivial identities for an anticommutative ternary operation:

$$
\begin{aligned}
&[[[abc]de]fg] + [[[bac]de]fg], \quad [[[abc]de]fg] + [[[acb]de]fg],\\
&[[[abc]de]fg] + [[[abc]ed]fg], \quad [[[abc]de]fg] + [[[abc]de]gf],\\
&[[abc][def]g] + [[bac][def]g], \quad [[abc][def]g] + [[acb][def]g],\\
&[[abc][def]g] + [[def][abc]g].
\end{aligned}
$$

They are represented by a matrix of size $7d \times 14d$ with a zero submatrix of size $7d \times 12d$ for association types 1–12.

There are $k = 2$ identities expressing the expansion of the anticommutative types $[[[abc]de]fg]$ and $[[abc][def]g]$ in terms of the nonassociative types. These expansions have $6^3 = 216$ terms in the first 12 association types, and one term in types 13 and 14, namely $-[[[abc]de]fg]$ (for the first expansion) and $-[[abc][def]g]$ (for the second expansion). They are represented by a matrix of size $2d \times 14d$.

2.4. Computer implementation

We performed all our computations using two independent computer programs. The first program was written in Maple by Bremner. The second program uses procedures written in C by Hentzel. Both programs perform the following computations:

1. Initialization of identities stored as lists of terms of the form $[p, t, c]$ where p is a permutation of the variables, t is an association type, and c is a coefficient.
2. Procedures to expand the anticommutative association types as sums of terms in the nonassociative association types.

3. Procedures to implement the group ring of the symmetric group:
   (a) generating the partitions;
   (b) computing the dimension of an irreducible representation using the hook formula;
   (c) generating the standard tableaux for a given partition;
   (d) computing the matrix representing a given permutation in the natural representation for a given partition.
4. The main loop which for each partition computes and reduces the matrices of
   (a) trivial identities;
   (b) totally (or partially) associative (or alternative or assosymmetric) identities;
   (c) expansion identities.

Both programs use arithmetic in characteristic p to control memory allocation during the row-reduction of large matrices. The Maple program uses $p = 101$ and the C program uses $p = 103$. Since both primes are much larger than the degree of the identities being studied, the group ring will be semisimple, and we can expect that the ranks we obtain will be the same as in the characteristic 0 case. This assumption is corroborated by the fact that we obtain the same ranks in all cases using two different primes.

3. The totally associative case

The totally associative case was studied by Bremner [1] using computational linear algebra but without the representation theory of the symmetric group. In this section we present equivalent results expressed in terms of group representations.

3.1. Degree 5

The ranks of the identities are displayed in Table 1.

Column 2 gives the partitions of 5 corresponding to the irreducible representations of the symmetric group S_5, and column 3 gives the dimensions of the representations.

Table 1
Totally associative case, degree 5

i	λ	d_λ	Totally associative	Associative + expansion	Type 4	Trivial	New
1	5	1	2	3	1	1	0
2	41	4	8	12	4	4	0
3	32	5	10	15	5	5	0
4	311	6	12	18	6	6	0
5	221	5	10	15	4	4	0
6	2111	4	8	12	3	3	0
7	11111	1	2	3	0	0	0

Column 4 gives the rank of the totally associative identities in each representation, and column 5 gives the rank for the stacked matrix containing the totally associative identities and the expansion identity.

Column 6 gives the rank in the last association type of the matrix from column 5: that is, we compute the row canonical form of the stacked matrix from column 5, delete all the rows which have a nonzero entry in the first three association types, and compute the rank of the resulting matrix. This tells us how many distinct irreducible identities are satisfied by the alternating sum in this representation; however, some of these identities will be trivial identities which are consequences of the anticommutativity of the alternating sum.

Column 7 gives the rank of the trivial identities, and column 8 gives the difference between columns 6 and 7: this is the number of new non-trivial identities satisfied by the alternating sum. From the last column of Table 1 we see that every identity in degree 5 is a consequence of the trivial identities.

3.2. Degree 7

The ranks of the identities are displayed in Table 2.

Column 2 gives the partitions of 7 and column 3 gives the dimensions of the representations.

Column 4 gives the rank of the totally associative identities (lifted to degree 7) in each representation, and column 5 gives the rank for the stacked matrix containing the totally associative identities and the expansion identities.

Column 6 gives the rank in the last two association types of the matrix from column 5.

Table 2
Totally associative case, degree 7

i	λ	d_λ	Totally associative	Associative + expansion	Types 13, 14	Trivial	New
1	7	1	11	13	2	2	0
2	61	6	66	78	12	12	0
3	52	14	154	182	28	28	0
4	511	15	165	195	30	30	0
5	43	14	154	182	28	28	0
6	421	35	385	455	70	70	0
7	4111	20	220	260	40	40	0
8	331	21	231	273	41	41	0
9	322	21	231	273	41	41	0
10	3211	35	385	455	67	67	0
11	31111	15	165	195	29	29	0
12	2221	14	154	182	25	25	0
13	22111	14	154	182	24	24	0
14	211111	6	66	78	10	9	1
15	1111111	1	11	13	1	0	1

Column 7 gives the rank of the trivial identities, and column 8 gives the difference between columns 6 and 7: this is the number of new non-trivial identities satisfied by the alternating sum. From the last column of Table 2 we see that there is one new identity in each of the last two representations. This confirms the following result from Bremner [1].

Theorem 1. *Every identity of degree* $\leqslant 7$ *satisfied by the alternating sum in every totally associative ternary algebra is a consequence of anticommutativity in degree* 3 *and the following two identities in degree* 7. *The identity for partition* 211111 *is the linearization of*

$$\sum_{\mathrm{alt}(c,d,e,f,g)} (3[[[acd]ae]fg] + 2[[[cde]af]ag] - [[acd][efg]a])$$

and the identity for partition 1111111 *is*

$$\sum_{\mathrm{alt}(a,b,c,d,e,f,g)} ([[[abc]de]fg] - [[abc][def]g]).$$

(*In both cases the sum is an alternating sum over the indicated variables*.)

Remark. In Bremner [1] it is shown that the two identities in Theorem 1 are together equivalent to the single identity

$$\sum_{\mathrm{alt}(b,c,d,e,f,g)} ([[[bcd]ae]fg] + [[abc][def]g]).$$

4. The totally alternative case

Every totally associative ternary algebra is totally alternative, and since there are no identities in degree 5 for the totally associative case, it follows that there are no identities in degree 5 for the totally alternative case.

For the same reason, in degree 7, the only representations in which the alternating sum can satisfy an identity in the totally alternative case are the representations for which an identity holds in the totally associative case: the last two representations. The ranks for these two representations are displayed in Table 3.

The columns of Table 3 have the same meaning as in the totally associative case, except that here we are using the liftings of the totally alternative identities to degree 7. These computations together with Theorem 1 establish the following result.

Theorem 2. *Every identity of degree* $\leqslant 7$ *satisfied by the alternating sum in every totally alternative ternary algebra is a consequence of anticommutativity and the first identity of Theorem* 1.

Table 3
Totally alternative case, degree 7

i	λ	d_λ	Totally alternative	Alternative + expansion	Types 13, 14	Trivial	New
14	211111	6	66	78	10	9	1
15	1111111	1	8	10	0	0	0

In the binary case, the commutator in an associative algebra satisfies the Jacobi identity in degree 3, and the commutator in an alternative algebra satisfies the Malcev identity in degree 4. The results in the ternary case are quite different: the alternating sum in a totally associative algebra satisfies two identities in degree 7, and the alternating sum in a totally alternative algebra satisfies exactly one of these identities (in the same degree).

5. The totally assosymmetric case

Every totally associative algebra is totally assosymmetric, and since there are no identities in degree 5 for the totally associative case, it follows that there are no identities in degree 5 for the totally assosymmetric case.

For the same reason, in degree 7, the only representations in which the alternating sum can satisfy an identity in the totally assosymmetric case are the representations for which an identity holds in the totally associative case: the last two representations. The ranks for these two representations are displayed in Table 4.

These computations together with Theorem 1 give the following result.

Theorem 3. *The alternating sum satisfies the same identities of degree* $\leqslant 7$ *in a totally assosymmetric algebra as it does in a totally associative algebra.*

This is not surprising in view of the fact that for binary algebras every assosymmetric algebra is Lie-admissible.

Table 4
Totally assosymmetric case, degree 7

i	λ	d_λ	Totally assosymm	Assosymm + expansion	Types 13, 14	Trivial	New
14	211111	6	66	78	10	9	1
15	1111111	1	11	13	1	0	1

6. The partially associative case

6.1. Degree 5

The ranks of the various identities are displayed in Table 5. From these ranks we see that there is a new identity in degree 5 which is not a consequence of the anticommutativity of the alternating sum.

Since this new identity lies in the last representation, and there is only one association type, the new identity must be a (nonzero) scalar multiple of the alternating sum

$$\sum_{\mathrm{alt}(a,b,c,d,e)} [[abc]de].$$

If we use the anticommutativity of the alternating sum to collect terms then we obtain (12 times) the alternating sum over all (3, 2)-shuffles:

$$\begin{aligned}&[[abc]de] - [[abd]ce] + [[abe]cd] + [[acd]be] - [[ace]bd]\\ &\quad + [[ade]bc] - [[bcd]ae] + [[bce]ad] - [[bde]ac] + [[cde]ab].\end{aligned}$$

This identity (and its n-ary generalization) appears in Gnedbaye [6] and Hanlon and Wachs [7].

6.2. Degree 7

The ranks of the identities are displayed in Table 6.

In this case we have a known identity in degree 5 which we lift to degree 7. If we stack together the matrices representing the trivial identities and the lifted identities, and compute the row canonical form of the stacked matrix, we obtain the same ranks as column 6 (the last two association types). This implies that all of the new identities in degree 7, although they are not consequences of the trivial identities for the alternating

Table 5
Partially associative case, degree 5

i	λ	d_λ	Partially associative	Associative + expansion	Type 4	Trivial	New
1	5	1	1	2	1	1	0
2	41	4	4	8	4	4	0
3	32	5	5	10	5	5	0
4	311	6	6	12	6	6	0
5	221	5	5	10	4	4	0
6	2111	4	4	8	3	3	0
7	11111	1	1	2	1	0	1

Table 6
Partially associative case, degree 7

i	λ	d_λ	Partially associative	Associative + expansion	Types 13, 14	Trivial	New
1	7	1	8	10	2	2	0
2	61	6	48	60	12	12	0
3	52	14	112	140	28	28	0
4	511	15	120	150	30	30	0
5	43	14	112	140	28	28	0
6	421	35	280	350	70	70	0
7	4111	20	160	200	40	40	0
8	331	21	168	210	41	41	0
9	322	21	168	210	41	41	0
10	3211	35	280	350	67	67	0
11	31111	15	120	150	29	29	0
12	2221	14	112	140	26	25	1
13	22111	14	112	140	26	24	2
14	211111	6	48	60	11	9	2
15	1111111	1	8	10	2	0	2

sum, are in fact consequences of the known identity in degree 5. We therefore have the following result.

Theorem 4. *Every identity of degree* $\leqslant 7$ *satisfied by the alternating sum in every partially associative ternary algebra is a consequence of anticommutativity and the shuffle identity in degree* 5.

7. The partially alternative case

Every partially associative algebra is partially alternative, and therefore, in degree 5, the only representation in which the alternating sum can satisfy an identity in the partially alternative case is the representation for which an identity holds in the partially associative case: the last representation. The ranks for this representation are displayed in Table 7.

From these computations we see that the alternating sum identity in degree 5 from the partially associative case is not satisfied in the partially alternative case.

Table 7
Partially alternative case, degree 5

i	λ	d_λ	Partially alternative	Alternative + expansion	Type 4	Trivial	New
7	11111	1	0	1	0	0	0

Table 8
Partially alternative case, degree 7

i	λ	d_λ	Partially alternative	Alternative + expansion	Types 13, 14	Trivial	New
12	2221	14	111	139	25	25	0
13	22111	14	106	134	24	24	0
14	211111	6	39	51	9	9	0
15	1111111	1	4	6	0	0	0

Similarly, in degree 7, the only representations in which the alternating sum can satisfy an identity in the partially alternative case are the representations for which an identity holds in the partially associative case: the last four representations. The ranks for these representations are displayed in Table 8.

Since the ranks in column 8 are all zero, we have the following result.

Theorem 5. *Every identity of degree $\leqslant 7$ satisfied by the alternating sum in every partially alternative ternary algebra is a consequence of anticommutativity. Hence the minimal identity in this case has degree $\geqslant 9$.*

8. The partially assosymmetric case

Every partially associative algebra is partially assosymmetric, and therefore in degree 5 the alternating sum can satisfy an identity in the partially assosymmetric case only in the last representation. The ranks for this representation are displayed in Table 9.

From this we see that the shuffle identity in degree 5 from the partially associative case is also satisfied in the partially assosymmetric case.

Similarly, in degree 7 the alternating sum can satisfy an identity in the partially assosymmetric case only in the last four representations. The ranks for these representations are displayed in Table 10.

These computations give the following result.

Theorem 6. *The alternating sum satisfies the same identities of degree $\leqslant 7$ in a partially assosymmetric algebra as it does in a partially associative algebra.*

Table 9
Partially assosymmetric case, degree 5

i	λ	d_λ	Partially assosym	Assosym + expansion	Type 4	Trivial	New
7	11111	1	1	2	1	0	1

Table 10
Partially assosymmetric case, degree 7

i	λ	d_λ	Partially assosymm	Assosymm + expansion	Types 13, 14	Trivial	New
12	2221	14	112	140	26	25	1
13	22111	14	112	140	26	24	2
14	211111	6	48	60	11	9	2
15	1111111	1	8	10	2	0	2

9. Conclusions

Our results suggest two possible directions for the definition of Lie and Malcev algebras in the ternary case, corresponding to total and partial associativity.

In the totally associative case, we may define a ternary Lie algebra to be an algebra with an anticommutative product satisfying the two identities of Theorem 1. Similarly, we may define a ternary Malcev algebra to be an algebra with an anticommutative product satisfying the identity of Theorem 2 (the first identity of Theorem 1).

In the partially associative case, we may define a ternary Lie algebra to be an algebra with an anticommutative product satisfying the shuffle identity in degree 5 (see Theorem 4). Similarly, a ternary Malcev algebra will be an algebra with an anticommutative product satisfying the minimal identity for the alternating sum; by Theorem 5 we know this identity will have degree at least 9.

Acknowledgments

Murray Bremner thanks NSERC (the Natural Sciences and Engineering Research Council of Canada) for supporting this research through a Discovery Grant, and IME-USP (the Institute of Mathematics and Statistics of the University of São Paulo) for its hospitality and financial support during his visit to Brazil in February 2005. This paper was written while Luiz Peresi held a grant from CNPq of Brazil.

References

[1] M.R. Bremner, Identities for the ternary commutator, J. Algebra 206 (2) (1998) 615–623, MR1637092 (99h:16071).
[2] M.R. Bremner, I.R. Hentzel, Identities for algebras of matrices over the octonions, J. Algebra 277 (1) (2004) 73–95, MR2059621 (2005b:17002).
[3] M.R. Bremner, I.R. Hentzel, Invariant nonassociative algebra structures on irreducible representations of simple Lie algebras, Exp. Math. 13 (2) (2004) 231–256, MR2068896.
[4] J.M. Clifton, A simplification of the computation of the natural representation of the symmetric group S_n, Proc. Am. Math. Soc. 83 (2) (1981) 248–250, MR0624907 (82j:20024).
[5] V.T. Filippov, n-Lie algebras, Sib. Mat. Zh. 26 (6) (1985) 126–140, MR0816511 (87d:08019).

[6] A.V. Gnedbaye, Opérades des algèbres (k+1)-aires, Contemporary Mathematics, vol. 202, Operads: Proceedings of Renaissance Conferences, American Mathematical Society, 1997, 83–113, MR1436918 (98a:17003).

[7] P. Hanlon, M. Wachs, On Lie k-algebras, Adv. Math. 113 (2) (1995) 206–236, MR1337108 (96h:17006).

[8] I.R. Hentzel, Applying group representation to nonassociative algebras, Ring Theory (Conference Proceedings, Ohio University, Athens, Ohio, 1976), Marcel Dekker, 1977, pp. 133–141, MR0435159 (55 #8120).

[9] I.R. Hentzel, Processing identities by group representation, Rings and Algebras (Special Session, 82nd Annual Meeting of the American Mathematical Society, San Antonio, Texas, 1976), Academic Press, 1977, pp. 13–40, MR0463251 (57 #3204).

[10] I.R. Hentzel, L.A. Peresi, Identities of Cayley–Dickson algebras, J. Algebra 188 (1) (1997) 292–309, MR1432358 (98a:17004).

[11] I.R. Hentzel, L.A. Peresi, A nonzero element of degree 7 in the center of the free alternative algebra, Commun. Algebra 31 (3) (2003) 1279–1299, MR1971063 (2004e:17028).

[12] G. James, A. Kerber, The representation theory of the symmetric group, Encyclopedia of Mathematics and its Applications, vol. 16, Addison-Wesley, 1981, MR0644144 (83k:20003).

[13] A.I. Malcev, On algebras defined by identities, Mat. Sb. 26 (68) (1950) 19–33, MR0033280 (11,414d).

[14] A.I. Malcev, Analytic loops, Mat. Sb. 36 (78) (1955) 569–576, MR0069190 (16,997g).

[15] J.M. Pérez-Izquierdo, I.P. Shestakov, An envelope for Malcev algebras, J. Algebra 272 (1) (2004) 379–393, MR2029038 (2004j:17040).

[16] A.P. Pozhidaev, n-ary Mal'tsev algebras, Algebra Log. 40 (3) (2001) 309–329, MR1857886 (2002g:17005).

[17] D.E. Rutherford, Substitutional Analysis, The University Press, Edinburgh, 1948, MR0027272 (10,280i).

[18] A.A. Sagle, Malcev algebras, Trans. Am. Math. Soc. 101 (1961) 426–458, MR0143791 (26 #1343).

[19] R.D. Schafer, An Introduction to Nonassociative Algebras, Corrected reprint of the 1966 original, Dover Publications, 1995, MR1375235 (96j:17001).

[20] I.P. Shestakov, Free Malcev superalgebra on one odd generator, J. Algebra Appl. 2 (4) (2003) 451–461, MR2020951 (2004h:17031).

[21] W. Specht, Gesetze in Ringen, I, Math. Z. 52 (1950) 557–589, MR0035274 (11,711i).

[22] A. Young, The collected papers of Alfred Young (1873–1940), with a foreword by G. de B. Robinson and a biography by H.W. Turnbull, Mathematical Expositions, No. 21, University of Toronto Press, 1977, MR0439548 (55 #12438).

[23] K.A. Zhevlakov, A.M. Slin'ko, I.P. Shestakov, A.I. Shirshov, Rings that are nearly associative, (translated from the Russian by Harry F. Smith), Academic Press, 1982, MR0668355 (83i:17001).