Presentations of Semigroups and Inverse Semigroups

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... Serenamente descobriu

Que afinal tudo tinha o seu sentido ...

Jorge Palma

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Declaration

I, Catarina Alexandra Santos Carvalho, hereby certify that this dissertation has been composed by myself, that it is a record of my own work, and that it has not been accepted in any previous application for any degree.

Signature :
Date :

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Abstract

It is known that a group is finitely presented as a group if and only if it is finitely presented as a monoid, and that a monoid is finitely presented as a monoid if and only if it is finitely presented as a semigroup.

A similar result does not hold for all inverse semigroups; the free inverse semigroup is an example of that. After describing the free inverse semigroup and see why it cannot be finitely presented as a semigroup, we look at two "classes" of inverse semigroups that are finitely presented as inverse semigroups if and only if they are finitely presented as semigroups, namely inverse monoids with finitely many left and right ideals and Bruck-Reilly extensions of groups.

In the last part of this dissertation we study Bruck-Reilly extensions of Clifford monoids and prove that they are finitely presented as inverse semigroups if and only if they are finitely presented as semigroups. We also show that in some specific cases the Bruck-Reilly extensions of a Clifford monoid, like the Clifford semigroups, are finitely presented if and only if its $\mathcal{D}$-classes are finitely presented.

Chapter 1

Free Inverse Semigroup

The free inverse semigroups "represent one of the most interesting and important classes of inverse semigroups" [9, p.355]. In this chapter we give a description of the free inverse semigroup, with the idea of understanding better an important example we will find latter in this dissertation.

1 Inverse Semigroups and Free Algebras

Let S be a semigroup. An element $a \in S$ is said to be *regular* if there exists $x \in S$ such that $a = axa$. The element x is an *inverse* of a if $a = axa$ and $x = xax$.

Note: If x is an inverse of a, then the elements ax and xa are idempotents in S, i.e. $ax = axax$ and $xa = xaxa$.

We say that the semigroup S is *inverse* if a unary operation $x \mapsto x^{-1}$ is defined on S, with the properties:

$$\forall x, y \in S \quad (x^{-1})^{-1} = x, \quad xx^{-1}x = x, \quad xx^{-1}yy^{-1} = yy^{-1}xx^{-1}.$$

Note that x^{-1} is the inverse of x and vice-versa. The following two results contain some properties of inverse semigroups that simplify our work with them, they can be found in [6, Section 5.1].

Proposition 1.1 *Let S be a semigroup. The following statements are equivalent:*
(i) *S is an inverse semigroup*;
(ii) *S is regular, and its idempotents commute*;
(iii) *every $\mathcal{L}-$class and every $\mathcal{R}-$class contain exactly one idempotent*;
(iv) *every element of S has a unique inverse.*

Proposition 1.2 *Let S be an inverse semigroup. Then:*
(i) $(a_1a_2\dots a_n)^{-1} = a_n^{-1}\dots a_2^{-1}a_1^{-1}$, *for all* $a_1, a_2, \dots, a_n \in S$,
(ii) $a\mathcal{L}b \;\Leftrightarrow\; a^{-1}a = b^{-1}b, \quad a, b \in S$,
(iii) $a\mathcal{R}b \;\Leftrightarrow\; aa^{-1} = bb^{-1}, \quad a, b \in S$,
(iv) *if e is an idempotent in S, then for any a in S, aea^{-1} and $a^{-1}ea$ are idempotents in S.*

Let $\mathcal{C}$ be a class of algebras, A an element of $\mathcal{C}$, X a non-empty set and φ a map from X into A. The pair (A, φ) is a *free $\mathcal{C}$-algebra* if for any C in $\mathcal{C}$ and any mapping $\psi : X \longrightarrow C$ there exists a unique homomorphism $\psi' : A \longrightarrow C$ making the following diagram commutative:

$$\begin{tikzcd} X \arrow[r, "\varphi"] \arrow[d, "\psi"'] & A \arrow[dl, "\psi'"] \\ C & \end{tikzcd}$$

It is clear, from the definition, that when such a structure exists it is unique.

Some very well known free algebras, on a non-empty set X, are the *free semigroup*, that is the set of all non-empty words with letters in X under the operation of concatenation, we denote it by X^+. Adjoining an identity, 1, to X^+ we obtain the *free monoid* on X, that we denote by X^*. The *free group* is the set of all reduced words in the alphabet $X \cup X^{-1}$, where $X^{-1} = \{x^{-1} : x \in X\}$ is a set in one-one correspondence with X and disjoint from it, we denote it by FG_X. We say that a word is *reduced* if, for each $x \in X$, it contains no occurrences of xx^{-1} or $x^{-1}x$.

We can think of inverse semigroups as a class of (2,1)-algebras, so it makes sense to try to define the free inverse semigroup and that is what we will do in the next sections. First we will follow a construction given in [6, Section 5.10] that defines it as a quotient of a semigroup by a congruence. Then we will define it by means of a P-semigroup, this construction can be found in [9, Section VIII.1] and [6, Section 5.10]. Finally we will define it in terms of birooted word trees, following [9, Section VIII.3].

2 Construction of the Free Inverse Semigroup

Let X be a non-empty set and $X^{-1} = \{x^{-1} : x \in X\}$ be a set in one-one correspondence with X and disjoint from it. Let $Y = X \cup X^{-1}$ and consider Y^+, the free semigroup on Y. Define inverses for the elements of Y^+ by the rules:

$$\begin{array}{rl} (x^{-1})^{-1}, & x \in X \\ (y_1 y_2 \dots y_n)^{-1} = y_n^{-1} \dots y_1^{-1}, & y_1, y_2, \dots, y_n \in Y, \end{array}$$

note that for any $w \in Y^+$ $(w^{-1})^{-1} = w$. Let τ be the congruence generated by the set

$$T = \{(ww^{-1}w, w) : w \in Y^+\} \cup \{(ww^{-1}zz^{-1}, zz^{-1}ww^{-1}) : w, z \in Y^+\},$$

Y^+/τ is a semigroup under the multiplication $(w\tau)(z\tau) = (wz)\tau$, $w, z \in Y^+$, see for example [6, Section 1.5]. By the definition of τ, for any $w \in Y^+$, we have

$$(ww^{-1}w)\tau = w\tau(w^{-1}\tau)w\tau = w\tau, \quad \text{and} \quad w^{-1}\tau = (w^{-1}\tau)w\tau(w^{-1}\tau),$$

so $w^{-1}\tau$ is an inverse of $w\tau$ in Y^+/τ. Hence any element of Y^+/τ has at least one inverse, so this semigroup is regular. Similarly we can prove that the idempotents of Y^+/τ commute, using the definition of τ and the fact that if $a\tau \in Y^+/\tau$ is an idempotent there exists and idempotent $e \in Y^+$ such that $a\tau = e\tau$, see [6, Lemma 2.4.3]. Thus Y^+/τ is an inverse semigroup.

The map $\varphi : X \longrightarrow Y^+/\tau, \;\; x \mapsto x\tau,$ is obviously well-defined and is the map that we associate to Y^+/τ to prove that this inverse semigroup is in fact the free inverse semigroup.

Let S be any inverse semigroup and ψ any map from X into S. We can extend ψ to Y by defining:

$$x^{-1}\psi = (x\psi)^{-1}, \quad x \in X,$$

where $(x\psi)^{-1}$ is the inverse of $x\psi$ in S. Since Y^+ is the free semigroup on Y, we can define a semigroup morphism $\hat{\psi} : Y^+ \longrightarrow S$ by the rule:

$$(y_1y_2\dots y_n)\hat{\psi} = y_1\psi y_2\psi \dots y_n\psi, \qquad y_1, y_2, \dots, y_n \in Y.$$

Since S is inverse we know that for all $w\hat{\psi} \in S$ there exists $(w\hat{\psi})^{-1} \in S$ such that $w\hat{\psi} = w\hat{\psi}(w\hat{\psi})^{-1}w\hat{\psi}$ and $(w\hat{\psi})^{-1} = (w\hat{\psi})^{-1}w\hat{\psi}(w\hat{\psi})^{-1}$. Let $w, z \in Y^+$ arbitrary, say $w = y_1y_2\dots y_n$, $z = x_1x_2\dots x_k$, with $y_j, x_i \in Y$, $i = 1, \dots, k$, $j = 1, \dots n$. We have

$$\begin{aligned}
w\hat{\psi} &= w\hat{\psi}(w\hat{\psi})^{-1}w\hat{\psi} && (S \text{ inverse})\\
&= w\hat{\psi}((y_1y_2\dots y_n)\hat{\psi})^{-1}w\hat{\psi} &&\\
&= w\hat{\psi}(y_1\psi y_2\psi \dots y_n\psi)^{-1}w\hat{\psi} && (\text{def. } \hat{\psi})\\
&= w\hat{\psi}((y_n\psi)^{-1}\dots(y_2\psi)^{-1}(y_1\psi)^{-1})w\hat{\psi} && (S \text{ inverse})\\
&= w\hat{\psi}(y_n^{-1}\psi \dots y_2^{-1}\psi y_1^{-1}\psi)w\hat{\psi} && (\text{def. } \psi)\\
&= w\hat{\psi}((y_n^{-1}\dots y_2^{-1}y_1^{-1})\hat{\psi})w\hat{\psi} && (\text{def. } \hat{\psi})\\
&= w\hat{\psi}((y_1y_2\dots y_n)^{-1}\hat{\psi})w\hat{\psi} && (\text{def. of inverses in } Y^+)\\
&= w\hat{\psi}(w^{-1}\hat{\psi})w\hat{\psi} &&\\
&= (ww^{-1}w)\hat{\psi}, && (\hat{\psi} \text{ morphism})
\end{aligned}$$

from this we can see that $w^{-1}\hat{\psi} = (w\hat{\psi})^{-1}$, since the inverses in S are unique. We also have

$$\begin{aligned}(ww^{-1}zz^{-1})\hat{\psi} &= w\hat{\psi}(w^{-1}\hat{\psi})z\hat{\psi}(z^{-1})\hat{\psi}) && (\hat{\psi} \text{ morphism})\\ &= w\hat{\psi}(w\hat{\psi})^{-1}z\hat{\psi}(z\hat{\psi})^{-1} && (\text{by above})\\ &= z\hat{\psi}(z\hat{\psi})^{-1}w\hat{\psi}(w\hat{\psi})^{-1} && (S \text{ inverse})\\ &= (zz^{-1}ww^{-1})\hat{\psi}. && (\hat{\psi} \text{ morphism})\end{aligned}$$

We know that the kernel of the homomorphism $\hat{\psi}$ is the congruence

$$Ker\hat{\psi} = \{(a,b) \in Y^+ \times Y^+ : a\hat{\psi} = b\hat{\psi}\},$$

see [6, Theorem 1.5.2], and by what we have just seen , $T \subseteq Ker\hat{\psi}$, so we must have $\tau \subseteq Ker\hat{\psi}$, since τ is the smallest congruence containing T. This implies, by [6, Theorem 1.5.3], that there exists a unique morphism $\psi' : Y^+/\tau \longrightarrow S$ such that $\tau^b\psi' = \hat{\psi}$, where $\tau^b : Y^+ \longrightarrow Y^+/\tau$, $y \mapsto y\tau$, $y \in Y^+$. Thus, we may conclude that there is a map $\psi' : Y^+/\tau \longrightarrow S$ such that $\varphi\psi' = \psi$, since

$$x\varphi\psi' = (x\tau)\psi' = x\hat{\psi} = x\psi, \quad \forall x \in X.$$

Suppose that there exists a morphism $\alpha : Y^+/\tau \longrightarrow S$ such that $\varphi\alpha = \psi$, then

$$x\varphi\alpha = x\psi \;\Leftrightarrow\; (x\tau)\alpha = x\psi, \quad \forall x \in X$$

but α is a morphism so

$$(x^{-1}\tau)\alpha = (x\tau)^{-1}\alpha = (x\tau\alpha)^{-1} = (x\psi)^{-1} = x^{-1}\psi, \quad \forall x \in X.$$

Hence, given $w \in Y^+$ arbitrary, say $w = y_1y_2\ldots y_n$, for some $y_1, y_2, \ldots, y_n \in Y$, we have

$$\begin{aligned}(w\tau)\alpha &= ((y_1y_2\ldots y_n)\tau)\alpha = (y_1\tau)\alpha(y_2\tau)\alpha\ldots(y_n\tau)\alpha\\ &= y_1\psi y_2\psi\ldots y_n\psi = (y_1y_2\ldots y_n)\hat{\psi} = w\hat{\psi},\end{aligned}$$

so $\tau^b\alpha = \hat{\psi}$, this implies that $\alpha = \psi'$, and we can conclude that ψ' is the unique morphism from Y^+/τ into S such that $\varphi\psi' = \psi$.

Thus Y^+/τ is the free inverse semigroup on X. We will denote it by FI_X.

3 P-Semigroups

3.1 Definitions

Given a non-empty set X with a partial order $\leq$, a non-empty subset Y of X is called an *ideal* of X if

$$\forall b \in Y \ \ \forall a \in X \quad a \leq b \ \Rightarrow \ a \in Y,$$

and is called a *subsemilattice* of X if

$$\forall a, b \in Y \ \ \exists a \wedge b \ \text{ and } \ a \wedge b \in Y,$$

where $a \wedge b$ represents the meet of a and b, i.e. $a \wedge b \leq a$, $a \wedge b \leq b$ and for all $c \in Y$ such that $c \leq a$ and $c \leq b$ we have $c \leq a \wedge b$. Given a group G, we say that G *acts on* X if for any element $g \in G$ there exists an order preserving automorphism $\varphi_g : X \longrightarrow X$, $a \mapsto ga$, such that, given $g, h \in G$ we have $\varphi_g \varphi_h = \varphi_{gh}$.

By saying that the bijection $\varphi_g : X \longrightarrow X$ is an order preserving automorphism we mean that

$$\forall \ a, b \in X \quad a \leq b \ \Leftrightarrow \ a\varphi_g \leq b\varphi_g \ \Leftrightarrow \ ga \leq gb.$$

Proposition 1.3 *Given a group G, with identity element 1_G, and a poset X, G acts on X if and only if*

$$\forall a, b \in X, \ \ \forall g, h \in G \qquad a \leq b \ \Rightarrow ga \leq gb, \quad (gh)a = g(ha), \quad 1_G a = a.$$

Proof. Suppose that the group G acts on the poset X, then we clearly have

$$\forall \ a, b \in X \ \ \forall g, h \in G \qquad a \leq b \ \Rightarrow ga \leq gb, \qquad (gh)a = g(ha).$$

For any $g \in G$ the map φ_g is a bijection, with inverse map $\varphi_{g^{-1}}$, then for any $a \in X$ we have

$$\begin{aligned} a\varphi_g\varphi_{g^{-1}} &= a\varphi_{gg^{-1}} = a\varphi_{1_G} = 1_G a, \\ a\varphi_g\varphi_{g^{-1}} &= a id_G = a, \end{aligned}$$

so $1_g a = a$. Conversely suppose that

$$\forall a, b \in X, \;\; \forall g, h \in G \quad\; a \leq b \;\Rightarrow ga \leq gb, \qquad (gh)a = g(ha), \qquad 1_G a = a.$$

Then, for all $g \in G$ and $a, b \in X$, we have

$$ga \leq gb \;\;\Rightarrow\;\; g^{-1}ga \leq g^{-1}gb \;\;\Leftrightarrow\;\; 1_G a \leq 1_G b \;\;\Leftrightarrow\;\; a \leq b,$$

so, the map $\varphi_g : X \longrightarrow X,\; a \mapsto ga$ is an order preserving automorphism. By the hypothesis, we know that for all $g, h \in G$ we have $\varphi_g \varphi_h = \varphi_{gh}$. ∎

Given a group G acting on a poset X, a subsemilattice and ideal Y of X, such that $GY = X$ and for all $g \in G \quad gY \cap Y \neq \emptyset$, we say that (G, X, Y) is a *McAlister triple*.

Proposition 1.4 *Given a McAlister triple* (G, X, Y)*, the set*

$$P(G, X, Y) = \{(a, g) \in Y \times G : g^{-1}a \in Y\},$$

under the multiplication $(a, g)(b, h) = (a \wedge gb, gh), \quad a, b \in Y,\; g, h \in G$*, is an inverse semigroup.*

For a proof see for example [6, Theorem 5.9.2]. The inverse semigroups defined in this proposition are called *P-semigroups*.

3.2 Construction of FI_X

We will see that the free inverse monoid can be described as a P-semigroup $P(FG_X, \mathcal{X}, E)$, where FG_X is the free group on the non-empty set X, E is a semilattice and $\mathcal{X}$ is a poset, both obtained from FG_X.

Note: Given two words v, w in Y^+, $Y = X \cup X^{-1}$, we will denote by vw their product in the semigroup Y^+ and by $v \cdot w$ their product in FG_X.

For any word $w = x_1x_2 \dots x_n,\ \ x_i \in X,\ i = 1, \dots, n$, we define $w^{\downarrow}$ to be the set of all left factors of w, $\{1,\ x_1,\ x_1x_2,\ x_1x_2x_3,\ \dots,\ x_1x_2\dots x_n\}$, where 1 represents the empty word. Let

$$E = \{A \in \mathcal{P}(FG_X) : 0 < |A| < \infty,\ \ w \in A \ \Rightarrow\ w^{\downarrow} \subseteq A\},$$

and define a partial order on E by the rule $A \leq B \ \Leftrightarrow\ B \subseteq A, \quad A, B \in E$. Since $\subseteq$ is a partial order we can see that $\leq$ is also a partial order. Clearly, given $A, B \in E$ we have $A \cup B \leq A$ and $A \cup B \leq B$. Considering a set $T \in E$ such that $T \leq A$ and $T \leq B$ we have $A \subseteq T$ and $B \subseteq T$ so $A \cup B \subseteq T$, and we can conclude that $A \wedge B = A \cup B$. Considering a word $w \in A \cup B$ we have

$$\begin{array}{ll} w \in A \ \Rightarrow\ w^{\downarrow} \subseteq A, & \quad (A \in E) \\ w \in B \ \Rightarrow\ w^{\downarrow} \subseteq B, & \quad (B \in E) \end{array}$$

so $w^{\downarrow} \subseteq A \cup B$. We know that $0 < |A|,\ |B| < \infty$ and $|A \cup B| = |A| + |B| - |A \cap B|$ so $0 < |A \cup B| < \infty$. Hence for all $A, B \in E\ \ A \wedge B \in E$, thus E is a subsemilattice of $\mathcal{P}_X$.

Given $A \in E$ we say that an element $w \in A$ is *maximal* if it is not a proper left factor of any element of A. Since all elements of E are finite we know that every element of E has at least one maximal element.

Lemma 1.5 *If $w_1, w_2, \dots, w_n$ are all maximal elements of A, where $A \in E$, then $A = w_1^{\downarrow} \cup w_2^{\downarrow} \cup \dots \cup w_n^{\downarrow}$.*

PROOF. Since $w_1, w_2, \dots, w_n \in A$ and $A \in E$ we clearly have $w_1^{\downarrow} \cup w_2^{\downarrow} \cup \dots \cup w_n^{\downarrow} \subseteq A$. Now consider $z \in A$ arbitrary, z is a left factor of some maximal element of A, w_i, for some $i = 1, \dots, n$, so $z \in w_i^{\downarrow}$ and we can conclude that $A = w_1^{\downarrow} \cup w_2^{\downarrow} \cup \dots \cup w_n^{\downarrow}$. ■

For any $g \in FG_X$ and $A \in E$ we define $g \cdot A$ to be the set $\{g \cdot w : w \in A\}$, and

$$\mathcal{X} = \{g \cdot A : g \in FG_X,\ A \in E\}.$$

Define a partial order in $\mathcal{X}$ by the rule $F_1 \leq F_2 \Leftrightarrow F_2 \subseteq F_1, \quad F_1, F_2 \in \mathcal{X}$. Let $F_1, F_2 \in \mathcal{X}$ and $g, h \in FG_X$ be arbitrary. We have $F_1 = g_1 \cdot A_1, \quad F_2 = g_2 \cdot A_2$ for some $g_1, g_2 \in FG_X$ and $A_1, A_2 \in E$, then

$$\begin{aligned} F_1 \leq F_2 \ &\Leftrightarrow \ g_1 \cdot A_1 \leq g_2 \cdot A_2 \ \Leftrightarrow \ g_2 \cdot A_2 \subseteq g_1 \cdot A_1 \\ &\Rightarrow \ g \cdot (g_2 \cdot A_2) \subseteq g \cdot (g_1 \cdot A_1) \ \Leftrightarrow \ g \cdot F_1 \leq g \cdot F_2, \end{aligned}$$

$$(g \cdot h) \cdot F_1 = (g \cdot h) \cdot (g_1 \cdot A_1) = (g \cdot h) \cdot g_1) \cdot A_1 = g \cdot (h \cdot (g_1 \cdot A_1) = g \cdot (h \cdot F_1),$$

$$1 \cdot F_1 = 1 \cdot (g_1 \cdot A_1) = (1 \cdot g_1) \cdot A_1 = g_1 \cdot A_1 = F_1,$$

where 1 is the identity in FG_X, the empty word. So, by Proposition 1.3, FG_X acts on the poset $\mathcal{X}$.

Now we want to check that E is an ideal of $\mathcal{X}$. For any $A \in \mathcal{X}$ we have $A = 1 \cdot A$ and $1 \cdot A$ belongs to $\mathcal{X}$, so $E \subseteq \mathcal{X}$. Consider $A \in E$, $F \in \mathcal{X}$ arbitrary, say $F = g \cdot B$ with $g \in FG_X$, $B \in E$, and suppose that $F \leq A$, i.e. $A \subseteq g \cdot B$. For any $w \in F$, $w = g \cdot w'$ for some $w' \in B$. Suppose that $g = g_1 g_2 \ldots g_r$ and $w' = w'_1 w'_2 \ldots w'_n$ with $g_i, w'_j \in Y$, $i = 1, \ldots, r$, $j = 1, \ldots, n$, then

$$w^{\downarrow} = \{1, \ g_1, \ g_1 g_2, \ \ldots, \ g, \ g w'_1, \ g w'_1 w'_2, \ \ldots, \ g w' = w\}.$$

We know that for all $v \in A \ \ v^{\downarrow} \subseteq A$, in particular $1 \in A$, but $A \subseteq g \cdot B$ by hypothesis, so there exists $z \in B$ such that $1 = g \cdot z$, then

$$1 = g \cdot z \ \Rightarrow \ g^{-1} = 1 \cdot z = z \ \Rightarrow \ g^{-1} \in B,$$

this implies that $(g^{-1})^{\downarrow} \subseteq B$, for $B \in E$.

Lemma 1.6 *$h^{-1} \cdot h^{\downarrow} = (h^{-1})^{\downarrow}$ for any $h \in FG_X$.*

Proof. Let $h = h_1 h_2 \ldots h_s$, for some $h_1, h_2, \ldots, h_s \in Y$, then $h^{-1} = h_s^{-1} \ldots h_2^{-1} h_1^{-1}$ and we have

$$\begin{aligned} h^{\downarrow} &= \{1, \ h_1, \ h_1 h_2, \ \ldots, \ h_1 h_2 \ldots h_s\}, \\ (h^{-1})^{\downarrow} &= \{1, \ h_s^{-1}, \ \ldots, \ h_s^{-1} \ldots h_2^{-1}, \ h_s^{-1} \ldots h_2^{-1} h_1^{-1}\}. \end{aligned}$$

Let $v \in h^{-1} \cdot h^{\downarrow}$ arbitrary, $v = (h_s^{-1} \dots h_2^{-1} h_1^{-1}) \cdot (h_1 h_2 \dots h_k)$ for some $0 \leq k \leq s$, but

$$(h_s^{-1} \dots h_2^{-1} h_1^{-1}) \cdot (h_1 h_2 \dots h_k) = h_r^{-1} \dots h_{k+1}^{-1}$$

so $v \in (h^{-1})^{\downarrow}$. Conversely, let $v \in (h^{-1})^{\downarrow}$ be arbitrary, then $v = h_s^{-1} \dots h_k^{-1}$ for some $1 \leq k \leq s$, but we can write v in the form $v = h^{-1} \cdot (h_1 h_2 \dots h_{k-1})$, hence $v \in h^{-1} \cdot h^{\downarrow}$. Thus $h^{-1} \cdot h^{\downarrow} = (h^{-1})^{\downarrow}$. ■

From $(g^{-1})^{\downarrow} \subseteq B$, we obtain $g^{-1} \cdot g^{\downarrow} \subseteq B$, this implies $g \cdot g^{-1} \cdot g^{\downarrow} \subseteq g \cdot B \Leftrightarrow g^{\downarrow} \subseteq F$. Clearly, by the definitions of these sets, we have $w^{\downarrow} \subseteq g^{\downarrow} \cup g \cdot (w')^{\downarrow}$ and

$$w' \in B \;\Rightarrow\; (w')^{\downarrow} \subseteq B \;\Rightarrow\; g \cdot (w')^{\downarrow} \subseteq g \cdot B \;\Leftrightarrow\; g \cdot (w')^{\downarrow} \subseteq F,$$

hence $g^{\downarrow} \cup g \cdot (w')^{\downarrow} \subseteq F$, and we conclude that $w^{\downarrow} \subseteq F$. Since $F = g \cdot B$ and $B \in E$ we know that $F \in \mathcal{P}_X$ and $0 < |F| < \infty$. By what we have just seen for all $w \in F$ $w^{\downarrow} \subseteq F$, so $F \in E$ and we conclude that

$$\forall A \in E \;\; \forall F \in \mathcal{X} \quad F \leq A \;\Rightarrow\; F \in E,$$

thus E is an ideal of $\mathcal{X}$.

By definition of $\mathcal{X}$ we have $FG_X \cdot E = \mathcal{X}$ so, to prove that $(FG_X, \mathcal{X}, E)$ is a McAlister triple we just need to check that for any $g \in FG_X$ we have $g \cdot E \cap E \neq \emptyset$. Let g be an arbitrary element of FG_X, the set $g^{\downarrow}$ belongs to E since for any $v \in g^{\downarrow}$ we clearly have $v^{\downarrow} \subseteq g^{\downarrow}$ and, by Lemma 1.6, $g^{\downarrow} = g \cdot (g^{-1})^{\downarrow}$. Similarly $(g^{-1})^{\downarrow} \in E$, so $g^{\downarrow} \in g \cdot E$, thus $g^{\downarrow} \in g \cdot E \cap E$ and it follows that $g \cdot E \cap E \neq \emptyset$. Hence $(FG_X, \mathcal{X}, E)$ is a McAlister triple.

The P-semigroup originated by this McAlister triple is the set

$$P(FG_X, \mathcal{X}, E) = \{(A, g) \in E \times FG_X : g^{-1} \cdot A \in E\}$$

with the multiplication defined by:

$$(A, g)(B, h) = (A \wedge g \cdot B,\; g \cdot h) = (A \cup g \cdot B,\; g \cdot h).$$

Lemma 1.7 $P(FG_X, \mathcal{X}, E) = \{(A, g) \in E \times FG_X : g \in A\}$.

PROOF. Let $M_X = \{(A, g) \in E \times FG_X : g \in A\}$. For an arbitrary element (A, g) in $P(FG_X, \mathcal{X}, E)$ we have $A \in E$, $g \in FG_X$ and $g^{-1} \cdot A \in E$. From $A \in E$ we know that $1 \in A$ then $g^{-1} \in g^{-1} \cdot A$ and we have

$$\begin{array}{rll} g^{-1} \in g^{-1} \cdot A & \Rightarrow (g^{-1})^{\downarrow} \subseteq g^{-1} \cdot A & (g^{-1} \cdot A \in E) \\ & \Leftrightarrow g^{-1} g^{\downarrow} \subseteq g^{-1} \cdot A & (\text{Claim 1}) \\ & \Rightarrow g^{\downarrow} \subseteq A & (1 \cdot A = A) \\ & \Rightarrow g \in A & (g \in g^{\downarrow}) \end{array}$$

hence $(A, g) \in M_X$. Conversely, let $(A, g) \in M_X$ arbitrary, we know that $g \in FG_X$ and $A \in E$ so $g^{-1} \cdot A \in \mathcal{P}_X$ and $0 < |g^{-1} \cdot A| < \infty$. Let $w \in g^{-1} \cdot A$ be arbitrary, $w = g^{-1} \cdot w'$ for some $w' \in A$, by the definitions of $w^{\downarrow}$, $(g^{-1})^{\downarrow}$ and $g^{-1} \cdot (w')^{\downarrow}$ we can see, like we did above, that $w^{\downarrow} \subseteq (g^{-1})^{\downarrow} \cup g^{-1} \cdot (w')^{\downarrow}$, and by Lemma 1.6 we have

$$w^{\downarrow} \subseteq g^{-1} \cdot g^{\downarrow} \cup g^{-1} \cdot (w')^{\downarrow}.$$

From $A \in E$ and $w' \in A$ we know that $(w')^{\downarrow} \subseteq A$ so $g^{-1} \cdot (w')^{\downarrow} \subseteq g^{-1} \cdot A$, similarly, since $g \in A$, we have $g^{-1} \cdot g^{\downarrow} \subseteq g^{-1} \cdot A$, then $w^{\downarrow} \subseteq g^{-1} \cdot A$ and we conclude that $g^{-1} \cdot A \in E$. Thus $(A, g) \in P(FG_X, \mathcal{X}, E)$ and it follows that $M_X = P(FG_X, \mathcal{X}, E)$. ■

From Proposition 1.4 we know that $P(FG_X, \mathcal{X}, E)$ is an inverse semigroup and we can easily check that $(1^{\downarrow}, 1)$ is the identity of $P(FG_X, \mathcal{X}, E)$. The next result gives us a generating set for this inverse monoid.

Lemma 1.8 *$P(FG_X, \mathcal{X}, E)$ is generated by the elements $(x^{\downarrow},\ x)$ with $x \in X$.*

PROOF. Let T_X be the inverse submonoid of $P(FG_X, \mathcal{X}, E)$ generated by the set $\{(x^{\downarrow},\ x) : x \in X\}$. For any $x \in X$ we have

$$\begin{aligned} & (x^{\downarrow},\ x)((x^{-1})^{\downarrow},\ x^{-1})(x^{\downarrow},\ x) = (x^{\downarrow} \cup x \cdot (x^{-1})^{\downarrow},\ x \cdot x^{-1})(x^{\downarrow},\ x) \\ = {} & (x^{\downarrow} \cup x \cdot (x^{-1})^{\downarrow} \cup 1 \cdot x^{\downarrow},\ 1 \cdot x) = (x^{\downarrow} \cup x \cdot x^{-1} \cdot x^{\downarrow} \cup x^{\downarrow}, x) \\ = {} & (x^{\downarrow} \cup 1 \cdot x^{\downarrow} \cup x^{\downarrow},\ x) = (x^{\downarrow},\ x), \end{aligned}$$

note that $x \cdot (x^{-1})^{\downarrow} = x \cdot x^{-1} \cdot x^{\downarrow}$, by Lemma 1.6. Similarly we can check that

$$((x^{-1})^{\downarrow},\ x^{-1})(x^{\downarrow},\ x)((x^{-1})^{\downarrow},\ x^{-1}) = ((x^{-1})^{\downarrow},\ x^{-1}),$$

so $((x^{-1})^{\downarrow},\ x^{-1})$ is the inverse of $(x^{\downarrow},\ x)$ in $P(FG_X, \mathcal{X}, E)$ and it belongs to T_X. This implies that

$$(x^{\downarrow},\ x)((x^{-1})^{\downarrow},\ x^{-1}) = (x^{\downarrow},\ 1) \in T_X$$

and $((x^{-1})^{\downarrow},\ 1) = (x^{\downarrow},\ 1)^{-1}$ also belongs to T_X since T_X is an inverse submonoid of $P(FG_X, \mathcal{X}, E)$.

Let w be any reduced word in Y^+, if $|w| = 1$ then $(w^{\downarrow},\ 1) \in T_X$ by what we have just seen. Now suppose that for any reduced word, w, in Y^+ with $|w| \leq k$ we have $(w^{\downarrow},\ 1) \in T_X$, and let $z \in Y^+$ be a reduced word with $|z| = k+1$, say $z = yz'$ for some $y \in Y$ and $z' \in Y^+$. We have

$$(z^{\downarrow},\ 1) = (y^{\downarrow},\ y)((z')^{\downarrow},\ 1)((y^{-1})^{\downarrow}, y^{-1})$$

and by hypothesis $((z')^{\downarrow},\ 1) \in T_X$, then $(z^{\downarrow},\ 1) \in T_X$. Note that $(y^{\downarrow},\ y)$ and $((y^{-1})^{\downarrow}, y^{-1})$ belong to T_X by its definition and by what we have seen above. Then, by induction, we conclude that for any reduced word $w \in Y^+$ $(w^{\downarrow},\ 1) \in T_X$.

Consider, w, a reduced word in Y^+ and let u be a left factor of w, say $w = y_1 y_2 \dots y_n$ and $u = y_1 y_2 \dots y_j$ for some $y_1, y_2, \dots, y_j \in Y$, and some $0 \leq j \leq n-1$. If $j = 0$, then

$$(w^{\downarrow},\ u) = (w^{\downarrow},\ 1) \in T_X.$$

Now suppose that for any $j \leq k$ $(w^{\downarrow},\ u) \in T_X$, we have

$$(w^{\downarrow},\ y_1 y_2 \dots y_k y_{k+1}) = (w^{\downarrow},\ y_1 y_2 \dots y_k)(y_{k+1}^{\downarrow},\ y_{k+1}).$$

By hypothesis $(w^{\downarrow},\ y_1 y_2 \dots y_k) \in T_X$, and $(y_{k+1}^{\downarrow}, y_{k+1}) \in T_X$ since $y_{k+1} \in Y$, then, by induction, $(w^{\downarrow},\ u) \in T_X$ for any left factor u of w.

Now, let (A, g) be an arbitrary element of $P(FG_X, \mathcal{X}, E)$. By Lemma 1.5 we know that $A = w_1^{\downarrow} \cup w_2^{\downarrow} \cup \cdots w_n^{\downarrow}$, where $w_1, w_2, \dots, w_n$ are the maximal elements of A. Since $g \in A$, g is a left factor of w_i for some $1 \leq i \leq n$, we can rewrite A in the form

$$A = w_1^{\downarrow} \cup \cdots \cup w_{i-1}^{\downarrow} \cup w_{i+1}^{\downarrow} \cup \cdots \cup w_n^{\downarrow} \cup w_i^{\downarrow},$$

it follows that

$$(A,g) = (w_1^{\downarrow},\ 1)\dots(w_{i-1}^{\downarrow},\ 1)(w_{i+1}^{\downarrow}, 1)\dots(w_n^{\downarrow},\ 1)(w_i^{\downarrow},\ g),$$

so (A,g) is a product of elements of T_X by what we have just seen. Thus $(A,g) \in T_X$ and we conclude that $P(FG_X, \mathcal{X}, E)$ is generated by the set $\{(x^{\downarrow},\ x) : x \in X\}$.

■

Finally we will prove that $P(FG_X, \mathcal{X}, E)$ is the free inverse monoid on X. Note that to obtain the free inverse semigroup we just need to remove the identity element $(1^{\downarrow},\ 1)$, see [9, Proposition 8.1.8].

We define a map $\varphi : X \longrightarrow P(FG_X, \mathcal{X}, E),\quad x \mapsto (x^{\downarrow},\ x)$. This map is obviously well-defined so now we need to check that for every inverse monoid S and every map $\psi : X \longrightarrow S$ there is a unique morphism $\psi' : P(FG_X, \mathcal{X}, E) \longrightarrow S$ such that $\varphi\psi' = \psi$. Let S be an inverse monoid and ψ any map from X into S. We can extend ψ to Y^+ by defining:

$$\begin{aligned}(x^{-1})\psi &= (x\psi)^{-1}, && x \in X\\ (y_1y_2\dots y_n)\psi &= y_1\psi y_2\psi\dots y_n\psi, && y_i \in Y,\ i = 1,\dots,n.\end{aligned}$$

For any finite subset Z of Y^+, with maximal elements $w_1, w_2, \dots, w_n$ we define

$$e_Z = ((w_1w_1^{-1})(w_2w_2^{-1})\dots(w_nw_n^{-1}))\psi,$$

e_Z is an element of the inverse monoid S and we have

Claim 1 *Let Z be a finite subset of Y^+ then e_Z is an idempotent of S.*

PROOF.

$$\begin{aligned}e_Ze_Z &= (w_1w_1^{-1}w_2w_2^{-1}\dots w_nw_n^{-1})\psi(w_1w_1^{-1}w_2w_2^{-1}\dots w_nw_n^{-1})\psi\\ &= w_1\psi(w_1\psi)^{-1}\dots w_n\psi(w_n\psi)^{-1}w_1\psi(w_1\psi)^{-1}\dots w_n\psi(w_n\psi)^{-1}\\ &= w_1\psi(w_1\psi)^{-1}\dots w_n\psi(w_n\psi)^{-1}w_n\psi(w_n\psi)^{-1}\dots w_1\psi(w_1\psi)^{-1}\\ &\quad (\ w_i\psi(w_i\psi)^{-1},\ i = 1,\dots,n,\ \text{are idempotents in } S \text{ so commute})\\ &= w_1\psi(w_1\psi)^{-1}\dots(w_{n-1}\psi(w_{n-1}\psi)^{-1})^2\dots w_1\psi(w_1\psi)^{-1}w_n\psi(w_n\psi)^{-1}\\ &\quad \dots\\ &= w_1\psi(w_1\psi)^{-1}\dots w_n\psi(w_n\psi)^{-1} = (w_1w_1^{-1}\dots w_nw_n^{-1})\psi\\ &= e_Z\end{aligned}$$

$\blacksquare$

Claim 2 *Let $A, B \in E$, then $e_A e_B = e_{A\cup B}$.*

Proof. By Lemma 1.5 we know that $A = w_1^{\downarrow} \cup w_2^{\downarrow} \cup \cdots \cup w_n^{\downarrow}$ and $B = z_1^{\downarrow} \cup z_2^{\downarrow} \cup \cdots \cup z_m^{\downarrow}$, where $w_1, \ldots, w_n$ and $z_1, \ldots, z_m$ are the maximal elements of A and B respectively. Then

$$\begin{aligned} e_A e_B &= ((w_1 w_1^{-1}) \ldots (w_n w_n^{-1}))\psi((z_1 z_1^{-1}) \ldots (z_m z_m^{-1}))\psi \\ &= ((w_1 w_1^{-1}) \ldots (w_n w_n^{-1})(z_1 z_1^{-1}) \ldots (z_m z_m^{-1}))\psi \\ &= e_{A\cup B}, \end{aligned}$$

note that $A \cup B = w_1^{\downarrow} \cup w_2^{\downarrow} \cup \cdots \cup w_n^{\downarrow} \cup z_1^{\downarrow} \cup z_2^{\downarrow} \cup \cdots z_m^{\downarrow}$ and if w_i is a left factor of z_j, for some $i = 1, \ldots, n,\ j = 1, \ldots, m$, then $w_i^{\downarrow} \cup z_j^{\downarrow} = z_j^{\downarrow}$ and writing w_i and z_j as a product of elements of Y , from definition of ψ and from the fact that $y\psi(y\psi)^{-1}$ is an idempotent of S for all $y \in Y$, we obtain

$$w_i\psi(w_i\psi)^{-1} z_j\psi(z_j\psi)^{-1} = z_j\psi(z_j\psi)^{-1}.$$

$\blacksquare$

Claim 3 *Let $A \in E$ and $g \in FG_X$ arbitrary, then $(g\psi)e_A = e_{g\cdot A}(g\psi)$.*

Proof. Let $A = w_1^{\downarrow} \cup w_2^{\downarrow} \cup \cdots \cup w_n^{\downarrow}$ like on Claim 2. We have

$$\begin{aligned} g\psi e_A &= g\psi((w_1 w_1^{-1}) \ldots (w_n w_n^{-1}))\psi \\ &= (g \cdot w_1 w_1^{-1})\psi((w_2 w_2^{-1}) \ldots (w_n w_n^{-1}))\psi \\ &= (g \cdot w_1 w_1^{-1} \cdot 1)\psi((w_2 w_2^{-1}) \ldots (w_n w_n^{-1}))\psi \\ &= (g \cdot w_1 w_1^{-1} \cdot g^{-1} \cdot g)\psi((w_2 w_2^{-1}) \ldots (w_n w_n^{-1}))\psi \\ &= (g \cdot w_1 w_1^{-1} \cdot g^{-1})\psi(g\psi)((w_2 w_2^{-1}) \ldots (w_n w_n^{-1}))\psi \\ &= ((g \cdot w_1)(g \cdot w_1)^{-1})\psi(g\psi)((w_2 w_2^{-1}) \ldots (w_n w_n{}^{-1}))\psi \\ &= (g \cdot w_1(g \cdot w_1)^{-1})\psi(g \cdot w_2(g \cdot w_2)^{-1})\psi(g\psi)((w_3 w_3^{-1}) \ldots (w_n w_n^{-1}))\psi \\ &\quad \ldots \\ &= (g \cdot w_1(g \cdot w_1)^{-1})\psi(g \cdot w_2(g \cdot w_2)^{-1})\psi \ldots (g \cdot w_n(g \cdot w_n)^{-1})\psi(g\psi) \\ &= ((g \cdot w_1(g \cdot w_1)^{-1})(g \cdot w_2(g \cdot w_2)^{-1}) \ldots (g \cdot w_n(g \cdot w_n)^{-1}))\psi(g\psi) \\ &= e_{g\cdot A}(g\psi), \end{aligned}$$

note that, by the definition of $g \cdot A$, w_i is a maximal element of A if and only if $g \cdot w_i$ is a maximal element of $g \cdot A$. ■

Defining a map $\psi' : P(FG_X, \mathcal{X}, E) \longrightarrow S$, $(A, g) \mapsto e_A(g\psi)$, for any $(A, g) \in P(FG_X, \mathcal{X}, E)$ we clearly have $e_A(g\psi) \in Y^+\psi$ so $(A, g)\psi' \in S$. Thus ψ' is obviously well-defined. Given (A, g), $(B, h) \in P(FG_X, \mathcal{X}, E)$ arbitrary we have

$$\begin{aligned}
((A,g)(B,h))\psi' &= (A \cup g \cdot B,\ g \cdot h)\psi' \\
&= e_{A \cup g \cdot B}((g \cdot h)\psi) \\
&= e_A e_{g \cdot B}(g\psi)(h\psi) && \text{(Claim 2)} \\
&= e_A(g\psi) e_B(h\psi) && \text{(Claim 3)} \\
&= (A,g)\psi'(B,h)\psi',
\end{aligned}$$

hence ψ' is a morphism. Let $x \in X$ be arbitrary, we know that $e_{x^\downarrow} = (xx^{-1})\psi$ and

$$x\varphi\psi' = (x^\downarrow,\ x)\psi' = e_{x^\downarrow}(x\psi) = ((xx^{-1})\psi)(x\psi) = x\psi(x\psi)^{-1}x\psi = x\psi,$$

so $\varphi\psi' = \psi$. Now suppose that there exists a morphism $\alpha : (FG_X, \mathcal{X}, E) \longrightarrow S$ such that $\varphi\alpha = \psi$. Then for any $x \in X$ $x\varphi\alpha = x\psi$, i.e. $(x^\downarrow,\ x)\alpha = x\psi$, so for all $x \in X$

$$(x^\downarrow,\ x)\alpha = (x^\downarrow,\ x)\psi,$$

this tell us that α coincides with ψ' in the generators of $P(FG_X, \mathcal{X}, E)$, then $\alpha = \psi'$ in this semigroup, hence $\alpha = \psi'$. We conclude that ψ' is the unique morphism from $P(FG_X, \mathcal{X}, E)$ into S such that $\varphi\psi' = \psi$. It follows that $P(FG_X, \mathcal{X}, E)$ is the free inverse monoid on X.

Remark 1 The free inverse semigroup is unique, so $P(FG_X, \mathcal{X}, E)\backslash\{(1^\downarrow,\ 1)\}$ must be isomorphic to Y^+/τ, a proof of this appears in [6, Section 5.10]. The given isomorphism maps $P(FG_X, \mathcal{X}, E)\backslash\{(1^\downarrow,\ 1)\}$ onto Y^+/τ in the following way:

$$(w_1^\downarrow \cup w_2^\downarrow \cup \cdots \cup w_n^\downarrow,\ g) \mapsto ((w_1 w_1^{-1}) \ldots (w_n w_n^{-1})g)\tau.$$

4 Birooted Word Trees

4.1 Definitions

A *graph* is a finite non-empty set of elements, that are called *vertices*, together with a set of unordered pairs of distinct vertices called *edges*.

The set of all vertices of a graph Γ is denoted by $V(\Gamma)$.

If two vertices, v_1, v_2, form an edge of the graph we say that they are *adjacent*.

A graph Γ' is a *subgraph* of the graph Γ if all vertices and all edges of Γ' are also vertices and edges of Γ.

A vertex is *extreme* if it belongs to exactly one edge.

A *walk* in the graph Γ is a sequence $(y_0, y_1, \ldots, y_n)$ of vertices of Γ such that y_{i-1}, y_i are adjacent for all $i = 0, \ldots, n$. This is a walk of length n, and we call it a (y_0, y_n)-walk.

A *path* is a walk in which all vertices are distinct.

The graph Γ is *connected* if every pair of vertices of Γ is joined by a path.

An (α, α)-walk is said to be *closed*.

A *cycle* is a closed walk all of whose vertices are distinct and with at least three vertices.

A *tree* is a connected graph without cycles.

Note: In a tree T, for any $\alpha, \beta \in V(T)$, there is a unique (α, β)-path, we denote it by $\Pi(\alpha, \beta)$.

A tree in which a vertex is distinguished is called a *rooted tree*.

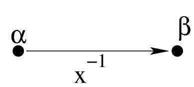
We say that a walk w *spans* the graph Γ, or is a *spanning walk* if all vertices of Γ occur among vertices of w.

An edge with vertices α and β is *oriented* if we consider the edge together with (α, β) as an oriented pair. In this case we write $\alpha \longrightarrow \beta$ and denote the edge by $\alpha\beta$.

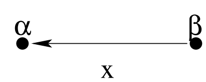
An edge is *labeled* if a symbol is associated to it.

A *word tree*, T, on a non-empty set X is a tree with at least one edge, where each edge is oriented and labeled by an element of X and with no subgraph of the form:

Note: We can extend the set of labels from X to $Y = X \cup X^{-1}$, making a convention that



means the same as



Let T and T' be word trees on X. An *isomorphism* of T onto T' is a bijection of $V(T)$ onto $V(T')$ which preserves adjacency, orientation and labeling of edges. If such a bijection exists we say that T is *isomorphic* to T', and write $T \cong T'$. Note that isomorphism is an equivalence relation on the class of all word trees on X.

4.2 Composition of word trees

Let $\mathcal{T}_X$ be a cross section of the isomorphism classes of word trees on X, i.e. a set intersecting each equivalence class (where the equivalence relation is isomorphism) in a single element.

Let $T, T' \in \mathcal{T}_X$ and $\alpha \in V(T),\ \alpha' \in V(T')$ be arbitrary. Let γ be an extreme vertex in T' and consider the path $\Pi(\alpha', \gamma) = (\alpha' = \gamma_0, \gamma_1, \ldots, \gamma_n = \gamma)$ in T'. There exists $\delta_m \in V(T)$, such that the path $\Pi(\alpha, \delta_m) = (\alpha = \delta_0, \delta_1, \ldots, \delta_m)$ in T, is isomorphic to $(\alpha' = \gamma_0, \gamma_1, \ldots, \gamma_m)$, and m is the greatest integer with this property $(m \leq n)$.

Note that $\Pi(\alpha, \delta_m)$ is obviously unique since we are working in trees.

To do the composition of T with T' we identify γ_i with δ_i, for $i = 0, \ldots, m$. If $m < n$ we attach the graph $(\gamma_m, \gamma_{m+1}, \ldots, \gamma_n)$ to the vertex $\gamma_m = \delta_m$. Repeating this process for all extreme vertices in $V(T')$, we obtain a word tree on X that is the composition of T with T'. We represent by $T(\alpha, \alpha')T'$ its representative in $\mathcal{T}_X$. It is convenient to identify the vertices of T and T' with the corresponding vertices of $T(\alpha, \alpha')T'$.

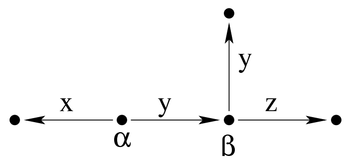
A triple (α, T, β) is a *birooted word tree* on X if $T \in \mathcal{T}_X$ and $\alpha, \beta \in V(T)$.

Considering the set of all birooted word trees on X we can define a multiplication in it by the rule:

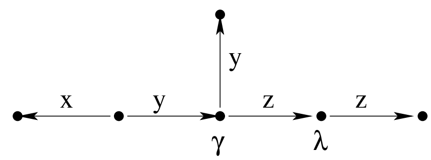
$$(\alpha, T, \beta)(\alpha', T', \beta') = (\alpha, T(\beta, \alpha')T', \beta').$$

We denote this set, together with this multiplication, by $\mathcal{B}_X$. Intuitively, given two birooted word trees we obtain their product by identifying the "second" root of the first tree with the "first" root of the second, making all the common edges coincide and attach to the common vertices all the other vertices of both trees, as we can see in the next two examples.

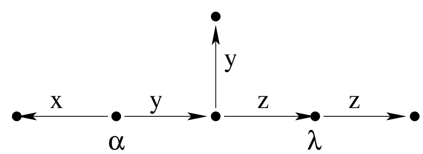
Example 1.1 Given the birooted word trees



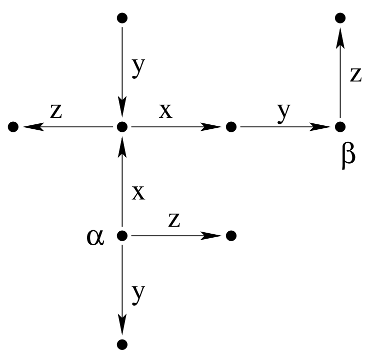
and



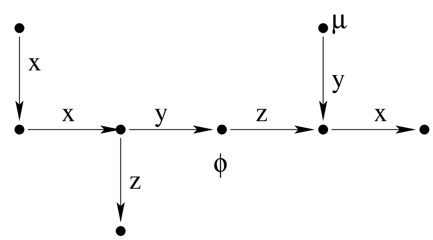
their composition is the birooted word tree



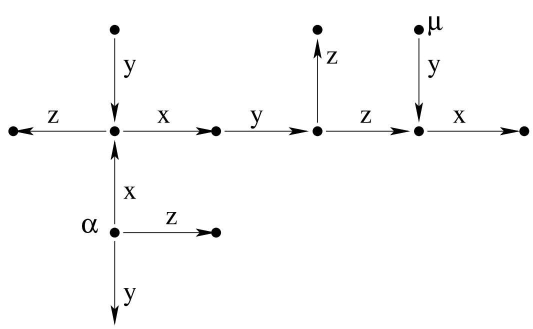
Example 1.2 The composition of the birooted word trees



and



is the birooted word tree



4.3 FI_X as the set of birooted word trees

Consider the free inverse semigroup on the non-empty set X as the P-semigroup $P'(FG_X, \mathcal{X}, E) = P(FG_X, \mathcal{X}, E) \backslash \{(1^{\downarrow},\ 1)\}$. Define a map

$$\varphi : P'(FG_X, \mathcal{X}, E) \longrightarrow \mathcal{B}_X$$

such that, for $(A, g) \in P'(FG_X, \mathcal{X}, E)$, $(A, g)\varphi$ is the birooted word tree (α, T, β), constructed in the following way:

Consider a word of length one, x, in A. Form an edge (α, γ) labeled by x. Fixing α, repeat the process obtaining edges of the form (α, δ). Assuming that we have assigned a path to each word in A, of length less then k, consider the word $x_1 x_2 \dots x_k \in A$. There exists a unique path $(\alpha = \gamma_0, \gamma_1, \dots, \gamma_{k-1})$ labeled $x_1, x_2, \dots, x_{k-1}$ in the graph already constructed, so we can attach to it an edge (γ_{k-1}, γ_k) labeled x_k. Doing this for all words of length k we have inductively constructed a word tree. Let T be its representative in $\mathcal{T}_X$ and β be the vertex of T for which the (α, β)-path is the one labeled by $x_1, x_2, \dots, x_n$, where $x_1, x_2, \dots, x_n$ are such that $g = x_1 x_2 \dots x_n$, in the reduced form.

By this construction we can see that (α, T, β) is the unique birooted word tree associated to (A, g), so the map φ is well-defined. Using an inverse construction we can check that φ is onto:

Let (α, T, β) be an arbitrary element of $\mathcal{B}_X$ and let A be the set of words which label the (α, γ)-paths of T, for all $\gamma \in V(T)$. Let g be the word that labels the path $\Pi(\alpha, \beta)$. Clearly g belongs to A so we just need to check that A is in the semilattice E. Let $w \in A$ arbitrary, then w labels a (α, γ_k)-path, $(\alpha, \gamma_k) = (\alpha = \gamma_0, \gamma_1, \dots, \gamma_k)$, so for any word z in $w^{\downarrow}$, we know that z labels a (α, γ_i)-path, for some $0 \le i \le k$. Hence $z \in A$ and we may conclude that $w^{\downarrow} \subseteq A$. It follows that $(A, g) \in P'(FG_X, \mathcal{X}, E)$. We know that $(A, g)\varphi$ is the unique birooted word tree on X whose set of all (α, γ)-paths bears the labels of words in A and the (α, β)-path is labeled by the letters in the word g, so we must have $(A, g)\varphi = (\alpha, T, \beta)$. Thus φ is onto.

Let $(A, g), (B, h) \in P'(FG_X, \mathcal{X}, E)$ and suppose that $(A, g)\varphi = (B, h)\varphi$. The construction of the birooted word tree from (A, g) is unique, so if it is the same as the one constructed from (B, h), we must have $A = B$ and $g = h$, thus φ is one-one.

Let $(A, g), (B, h)$ be arbitrary elements of $P'(FG_X, \mathcal{X}, E)$. Let $(A, g)\varphi = (\alpha, T, \beta)$ and $(B, h)\varphi = (\alpha', T', \beta')$, then

$$((A, g)(B, h))\varphi = (A \cup g \cdot B, gh)\varphi,$$
$$(A, g)\varphi(B, h)\varphi = (\alpha, T, \beta)(\alpha', T', \beta') = (\alpha, T(\beta, \alpha')T', \beta').$$

We know that $(A \cup g \cdot B, gh)\varphi$ is the birooted word tree constructed with the words of $A \cup g \cdot B$. If we do the composition of T and T', identifying β with α' we obtain a tree "reading" the words of $A \cup g \cdot B$, and this composition is $T(\beta, \alpha')T'$. The path $\Pi(\alpha, \beta')$ is obtained by following $\Pi(\alpha, \beta)$ by $\Pi(\alpha', \beta')$, so the word gh labels the path $\Pi(\alpha, \beta')$. Hence, we must have

$$(A \cup g \cdot B, gh)\varphi = (\alpha, T(\beta, \alpha')T', \beta'),$$

so φ is a morphism. We conclude that $P'(FG_X, \mathcal{X}, E)$ is isomorphic to $\mathcal{B}_X$, this tells us that we can define the free inverse semigroup on a non-empty set X as the set of all birooted word trees $\mathcal{B}_X$.

Chapter 2

Presentations

Let $\mathcal{C}$ be a class of algebras and C an algebra in $\mathcal{C}$. A presentation for C defines it as a homomorphic image of the free $\mathcal{C}$-algebra. In this chapter we will focus on semigroup and inverse semigroup presentations. The definitions, examples and methods described in this chapter can be found, when not stated otherwise, in [8] and [10].

1 Writing Presentations

Let A be an alphabet. A *semigroup presentation* is a pair $< A \mid \mathfrak{R} >$, where $\mathfrak{R} \subseteq A^+ \times A^+$. The elements of A are called *generating symbols* or simply *generators*, and the elements of $\mathfrak{R}$ are called *defining relations*. A pair $(u, v) \in \mathfrak{R}$ is usually represented by $u = v$. The semigroup defined by the presentation $< A \mid \mathfrak{R} >$ is the semigroup A^+/ρ, where ρ is the smallest congruence on A^+ containing $\mathfrak{R}$.

For $w_1, w_2 \in A^+$ we write $w_1 \equiv w_2$ if w_1 and w_2 are identical words in A^+, and $w_1 = w_2$ if they represent the same element of S, i.e. $(w_1, w_2) \in \rho$. In this last case, we say that S *satisfies the relation* $w_1 = w_2$.

Let T be a semigroup generated by a set B, and $\phi : A \longrightarrow B$ an onto mapping. We can extend ϕ in a unique way to an epimorphism $\phi' : A^+ \longrightarrow T$, see for example [10, Proposition 1.1]. We say that T *satisfies relations* $\mathfrak{R}$ if for each relation $u = v$ in $\mathfrak{R}$ we have $u\phi = v\phi$. We can now state the following result:

Proposition 2.1 *Let* $< A \mid \mathfrak{R} >$ *be a presentation,* S *the semigroup defined by it and* T *a semigroup satisfying* $\mathfrak{R}$*. Then* T *is a homomorphic image of* S*.*

Proof. We know that $S = A^+/\rho$, where ρ is the smallest congruence containing $\mathfrak{R}$. Since T satisfies $\mathfrak{R}$ we know that there exists an epimorphism $\phi : A^+ \longrightarrow T$, such that for any $(u = v) \in \mathfrak{R}$ we have $u\phi = v\phi$. Hence, $\mathfrak{R} \subseteq Ker\phi$ and $Ker\phi$ is a congruence, so we must have $\rho \subseteq Ker\phi$. Then, by [6, Theorem 1.5.4],

$$A^+/Ker\phi \cong (A^+/\rho)/(Ker\phi/\rho)$$

and by the Homomorphism Theorem [6, Theorem 1.5.2], we have $A^+/Ker\phi \cong T$, so $T \cong (A^+/\rho)/(Ker\phi/\rho)$. Hence T is a homomorphic image of S. ■

Given $w_1, w_2 \in A^+$, we say that w_2 is *obtained from* w_1 *by one application of one relation* from $\mathfrak{R}$ if there exists α, β in A^* and a relation $u = v$ in $\mathfrak{R}$ such that $w_1 \equiv \alpha u \beta$ and $w_2 \equiv \alpha v \beta$. We say that w_2 *can be deduced from* w_1 if there exists a sequence

$$w_1 \equiv \alpha_1,\ \alpha_2, \ldots, \alpha_{k-1},\ \alpha_k \equiv w_2$$

of words from A^+ such that α_{i+1} is obtained from α_i by one application of one relation from $\mathfrak{R}$. We also say that $w_1 = w_2$ is a *consequence* of $\mathfrak{R}$.

Proposition 2.2 *Let* S *be a semigroup generated by a set* A *and* $\mathfrak{R}$ *a subset of* $A^+ \times A^+$*. Then* $< A \mid \mathfrak{R} >$ *is a presentation for* S *if and only if*

(*i*) S *satisfies all relations from* $\mathfrak{R}$,

(*ii*) *if* u, v *are any two words in* A^+ *such that* S *satisfies* $u = v$ *then* $u = v$ *is a consequence of* $\mathfrak{R}$.

For a proof see [10, Proposition 2.3]. Now we will look at some examples of semigroup presentations.

Example 2.1 The presentation $< A \mid \ >$ defines the free semigroup A^+, for the smallest congruence on A^+ containing the empty set is the diagonal relation $\triangle = \{(w, w) : w \in A^+\}$, and $A^+/\triangle \cong A^+$.

Example 2.2 Consider the subset $\mathfrak{R} = \{(a, a^2)\}$ of $\{a\}^+ \times \{a\}^+$ and let ρ be the smallest congruence on $\{a\}^+$ containing $\mathfrak{R}$, then

$$\begin{array}{l}
a\rho a^2 \ \Rightarrow \ a^2\rho a^3 \\
a\rho a^2 \ \wedge \ a^2\rho a^3 \ \Rightarrow \ a\rho a^3 \\
a\rho a^3 \ \Rightarrow \ a^2\rho a^4 \\
a\rho a^2 \ \wedge \ a^2\rho a^4 \ \Rightarrow \ a\rho a^4 \\
\dots \\
a\rho a^2 \ \wedge \ a^2\rho a^n \ \Rightarrow \ a\rho a^{n+1}, \quad \forall n \in \mathbb{N}
\end{array}$$

so we have $a\rho = \{a\}^+$, hence $\{a\}^+/\rho$ is trivial. We may conclude that the presentation

$$< a \mid a = a^2 >$$

defines the trivial semigroup.

Example 2.3 The presentation $< a \mid a^{n+r} = a^r >$ defines the monogenic semigroup of order $n + r - 1$ and period n. For definitions related with monogenic semigroup see for example [6, Section 1.1]

Proof. Let $M = \{a, a^2, \dots, a^n, \dots, a^{n+r-1}\}$ be the monogenic semigroup of order $n+r-1$ and period n, generated by a. We know that r is the least positive integer, k, such that a^k is repeated, and $n + r$ is the power of the first repetition of a^r, so M satisfies the relation $a^r = a^{n+r}$. Suppose that M satisfies the relation $a^{p_1} = a^{p_2}$, we can assume that a^{p_2} is the first repetition of a^{p_1}. We want to show that this relation is a consequence of $a^r = a^{n+r}$. If $p_1 = p_2$ then $a^{p_1} \equiv a^{p_2}$ and the result follows, so we can suppose without loss of generality that $p_2 > p_1$.

Claim 4 *If $p_1, p_2 \geq r$ and $p_1 \equiv p_2 \pmod{n}$ then $a^{p_1} = a^{p_2}$ can be deduced from $a^{n+r} = a^r$.*

PROOF. In this case we can write $p_2 - p_1 = kn$ for some $k \in \mathbb{N}$, then

$$\begin{aligned} a^{p_1} &\equiv a^{kn+p_2} \equiv a^{kn+r}a^{p_2-r} \equiv a^{(k-1)n}a^{n+r}a^{p_2-r} = a^{(k-1)n}a^r a^{p_2-r} \\ &\equiv a^{(k-2)n}a^{n+r}a^{p_2-r} = a^{(k-2)n}a^r a^{p_2-r} = \ldots = a^r a^{p_2-r} \equiv a^{p_2} \end{aligned}$$

so we can conclude that $a^{p_1} = a^{p_2}$ can be deduced from $a^{n+r} = a^r$. ■

Since r is the least power of a to be repeated in M we must have $p_1 \geq r$, thus $p_1, p_2 \geq r$. Suppose that $n \not| \, p_2 - p_1$, then $p_2 - p_1 = kn + q$ for some $k \in \mathbb{N}$ and some $0 < q < n$, and we have

$$a^{p_1} \equiv a^{p_1-r}a^r = a^{p_1-r}a^{n+r} = \ldots = a^{p_1-r}a^{kn+r} \equiv a^{p_1+kn} \equiv a^{p_2-q},$$

then $a^{p_1} = a^{p_2-q}$ and $p_2 - q < p_2$, this contradicts the fact of a^{p_2} being the first repetition of a^{p_1}. So we must have $p_2 \equiv p_1 \pmod{n}$ and we conclude that $a^{p_1} = a^{p_2}$ is a consequence of $a^r = a^{n+r}$. Thus, by Proposition 2.2, the presentation $< a \mid a^{n+r} = a^r >$ defines M. ■

The following result shows that we can always obtain a presentation for a semigroup by means of its multiplication table.

Proposition 2.3 *Any semigroup can be defined by a presentation.*

PROOF. Let S be any semigroup and define an alphabet $A = \{a_s : s \in S\}$. A is obviously in one-one correspondence with S. The set

$$\mathfrak{R} = \{a_x a_y = a_{xy} : x, y \in S\}$$

is contained in $A^+ \times A^+$ so we can consider the presentation $< A \mid R >$. Let T be the semigroup defined by this presentation. S satisfies all the relations of

$\mathfrak{R}$ (by the definition of semigroup) so, by Proposition 2.1, S is an homomorphic image of T, i.e. there exists an epimorphism $\phi : T \longrightarrow S,\ a_s \mapsto s$.

Let $u, v \in A^+$ be such that $u\phi = v\phi$, then there exists $x, y \in S$ such that $u = a_x$ and $v = a_y$ hold in T and we have

$$a_x\phi = a_y\phi \ \Leftrightarrow \ x = y,$$

this obviously implies $a_x = a_y$, so $u = v$ in T. Thus ϕ is one-one and we can conclude that S is isomorphic to T, thus S is defined by the presentation $< A \mid \mathfrak{R} >$. ■

Let A be an alphabet. We define a *monoid presentation* just like a semigroup presentation, replacing A^+ by A^*. An *inverse semigroup presentation* is a pair $< B \mid Q >$ where B is an alphabet, $B^{-1} = \{b^{-1} : b \in B\}$ is another alphabet disjoint from B and in one-one correspondence with it, and Q is a subset of $(B \cup B^{-1})^+ \times (B \cup B^{-1})^+$. Similarly we can define a *group presentation* and an *inverse monoid presentation.*

Remark 2 If S is a monoid defined by a monoid presentation $< A \mid \mathfrak{R} >$ then S is defined by the semigroup presentation $< A,\ e \mid \mathfrak{R},\ ae = ea = a\ (a \in A) >$.

Let S be the monoid defined by the semigroup presentation $< A \mid \mathfrak{R} >$. There exists a word w in A^+ representing the identity of S and S is defined by the monoid presentation $< A \mid \mathfrak{R},\ w = 1 >$.

Remark 3 The inverse semigroup defined by the (inverse semigroup) presentation $< B \mid Q >$ is the semigroup defined by the presentation

$$< B,\ B^{-1} \mid Q,\ ww^{-1}w = w,\ ww^{-1}zz^{-1} = zz^{-1}ww^{-1},\ (w, z \in (B \cup B^{-1})^+) > .$$

Remark 4 The group defined by the group presentation $< B \mid Q >$ is defined by the monoid presentation $< B,\ B^{-1} \mid Q,\ bb^{-1} = b^{-1}b = 1\ (b \in B) >$.

Given a semigroup S, one way of obtaining a presentation for it consists in the following stn:

- find a generating set A for S;
- find a set $\mathfrak{R}$ of relations which are satisfied by the generators A, and which seem to be sufficient to define S;
- find a set $W \subseteq A^+$, such that each word from A^+ can be transformed to a word from W by applying relations from $\mathfrak{R}$;
- prove that distinct words from W represent distinct elements in S.

The set W described above is called a set of *canonical* or *normal forms* for S. This method for finding a presentation is described in [10], and the next result shows that the presentation $< A \mid \mathfrak{R} >$ that we obtain is in fact a presentation for S.

Proposition 2.4 *Let S be a semigroup, A a generating set for S, $\mathfrak{R} \subseteq A^+ \times A^+$ a set of relations and W a subset of A^+. Assume that:*
(i) *the generators A of S satisfy all the relations from $\mathfrak{R}$;*
(ii) *for each word $w \in A^+$ there exists a word $w' \in W$ such that $w = w'$ is a consequence of $\mathfrak{R}$;*
(iii) *if $u, v \in W$ are such that $u \not\equiv v$ then $u \neq v$ in S;*
then $< A \mid \mathfrak{R} >$ defines S in terms of generators A.

PROOF. The set A generates the semigroup S and $\mathfrak{R}$ holds in S, so we just need to show that any relation in S is a consequence of $\mathfrak{R}$. Let w_1, w_2 be arbitrary elements of S, such that $w_1 = w_2$ holds in S. Then there exists $w_1', w_2' \in W$, such that the relations $w_1 = w_1', \ \ w_2 = w_2'$ are a consequence of $\mathfrak{R}$. From $w_1 = w_2$ we have, by (iii), $w_1' \equiv w_2'$. So

$$w_1 = w_1' \equiv w_2' = w_2$$

is a consequence of $\mathfrak{R}$. Thus S is defined by the presentation $< A \mid \mathfrak{R} >$. ■

Note: When S is a finite semigroup the condition (iii) in Proposition 2.4 can be substituted by the condition $|W| \leq |S|$, see [10, Proposition 2.2].

A way of relating two different presentations for the same semigroup (inverse semigroup, monoid, group, etc.) is by *Tietze Transformations*. These are four operations that applied to a presentation allow us to obtain a different presentation defining the same structure. Given a presentation $< A \mid \mathfrak{R} >$ we can:

- T1) add a relation;
  Given $u, v \in A^+$ such that $u = v$ is not in $\mathfrak{R}$, but it is a consequence of the relations in $\mathfrak{R}$, the presentation
  $$< A \mid \mathfrak{R},\ u = v >$$
  defines the same structure as $< A \mid \mathfrak{R} >$.

- T2) remove a relation;
  If $u = v$ is a relation in $\mathfrak{R}$ that is a consequence of the relations in $\mathfrak{R}\backslash\{(u, v)\}$, then the structure defined by $< A \mid \mathfrak{R} >$ can be defined by the presentation
  $$< A \mid \mathfrak{R}\backslash\{u = v\} > .$$

- T3) add a generator;
  Given a symbol b not in A and a word w in A^+ we can define a relation $b = w$, and the presentations
  $$< A,\ b \mid \mathfrak{R},\ b = w >$$
  and $< A \mid \mathfrak{R} >$ define the same structure.

- T4) remove a generator;
  Given $a \in A$, $u \in (A\backslash\{a\})^+$ such that $a = u$ is in $\mathfrak{R}$, we can replace a by u in

all the relations of $\mathfrak{R}$ where a appears, remove a from the set of generators and remove the relation $a = u$ from $\mathfrak{R}$. We obtain the presentation

$$< A \backslash \{b\} \mid \mathfrak{R}' \backslash \{a = u\} >,$$

defining the same structure as $< A \mid \mathfrak{R} >$, where $\mathfrak{R}'$ is $\mathfrak{R}$ with all occurrences of a replaced by u.

Proposition 2.5 *Two finite presentations define the same semigroup if and only if one can be obtained from the other by a finite number of applications of Tietze Transformations.*

For a proof see for example [10, Proposition 2.5]. One example where we can use Tietze Transformations is the following:

Example 2.4 The bicyclic monoid is defined by the monoid presentation

$$< a, b \mid ab = 1 >,$$

and as a semigroup it admits the presentation

$$< a, b \mid aba = a^2b = a, \ bab = ab^2 = b > .$$

Proof. Let B be the bicyclic monoid, it is defined as a transformation monoid in $\mathbb{N}_0$ by the following graph

so B is generated by x and y, where x is the transformation defined by

$$nx = n + 1, \quad \forall n \in \mathbb{N}_0,$$

and y is the transformation defined by

$$0y = 0, \quad ny = n - 1, \quad \forall n \in \mathbb{N}.$$

The transformation xy is the identity transformation in $\mathbb{N}_0$ since $0xy = (0x)y = 1y = 0$ and $nxy = (n+1)y = n$. Next, for all $j, k \in \mathbb{N}_0$ we have

$$x^k y^j = \begin{cases} y^{j-k} & \text{if } j \geq k \\ x^{k-j} & \text{if } k > j, \end{cases}$$

hence any element of B can be written in the form $y^m x^n$, for some m and n in $\mathbb{N}_0$. It follows that a relation holding in B that is not a consequence of $xy = 1$ can always be taken to be of the form $y^m x^n = y^j x^k$, for some $m, n, j, k \in \mathbb{N}_0$. Now

$$\begin{array}{l} 0y^m x^n = 0x^n = n, \\ 0y^j x^k = 0x^k = k, \end{array}$$

so $k = n$, and considering an integer i such that $i > max(m, j)$ we have

$$\begin{array}{l} iy^m x^n = (i - m)x^n = i - m + n, \\ iy^j x^n = (i - j)x^n = i - j + n, \end{array}$$

this implies that $m = j$, thus

$$y^m x^n = y^j x^k \quad \Rightarrow \quad m = j \text{ and } n = k.$$

So all the relations satisfied by B are consequences of $xy = 1$. Considering an alphabet $A = \{a, b\}$ and making a correspondence between a, b and x, y respectively we may conclude that B admits the monoid presentation $< a, b \mid ab = 1 >$.

Let M be the semigroup defined by the presentation

$$< a, b \mid aba = a^2 b = a, \; bab = ab^2 = b > .$$

From the defining relations of M we have

$$(ab)a = a, \quad a(ab) = a, \quad b(ab) = b, \quad (ab)b = b,$$

so ab acts like an identity in the generators of M, hence M is defined by the monoid presentation

$$< a, b \mid aba = a^2 b = a, \; bab = ab^2 = b, \; ab = 1 > .$$

From the relation $ab = 1$ we can obtain the other four relations in this presentation so, applying Tietze Transformations (T2) we obtain

$$M \cong < a, b \mid ab = 1 >,$$

thus M is the bicyclic monoid. ■

2 Rewriting Presentations

2.1 Subsemigroups of semigroups

Let S be a semigroup defined by the presentation $< A \mid \mathfrak{R} >$, T a subsemigroup of S generated by the set

$$X = \{\xi_i : i \in I\}$$

where ξ_i, are words from A^+. A natural question to put is how to obtain a presentation for T from the presentation of S. We are going to describe a method, given in [4], that answers this question.

Let $B = \{b_i : i \in I\}$ be a set in one-one correspondence with X, define a map from B into A^+, mapping b_i to ξ_i, and let $\psi : B^+ \longrightarrow A^+$ be the natural homomorphism defined by it. Clearly the image of ψ is T so we can think of ψ as interpreting each word in B^+ as an element of T. We call ψ the *interpretation map*.

We denote by $\mathcal{L}(A, T)$ the set of words in A^+ representing elements of T. Any word in $\mathcal{L}(A, T)$ is associated to a word in B^+, so there exists a map $\phi : \mathcal{L}(A, T) \longrightarrow B^+$ with the property that $(w\phi)\psi = w$ in S, for any $w \in \mathcal{L}(A, T)$. The map ϕ rewrites the elements of T as a product of the given generators for T, we call it a *rewriting map*. The next result give us a presentation for T in terms of generators B.

Theorem 2.6 *With the notation above, T is defined by the generators B subject to the relations:*

$$\begin{aligned} b_i &= \xi_i\phi, \quad i \in I, & (2.1)\\ (w_1w_2)\phi &= w_1\phi w_2\phi, \quad w_1, w_2 \in \mathcal{L}(A,T), & (2.2)\\ (w_3uw_4)\phi &= (w_3vw_4)\phi, \quad u = v \in \mathfrak{R}, \quad w_3, w_4 \in A^*, & (2.3) \end{aligned}$$

where w_3 and w_4 are any words such that $w_3uw_4 \in \mathcal{L}(A,T)$.

Proof. First we check that the relations (2.1), (2.2) and (2.3) hold in T. Note that if $(\alpha)\psi = (\beta)\psi$ holds in S, with $\alpha, \beta \in \mathcal{L}(A,T)$, then, since ψ interprets each word in B^+ as an element of T, the relation $\alpha = \beta$ holds in T. We have $(b_i)\psi = \xi_i$ and $((\xi_i)\phi)\psi = \xi_i$, then $(b_i)\psi = ((\xi_i)\phi)\psi$ so

$$b_i = \xi_i\phi, \quad i \in I,$$

holds in T. Given $w_1, w_2 \in \mathcal{L}(A,T)$ we have

$$\begin{aligned} ((w_1w_2)\phi)\psi &= w_1w_2 && (\text{def. } \phi)\\ &= ((w_1\phi)\psi((w_2\phi)\psi && (\text{def. } \phi)\\ &= ((w_1\phi)(w_2\phi))\psi, && (\psi \text{ morphism}) \end{aligned}$$

so relation (2.2) holds in T. Given an arbitrary relation $u = v$ in $\mathfrak{R}$ and words $w_3, w_4 \in A^*$ such that $w_3uw_4 \in \mathcal{L}(A,T)$, we have

$$((w_3uw_4)\phi)\psi = w_3uw_4, \quad \text{and} \quad ((w_3vw_4)\phi)\psi = w_3vw_4,$$

but $(u = v) \in \mathfrak{R}$, so the relation $w_3uw_4 = w_3vw_4$ is a consequence of $\mathfrak{R}$, i.e. it holds in S, then $(w_3uw_4)\phi = (w_3vw_4)\phi$ holds in T. Now, we need to show that any relation in T is a consequence of (2.1), (2.2) and (2.3). Let $\alpha, \beta \in B^+$ be such that $\alpha = \beta$ holds in T, then the relation $(\alpha)\psi = (\beta)\psi$ holds in S so it can be deduced from the relations $\mathfrak{R}$, and by (2.3) we have $((\alpha)\psi)\phi = ((\beta)\psi)\phi$. We can write

$$\alpha \equiv a_{j,1}a_{j,2}\ldots a_{j,l}, \quad \beta \equiv b_{i,1}b_{i,2}\ldots b_{i,k},$$

where $a_{j,n}, b_{i,m} \in B, \quad m = 1,\ldots,k, \; n = 1,\ldots,l$. Then

$$(\beta)\psi \equiv \xi_{i,1}\xi_{i,2}\ldots\xi_{i,k}, \quad (\alpha)\psi \equiv \xi_{j,1}\xi_{j,2}\ldots\xi_{j,l},$$

and we obtain

$$\begin{array}{lll} \beta & \equiv b_{i,1}b_{i,2}\ldots b_{i,k} & \\ & = (\xi_{i,1})\phi(\xi_{i,2})\phi\ldots(\xi_{i,k})\phi & (2.1) \\ & \equiv ((b_{i,1})\psi)\phi((b_{i,2})\psi)\phi\ldots((b_{i,k})\psi)\phi & (\text{def. } \psi) \\ & = ((b_{i,1})\psi(b_{i,2})\psi\ldots(b_{i,k})\psi)\phi & (2.2) \\ & \equiv ((b_{i,1}b_{i,2}\ldots b_{i,k})\psi)\phi & (\psi \text{ morphism}) \\ & \equiv ((\beta)\psi)\phi, & \end{array}$$

similarly $\alpha = ((\alpha)\psi)\phi$. So $\alpha = \beta$ is a consequence of (2.1), (2.2) and (2.3). We conclude that T is defined by the presentation $< B \mid (2.1),\ (2.2),\ (2.3) >$. ■

In the case where S is an inverse semigroup, defined by the (inverse semigroup) presentation $< A \mid \mathfrak{R} >$, the presentation $< C \mid \mathfrak{Q} >$ where $C = A \cup A^{-1}$ and

$$\mathfrak{Q} = \mathfrak{R} \cup \{(w, ww^{-1}w) : w \in C^+\} \cup \{(ww^{-1}zz^{-1}, zz^{-1}ww^{-1}) : w, z \in C^+\},$$

defines S as a semigroup. If T is an (inverse) subsemigroup generated by a set

$$X = \{\xi_i : i \in I\},$$

where ξ_i, are words from C^+, then applying the result above we obtain the presentation

$$\begin{array}{lll} < B \mid & b_i = (\xi_i)\phi, & (i \in I) \\ & (w_1w_2)\phi = (w_1)\phi(w_2)\phi, & (w_1, w_2 \in \mathcal{L}(C,T)) \\ & (w_3uw_4)\phi = (w_3vw_4)\phi, & ((u = v) \in \mathfrak{Q})\ >, \end{array}$$

where w_3, w_4 are any words in C^* such that $w_3uw_4 \in \mathcal{L}(C,T)$, that defines T as a semigroup, in terms of generators B, where, like above, B is a set in one-one correspondence with X. We can decompose the last relation in the presentation to obtain relations for T obtained from $\mathfrak{R}$, the presentation becomes

$$\begin{aligned} < \ B \ \mid \ & b_i = (\xi_i)\phi, \qquad (i \in I) \\ & (w_1w_2)\phi = (w_1)\phi(w_2)\phi, \qquad (w_1, w_2 \in \mathcal{L}(C,T)) \\ & (w_3uw_4)\phi = (w_3vw_4)\phi, \quad ((u = v \in \mathfrak{R}) \\ & (w_5u_1u_1^{-1}u_1w_6)\phi = (w_5u_1w_6)\phi, \\ & (w_7u_1u_1^{-1}u_2u_2^{-1}w_8)\phi = (w_7u_2u_2^{-1}u_1u_1^{-1}w_8)\phi, \quad (u_1, u_2 \in C^+) \ > \end{aligned}$$

where $w_3,\ w_4,\ w_5,\ w_6,\ w_7,\ w_8 \in C^*$ are such that w_3uw_4, $w_5u_1w_6$, and $w_7(u_1u_1^{-1}u_2u_2^{-1})w_8$ belong to $\mathcal{L}(C,T)$.

2.2 Subgroups of inverse monoids

A subgroup T of a semigroup S is clearly a subsemigroup of S, so the results in the last section clearly hold in this case. But T being a group, it makes sense to look for a simpler method to obtain a presentation for it. In [11], the Reidemeister-Schreier Theorem, giving a presentation for a subgroup of a group, was generalized to subgroups of monoids. We will look at these results, for subgroups of inverse monoids, that also appear in [11].

Let S be an inverse monoid, X a non-empty subset of S. The *cosets* of X in S are the sets of the form Xs, $s \in S$ such that there exists $t \in S$ such that $Xst = X$. We represent by $\mathcal{C} = \{C_i : i \in I\}$ the collection of all cosets of X in S. The number of elements of $\mathcal{C}$ is called the *index* of X and we denote it by $[S : X]$.

Lemma 2.7 *S acts on $\{Xs : s \in S\}$ by multiplication on the right. This action induces an action of S on $\mathcal{C} \cup \{C_0\}$, $0 \notin I$, defining $C_0s = C_0$ and $C_is = C_0$ if and only if $C_is \notin \mathcal{C}$, $s \in S$.*

PROOF. Given $Xs \notin \mathcal{C}$, $s \in S$, suppose that there exists $t \in S$ such that $Xst \in \mathcal{C}$, then there must exist $v \in S$ such that $Xstv = X$. But $tv \in S$ and $Xs(tv) = X$, this contradicts the fact that $Xs \notin \mathcal{C}$, hence

$$Xs \notin \mathcal{C} \;\Rightarrow\; Xst \notin \mathcal{C} \quad \forall t \in S.$$

■

The action of S on $\mathcal{C} \cup \{C_0\}$ is equivalent to the action of S on $I \cup \{0\}$ given by $C_i s = C_{is},\ i \in I,\ s \in S$. We now look at the case where $X = G$ is a subgroup of S. We will denote the identity of S by 1 and the identity of the group G by e. Note that e is an idempotent of S not necessarily equal to 1. The following results hold:

Proposition 2.8 *If $i, j \in I$ with $i \neq j$, then $C_i \cap C_j = \emptyset$.*

PROOF. Since $i \neq j$ we know that $C_i \neq C_j$, by definition of $\mathcal{C}$. We have $C_i = Gs$ and $C_j = Gt$ for some $s, t \in S$, suppose that $C_i \cap C_j \neq \emptyset$ and let x be an element in this intersection, we can write $x = g_1 s = g_2 t$ for some $g_1, g_2 \in G$. Let y be an arbitrary element of C_i, then $y = g_3 s$ for some $g_3 \in G$, it follows that

$$y = g_3 s = (g_3 e)s = (g_3(g_1^{-1}g_1))s = (g_3 g_1^{-1})(g_1 s) = (g_3 g_1^{-1})(g_2 t) = (g_3 g_1^{-1} g_2)t,$$

so $y \in Gt = C_j$, this implies that $C_i \subseteq C_j$. Similarly we can show that $C_j \subseteq C_i$, and we obtain $C_i = C_j$, but this contradicts our assumption, so we conclude that $C_i \cap C_j = \emptyset$. ■

Proposition 2.9 *For each $i \in I$ there exists $r_i, r_i' \in S$ such that $Gr_i = C_i$ and $gr_i r_i' = g$ for all $g \in G$.*

PROOF. Let $i \in I$ arbitrary, $C_i = Gr_i$ for some $r_i \in S$ and, since C_i is a coset, there exists $q_i \in S$ such that $C_i q_i = Gr_i q_i = G$. Let us fix an $h \in G$ and let $h' \in G$ be such that $h' = hr_i q_i$. Let $r'_i = q_i h'^{-1} h$, then for any $g \in G$ we have

$$\begin{aligned} gr_i r'_i &= gr_i q_i h'^{-1} h = g(h^{-1}h) r_i q_i h'^{-1} h \\ &= gh^{-1}(hr_i q_i) h'^{-1} h = gh^{-1} h' h'^{-1} h = gh^{-1} h = g, \end{aligned}$$

hence, there exist $r_i, r'_i \in S$ such that $Gr_i = C_i$ and $gr_i r'_i = g \quad \forall g \in G$. ■

A collection of elements r_i, r'_i is a *system of coset representatives* if

- $Gr_i = C_i$,
- $gr_i r'_i = g$,
- $r_1 = r'_1 = 1, \qquad \forall i \in I, \quad \forall g \in G.$

Given any system of coset representatives r_i, r'_i, $i \in I$, we have $er_i r'_i = e$, then

$$e = (er_i) r'_i, \quad er_i = (e) r_i \;\Rightarrow\; e\mathcal{R}er_i$$

but S is inverse so

$$e\mathcal{R}er_i \;\Leftrightarrow\; ee^{-1} = (er_i)(er_i)^{-1} \;\Leftrightarrow\; e = er_i r_i^{-1} e \;\Leftrightarrow\; e = er_i r_i^{-1}$$

and for any $g \in G$ we have

$$g = ge = g(er_i r_i^{-1}) = (ge) r_i r_i^{-1} = gr_i r_i^{-1},$$

so we can take r'_i to be r_i^{-1}. Note that r_i belongs to S not necessarily to G so r_i^{-1} is the inverse of r_i in the sense of inverse semigroup inverse, i.e

$$r_i = r_i r_i^{-1} r_i, \quad \text{and} \quad r_i^{-1} = r_i^{-1} r_i r_i^{-1}.$$

Clearly, for an element $g \in G$, the inverse of g in the group coincides with its inverse in S, since the inverses in S are unique and $gg^{-1} = e \;\Rightarrow\; gg^{-1} g = g$.

Lemma 2.10 *The elements of a coset are $\mathcal{R}$ related to the elements of G.*

PROOF. Let $C_i = Gr_i$ be any coset of G in S. Consider x an arbitrary element of C_i, we know that $x = mr_i$ for some $m \in G$. For any $y \in G$ we have

$$\begin{aligned}
&x = mr_i = (em)r_i = ((yy^{-1})m)r_i = y(y^{-1}mr_i),\\
&y = ey = mm^{-1}y = (mr_ir_i^{-1})m^{-1}y = (mr_i)(r_i^{-1}m^{-1}y) = x(r_i^{-1}m^{-1}y),
\end{aligned}$$

so $x\mathcal{R}y$. ■

Lemma 2.11 *The map $\varphi_{r_i} : G \longrightarrow C_i,\ \ m \mapsto mr_i,$ is a bijection with inverse $\varphi_{r_i^{-1}}$.*

PROOF. φ_{r_i} is obviously well-defined, and considering $m, n \in G$ arbitrary, we have

$$m\varphi_{r_i} = n\varphi_{r_i} \ \Leftrightarrow\ mr_i = nr_i \ \Rightarrow\ mr_ir_i^{-1} = nr_ir_i^{-1} \ \Leftrightarrow\ m = n$$

so φ_{r_i} is one-one. $C_i = Gr_i$ so φ_{r_i} is clearly onto. Defining $\varphi_{r_i^{-1}} : C_i \longrightarrow G$, $m \mapsto mr_i^{-1}$, we clearly have $\varphi_{r_i}\varphi_{r_i^{-1}} = id_G$ and $\varphi_{r_i^{-1}}\varphi_{r_i} = id_{C_i}$, where id_G and id_{C_i} represent the identity map in G and C_i respectively. ■

From this last lemma we can see that for any x in C_i, $i \in I$, we have $xr_i^{-1}r_i = x$.

Lemma 2.12 *For any coset of G , C_i, and $s \in S$ such that $is \neq 0$, we have $C_iss^{-1} = C_i$.*

PROOF. $C_i = Gr_i$ so for any $s \in S$, such that $is \neq 0$ and for any $g \in G$, gr_iss^{-1} belongs to $Gr_iss^{-1} = C_iss^{-1} = C_{iss^{-1}}$. Since $is \neq 0$ we know that $iss^{-1} \neq 0$, then $C_{iss^{-1}}$ is a coset of G so , by Lemma 2.10, $gr_iss^{-1}\ \mathcal{R}\ g$, but S is inverse so

$$\begin{aligned}
&gg^{-1} = gr_iss^{-1}ss^{-1}r_i^{-1}g^{-1} \ \Leftrightarrow\ g = gr_iss^{-1}r_i^{-1}g^{-1}g\\
\Leftrightarrow\ &g = g(r_iss^{-1}r_i^{-1})e \ \Leftrightarrow\ g = ge(r_iss^{-1}r_i^{-1})\\
\Leftrightarrow\ &g = gr_iss^{-1}r_i^{-1} \ \Rightarrow\ gr_i = gr_i(ss^{-1})r_i^{-1}r_i\\
\Leftrightarrow\ &gr_i = gr_ir_i^{-1}r_i(ss^{-1}) \ \Leftrightarrow\ gr_i = gr_iss^{-1},
\end{aligned}$$

hence, for any $g \in G \;\; gr_i = gr_iss^{-1}$, thus $C_i = C_iss^{-1} = C_{iss^{-1}}$. ∎

This result shows that when we have a coset of G with index iss^{-1}, for some $s \in S$ and $i \in I$, we can replace this index by the index i, and vice-versa. It is also clear that for any element g in G, the index $1g$ can be replaced by the index 1 and vice-versa, for $C_1 = G = Gg = C_1g = C_{1g}$.

Proposition 2.13 *If G is a maximal subgroup of S the index of G in S equals the number of $\mathcal{H}$-classes in the $\mathcal{R}$-class of G.*

Proof. Let $C_i = Gr_i$ be any coset of G in S. We have seen that $e\mathcal{R}er_i$ so, by Green's Lemma [6, Lemma 2.2.1 and 2.2.2], the map $\varphi_{r_i} : He \longrightarrow Her_i, \; x \mapsto xr_i$, where He represents the $\mathcal{H}$-class of e, is a bijection with inverse map $\varphi_{r_i^{-1}}$. Since e is an idempotent its $\mathcal{H}$-class, He, is a group, see [6, Theorem 2.2.5]. Given $m \in G$ arbitrary

$$mm^{-1} = e, \quad m = em, \quad \text{and} \quad m^{-1}m = e, \quad m = me,$$

so $m\mathcal{H}e$. Hence $G \subseteq He$, but G is maximal so $G = He$. Then $\varphi_{r_i} : G \longrightarrow Her_i$ is a bijection, and we know that $\varphi_{r_i} : G \longrightarrow Gr_i$ is a bijection, so we must have $Gr_i = Her_i$. Hence the cosets of G are $\mathcal{H}$-classes that are in the $\mathcal{R}$-class of G.

Conversely, let H be any $\mathcal{H}$-class in the $\mathcal{R}$-class of G, and $a \in H$ be arbitrary. The element a is $\mathcal{R}$ related with e so there exist $s, t \in S$ such that $as = e$ and $et = a$. By Green's Lemma the map $\varphi_t : He \longrightarrow Ha, \;\; x \mapsto xt$ is a bijection with inverse map φ_s. So the map $\varphi : G \longrightarrow H, \;\; g \mapsto gt,$ is a bijection and $Gts = G$. This tell us that Gt is a coset of G, hence $H(= Gt)$ is a coset of G. We conclude that $[S : G]$ equals the number of $\mathcal{H}$-classes in the $\mathcal{R}$-class of G ∎

Now we give a generating set for G using a system of coset representatives.

Proposition 2.14 *Let S be generated, as an inverse monoid, by the set A. Then,* *the set*

$$Y = \{er_i a r_{ia}^{-1} : i \in I, \quad a \in A \cup A^{-1}, \quad ia \neq 0\}$$

generates G as a monoid.

Proof. We note that since S is generated as an inverse monoid by A the set $A \cup A^{-1}$ generates S as a monoid. Let

$$M = \{er_i s r_{is}^{-1} : i \in I, \quad s \in S, \ is \neq 0\},$$

for any $s \in S$ we have $er_i s r_{is}^{-1} \in Gr_i s r_{is}^{-1}$ and

$$Gr_i s r_{is}^{-1} = C_i s r_{is}^{-1} = C_{is} r_{is}^{-1} = G,$$

so $M \subseteq G$. Consider the set

$$\{er_1 g r_{1g}^{-1} : g \in G\}$$

noting that $G \subseteq S$, we can see that this set is a subset of M. Let g be an arbitrary element of G

$$C_{1g} = C_1 g = Gg = G = G1 = Gr_1 = C_1$$

so $r_1 = r_{1g} = 1$, then $r_1^{-1} = r_{1g}^{-1} = 1$ and we have

$$g = eg = e1g1 = er_1 g r_{1g}^{-1},$$

hence $G \subseteq \{er_1 g r_{1g}^{-1} : g \in G\} \subseteq M$. We conclude that

$$G = \{er_i s r_{is}^{-1} : i \in I, \quad s \in S, \ is \neq 0\}.$$

Now let s be any element of S, we can write $s = a_1 a_2 \ldots a_n$ for some $a_1, a_2, \ldots, a_n \in A \cup A^{-1}$, and some $n \in \mathbb{N}$. If $n = 1$ we have $s = a_1$ and $er_i s r_{is}^{-1} \in Y$. Assume that for $n \leq k$ the element $er_i s r_{is}^{-1}$ belongs to the monoid generated by Y, and let $s = a_1 a_2 \ldots a_{k+1}$. We can write $s = a_1 t$ where $t = a_2 \ldots a_{k+1}$, and it follows that

$$er_i s r_{is}^{-1} = er_i a_1 t r_{ia_1 t}^{-1} = er_i a_1 r_{ia_1}^{-1} er_{ia_1} t r_{ia_1 t}^{-1}$$

since $er_i a_1 \in G r_i a_1 = C_i a_1 = C_{ia_1}$, so $er_i a_1 r_{ia_1}^{-1} \in C_{ia_1} r_{ia_1}^{-1} = G$, hence

$$(er_i a_1 r_{ia_1}^{-1})e = er_i a_1 r_{ia_1}^{-1}, \quad \text{and} \quad (er_i a_1) r_{ia_1}^{-1} r_{ia_1} = er_i a_1.$$

We know that $er_i a_1 r_{ia_1}^{-1}$ belongs to the monoid generated by Y, for $a_1 \in A \cup A^{-1}$. And $er_{ia_1} t r_{ia_1 t}^{-1}$ belongs to the monoid generated by Y by hypothesis. Then, $er_i s r_{is}^{-1}$ belongs to the monoid generated by Y. By induction, we conclude that $er_i s r_{is}^{-1}$ belongs to the monoid generated by Y for all $s \in S$. It follows that G is contained in the monoid generated by Y, but Y is clearly contained in G, so Y generates G as a monoid. ■

Let $< A \mid \mathfrak{R} >$ be an inverse monoid presentation defining S, we know that the presentation

$$< \ A, \ A^{-1} \mid \mathfrak{R}, \ w = ww^{-1}w, \ ww^{-1}zz^{-1} = zz^{-1}ww^{-1}, \ \ (w, z \in (A \cup A^{-1})^*) >$$

defines S as a monoid. We denote by $\mathfrak{Q}$ the union of $\mathfrak{R}$ with the sets

$$\{(w, ww^{-1}w) : w \in (A \cup A^{-1})^*\} \cup \{(ww^{-1}zz^{-1}, zz^{-1}ww^{-1}) : w, z \in (A \cup A^{-1})^*\}.$$

We already have a generating set for the subgroup G, given by Proposition 2.14, so we need a set of defining relations for it. We define an alphabet

$$B' = \{[i, a] : i \in I, \ a \in A \cup A^{-1}, \ ia \neq 0\},$$

and a map $\phi' : \{(i, w) : i \in I, \ w \in (A \cup A^{-1})^*, \ iw \neq 0\} \longrightarrow (B')^*$ by the rules

$$(i, 1)\phi' = 1, \qquad (i, aw)\phi' = [i, a]((ia, w)\phi'),$$

for any $i \in I$, $a \in A \cup A^{-1}$, $w \in (A \cup A^{-1})^*$ such that $iaw \neq 0$. Note that the definition of ϕ' can be extended to

$$(i, w_1 w_2)\phi' \equiv (i, w_1)\phi'(iw_1, w_2)\phi'$$

for any $i \in I$, $w_1, w_2 \in (A \cup A^{-1})^*$ with $iw_1 w_2 \neq 0$, since writing w_1 as a product of elements of $A \cup A^{-1}$, say $w_1 \equiv a_1 a_2 \ldots a_n$, we obtain

$$\begin{aligned} (i, w_1 w_2)\phi' &\equiv [i, a_1](ia_1, a_2 \ldots a_n w_2)\phi' \equiv \cdots \\ &\equiv [i, a_1][i, a_2] \ldots [i, a_n](iw_1, w_2)\phi' \equiv (i, w_1)\phi'(iw_1, w_2)\phi'. \end{aligned}$$

Lemma 2.15 *Let $w_1, w_2 \in (A \cup A^{-1})^*$ be such that $w_1 = w_2$ holds in S, and let $i \in I$ be such that $iw_1 \neq 0$. Then the relation $(i, w_1)\phi' = (i, w_2)\phi'$ is a consequence of the relations*

$$(i,u)\phi' = (i,v)\phi', \quad i \in I, \quad (u = v) \in \mathfrak{Q}, \quad iu \neq 0.$$

Proof. The relation $w_1 = w_2$ holds in S, so we can obtain w_2 from w_1 by applying relations from $\mathfrak{Q}$. Suppose, without loss of generality, that we only need to apply one relation from $\mathfrak{Q}$, then $w_1 = \alpha u \beta$ and $w_2 = \alpha v \beta$ for some $\alpha, \beta \in (A \cup A^{-1})^*$ and some relation $u = v$ in $\mathfrak{Q}$. It follows that

$$\begin{aligned} (i, w_1)\phi' &\equiv (i,\alpha)\phi'(i\alpha, u)\phi'(i\alpha u, \beta) \\ &= (i,\alpha)\phi'(i\alpha, v)\phi'(i\alpha v, \beta) \quad \text{(by hypothesis)} \\ &\equiv (i, w_2)\phi' \end{aligned}$$

note that $i\alpha u \neq 0$, since $iw_1 \neq 0$. ■

We can now give a presentation for G.

Theorem 2.16 *The presentation*

$$\begin{aligned} < \; B' \; \mid \; & (i,u)\phi' = (i,v)\phi' \qquad (i \in I, \quad (u = v) \in \mathfrak{R}, \quad iu \neq 0), && (2.4) \\ & (i, \alpha\alpha^{-1}\alpha)\phi' = (i,\alpha)\phi' && (2.5) \\ & \qquad (i \in I, \quad \alpha \in (A \cup A^{-1})^*, \quad i\alpha \neq 0), \\ & (i, \alpha\alpha^{-1}\beta\beta^{-1})\phi' = (i, \beta\beta^{-1}\alpha\alpha^{-1})\phi' && (2.6) \\ & \qquad (i \in I, \quad \alpha, \beta \in (A \cup A^{-1})^*, \quad i\alpha\alpha^{-1}\beta\beta^{-1}, \neq 0), \\ & (1, er_i a r_{ia}^{-1})\phi' = [i,a] \qquad (i \in I, \quad a \in A \cup A^{-1}, \quad ia \neq 0), && (2.7) \\ & \qquad\qquad (1,e)\phi' = 1 \; > && (2.8) \end{aligned}$$

defines G as a monoid.

Proof. Define a map $\psi : B' \longrightarrow G$, $[i,a] \mapsto er_i a r_{ia}^{-1}$. This map can be extended to a homomorphism $\psi : (B')^* \longrightarrow G$, by the rule:

$$([i,w])\psi = ([i,a_1])\psi([i,a_2])\psi \ldots ([i,a_n])\psi,$$

where $w \equiv a_1 a_2 \ldots a_n$, with $a_1, a_2, \ldots, a_n \in B'$. We can think of ψ as interpreting the elements of $(B')^*$ as elements of G, so we say that the relation $\gamma = \delta$ holds in G if $(\gamma)\psi = (\delta)\psi$ holds in S.

Let $u = v$ be any relation in $\mathfrak{R}$ and $i \in I$ be such that $iu \neq 0$, we can write $u \equiv u_1 u_2 \ldots u_n$, for some $u_1, u_2, \ldots, u_n \in A \cup A^{-1}$, and we have

$$\begin{aligned}
((i,u)\phi')\psi &= ([i,u_1][iu_1,u_2]\ldots[iu_1u_2\ldots u_{n-1},u_n])\psi \\
&= (er_iu_1r_{iu_1}^{-1})(er_{iu_1}u_2r_{iu_1u_2}^{-1})\ldots(er_{iu_1\ldots u_{n-1}}u_nr_{iu}^{-1}) \\
&= er_iu_1r_{iu_1}^{-1}r_{iu_1}u_2r_{iu_1u_2}^{-1}\ldots er_{iu_1\ldots u_{n-1}}u_nr_{iu}^{-1} && (er_iu_1r_{iu_1}^{-1} \in G) \\
&= er_iu_1u_2r_{iu_1u_2}^{-1}\ldots er_{iu_1\ldots u_{n-1}}u_nr_{iu}^{-1} && (er_iu_1 \in C_{iu_1}) \\
&\quad \cdots \\
&= er_iur_{iu}^{-1},
\end{aligned}$$

similarly we obtain $((i,v)\phi')\psi = er_ivr_{iv}^{-1}$. The relation $u = v$ holds in S so the relation

$$er_iur_{iu}^{-1} = er_ivr_{iv}^{-1} \;\Leftrightarrow\; ((i,u)\phi')\psi = ((i,v)\phi')\psi$$

holds in S, thus $(i,u)\phi' = (i,v)\phi'$ holds in G. Similarly we can check that the relations (2.5) and (2.6) hold in G. Now

$$\begin{aligned}
((1,er_iar_{ia}^{-1})\phi')\psi &= er_1er_iar_{ia}^{-1}r_{1er_iar_{ia}^{-1}}^{-1} && \text{(by above)} \\
&= er_iar_{ia}^{-1} && (er_iar_{ia}^{-1} \in G)
\end{aligned}$$

and $([i,a])\psi = er_iar_{ia}^{-1}$, so relation (2.7) holds in G. We have $((1,e)\phi')\psi = e$ and, since ψ is a morphism, $(1)\psi = e$ hence $((1,e)\phi')\psi = (1)\psi$ holds in S, it follows that (2.8) holds in G.

Note that ϕ' can be seen as a rewriting mapping, since for any $w \in \mathcal{L}(A \cup A^{-1}, G)$, with $w \equiv w_1 w_2 \ldots w_n$ for some $w_1, w_2, \ldots, w_n \in A \cup A^{-1}$, we have

$$\begin{aligned}
((1,w)\phi')\psi &= ([1,w_1][w_1,w_2][w_1w_2,w_3]\ldots[w_1\ldots w_{n-1},w_n])\psi \\
&= ([1,w_1])\psi([w_1,w_2])\psi\ldots([w_1\ldots w_{n-1},w_n])\psi \\
&= (er_1w_1r_{1w_1}^{-1})((er_{w_1}w_2r_{1w_1w_2}^{-1})\ldots(er_{w_1\ldots w_{n-1}}w_nr_{1w}^{-1}) \\
&= er_1w_1r_{1w_1}^{-1}r_{w_1}w_2r_{1w_1w_2}^{-1}\ldots(er_{w_1\ldots w_{n-1}}w_nr_{1w}^{-1}) \\
&= er_1w_1w_2r_{1w_1w_2}^{-1}\ldots(er_{w_1\ldots w_{n-1}}w_nr_{1w}^{-1}) \\
&\quad \cdots \\
&= er_1w_1w_2\ldots w_nr_{1w}^{-1} \\
&= w,
\end{aligned}$$

then the map $\overline{\phi'} : \mathcal{L}(A \cup A^{-1}, G) \longrightarrow (B')^*$, $w \mapsto (1,w)\phi'$ satisfies $((w)\overline{\phi'})\psi = w$, so it is a rewriting mapping. In this case $\mathcal{L}(A \cup A^{-1}, G)$ is the set of all words in $(A \cup A^{-1})^*$ representing elements of G. Applying Theorem 2.6 we obtain the presentation

$$< B' \quad | \quad (1, w_1uw_2)\phi' = (1, w_1vw_2)\phi', \tag{2.9}$$

$$(w_1, w_2 \in (A \cup A^{-1})^*,\ (u = v) \in \mathfrak{R},\ w_1uw_2 \in \mathcal{L}(A \cup A^{-1}, G))$$

$$(1, w_3ww_4)\phi' = (1, w_3ww^{-1}ww_4)\phi', \tag{2.10}$$

$$(1, w_5ww^{-1}zz^{-1}w_6)\phi' = (1, w_5zz^{-1}ww^{-1}w_6)\phi', \tag{2.11}$$

$$(w_3, w_4, w_5, w_6 \in (A \cup A^{-1})^*, \quad w, z \in (A \cup A^{-1})^*,$$

$$w_3ww_4,\ w_5ww^{-1}zz^{-1}w_6 \in \mathcal{L}(A \cup A^{-1}, G))$$

$$(1, er_iar_{ia}^{-1})\phi' = [i, a], \quad (i \in I,\ a \in A \cup A^{-1}, \quad ia \neq 0) \tag{2.12}$$

$$(1, u_1u_2)\phi' = (1, u_1)\phi'(1, u_2)\phi', \quad (u_1, u_2 \in \mathcal{L}(A \cup A^{-1}, G)) \ > \tag{2.13}$$

that defines G as a semigroup. Adding to this presentation the relation (2.8) we obtain a presentation defining G as a monoid. Now we will see that the relations (2.4) to (2.8) imply relations (2.8) to (2.13). By Lemma 2.15, we know that the relation

$$(i, x)\phi' = (i, y)\phi', \quad i \in I,\ (x = y) \in \mathfrak{Q},\ ix \neq 0$$

implies

$$(i, \alpha)\phi' = (i, \beta)\phi', \quad i \in I,$$

if $\alpha = \beta$ is a relation in S. But for any relation $x = y$ in $\mathfrak{Q}$ and $w_1, w_2 \in (A \cup A^{-1})^*$ the relation $w_1xw_2 = w_1yw_2$ holds in S, so the relations (2.9) to (2.11) are a consequence of relations (2.4) to (2.6). Let u_1, u_2 be arbitrary elements of $\mathcal{L}(A \cup A^{-1}, G)$, by definition of ϕ' we have

$$(1, u_1u_2)\phi' \equiv (1, u_1)\phi'(1u_1, u_2)\phi' \equiv (1, u_1)\phi'(1, u_2)\phi,$$

so relation (2.13) is redundant and can be removed from the presentation. Relation (2.12) is the same relation as (2.7), so we conclude that the set B' subject to the relations (2.4) to (2.8) defines G as a monoid. ■

Now we define a new alphabet

$$B = \{[i, a] : i \in I,\ a \in A,\ ia \neq 0\},$$

and a map $\phi : \{(i,w) : i \in I,\ w \in (A \cup A^{-1})^*,\ iw \neq 0\} \longrightarrow (B \cup B^{-1})^*$ by the rules $(i,1)\phi = 1$ and

$$(i,aw)\phi = \begin{cases} [i,a](ia,w)\phi & \text{if } a \in A \\ [ia,a^{-1}]^{-1}(ia,w)\phi & \text{if } a \in A^{-1}, \end{cases}$$

we can check, like we did to ϕ', that ϕ is a rewriting mapping and using it we can obtain a simpler presentation for the group G.

Theorem 2.17 *The presentation*

$$\begin{aligned} < \ B \ \mid \ & (i,u)\phi = (i,v)\phi, \qquad (i \in I,\ (u=v) \in \mathfrak{R},\ iu \neq 0) \\ & (1, er_i a r_{ia}^{-1})\phi = [i,a], \qquad (i \in I,\ a \in A,\ ia \neq 0) \ > \end{aligned}$$

defines G as a group.

Proof. The presentation for G given in Theorem 2.16 defines it as a monoid so it also defines G as a group. Let $i \in I$ and $a \in A \cup A^{-1}$ arbitrary, be such that $ia \neq 0$, we have

$$\begin{aligned} ([i,a][ia,a^{-1}])\psi &\equiv ([i,a])\psi([ia,a^{-1}])\psi \\ &\equiv (er_i a r_{ia}^{-1})(er_{ia}a^{-1}r_{iaa^{-1}}) \\ &\equiv (er_i a r_{ia}^{-1})(er_{ia}a^{-1}r_i). \qquad \text{(Lemma 2.12)} \end{aligned}$$

We know that er_ia belongs to the coset C_{ia}, so, by Lemma 2.10, we have $e\mathcal{R}er_ia$, and since S is inverse we obtain

$$\begin{aligned} & ee^{-1} = er_iaa^{-1}r_i^{-1}e^{-1} \ \Leftrightarrow \ e = e(r_iaa^{-1}r_i^{-1})e \\ \Leftrightarrow \ & e = ee(r_iaa^{-1}r_i^{-1}) \ \Leftrightarrow \ e = er_iaa^{-1}r_i^{-1}, \end{aligned}$$

it follows that

$$\begin{aligned} ([i,a][ia,a^{-1}])\psi &\equiv (er_iar_{ia}^{-1})er_{ia}a^{-1}r_i \\ &\equiv (er_ia)r_{ia}^{-1}r_{ia}a^{-1}r_i \qquad (er_iar_{ia}^{-1} \in G) \\ &\equiv er_iaa^{-1}r_i \qquad \text{(Lemma 2.11)} \\ &= e. \qquad \text{(by above)} \end{aligned}$$

We know that $(1)\psi = e$ and we have just seen that $([i,a][ia,a^{-1}])\psi = e$, so the relation $([i,a][ia,a^{-1}])\psi = (1)\psi$ holds in S. It follows that

$$[i,a][ia,a^{-1}] = 1, \qquad (i \in I,\ a \in A \cup A^{-1},\ ia \neq 0) \tag{2.14}$$

holds in G so we can add it to the presentation given in Theorem 2.16. Let α be an arbitrary element of $(A \cup A^{-1})^*$ such that $i\alpha \neq 0, \quad i \in I$, and let $a_1, a_2, \ldots, a_n \in A \cup A^{-1}$ be such that $\alpha \equiv a_1 a_2 \ldots a_n$. Supposing, without loss of generality, that $a_1, a_2, \ldots, a_n \in A$, we obtain

$$\begin{aligned}
(i,\alpha)\phi(i\alpha,\alpha^{-1})\phi \; &\equiv (i, a_1 a_2 \ldots a_n)\phi(i\alpha, a_n^{-1}a_{n-1}^{-1}\ldots a_1^{-1})\phi \\
&\equiv [i,a_1]((ia_1, a_2\ldots a_n)\phi)[i\alpha a_n^{-1}, a_n]^{-1}((i\alpha a_n^{-1}, a_{n-1}^{-1}\ldots a_1^{-1})\phi) \\
&\quad \cdots \\
&\equiv [i,a_1][ia_1,a_2]\ldots[ia_1a_2\ldots a_{n-1},a_n][i\alpha a_n^{-1},a_n]^{-1} \\
&\qquad\qquad [i\alpha a_n^{-1}a_{n-1}^{-1}, a_{n-1}]^{-1}[i\alpha a_n^{-1}\ldots a_1^{-1}, a_1]^{-1} \\
&\equiv [i,a_1][ia_1,a_2]\ldots[ia_1a_2\ldots a_{n-1},a_n] \\
&\qquad\qquad [ia_1\ldots a_{n-1},a_n]^{-1}\ldots[i,a_1]^{-1},
\end{aligned}$$

then

$$\begin{aligned}
((i,\alpha)\phi(i\alpha,\alpha^{-1})\phi)\psi \; &\equiv ((i,\alpha)\phi)\psi((i\alpha,\alpha^{-1})\phi)\psi \\
&\equiv ([i,a_1])\psi\ldots([ia_1a_2\ldots a_{n-1},a_n])\psi \\
&\qquad ([ia_1\ldots a_{n-1},a_n]^{-1})\psi\ldots([i\alpha a_n^{-1}\ldots a_1^{-1},a_1]^{-1})\psi \\
&\equiv ([i,a_1])\psi\ldots([ia_1a_2\ldots a_{n-1},a_n])\psi \\
&\qquad (([ia_1\ldots a_{n-1},a_n])\psi)^{-1}\ldots(([i\alpha a_n^{-1}\ldots a_1^{-1},a_1)\psi)^{-1} \\
&\equiv e,
\end{aligned}$$

so the relation $((i,\alpha)\phi(i\alpha,\alpha^{-1})\phi)\psi = (1)\psi$ holds in S, hence

$$(i,\alpha)\phi(i\alpha,\alpha^{-1})\phi = 1, \qquad (i \in I,\ \alpha \in (A\cup A^{-1})^*,\ i\alpha \neq 0) \tag{2.15}$$

holds in G. Adding this relation, we obtain the following presentation for G:

$$< B' \mid (2.4),\ (2.5),\ (2.6),\ (2.7),\ (2.8),\ (2.14),\ (2.15) > .$$

From $[i,a][ia,a^{-1}] = 1$ we obtain $[ia,a^{-1}] = [i,a]^{-1}$ and $[i,a] = [ia,a^{-1}]^{-1}$, so

$$[ia^{-1},a]^{-1} = [i,a^{-1}], \quad \forall a \in A\cup A^{-1}, \quad i \in I, \quad ia \neq 0,$$

and we can write the set $B^{-1} = \{[i,a]^{-1} : i \in I,\ a \in A,\ ia \neq 0\}$ in the form

$$B^{-1} = \{[ia,a^{-1}] : i \in I,\ a \in A,\ ia \neq 0\}$$

and the set B' becomes

$$\begin{aligned}
B' \; &= B \cup \{[i,a^{-1}] : i \in I,\ a \in A,\ ia \neq 0\} \\
&= B \cup \{[ia^{-1},a]^{-1} : i \in I,\ a \in A,\ ia \neq 0\},
\end{aligned}$$

then $B' \subseteq B \cup B^{-1}$. For $a \in A^{-1}$, $w \in (A \cup A^{-1})^*$, with $iaw \neq 0$, $i \in I$, we have

$$(i, aw)\phi' = [i, a](ia, w)\phi' = [ia, a^{-1}]^{-1}(ia, w)\phi',$$

hence, relation (2.14) allow us to replace ϕ' by ϕ, substituting the generating set B' by B. Note that we substitute B' by B following the rule

$$[ia^{-1}, a]^{-1} = [i, a^{-1}], \quad \forall a \in A \cup A^{-1}, \quad i \in I, \quad ia \neq 0.$$

Relation (2.15) is equivalent to $(i, \alpha\alpha^{-1})\phi = 1$, and from this relation we obtain relation (2.6), since

$$\begin{aligned} (i, \alpha\alpha^{-1})\phi &\equiv (i, \alpha\alpha^{-1})\phi(i\alpha\alpha^{-1}, \beta\beta^{-1})\phi & \\ &= 1(i\alpha\alpha^{-1}, \beta\beta^{-1})\phi & (2.15) \\ &\equiv (i, \beta\beta^{-1})\phi & (\text{Lemma } 2.12) \\ &= 1 & (2.15) \end{aligned}$$

and, similarly, $(i, \beta\beta^{-1}\alpha\alpha^{-1})\phi = 1$. Thus, we can remove relation (2.6) from the presentation of G. For $i \in I$ and $\alpha \in (A \cup A^{-1})^*$ with $i\alpha \neq 0$ we have

$$\begin{aligned} (i, \alpha\alpha^{-1}\alpha)\phi &\equiv (i, \alpha)\phi(i\alpha, \alpha^{-1})\phi(i\alpha\alpha^{-1}, \alpha)\phi & \\ &= 1(i\alpha\alpha^{-1}, \alpha)\phi & (2.15) \\ &\equiv (i, \alpha)\phi, & (\text{Lemma } 2.12) \end{aligned}$$

hence relation (2.5) can be deduced from relation (2.15). By the definition of ϕ we can deduce relation (2.14) from relation (2.15), so we have

$$G \cong < B \mid (2.4),\ (2.7),\ (2.8),\ (2.15) > .$$

The element $(1, e)\phi$ of $(B \cup B^{-1})^*$ is an idempotent in G, since

$$\begin{aligned} (1, e)\phi(1, e)\phi &\equiv (1, e)\phi(1e, e)\phi & \\ &\equiv (1, ee)\phi & \\ &= (1, e)\phi & (\text{Lemma } 2.15) \end{aligned}$$

so, considering our presentation for G as a group presentation, the relation (2.8) is redundant. Since we changed the generators B' to B, the relation $[i, a]^{-1} = [ia, a^{-1}]$, $a \in A$, holds naturally in G, and we have

$$\begin{aligned} &(i, \alpha)\phi(i\alpha, \alpha^{-1})\phi \\ \equiv\ &[i, a_1][ia_1, a_2] \dots [ia_1 \dots a_{n-1}, a_n][ia_1 \dots a_n, a_n^{-1}] \dots [ia_1, a_1^{-1}] = 1 \end{aligned}$$

hence, we can remove relation (2.15). We conclude that the presentation

$$\begin{aligned} < \ B \ \ | \ \ & (i,u)\phi = (i,v)\phi, \quad (i \in I, \ (u=v) \in \mathfrak{R}, \ iu \neq 0) \\ & (1, er_i a r_{ia}^{-1})\phi = [i,a], \quad (i \in I, \ a \in A, \ ia \neq 0) \ > \end{aligned}$$

defines G as a group. ■

Chapter 3

Finite Presentability

Finite presentations facilitate the study of infinite semigroups when they are finitely presented, but this not always happen, as we will see in section 2. Our main purpose in this chapter is to study some necessary and (or) sufficient conditions for a semigroup to be finitely presented, we will continue this topic in chapter 4 with Bruck-Reilly extensions. We also try to relate, in the 'finite presentability' sense, inverse semigroup and semigroup presentations.

1 Definition and Examples

A semigroup is said to be *finitely presented* if it can be defined by a presentation $< A \mid \mathfrak{R} >$ where A and $\mathfrak{R}$ are finite. Note that the property of being finitely presented is invariant of generating set, see for example [10, Proposition 3.1]. This definition can be extended to inverse semigroups (monoids, groups, etc.), and we say, for example, that the group G is finitely presented as a monoid if it is defined by a monoid presentation $< B \mid \mathfrak{T} >$, where B and $\mathfrak{T}$ are finite.

Example 3.1 The semigroups defined in Examples 2.1 to 2.4 are examples of finitely presented semigroups.

More generally we have:

Example 3.2 Every finite semigroup is finitely presented.

We just need to notice that when a semigroup is finite we can always choose a finite generating set and a finite set of defining relations when constructing the presentation given by Proposition 2.3.

By the definition of semigroup, monoid and group presentation, and by Remarks 2 and 4 we can see that the following holds:

Proposition 3.1 *A monoid is finitely presented as a monoid if and only if it is finitely presented as a semigroup.*

Proposition 3.2 *A group is finitely presented as a group if and only if it is finitely presented as a monoid.*

In the next section we will see that in the case of inverse semigroups a similar result may not hold.

2 Free Inverse Semigroup

We have seen that the free inverse semigroup, FI_X, on the non-empty set X, is the semigroup Y^+/τ, where $Y = X \cup X^{-1}$ and τ is the congruence generated by the set

$$\{(ww^{-1}w, w) : w \in Y^+\} \cup \{(ww^{-1}zz^{-1}, zz^{-1}ww^{-1}) : w, z \in Y^+\},$$

so FI_X is defined by the semigroup presentation

$$< X,\ X^{-1} \mid ww^{-1}w = w,\ ww^{-1}zz^{-1} = zz^{-1}ww^{-1},\ (w, z \in Y^+) > .$$

From this presentation we can see that FI_X, as an inverse semigroup, is defined by the presentation $< X \mid >$. So it is clear that when X is finite, FI_X is finitely presented as an inverse semigroup. The question that arises from this is if in this case FI_X is finitely presented as a semigroup. We will now answer this question, following the work of Schein [12], that also appears in [9, Section IX.4].

Lemma 3.3 *Let S be the semigroup generated by the set $\{u, v\}$, subject to the relations*

$$\begin{array}{lc} & uvu = u, \qquad v = vuv, \\ A_{m,n}) & u^m v^{m+n} u^n = v^n u^{n+m} v^m, \qquad \forall m, n \in \mathbb{N}. \end{array}$$

S is the free monogenic inverse semigroup.

Proof. The free monogenic inverse semigroup is the free inverse semigroup on the set with one element, $\{x\}$, we denote it by FI_x. Let ρ be the congruence generated by the set

$$\{(uvu, u),\ (vuv, v)\} \cup \{A_{m,n} : m, n \in \mathbb{N}\},$$

then $S = \{u, v\}^+/\rho$. We define a map $\varphi : \{u, v\}^+ \longrightarrow FI_x$ by the rules

$$u\varphi = x, \qquad v\varphi = x^{-1},$$
$$(w_1 w_2 \dots w_n)\varphi = w_1\varphi w_2\varphi \dots w_n\varphi,$$

for $w_1, w_2, \dots, w_n \in \{u, v\}$. Clearly φ is a morphism and its kernel

$$Ker\varphi = \{(a, b) \in \{u, v\}^+ \times \{u, v\}^+ : a\varphi = b\varphi\}$$

is a congruence in $\{u, v\}^+$, see [6, Theorem 1.5.2]. In FI_x we have $xx^{-1}x = x$ and $x^{-1}xx^{-1} = x^{-1}$ so

$$v\varphi = (vuv)\varphi \quad \text{and} \quad u\varphi = (uvu)\varphi,$$

for any $m, n \in \mathbb{N}$, since φ is a morphism. For any $k \in \mathbb{N}$ the word $x^k(x^{-1})^k = x^k(x^k)^{-1}$ is an idempotent in FI_x, then

$$x^m(x^m)^{-1}(x^n)^{-1}x^n = (x^n)^{-1}x^n x^m (x^m)^{-1}$$

since the idempotents in FI_x commute, it follows that

$$
\begin{aligned}
& x^m(x^m)^{-1}(x^n)^{-1}x^n = (x^n)^{-1}x^n x^m(x^m)^{-1} \\
\Leftrightarrow\ & x^m(x^{-1})^m(x^{-1})^n x^n = (x^{-1})^n x^{n+m}(x^{-1})^m \\
\Leftrightarrow\ & x^m(x^{-1})^{m+n}x^n = (x^{-1})^n x^{n+m}(x^{-1})^m \\
\Leftrightarrow\ & (u\varphi)^m(v\varphi)^{m+n}(u\varphi)^n = (v\varphi)^n(u\varphi)^{n+m}(v\varphi)^m \\
\Leftrightarrow\ & (u^m v^{m+m} u^n)\varphi = (v^n u^{n+m} v^m)\varphi.
\end{aligned}
$$

Thus (u, uvu), (v, vuv) and $(u^m v^{m+m} u^n, v^n u^{n+m} v^m)$ belong to $Ker\varphi$ for any $m, n \in \mathbb{N}$. So $\rho \subseteq ker\varphi$.

Claim 5 *Let $m, n, p \in \mathbb{N}_0$ be arbitrary. The following holds:*

$$
(u^m v^n u^p)\rho = \begin{cases} (u^{m+p-n})\rho & \text{if } n \le m,\ n \le p \\ (u^m v^{n-p})\rho & \text{if } m \ge n \ge p \\ (v^{n-m} u^p)\rho & \text{if } m \le n \le p \\ (v^{n-m} u^n v^{n-p})\rho & \text{if } n \ge m,\ n \ge p. \end{cases}
$$

Proof. First suppose that $m \ge n$,

$$
\begin{aligned}
& (u^m v^n u^p)\rho = (u^{m-n+1}(u^{n-1}v^n u)u^{p-1})\rho \\
&= (u^{m-n+1})\rho(vu^n v^{n-1})\rho(u^{p-1})\rho = (u^{m-n+1}vu^n v^{n-1}u^{p-1})\rho \\
&= (u^{m-n}uvuu^{n-1}v^{n-1}u^{p-1})\rho = (u^{m-n})\rho(uvu)\rho(u^{n-1}v^{n-1}u^{p-1})\rho \\
&= (u^{m-n})\rho u\rho(u^{n-1}v^{n-1}u^{p-1})\rho = (u^{m-n+1+n-1}v^{n-1}u^{p-1})\rho \\
&= (u^m v^{n-1}u^{p-1})\rho = \cdots = (u^m v^{n-2}u^{p-2})\rho = \cdots
\end{aligned}
$$

if $n \ge p$ we obtain

$$
\cdots = (u^m v^{n-p} u^{p-p})\rho = (u^m v^{n-p})\rho,
$$

and, if $p \ge n$ we obtain

$$
\cdots = (u^m v^{n-n} u^{p-n})\rho = (u^{m+p-n})\rho.
$$

Secondly, suppose that $p \ge n \ge m$, then

$$
\begin{aligned}
& (u^m v^n u^p)\rho = (u^{m-1}(uv^n u^{n-1}u)u^{p-n+1})\rho \\
&= (u^{m-1})\rho(uv^n u^{n-1})\rho(u^{p-n+1})\rho = (u^{m-1})\rho(v^{n-1}u^n v)\rho(u^{p-n+1})\rho \\
&= (u^{m-1}v^{n-1}u^{n-1})\rho(uvu)\rho(u^{p-n})\rho = ((u^{m-1}v^{n-1}u^{n-1})u(u^{p-n}))\rho \\
&= (u^{m-1}v^{n-1}u^p)\rho = \cdots = (u^{m-m}v^{n-m}u^{p-m})\rho \\
&= (v^{n-m}v^{p-m})\rho.
\end{aligned}
$$

Finally, suppose that $n \ge m$ and $n \ge p$, then

$$
(u^m v^n u^p)\rho = (u^m v^m v^{n-m} u^p)\rho,
$$

if $p \geq n-m$ we obtain

$$\begin{aligned}
&(u^m v^m v^{n-m} u^{n-m} u^{p-n+m})\rho = (u^m v^m v^{n-m} u^{n-m})\rho (u^{p-n+m})\rho \\
&= (v^{n-m} u^n v^m)\rho (u^{p-n+m})\rho = (v^{n-m})\rho (u^n v^m u^{p-n+m})\rho \\
&\overset{\star}{=} (v^{n-m})\rho (u^n v^{m-(p-n+m)}\rho = (v^{n-m} u^n v^{n-p})\rho
\end{aligned}$$

($\star$ - we have $n \geq m$ and $n \geq m$, this implies $p - n \leq 0 \;\Rightarrow\; m + p - n \leq m$, so $n \geq m \geq p - n + m$ and we apply what we proved above),

if $p < n - m$, defining $m' = n - m$ and $p' = n - p$ we have $p' > n - m'$, and applying the last case backwards we obtain

$$(u^{n-m'} v^n u^{n-p'})\rho = (v^{m'} u^n v^{p'})\rho = (v^{n-m} u^n v^{n-p})\rho.$$

■

Dually we can show that

$$(v^m u^n v^p)\rho = \begin{cases} (v^{m+p-n})\rho & \text{if } n \leq m,\ n \leq p \\ (v^m u^{n-p})\rho & \text{if } m \geq n \geq p \\ (u^{n-m} v^p)\rho & \text{if } m \leq n \leq p \\ (u^{n-m} v^n u^{n-p})\rho & \text{if } n \geq m,\ n \geq p. \end{cases}$$

Claim 6 *Every element of $\{u, v\}^+/\rho$ can be written in the form $(u^m v^n u^p)\rho$ for some $m, p \in \mathbb{N}_0$, $n \in \mathbb{N}$, with $m, p \leq n$.*

Proof. Let $w \in \{u, v\}^+/\rho$ arbitrary. If $w = (u^n)\rho$ for some $n \in \mathbb{N}$ we have

$$w = (u^{n+n-n})\rho = (u^n v^n u^n)\rho, \qquad \text{(Claim 5)}$$

if $w = (v^n)\rho$ for some $n \in \mathbb{N}$, then

$$w = (v^{0+n-0})\rho = (u^0 v^n u^0)\rho, \qquad \text{(Claim 5)}$$

suppose now that $w = (u^m v^n)\rho$ for some $m, n \in \mathbb{N}_0$ not both zero, if $m > n$ then $0 \leq m - n \leq m$ and $m > 0$, and by Claim 5, we obtain

$$w = (u^m v^{m-(m-n)})\rho = (u^m v^m u^{m-n})\rho,$$

if $n \geq m$ then

$$w = (u^m v^n u^0)\rho$$

where $0 \leq m \leq n$ and $n > 0$. Suppose that $w = (v^m u^n)\rho$ for some $m, n \in \mathbb{N}_0$ not both zero, if $m \geq n$ then

$$w = (u^0 v^m u^n)\rho, \quad \text{with } 0 \leq n \leq m,\ 0 < m,$$

if $n > m$ then $0 \leq n - m \leq n,\ 0 < n$ and by Claim 5 we have

$$w = (v^{m+n-n} u^n)\rho = (v^{n-(n-m)} u^n)\rho = (u^{n-m} v^n u^n)\rho.$$

Suppose that $w = (u^m v^n u^p)\rho$, if $0 \leq m, p \leq n$ then w is already in the form we want, if $p \leq n < m$ then $0 \leq m - n + p \leq m,\ \ 0 < m$ and using the first cases we considered, we obtain

$$w = (u^m v^n)\rho(u^p)\rho = (u^m v^m u^{m-n})\rho(u^p)\rho = (u^m v^m u^{m-n+p})\rho,$$

if $m \leq n < p$ then $0 \leq m - n + p \leq p,\ \ p > 0$ and by what we have seen above

$$w = (u^m)\rho(v^n u^p)\rho = (u^m)\rho(u^{p-n} v^p u^p)\rho = (u^{m+p-n} v^p u^p)\rho,$$

if $n < m,\ n < p$, by Claim 5 we obtain

$$w = (u^{m+p-n})\rho = (u^{m+p-n} v^{m+p-n} u^{m+p-n})\rho.$$

Suppose that $w = (v^m u^n v^p)\rho$, then using the dual of Claim 5 we can show, like we did above, that w can be written in the form $(u^\alpha v^\beta u^\gamma)\rho$ for some $0 \leq \alpha, \gamma \leq \beta,\ \ 0 < \beta$. Suppose that $w = (u^q v^m u^n v^p)\rho$ for some $p, q, n, m \in \mathbb{N}_0$ not all zero, then

$$\begin{aligned} w &= (u^q)\rho(v^m u^n v^p)\rho \\ &= (u^q)\rho(u^{m'} v^{n'} u^{p'})\rho \quad \text{(for some } 0 \leq m', p' \leq n',\ \text{by above)} \\ &= (u^{q+m'} v^{n'} u^{p'})\rho \\ &= (u^\alpha v^\beta u^\gamma)\rho, \qquad\quad \text{(for some } 0 \leq \alpha, \gamma \leq \beta,\ \text{by above)} \end{aligned}$$

similarly we can write w in the same form if $w = (v^m u^n v^p u^q)\rho$. This shows that we can reduce any element of $\{u, v\}^+/\rho$ to an element of the form $(u^m v^n u^p)\rho$ where $0 \leq m, p \leq n$ and $n > 0$. ■

Conversely, we will see that $Ker\varphi \subseteq \rho$. Let $b, b' \in \{u,v\}^+$ be such that $(b,b') \in Ker\varphi$, i.e. $b\varphi = b'\varphi$. By Claim 6 we know that $b\rho = (u^m v^n u^p)\rho$ and $b'\rho = (u^\alpha v^\beta u^\gamma)\rho$, for some $0 \leq m,p \leq n,\ 0 < n$ and $0 \leq \alpha,\gamma \leq \beta,\ 0 < \beta$. Since $\rho \subseteq Ker\varphi$ we have

$$b\varphi = (u^m v^n u^p)\varphi \quad \text{and} \quad b'\varphi = (u^\alpha v^\beta u^\gamma)\varphi,$$

but $b\varphi = b'\varphi$ by hypothesis, so

$$\begin{array}{ll} & (u^m v^n u^p)\varphi = (u^\alpha v^\beta u^\gamma)\varphi \\ \Rightarrow & (u\varphi)^m (v\varphi)^n (u\varphi)^p = (u\varphi)^\alpha (v\varphi)^\beta (u\varphi)^\gamma \\ \Rightarrow & x^m (x^{-1})^n x^p = x^\alpha (x^{-1})^\beta x^\gamma. \end{array}$$

We can consider the free inverse semigroup as the P-semigroup

$$P(FG_X, \mathcal{X}, E)\backslash\{(1^\downarrow,\ 1)\}$$

and using the isomorphism described in Remark 1, we rewrite the equality above in the following way:

$$(x^\downarrow,\ x)^m((x^{-1})^\downarrow,\ x^{-1})^n(x^\downarrow,\ x)^p = (x^\downarrow,\ x)^\alpha((x^{-1})^\downarrow,\ x^{-1})^\beta(x^\downarrow,\ x)^\gamma.$$

We have

$$\begin{array}{l} (y^\downarrow,\ y)(y^\downarrow,\ y) = (y^\downarrow \cup y \cdot y^\downarrow,\ y^2), \\ (y^\downarrow,\ y)^2(y^\downarrow,\ y) = (y^\downarrow \cup y \cdot y^\downarrow \cup y^2 \cdot y^\downarrow,\ y^3), \\ \cdots \end{array}$$

so we can write $(y^\downarrow,\ y)^k = (\cup_{i=0}^{k-1} y^i \cdot x^\downarrow,\ y^k)$, for any $k \in \mathbb{N}$ and $y \in \{x, x^{-1}\}$. It follows that

$$\begin{array}{l} \quad (x^\downarrow,\ x)^m((x^{-1})^\downarrow,\ x^{-1})^n(x^\downarrow,\ x)^p \\ = (\cup_{i=0}^{m-1} x^i \cdot x^\downarrow,\ x^m)(\cup_{i=0}^{n-1}(x^{-1})^i \cdot (x^{-1})^\downarrow,\ (x^{-1})^n)(\cup_{i=0}^{p-1} x^i \cdot x^\downarrow,\ x^p) \\ = (\cup_{i=0}^{m-1} x^i \cdot x^\downarrow \cup x^m \cdot (\cup_{i=0}^{n-1}(x^{-1})^i \cdot (x^{-1})^\downarrow),\ x^m \cdot (x^{-1})^n)(\cup_{i=0}^{p-1} x^i \cdot x^\downarrow,\ x^p) \\ = (\cup_{i=0}^{m-1} x^i \cdot x^\downarrow \cup (\cup_{i=0}^{n-1} x^{m-i} \cdot (x^{-1})^\downarrow) \cup x^{m-n} \cdot (\cup_{i=0}^{p-1} x^i \cdot x^\downarrow),\ x^{m-n+p}) \\ = (\cup_{i=0}^{m-1} x^i \cdot x^\downarrow \cup (\cup_{i=0}^{n-1} x^{m-i} \cdot (x^{-1})^\downarrow) \cup (\cup_{i=0}^{p-1} x^{m-n+i} \cdot x^\downarrow),\ x^{m-n+p}) \\ = (\cup_{i=0}^{m-1} x^i \cdot x^\downarrow \cup (\cup_{i=0}^{n-1} x^{m-i-1} \cdot x^\downarrow) \cup (\cup_{i=0}^{p-1} x^{m-n+i} \cdot x^\downarrow),\ x^{m-n+p}) \\ \hfill \text{(Lemma 1.6)}. \end{array}$$

We know that $x^\downarrow = \{1, x\}$, so

$$\begin{array}{l} \quad \cup_{i=0}^{m-1} x^i \cdot x^\downarrow \cup (\cup_{i=0}^{n-1} x^{m-i} \cdot (x^{-1})^\downarrow) \cup (\cup_{i=0}^{p-1} x^{m-n+i} \cdot x^\downarrow) = \\ = \{1, x\} \cup \{x, x^2\} \cup \cdots \cup \{x^{m-1}, x^m\} \cup \{x^{m-1}, x^m\} \cup \{x^{m-2}, x^{m-1}\} \cup \cdots \\ \qquad \cdots \cup \{x^{m-n}, x^{m-n+1}\} \cup \{x^{m-n}, x^{m-n+1}\} \cup \{x^{m-n+1}, x^{m-n+2}\} \cup \cdots \\ \hfill \cdots \cup \{x^{m-n+p+1}, x^{m-n+p}\} \end{array}$$

if $m+p \leq n$ this set becomes

$$\{x^{m-n}, x^{m-n+1}, \ldots, x^{m-n+p}, x^{m-n+p+1}, \ldots, 1, x, \ldots, x^m\},$$

and if $m+p > n$ we obtain

$$\{x^{m-n}, x^{m-n+1}, \ldots, 1, x, \ldots, x^{m+p-n}, x^{m-n+p+1}, \ldots, x^m\}.$$

From

$$(x^{\downarrow},\ x)^m((x^{-1})^{\downarrow},\ x^{-1})^n(x^{\downarrow},\ x)^p = (x^{\downarrow},\ x)^{\alpha}((x^{-1})^{\downarrow},\ x^{-1})^{\beta}(x^{\downarrow},\ x)^{\gamma}$$

we get $x^{m+p-n} = x^{\alpha+\gamma-\beta}$, from the decomposition above. This implies that $m+p-n = \alpha+\gamma-\beta$, so if $m+p \leq n$ we must have $\alpha+\gamma \leq \beta$ and if $m+p > n$ then $\alpha+\gamma > \beta$. Suppose without loss of generality that $m+p > n$, then

$$\begin{aligned} &\{x^{m-n}, x^{m-n+1}, \ldots, 1, x, \ldots, x^{m+p-n}, \ldots, x^m\} \\ =\ &\{x^{\alpha-\beta}, x^{\alpha-\beta+1}, \ldots, 1, x, \ldots, x^{\alpha+\gamma-\beta}, \ldots, x^{\alpha}\}, \end{aligned}$$

so $x^m = x^{\alpha}$ and $m-n = \alpha-\beta$, this implies $\alpha = m$ and $\beta = n$, then from $m+p-n = \alpha+\gamma-\beta$ we obtain $p = \gamma$, hence $u^m v^n u^p = u^{\alpha} v^{\beta} u^{\gamma}$. Thus, since ρ is reflexive, we have

$$b\rho = (u^m v^n u^p)\rho = (u^{\alpha} v^{\beta} u^{\gamma})\rho = b'\rho,$$

i.e. $(b, b') \in \rho$, then $Ker\varphi \subseteq \rho$ and we can conclude that $Ker\varphi = \rho$. From the Homomorphism Theorem, see for example [6, Theorem 1.5.2], we know that

$$\{u, v\}^+/Ker\varphi \cong (FI_x)\varphi,$$

but φ is onto and $Ker\varphi = \rho$, so

$$\{u, v\}^+/\rho \cong FI_x,$$

this shows that FI_x is defined by the semigroup presentation

$$< u, v \mid u = uvu,\ v = vuv,\ A_{m,n},\ (m, n \in \mathbb{N}) > .$$

$\blacksquare$

Lemma 3.4 *Consider the free monogenic inverse semigroup, FI_x, defined by the presentation*

$$< u, v \mid u = uvu,\ v = vuv,\ A_{m,n},\ (m, n \in \mathbb{N}) >,$$

given in Lemma 3.3. The set of defining relations in this presentation is not equivalent to any finite subset of these defining relations.

Proof. Let $A = \{0, 1, 2, \ldots, n\}$ be a finite set and consider the two partial transformations of A:

$$\alpha = \begin{pmatrix} 0 & 1 & 2 & \ldots & n-2 & n-1 \\ 1 & 2 & 3 & \ldots & n-1 & n \end{pmatrix} \qquad \beta = \begin{pmatrix} 0 & 1 & 2 & \ldots & n-1 & n \\ 0 & 0 & 1 & \ldots & n-2 & n-1 \end{pmatrix}$$

these transformations satisfy the defining relations of FI_x:

$$\alpha\beta\alpha = \begin{pmatrix} 0 & 1 & 2 & \ldots & n-2 & n-1 & n \\ 1 & 2 & 3 & \ldots & n-1 & n & - \\ 0 & 1 & 2 & \ldots & n-2 & n-1 & - \\ 1 & 2 & 3 & \ldots & n-1 & n & - \end{pmatrix} = \alpha,$$

where α is the transformation from the first row to the second, β is the transformation from the second row to the third, the second α is the transformation from the third row to the fourth, their composition, $\alpha\beta\alpha$, is the transformation from the first row to the fourth and we can see that it is equal to α,

$$\beta\alpha\beta = \begin{pmatrix} 0 & 1 & 2 & \ldots & n-2 & n-1 & n \\ 0 & 0 & 1 & \ldots & n-3 & n-2 & n-1 \\ 1 & 1 & 2 & \ldots & n-2 & n-1 & n \\ 0 & 0 & 1 & \ldots & n-3 & n-2 & n-1 \end{pmatrix} = \beta,$$

and

$$\alpha^2 = \begin{pmatrix} 0 & 1 & 2 & \ldots & n-2 & n-1 \\ 1 & 2 & 3 & \ldots & n-1 & n \\ 2 & 3 & 4 & \ldots & n & - \end{pmatrix},\ \alpha^3 = \begin{pmatrix} 0 & 1 & 2 & \ldots & n-2 & n-1 \\ 2 & 3 & 4 & \ldots & n & - \\ 3 & 4 & 5 & \ldots & - & - \end{pmatrix},$$

if we keep doing powers of α we see that

$$\alpha^k = \begin{pmatrix} 0 & 1 & 2 & \ldots & n-k \\ k & k+1 & k+2 & \ldots & n \end{pmatrix},\ \text{if } k \leq n$$

and α^k is the null transformation if $k > n$, we will denote it by $\overline{\emptyset}$. Similarly we can see that

$$\beta^k = \begin{pmatrix} 0 & 1 & 2 & \dots & k & k+1 & \dots & n-1 & n \\ 0 & 0 & 0 & \dots & 0 & 1 & \dots & n-k-1 & n-k \end{pmatrix}, \text{ if } k \le n, \text{ and}$$

$$\beta^k = \begin{pmatrix} 0 & 1 & 2 & \dots & n \\ 0 & 0 & 0 & \dots & 0 \end{pmatrix} = 0, \text{ if } k > n.$$

Let $A_{i,j} = \{u^i v^{i+j} u^j = v^j u^{j+i} y^i\}$, $i, j \in \mathbb{N}$, be a subset of the defining relations of FI_x. For $i, j > n$ we have

$$\alpha^i \beta^{i+j} \alpha^j = \overline{\emptyset}, \text{ and } \beta^j \alpha^{j+i} \beta^i = 0\overline{\emptyset}0 = \overline{\emptyset}.$$

If $i, j \le n$ then

$$\alpha^i \beta^i = \begin{pmatrix} 0 & 1 & 2 & \dots & n-i \\ i & i+1 & i+2 & \dots & n \\ 0 & 1 & 2 & \dots & n-i \end{pmatrix}, \text{ and}$$

$$\beta^j \alpha^j = \begin{pmatrix} 0 & 1 & 2 & \dots & j & j+1 & \dots & n-1 & n \\ 0 & 0 & 0 & \dots & 0 & 1 & \dots & n-j-1 & n-j \\ j & j & j & \dots & j & j+1 & \dots & n-1 & n \end{pmatrix},$$

so if $n - i < j$ we obtain

$$\alpha^i \beta^{i+j} \alpha^j = \begin{pmatrix} 0 & 1 & \dots & n-i \\ j & j & \dots & j \end{pmatrix}, \text{ and } \beta^j \alpha^{j+i} \beta^i = \overline{\emptyset},$$

if $n - i \ge j$ then

$$\alpha^i \beta^{i+j} \alpha^j = \begin{pmatrix} 0 & 1 & \dots & j & j+1 & \dots & n-i \\ j & j & \dots & j & j+1 & \dots & n-i \end{pmatrix} = \beta^j \alpha^{j+i} \beta^i.$$

We can resume this in the following way:

$$\alpha^i \beta^{i+j} \alpha^j = \begin{cases} \overline{\emptyset} & \text{if } i, j > n \\ \begin{pmatrix} 0 & 1 & \dots & n-i \\ j & j & \dots & j \end{pmatrix} & \text{if } n < i+j,\ i, j \le n \\ \begin{pmatrix} 0 & 1 & \dots & j & j+1 & \dots & n-i \\ j & j & \dots & j & j+1 & \dots & n-i \end{pmatrix} & \text{if } i+j \le n \end{cases}$$

$$\beta^j \alpha^{j+i} \beta^i = \begin{cases} \overline{\emptyset} & \text{if } i+j > n \\ \begin{pmatrix} 0 & 1 & \dots & j & j+1 & \dots & n-i \\ j & j & \dots & j & j+1 & \dots & n-i \end{pmatrix} & \text{if } i+j \le n \end{cases}$$

so u and v satisfy $A_{i,j}$ if and only if $\alpha^i\beta^{i+j}\alpha^j = \beta^j\alpha^{j+i}\beta^i$ if and only if $j+i \leq n$ or $i,j \geq n$. Let S_n be the semigroup of partial transformations of A generated by α and β and suppose that the set of defining relations

$$\alpha\beta\alpha = \alpha, \quad \beta\alpha\beta = \beta, \quad A_{i,j}, \ (i,j \in \mathbb{N})$$

is equivalent to a finite subset $\mathfrak{B}$ of these relations. Define n in the following way:

$$n = \begin{cases} max\{i+j : A_{i,j} \in \mathfrak{B}\} & \text{if } \{i+j : A_{i,j} \in \mathfrak{B}\} \neq \emptyset \\ \text{any natural number} & \text{otherwise,} \end{cases}$$

note that, since $\mathfrak{B}$ is finite, if $\{i+j : A_{i,j} \in \mathfrak{B}\} \neq \emptyset$ this set must have a maximal element. If $A_{i,j} \in \mathfrak{B}$ then

$$i+j \leq max\{i+j : A_{i,j} \in \mathfrak{B}\},$$

i.e. $i+j \leq n$ then, by what we have seen above, $A_{i,j}$ holds in S_n. It follows that all relations in $\mathfrak{B}$ hold in S_n. Since $\mathfrak{B}$ is equivalent to the relations

$$\alpha\beta\alpha = \alpha, \quad \beta\alpha\beta = \beta, \quad A_{i,j}, \quad (i,j \in \mathbb{N}),$$

all the relations in this set must hold in S_n, but we know that for example $A_{1,n}$ does not hold in S_n, so we have a contradiction. We may conclude that the set of defining relations of FI_x is not equivalent to any finite subset of itself. ∎

We can now prove the following result:

Proposition 3.5 *The free monogenic inverse semigroup is not finitely presented as a semigroup.*

Proof. By Lemma 3.3, we know that FI_x is defined by the semigroup presentation

$$< u, v \mid u = uvu, \ v = vuv, \ A_{m,n}, \ (m,n \in \mathbb{N}) > .$$

Suppose that FI_x is finitely presented, then $FI_x \cong < X \mid \mathfrak{R} >$, where X and $\mathfrak{R}$ are finite. Since the presentations $< X \mid \mathfrak{R} >$ and

$$< u, v \mid u = uvu, \ v = vuv, \ A_{m,n}, \ (m,n \in \mathbb{N}) >,$$

define the same semigroup, every element of X is equivalent to an expression of products of u and v, so we can replace X by $\{u, v\}$ and the elements of X in the relations $\mathfrak{R}$ by their expression as products of u and v. Let $\mathfrak{D}$ be this new set of relations, $\mathfrak{D}$ is obviously finite, and the presentation $< u, v \mid \mathfrak{D} >$ defines the same semigroup as

$$< u, v \mid u = uvu,\ v = vuv,\ A_{m,n},\ (m, n \in \mathbb{N}) >,$$

so $\mathfrak{D}$ is a finite set that can be deduced from the relations

$$u = uvu, \quad v = vuv, \quad A_{m,n}, \quad (m, n \in \mathbb{N}),$$

and vice-versa. To obtain $\mathfrak{D}$ from this set of relations we can only use a finite number of relations from it , let $\mathfrak{T}$ be the finite set of relations used. Then $< u, v \mid \mathfrak{D} >$ and $< u, v \mid \mathfrak{T} >$ define the same semigroup. This implies that $\mathfrak{T}$ is a subset of

$$u = uvu, \quad v = vuv, \quad A_{m,n}, \quad (m, n \in \mathbb{N}),$$

equivalent to it, but this contradicts Lemma 3.4. We conclude that FI_x cannot be finitely presented. ■

Finally, we generalize this result to any free inverse semigroup.

Proposition 3.6 *No free inverse semigroup is finitely presented as a semigroup.*

PROOF. Let X be a non-empty set and assume that FI_X is defined by the semigroup presentation $< Y \mid \mathfrak{R} >$, where Y and $\mathfrak{R}$ are finite. If X is infinite then some elements of X do not occur in the relations from $\mathfrak{R}$ since this set is finite. Let x be an element of X not occurring in the relations of $\mathfrak{R}$, then the relation $x = x^{-1}x$ does not hold in FI_X, this is a contradiction, so X must be finite. We may express each element in Y as a product of elements in X, so we can assume that $Y = X$. Let us add to $\mathfrak{R}$ the finite set of relations

$$\{x_i = x_j : x_i, x_j \in X,\ i \neq j\},$$

we are identifying all the elements of X as a unique element so we obviously obtain the free monogenic inverse semigroup, but we have already seen that this semigroup is not finitely presented so we can conclude that FI_X is not finitely presented. ■

3 Some Finite Presentability Conditions

We start by giving sufficient conditions for a subgroup of a monoid to be finitely presented. These first two results follow from results in chapter 2 and can be found in [11].

Proposition 3.7 *A subgroup of finite index in a finitely generated inverse monoid is itself finitely generated.*

Proof. Proposition 2.14 give us the generating set

$$Y = \{er_i a r_{ia}^{-1} : i \in I,\ a \in A \cup A^{-1},\ ia \neq 0\}$$

for a subgroup G, of an inverse monoid S, where the set $A \cup A^{-1}$ generates S as a monoid, and the cardinality of I equals the number of cosets of G in S. It follows that if S is finitely generated and the index of G in S is finite, G is finitely generated. ■

Note that in this result, as in Proposition 2.14, the condition of being inverse is not necessary. The results also hold for semigroups, with an appropriate system of coset representatives, see [11].

Theorem 3.8 *A subgroup of finite index in a finitely presented inverse monoid is also finitely presented.*

PROOF. Let S be a finitely presented inverse monoid, defined by the presentation $< A \mid \mathfrak{R} >$, and G a subgroup of S, such that $[S : G] = q$, for some $q \in \mathbb{N}$. By theorem 2.17, we know that G is defined by the group presentation

$$\begin{aligned} < \ B \ \mid \ & (i,u)\phi = (i,v)\phi, \quad (i \in I,\ (u=v) \in \mathfrak{R},\ iu \neq 0) \\ & (1, er_i a r_{ia}^{-1})\phi = [i,a], \quad (i \in I,\ a \in A,\ ia \neq 0)\ > . \end{aligned}$$

Since A and $\mathfrak{R}$ are finite and $|I| = q$ we conclude that G is finitely presented. ■

A semigroup without zero is called *simple* if it has no proper ideals. A semigroup, S, with zero is called *0-simple* if $\{0\}$ and S are its only ideals and $S^2 \neq \{0\}$. A 0-simple (simple) semigroup is said to be *completely 0-simple (completely simple)* if it contains a minimal idempotent within the set of non-zero idempotents.

We will give a necessary and sufficient condition for a completely 0-simple semigroup to be finitely presented. The properties of these semigroups appear in [6, Chapter 3], but the only result we need about them is the following:

Proposition 3.9 *Let G^0 (G) be a 0-group (group). Let I, Λ be non-empty sets, and let $P = (p_{\lambda i})$ be a $I \times \Lambda$ matrix with entries in G^0 (G). Suppose that no row or column of P consists entirely of zeros. Let $S = (I \times G \times \Lambda) \cup \{0\}$ $(S = (I \times G \times \Lambda))$ and define a multiplication on S in the following way:*

$$(i,g,\lambda)(j,h,\mu) = \begin{cases} (i, gp_{\lambda j}h, \mu) & \text{if } p_{\lambda j} \neq 0 \\ 0 & \text{if } p_{\lambda j} = 0, \end{cases}$$
$$0(i,g,\lambda) = (i,g,\lambda)0 = 00 = 0.$$

($(i,g,\lambda)(j,h,\mu) = (i, gp_{\lambda j}h, \mu)$). Then S is a completely 0-simple (simple) semigroup. We denote it by $\mathcal{M}^0[G, I, \Lambda; P]$ ($\mathcal{M}[G, I, \Lambda; P]$).

Conversely, every completely 0-simple (simple) semigroup is isomorphic to a semigroup constructed in this way.

For a proof see [6, Theorem 3.2.3 (3.3.1)].

Proposition 3.10 *A completely 0-simple semigroup $S = \mathcal{M}^0[G, I, \Lambda; P]$ is finitely presented if and only if G is finitely presented and both I and Λ are finite.*

Proof. S is the semigroup $(I \times G \times \Lambda) \cup \{0\}$ subject to the multiplication

$$\begin{array}{l} (i, g, \lambda)(j, h, \mu) = \begin{cases} (i, gp_{\lambda j}h, \mu) & \text{if } p_{\lambda j} \neq 0 \\ 0 & \text{if } p_{\lambda j} = 0 \end{cases} \\ 0(i, g, \lambda) = (i, g, \lambda)0 = 00 = 0. \end{array}$$

Suppose that S is finitely presented, then S is finitely generated, let

$$\{(i_1, g_1, \lambda_1), (i_2, g_2, \lambda_2), \ldots, (i_k, g_k, \lambda_k)\} \cup \{0\}$$

be a generating set for it. By the multiplication defined on S we can see that I and Λ are the sets

$$I = \{i_1, i_2, \ldots, i_k\} \quad \text{and} \quad \Lambda = \{\lambda_1, \lambda_2, \ldots, \lambda_k\},$$

so they are finite. Let us fix a non-zero element $p_{\lambda_0 i_0}$ in P, note that $p_{\lambda_0 i_0} \in G$.

Claim 7 *(i_0, G, λ_0) is a maximal subgroup of S.*

Proof. The map $\varphi : (i_0, G, \lambda_0) \longrightarrow G$, $(i_0, g, \lambda_0)\varphi = p^{-1}_{\lambda_0 i_0} g p^2_{\lambda_0 i_0}$, is obviously well-defined. Let (i_0, g, λ_0), $(i_0, h, \lambda_0) \in (i_0, G, \lambda_0)$ be arbitrary. Since $p_{\lambda_0 i_0} \neq 0$ we have

$$\begin{array}{l} ((i_0, g, \lambda_0)(i_0, h, \lambda_0))\varphi = (i_0, gp_{\lambda_0 i_0}h, \lambda_0)\varphi = p^{-1}_{\lambda_0 i_0} g p_{\lambda_0 i_0} h p^2_{\lambda_0 i_0}, \\ (i_0, g, \lambda_0)\varphi(i_0, h, \lambda_0)\varphi = p^{-1}_{\lambda_0 i_0} g p^2_{\lambda_0 i_0} p^{-1}_{\lambda_0 i_0} h p^2_{\lambda_0 i_0} = p^{-1}_{\lambda_0 i_0} g p_{\lambda_0 i_0} h p^2_{\lambda_0 i_0} \end{array}$$

so φ is a morphism. Supposing that $(i_0, g, \lambda_0)\varphi = (i_0, h, \lambda_0)\varphi$ we obtain

$$\begin{array}{ll} & p^{-1}_{\lambda_0 i_0} g p^2_{\lambda_0 i_0} = p^{-1}_{\lambda_0 i_0} h p^2_{\lambda_0 i_0} \\ \Rightarrow & p_{\lambda_0 i_0} p^{-1}_{\lambda_0 i_0} g p^2_{\lambda_0 i_0} p^{-2}_{\lambda_0 i_0} = p_{\lambda_0 i_0} p^{-1}_{\lambda_0 i_0} h p^2_{\lambda_0 i_0} p^{-2}_{\lambda_0 i_0} \\ \Rightarrow & g = h \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad (G \text{ group}) \\ \Rightarrow & (i_0, g, \lambda_0) = (i_0, h, \lambda_0) \end{array}$$

so φ is one-one. For any $g \in G$ we have

$$\begin{array}{rl} g & = (p^{-1}_{\lambda_0 i_0} p_{\lambda_0 i_0}) g (p^{-2}_{\lambda_0 i_0} p^2_{\lambda_0 i_0}) \\ & = p^{-1}_{\lambda_0 i_0} (p_{\lambda_0 i_0} g p^{-2}_{\lambda_0 i_0}) p^2_{\lambda_0 i_0} = (i_0, p_{\lambda_0 i_0} g p^{-2}_{\lambda_0 i_0}, \lambda_0)\varphi \end{array}$$

and $(i_0, p_{\lambda_0 i_0} g p^{-2}_{\lambda_0 i_0}, \lambda_0)$ belongs to (i_0, G, λ_0), so φ is onto. Hence G is isomorphic to (i_0, G, λ_0), so (i_0, G, λ_0) is a subgroup of S. Let T be a subgroup of S and suppose that $(i_0, G, \lambda_0) \subseteq T$, then T is of the form (I', G, Λ'), with $i_0 \in I'$ and $\lambda_0 \in \Lambda'$. Let (i, e, λ) be the identity of T, we have

$$(i_0, g, \lambda_0)(i, e, \lambda) = (i_0, g, \lambda_0) \;\Leftrightarrow\; (i_0, g p_{\lambda_0 i} e, \lambda) = (i_0, g, \lambda_0) \;\Rightarrow\; \lambda = \lambda_0,$$

and

$$g p_{\lambda_0 i} e = g \;\Leftrightarrow\; g^{-1} g p_{\lambda_0 i_0} e = g^{-1} g \;\Leftrightarrow\; p^{-1}_{\lambda_0 i_0} = e,$$

similarly, from $(i, e, \lambda)(i_0, g, \lambda_0) = (i_0, g, \lambda_0)$, we obtain $i = i_0$, so $(i_0, p^{-1}_{\lambda_0 i_0}, \lambda_0)$ is the identity of T. For any (i, g, λ) in T we have

$$(i, g, \lambda)(i_0, p^{-1}_{\lambda_0 i_0}, \lambda_0) = (i, g, \lambda) \;\Leftrightarrow\; (i, g p_{\lambda i_0} p^{-1}_{\lambda_0 i_0}, \lambda_0) = (i, g, \lambda),$$

so $\lambda = \lambda_0$, and similarly we obtain $i = i_0$, so $T = (i_0, G, \lambda_0)$. Hence (i_0, G, λ_0) is a maximal subgroup of S. ■

Claim 8 *(i_0, G, λ_0) has a finite number of cosets.*

PROOF. Let $(i, g, \lambda) \in S$ be such that $p_{\lambda_0 i} \neq 0$. Considering an element (j, h, λ_0) in S, we have

$$(i_0, G, \lambda_0)(i, g, \lambda)(j, h, \lambda_0) = (i_0, G, \lambda)(j, h, \lambda_0) = (i_0, G, \lambda_0)$$

so, for any $(i, g, \lambda) \in S$ such that $p_{\lambda_0 i} \neq 0$, the set $(i_0, G, \lambda_0)(i, g, \lambda)$ is a coset of (i_0, G, λ_0) in S. By the multiplication defined on S we know that if $p_{\lambda_0 i} = 0$ then $(i_0, G, \lambda_0)(i, g, \lambda)$ cannot be a coset , so the cosets of (i_0, G, λ_0) are the sets (i_0, G, λ) for any $\lambda \in \Lambda$. Since Λ is finite we conclude that (i_0, G, λ_0) has only finitely many cosets in S. ■

We know that (i_0, G, λ_0) is a subgroup of S with finite index, then, adapting Theorem 3.8 to semigroups, see [11, Corollary 2.11], we know that (i_0, G, λ_0) is finitely presented. Since G is isomorphic to this group we conclude that G is finitely presented.

Conversely, suppose that I and Λ are finite and that G is finitely presented. By Propositions 3.1 and 3.2 we know that G is finitely presented as a group if and only if is finitely presented as a semigroup, so let $< A \mid \mathfrak{R} >$ be finite presentation, defining G as a semigroup. We can rearrange the elements of P so that p_{11} is the identity of G. Let $e \in A^+$ be a word representing the identity of G and define a set

$$Y = A \cup \{y_i : i \in I \backslash \{1\}\} \cup \{z_\lambda : \lambda \in \Lambda \backslash \{1\}\}.$$

By [7, Theorem 6.2], we know that a presentation for S is

$$\begin{aligned} < Y \mid \mathfrak{R},\ & y_i e = y_i,\ \ e y_i = p_{1i},\ \ z_\lambda e = p_{\lambda 1},\ \ e z_\lambda = z_\lambda, \\ & z_\lambda y_i = p_{\lambda i}, \qquad (i \in I \backslash \{1\},\ \ \lambda \in \Lambda \backslash \{1\}) > \end{aligned}$$

and, since A, $\mathfrak{R}$, I and Λ are finite, we conclude that S is finitely presented. ■

Let S and T be disjoint semigroups, T having a zero. A semigroup M will be called an *ideal extension* of S by T if it contains S as an ideal and if $M/S \cong T$. Note that if I is an ideal of a semigroup S, then S is an ideal extension of I by S/I. The following result was proved in [11], and gives us a sufficient condition for an ideal extension of a semigroup, by another, to be finitely presented.

Proposition 3.11 *An ideal extension of a finitely presented semigroup by another finitely presented semigroup is finitely presented.*

Proof. Let T and U be semigroups defined by the finite presentations $< A \mid \mathfrak{R} >$ and $< B \mid \mathfrak{Q} >$ respectively. Let S be an ideal extension of T by U, i.e. T is (isomorphic to) an ideal of S and $S/T \cong U$. Let B_0 be the set of all generators from B representing the zero of U. We can look at S/T as the set $S \backslash T \cup \{0\}$, where all products not falling in $S \backslash T$ are zero, this way B generates

$S\backslash T \cup \{0\}$, so $B\backslash B_0$ generates $S\backslash T$, then $B\backslash B_0 \cup A$ generates S. Define $\mathfrak{Q}_0$ as the set of relations

$$\{u = v \in \mathfrak{Q} : u \text{ represents the zero of } U\}.$$

For all $u \in (B\backslash B_0)^+$ representing the zero of U fix a word $\rho(u) \in A^+$ such that $u = \rho(u)$ holds in S. For all pair of letters $a \in A, \quad b \in B\backslash B_0$ fix words $\sigma(a,b), \ \tau(b,a) \in A^+$ such that

$$ab = \sigma(a,b) \quad \text{ and } \quad ba = \tau(b,a)$$

hold in S. We will see that S is defined by the presentation

$$< A, \ B\backslash B_0 \ \ | \ \ \mathfrak{R}, \ \ \mathfrak{Q}\backslash\mathfrak{Q}_0, \tag{3.1}$$

$$u = \rho(u), \qquad ((u = v) \in \mathfrak{Q}, \ \ u \in (B\backslash B_0)^*) \tag{3.2}$$

$$ab = \sigma(a,b), \ \ ba = \tau(b,a), \qquad (a \in A, \ \ b \in B\backslash B_0) \ > . \tag{3.3}$$

Since T is an ideal of S we know that S satisfies $\mathfrak{R}$, and since U can be seen as $S\backslash T \cup \{0\}$ we know that S must satisfy $\mathfrak{Q}\backslash\mathfrak{Q}_0$. S obviously satisfies (3.2) and (3.3). Now let $w_1 = w_2$ be any relation holding in S. If w_1 represents a non-zero element of U then w_2 represents a non-zero element of U and $w_1 = w_2$ holds in U, with $w_1, w_2 \in (B\backslash B_0)^*$, so this relation can be deduced from $\mathfrak{Q}\backslash\mathfrak{Q}_0$. If w_1 (and then w_2) represents an element of T, we can write $w_1 \equiv a_1 a_2 \ldots a_k$ with $a_1, a_2, \ldots, a_k \in A \cup B\backslash B_0$. If any product of $a_1, a_2, \ldots, a_k$ represents the zero of U we use relation (3.2) to transform the product in a word from A^+, then we use relation (3.3) to obtain from w_1 a word $\overline{w_1}$ in A^+, such that $w_1 = \overline{w_1}$ holds in S. Similarly we obtain a word $\overline{w_2}$ in A^+, and we have

$$\overline{w_1} = \overline{w_2}, \qquad \overline{w_1}, \overline{w_2} \in A^+$$

holding in S, so this relation holds in T, then it can be deduced from $\mathfrak{R}$. We conclude that the presentation

$$< A, \ B\backslash B_0 \mid (3.1), \ (3.2), \ (3.3) >$$

defines S, and since A, B, $\mathfrak{R}$ and $\mathfrak{Q}$ are finite we know that S is finitely presented. ■

Let I, J be ideals of a semigroup S, such that $I \neq J$ and J is a maximal ideal in I. For any a in $I \backslash J$ define

$$I(a) = \{x \in S^1 a S^1 \ : \ S^1 x S^1 \subseteq S^1 a S^1\}.$$

The *principal factors* of S are its subsemigroups $S^1 a S^1 / I(a)$, and the minimal ideal of S, if it exists, that we represent by $K(S)$. We can now prove the following results, that can be found in [11].

Theorem 3.12 *Let S be a regular monoid with finitely many left and right ideals. Then S is finitely presented if and only if all maximal subgroups of S are finitely presented.*

PROOF. Saying that S has finitely many left and right ideals is equivalent to say that S has finitely many $\mathcal{R}$ and $\mathcal{L}$-classes. This implies that S contains finitely many $\mathcal{H}$-classes, then, by Proposition 2.13, every maximal subgroup of S has a finite number of cosets, i.e. it has finite index. Suppose that S is finitely presented, then, by Theorem 3.8, we know that all maximal subgroups of S are finitely presented.

Conversely, suppose that all maximal subgroups of S are finitely presented. The principal factors of a semigroup are null, 0-simple or simple semigroups, see for example [8, Proposition 1.13]. In this case S cannot have null principal factors since it is regular, the principal factors are subsemigroups of S and the null semigroup is not regular.

Claim 9 *Every 0-simple (simple) subsemigroup of S is completely 0-simple (simple).*

Proof. Suppose that S has an infinite descending chain of idempotents

$$f_1 > f_2 > f_3 > \cdots > f_k > \cdots$$

we recall that $f_i \leq f_j \ \Leftrightarrow \ f_i = f_i f_j = f_j f_i$, then the ideal Sf_i equals the ideal $Sf_i f_j$ that is contained in the ideal Sf_j, and we obtain a descending chain of left ideals

$$Sf_1 \supseteq Sf_2 \supseteq Sf_3 \supseteq \cdots \supseteq Sf_k \supseteq \cdots$$

Suppose that $Sf_i = Sf_{i+1}$, for some $i \in \mathbb{N}$, we know that $f_i \in Sf_i$, since S is regular, so there exists $a \in S$ such that $f_i = af_{i+1}$, then

$$\begin{aligned} & f_i = af_{i+1} \ \Rightarrow \ f_i f_{i+1} = af_{i+1} f_{i+1} \\ \Leftrightarrow \ & f_i f_{i+1} = af_{i+1} \ \Leftrightarrow \ f_i f_{i+1} = f_i \ \Leftrightarrow \ f_i \leq f_{i+1}, \end{aligned}$$

this implies $f_i = f_{i+1}$, but this contradicts $f_i > f_{i+1}$, so we must have an infinite descending chain of left ideals

$$Sf_1 \supset Sf_2 \supset Sf_3 \supset \cdots \supset Sf_k \supset \cdots$$

that contradicts our assumption. We conclude that S cannot have an infinite descending chain of idempotents, so it must contain a minimal idempotent within the set of non-zero idempotents, hence, every 0-simple (simple) subsemigroup of S is completely 0-simple (simple). ■

This result implies that all principal factors of S are completely 0-simple or completely simple semigroups. Let T be any principal factor of S that is a completely 0-simple semigroup, then, by Proposition 3.9,

$$T \cong \mathcal{M}^0[G, I, \Lambda; P]$$

where G is a group isomorphic to any maximal subgroup of S, I is a set in one-one correspondence with the set of all 0-minimal right ideals of S and Λ is a set in one-one correspondence with the set of all 0-minimal left ideals, see [6, Proof of Theorem 3.2.3]. From the fact that S contains only finitely many left and right ideals we know that I and Λ are finite. The group G is finitely presented by hypothesis, so, by Proposition 3.10, T is finitely presented. We may conclude that all principal factors of S are finitely presented, since we can clearly adapt Proposition 3.10 to show that $K(S)$, the principal factor of S that is completely simple, see [6, Proposition 3.1.4], is finitely presented. Considering a principal series of S

$$S_1 = S \supset S_2 \supset \cdots \supset S_m = K(S)$$

the factors

$$S_1/S_2, \quad S_2/S_3, \; \ldots, \; S_{m-1}/S_m$$

are isomorphic, in some order, to the principal factors of S, see [6, Exc.4,Chap. 3]. We have seen that S_m is finitely presented, and S_{m-1}/S_m is isomorphic to a principal factor of S, so S_{m-1} is an ideal extension of $K(S)$ by a principal factor of S, that we have seen to be finitely presented, it follows, from Proposition 3.11, that S_{m-1} is finitely presented. S_{m-2} is an ideal extension of S_{m-1} by a principal factor of S, so S_{m-2} is finitely presented, by Proposition 3.11. We keep repeating this argument until the beginning of the principal series and we obtain that $S_1 = S$ is finitely presented. ■

Finally, we will show that a similar result to Propositions 3.1 and 3.2 holds for inverse monoids, when they contain only finitely many left and right ideals.

Theorem 3.13 *Let S be an inverse monoid with finitely many left and right ideals. Then S is finitely presented as an inverse monoid if and only if it is finitely presented as a monoid.*

Proof. Suppose that S is finitely presented as an inverse monoid. From the fact that S contains only finitely many left and right ideals we know that any maximal subgroup of S has finite index, using the same argument as in the last result. Then, by Theorem 3.8, every maximal subgroup of S is finitely presented. Since S is inverse we know that S is regular then, by Theorem 3.12, S is finitely presented.

Conversely, suppose that the finite presentation $< A \mid \mathfrak{R} >$ defines S as a monoid. Then this presentation also defines S as an inverse monoid, so S is finitely presented as an inverse monoid. $\blacksquare$

Chapter 4

Bruck-Reilly Extensions

Our main aim in this chapter is to present necessary and (or) sufficient conditions for a Bruck-Reilly extension, of certain classes of monoids, to be finitely presented. We also relate the finite presentability of a Bruck-Reilly extension, of this classes of monoids, as an inverse monoid with its finite presentability when defined by a monoid presentation.

We look at Bruck-Reilly extensions of groups, following the work done in [2], and generalize some of this results for Bruck-Reilly extensions of monoids, stating some results from [1].

We will study the Bruck-Reilly extension of a Clifford monoid that is a union of two groups, considering two different cases. First we consider two copies of the same group, and the morphism linking them is similar to the identity map. In the second case we consider two arbitrary groups, linked by the morphism that maps all the elements of one group to the identity of the other.

Finally, we look at a Bruck-Reilly extension, $BR(S, \theta)$, of an arbitrary Clifford semigroup, S, determined by the morphism θ, that maps all elements of S to its identity.

1 Introduction - Bruck-Reilly Extensions of Monoids

Let S be a monoid, $x \in S$ is said to be a *unit* in S if there exist $p, q \in S$ such that $xp = 1$ and $qx = 1$, where 1 is the identity of S.

The set of all units of S is a subgroup of S, we will call it the *group of units* of S and represent it by $U(S)$. Note that every subgroup of S containing its identity, 1, is contained in $U(S)$, see [5, Theorem 1.10].

Let θ a morphism from S into $U(S)$. We define a multiplication on $\mathbb{N}_0 \times S \times \mathbb{N}_0$ in the following way:

$$(m, a, n)(p, b, q) = (m - n + t, (a\theta^{t-n})(b\theta^{t-p}), q - p + t)$$

where $t = max(n, p)$ and θ^0 is interpreted as the identity map in S. We denote $\mathbb{N}_0 \times S \times \mathbb{N}_0$ together with this multiplication by $BR(S, \theta)$ and call it the *Bruck-Reilly extension* of S determined by θ. The following results help us to characterize $BR(S, \theta)$:

Proposition 4.1
1. $BR(S, \theta)$ is a semigroup with identity $(0, 1, 0)$.
2. $(m, a, n)\ \mathcal{R}_{BR(S,\theta)}\ (p, b, q) \;\Leftrightarrow\; m = p$ and $a\ \mathcal{R}_S\ b$.
3. $(m, a, n)\ \mathcal{L}_{BR(S,\theta)}\ (p, b, q) \;\Leftrightarrow\; n = q$ and $a\ \mathcal{L}_S\ b$.
4. $(m, a, n)\ \mathcal{H}_{BR(S,\theta)}\ (p, b, q) \;\Leftrightarrow\; m = p,\ n = q$ and $a\ \mathcal{H}_S\ b$.
5. $(m, a, n)\ \mathcal{D}_{BR(S,\theta)}\ (p, b, q) \;\Leftrightarrow\; a\ \mathcal{D}_S\ b$.
6.The set of idempotents of $BR(S, \theta)$ is:

$$E(BR(S, \theta)) = \{(m, a, n) \in BR(S, \theta) : m = n,\ a \in E(S)\}.$$

7.$BR(S, \theta)$ is regular if and only if S is regular, in particular if a^{-1} is one inverse of a in S then (n, a^{-1}, m) is one inverse of (m, a, n) in $BR(S, \theta)$.
8.$BR(S, \theta)$ is inverse if and only if S is inverse.

For a proof see [6, Proposition 5.6.6]. Let $< A \mid \mathfrak{R} >$ be a presentation for the monoid S, we can define $BR(S, \theta)$ by means of a presentation containing the generators and defining relations of S.

Proposition 4.2 *The monoid $BR(S, \theta)$ is defined by the presentation*

$$< A,\ b,\ c \mid \mathfrak{R},\ bc = 1,\ ba = (a\theta)b,\ ac = c(a\theta),\ (a \in A) > .$$

This result appears in [7], where we can find a proof for it. The following result is a consequence of the presentation obtained for $BR(S, \theta)$.

Proposition 4.3 *If S is finitely presented then $BR(S, \theta)$ is finitely presented.*

The converse does not always hold. We can find an example of a Bruck-Reilly extension of a, not finitely presented, group, that is finitely presented in [11, Proposition 3.3].

Considering the presentation given in Proposition 4.2 as the definition of a Bruck-Reilly extension, we will rewrite some known properties of these monoids, using the elements of the presentation.

Lemma 4.4 *For all $i, j, k, l \in \mathbb{N}_0$ and $\alpha, \beta \in A^*$, the relation $c^i\alpha b^j = c^k\beta b^l$ holds in $BR(S, \theta)$ if and only if $i = k,\ j = l$ and $\alpha = \beta$ holds in S.*

PROOF. Let $\phi : (A \cup \{b, c\})^* \longrightarrow BR(S, \theta)$ be the monoid morphism extending the mapping

$$b\phi = (0, 1_S, 1),\quad c\phi = (1, 1_S, 0),\quad a\phi = (0, a, 0),\ a \in A,$$

where 1_S is the identity of S. The map ϕ is an epimorphism, see [7, Lemma 4.1]. So, if $c^i\alpha b^j = c^k\beta b^l$ holds in $BR(S, \theta)$ we have $(c^i\alpha b^j)\phi = (c^k\beta b^l)\phi$, then

$$(c^i)\phi(\alpha)\phi(b^j)\phi = (c^k)\phi(\beta)\phi(b^l)\phi \ \Leftrightarrow\ (i, \alpha, j) = (k, \beta, l),$$

it follows that $i = k$, $j = l$ and $\alpha = \beta$ in S. ∎

Lemma 4.5 *In $BR(S, \theta)$ we have:*
(i) $bw = (w\theta)b$, *for all* $w \in A^*$;
(ii) $wc = c(w\theta)$, *for all* $w \in A^*$;
(iii) $b^n a = (a\theta^n)b^n$, *for all* $n \in \mathbb{N}$, *and all* $a \in A$;
(iv) $ac^n = c^n(a\theta^n)$, *for all* $n \in \mathbb{N}$, *and all* $a \in A$.

Proof. Let $w \in A^*$ be arbitrary, say $w \equiv a_1 a_2 \dots a_r$, where $a_i \in A$, $i = 1, \dots, r$. Then

$$\begin{aligned} bw &\equiv b(a_1 a_2 \dots a_r) \equiv (ba_1)a_2 \dots a_r = (a_1\theta)ba_2 \dots a_r = (a_1\theta)(a_2\theta)ba_3 \dots a_r \\ &= \dots = (a_1\theta)(a_2\theta)(a_3\theta)\dots(a_r\theta)b = ((a_1 a_2 \dots a_r)\theta)b \equiv (w\theta)b, \end{aligned}$$

similarly we can see that $wc = c(w\theta)$. Let a be an arbitrary element in A, for any $n \in \mathbb{N}$ we have

$$\begin{aligned} b^n a &\equiv b^{n-1}(ba) = b^{n-1}(a\theta)b \equiv b^{n-2}b(a\theta)b \\ &\overset{\star}{=} b^{n-2}((a\theta)\theta)b^2 \equiv b^{n-2}(a\theta^2)b^2 = \dots = (a\theta^n)b \end{aligned}$$

($\star$ - $a\theta$ belongs to A^* so we can use (i)), similarly we can see that $ac^n = c^n(a\theta^n)$. ∎

Lemma 4.6 *Every word $w \in (A \cup \{b, c\})^*$ is equal, in $BR(S, \theta)$, to a word of the form $c^i \alpha b^j$, where $\alpha \in A^*$ and $i, j \in \mathbb{N}_0$.*

Proof. Let $w \in (A \cup \{b, c\})^*$. If $w = b^i c^j$ for some $i, j \in \mathbb{N}$, using the relation $bc = 1$, holding in $BR(S, \theta)$, we obtain

$$w = b^i c^j = \begin{cases} b^{i-j} & \text{if } i \geq j \\ c^{j-i} & \text{if } i < j. \end{cases}$$

If $w = \alpha c^i$ for some $\alpha \in A^*$ and $i \in \mathbb{N}_0$, by Lemma 4.5 (ii) and (iv) we can write w in the form $c^i(\alpha\theta^i)$, and $(\alpha\theta^i)$ belongs to A^*. If $w = b^i \alpha$ for some $\alpha \in A^*$ and $i \in \mathbb{N}_0$, using Lemma 4.5 (i) and (iii) we can write w in the form $(\alpha\theta^i)b^i$ where

$(\alpha\theta^i)$ belongs to A^*. We conclude that in $BR(S,\theta)$ every word can be written in the form $c^i\alpha b^j$ for some $\alpha \in A^*$ and $i,j \in \mathbb{N}_0$. ■

Proposition 4.7 *For any* $c^i\alpha b^j,\ c^k\beta b^l \in BR(S,\theta)$ *we have:*

(i) $c^i\alpha b^j \ \mathcal{R}_{BR(S,\theta)}\ c^k\beta b^l \ \Leftrightarrow\ i=k$ and $\alpha\ \mathcal{R}_S\ \beta$;

(ii) $c^i\alpha b^j \ \mathcal{L}_{BR(S,\theta)}\ c^k\beta b^l \ \Leftrightarrow\ j=l$ and $\alpha\ \mathcal{L}_S\ \beta$;

(iii) $c^i\alpha b^j \ \mathcal{H}_{BR(S,\theta)}\ c^k\beta b^l \ \Leftrightarrow\ i=k,\ j=l$ and $\alpha\ \mathcal{H}_S\ \beta$.

Proof. Suppose that $c^i\alpha b^j$ and $c^k\beta b^l$ are any two elements in $BR(S,\theta)$ that are $\mathcal{R}$ related. By Lemma 4.6, we know that we can define the Green's equivalence $\mathcal{R}$, in $BR(S,\theta)$, in the following way:

$$\begin{gathered} c^i\alpha b^j\ \mathcal{R}\ c^k\beta b^l \ \Leftrightarrow \\ \Leftrightarrow\ \exists\ c^{m_1}\alpha_1 b^{m_2},\ c^{m_3}\alpha_2 b^{m_4} \in BR(S,\theta) \quad : \\ c^i\alpha b^j c^{m_1}\alpha_1 b^{m_2} = c^k\beta b^l \quad \text{and} \quad c^k\beta b^l c^{m_3}\alpha_2 b^{m_4} = c^i\alpha b^j. \end{gathered}$$

If $j \geq m_1$ we have

$$\begin{aligned} & c^i\alpha b^j c^{m_1}\alpha_1 b^{m_2} = c^k\beta b^l \\ \Leftrightarrow\ & c^i\alpha b^{j-m_1}\alpha_1 b^{m_2} = c^k\beta b^l \\ \Leftrightarrow\ & c^i\alpha(\alpha_1\theta^{j-m_1})b^{j-m_1+m_2} = c^k\beta b^l \\ \Rightarrow\ & i=k,\ \ \alpha(\alpha_1\theta^{j-m_1}) = \beta,\ \ j-m_1+m_2 = l, \end{aligned}$$

if $m_1 > j$ we obtain

$$\begin{aligned} & c^i\alpha b^j c^{m_1}\alpha_1 b^{m_2} = c^k\beta b^l \\ \Leftrightarrow\ & c^i\alpha c^{m_1-j}\alpha_1 b^{m_2} = c^k\beta b^l \\ \Leftrightarrow\ & c^i c^{m_1-j}(\alpha\theta^{m_1-j})\alpha_1 b^{m_2} = c^k\beta b^l \\ \Rightarrow\ & m_1 = k-i+j \quad \Rightarrow \quad i > k. \end{aligned}$$

Suppose that $i > k$, from

$$c^k\beta b^l c^{m_3}\alpha_2 b^{m_4} = c^i\alpha b^j,$$

if $l \geq m_3$ we obtain

$$\begin{aligned} & c^k\beta b^{l-m_3}\alpha_2 b^{m_4} = c^i\alpha b^j \\ \Leftrightarrow\ & c^k\beta(\alpha_2\theta^{l-m_3})b^{l-m_3+m_4} = c^i\alpha b^j \\ \Rightarrow\ & i=k \end{aligned}$$

this contradicts our assumption, so we must have $l < m_3$, and in this case we obtain

$$\begin{array}{rl} & c^k\beta c^{m_3-l}\alpha_2 b^{m_4} = c^i\alpha b^j \\ \Leftrightarrow & c^{k+m_3-l}(\beta\theta^{m_3-l})\alpha_2 b^{m_4} = c^i\alpha b^j \\ \Rightarrow & i = k + m_3 - l, \end{array}$$

but $i > k$, so we must have $m_3 > l$, that is a contradiction. Hence i must be equal to k, and looking at the cases where we did not obtain a contradiction we see that we must have $j \geq m_1$, and similarly $l \geq m_3$, this implies that

$$\alpha(\alpha_1\theta^{j-m_1}) = \beta \quad \text{and} \quad \beta(\alpha_2\theta^{l-m_3}) = \alpha,$$

so we can obtain α by multiplying β by an element of A^* on the right and vice-versa, this is equivalent to say that α and β must be $\mathcal{R}$ related in S.

Conversely, consider the elements $c^j\alpha_1 b^l,\ c^l\alpha_2 b^j \in BR(S,\theta)$, where $\alpha_1, \alpha_2 \in A^*$ are such that $\alpha\alpha_1 = \beta$ and $\beta\alpha_2 = \alpha$. Then

$$\begin{array}{l} c^i\alpha b^j c^j\alpha_1 b^l = c^i\alpha\alpha_1 b^l = c^i\beta b^l, \\ c^i\beta b^l c^l\alpha_2 b^j = c^i\beta\alpha_2 b^j = c^i\alpha b^j, \end{array}$$

so $c^i\beta b^l$ is $\mathcal{R}$ related with $c^i\alpha b^j$ in $BR(S,\theta)$. We conclude that

$$c^i\alpha b^j\ \mathcal{R}_{BR(S,\theta)}\ c^k\beta b^l \ \Leftrightarrow\ i = k \text{ and } \alpha\ \mathcal{R}_S\ \beta.$$

Similarly we can see that (ii) holds, and since $\mathcal{H} = \mathcal{R} \cap \mathcal{L}$ we conclude that (iii) holds. ■

2 Bruck-Reilly Extensions of Groups

Let G be a group and θ an endomorphism in G. Considering the Bruck-Reilly extension, $BR(G,\theta)$, we can simplify some results of section 4.1 using the properties of the groups. Note that, since G is a group, $BR(G,\theta)$ is an inverse semigroup. We represent by 1 the identity of $BR(G,\theta)$ and by 1_G the identity of G. Let $< A \mid \mathfrak{R} >$ be a presentation defining G as a monoid.

Lemma 4.8 *In $BR(G,\theta)$, for all $i,j,k,l \in \mathbb{N}_0$ and $\alpha, \beta \in A^*$ we have:*
(i) $c^i \alpha b^j \;\mathcal{R}\; c^k \beta b^l \;\Leftrightarrow\; i = k;$
(ii) $c^i \alpha b^j \;\mathcal{L}\; c^k \beta b^l \;\Leftrightarrow\; j = l;$
(iii) $c^i \alpha b^j \;\mathcal{H}\; c^k \beta b^l \;\Leftrightarrow\; i = k,\; j = l.$

Proof. It follows from Lemma 4.7 and from the fact that in (a group) G we have $\mathcal{R} = \mathcal{L} = \mathcal{H} = G \times G$, see [6, Section 2.1] ■

Lemma 4.9 *There is a (unique) epimorphism, π, from $BR(G,\theta)$ onto the bicyclic monoid B, such that $b\pi = b$, $c\pi = c$ and $a\pi = 1_B$, $a \in A$, where 1_B represents the identity of B.*

Proof. In Example 2.4 we saw that the presentation

$$< \; b, \; c \mid bc = 1 \; >$$

defines B as a monoid. Define a map $\pi : A \cup \{b,c\} \longrightarrow B$ by the rules:

$$b\pi = b, \quad c\pi = c \;\; \text{and} \;\; a\pi = 1_B,$$

for any $a \in A$. The map π is obviously well-defined and we can extend it to a morphism from $BR(G,\theta)$ into B by the rule

$$(x_1 x_2 \ldots x_r)\pi = x_1\pi x_2\pi \ldots x_r\pi, \qquad x_i \in A \cup \{b,c\}, \;\; i = 1, \ldots, r.$$

Let $w \in B$ be arbitrary, noticing that the bicyclic monoid can be seen as the Bruck-Reilly extension of the trivial group, we can write $w = c^k b^l$ for some $k, l \in \mathbb{N}_0$. Then

$$w = c^k b^l = (c\pi)^k (b\pi)^l = (c^k)\pi(b^l)\pi = (c^k b^l)\pi = (c^k b^l)\pi$$

and $c^k b^l$ belongs to $BR(G,\theta)$, hence π is onto. Since we defined π over the generators of $BR(G,\theta)$, we can conclude that π is the unique epimorphism from $BR(G,\theta)$ onto the bicyclic monoid. ■

Lemma 4.10 *G is isomorphic to the group of units of $BR(G,\theta)$. A word w in $(A \cup \{b,c\})^*$ represents an element of the group of units of $BR(G,\theta)$ if and only if it is equal to some word from A^*, i.e. if and only if $w\pi = 1_B$, where π is defined above.*

PROOF. Let $c^i\alpha b^l$ be a unit in $BR(G,\theta)$, there exists $c^j\beta b^k \in BR(G,\theta)$ such that $c^i\alpha b^l c^j\beta b^k = 1$. If $l \geq j$ this implies

$$\begin{array}{rl} & c^i\alpha b^{l-j}\beta b^k = 1 \quad \Leftrightarrow \quad c^i\alpha(\beta\theta^{l-j})b^{l-j+k} = 1 \\ \Rightarrow & i = 0, \quad \alpha(\beta\theta^{l-j}) = 1_G, \quad j = l+k, \end{array}$$

if $j > l$ we obtain

$$c^i\alpha c^{j-l}\beta b^k = 1 \quad \Leftrightarrow \quad c^i c^{j-l}(\alpha\theta^{j-l})\beta b^k = 1 \quad \Rightarrow \quad i + j = l,$$

so in this case we have a contradiction. Hence $i = 0$ and $j = l + k$, then

$$\begin{array}{rl} & c^i\alpha b^l c^j\beta b^k = 1 \quad \Leftrightarrow \quad \alpha b^l c^{l+k}\beta b^k = 1 \\ \Leftrightarrow & \alpha c^k\beta b^k = 1 \quad \Leftrightarrow \quad c^k(\alpha\theta^k)\beta b^k = 1 \quad \Rightarrow \quad k = 0. \end{array}$$

Thus w is a unit if and only if w is of the form $c^0\alpha b^0$ with $\alpha \in A^*$, i.e. if and only if w belongs to A^*. Since A generates G we conclude that the map $U(BR(G,\theta)) \longrightarrow G, \ \alpha \mapsto \alpha,$ is an isomorphism. ∎

To the words in $(A \cup \{b,c\})^*$ which represent elements of G we will call *group words*. We are now able to follow the proof of the next result, given in [2].

Proposition 4.11 *$BR(G,\theta)$ is finitely generated if and only if there exists a finite subset, A_0, of G such that G is generated, as a monoid, by the set $\bigcup_{i\geq 0} A_0\theta^i$.*

PROOF. Let A_0 be a finite subset of G such that the presentation

$$< \bigcup_{i\geq 0} A_0\theta^i \mid \mathfrak{R} >$$

defines G as a monoid. By Proposition 4.2, we know that $BR(G,\theta)$ is defined by the monoid presentation

$$< \bigcup_{i\geq 0} A_0\theta^i,\ b,\ c \ \ | \ \ \mathfrak{R},\ \ bc=1,\ \ ba=(a\theta)b,$$
$$ac=c(a\theta), \qquad (a\in \bigcup_{i\geq 0} A_0\theta^i) \ > .$$

Let $a \in A_0$ be arbitrary, note that $a = a\theta^0$, so $a \in \bigcup_{i\geq 0} A_0\theta^i$. For any $i \in \mathbb{N}_0$ we have

$$\begin{array}{lll} & (a\theta^i)b^i = b^i a & \text{(Lemma 4.5)} \\ \Rightarrow & (a\theta^i)b^i c^i = b^i a c^i & \\ \Leftrightarrow & a\theta^i = b^i a c^i & (bc=1) \end{array}$$

it follows that $\bigcup_{i\geq 0} A_0\theta^i \subseteq (A_0 \cup \{b,c\})^*$. Thus $A_0 \cup \{b,c\}$ generates $BR(G,\theta)$, so this monoid is finitely generated.

Conversely, suppose that $BR(G,\theta)$ is finitely generated. We know that if A generates G, then $A \cup \{b,c\}$ generates $BR(G,\theta)$, so there exists a finite subset A_0 of A such that $A_0 \cup \{b,c\}$ generates $BR(G,\theta)$. Let U be the group of units of $BR(G,\theta)$, clearly the identity of U is the identity of $BR(G,\theta)$. Suppose that T is a subgroup of $BR(G,\theta)$ that contains U, then $1 \in T$ and we obviously have

$$1g = g = g1,$$

for all $g \in T$, so 1 is the identity of T. Then all elements of T are units of $BR(G,\theta)$ and we must have $U = T$. Hence U is maximal. By Lemma 4.10, we know that $G \cong U$ so G is a maximal subgroup of $BR(G,\theta)$, then, by Proposition 2.13, the cosets of G in $BR(G,\theta)$ are the $\mathcal{H}$-classes in the $\mathcal{R}$-class of G. By Lemma 4.8, we can see that the $\mathcal{R}$-class of G is the set $Gb^* = \{\alpha b^i : i \geq 0,\ \ \alpha \in G\}$ and that the $\mathcal{H}$-classes in this $\mathcal{R}$-class are the sets $H_i = Gb^{i-1},\ \ i \geq 1$. Proposition 2.14 give us the following generating set for G:

$$Y = \{1_G r_i a r_{ia}^{-1} : i \in I,\ a \in A_0 \cup \{b,c\},\ ia \in I\}$$

where $r_i,\ r_i^{-1},\ \ i \in I$, is a system of coset representatives. We have

$$H_i c^{i-1} = Gb^{i-1}c^{i-1} = G, \qquad i \geq 1,$$

so we can take $r_i = b^{i-1}, \;\; r_i^{-1} = c^{i-1}$, to be a system of coset representatives. For any $a \in A_0$ we have

$$Gba = G(a\theta)b = Gb, \qquad (a\theta \in G)$$

so $r_{ia} = r_i$ for $a \in A_0$, and

$$\begin{array}{c} Gb^ib = Gb^{i+1} \quad \Rightarrow \quad r_{ib} = r_ib \quad \Rightarrow \quad r_{ib}^{-1} = b^{-1}r_i^{-1} = cr_i^{-1}, \\ Gb^ic = Gb^{i-1} \quad \Rightarrow \quad r_{ic} = r_{i-1}, \end{array}$$

then our generating set becomes

$$\begin{aligned} &\{1_G b^i a r_{ia}^{-1} : i \geq 0, \; a \in A_0 \cup \{b, c\}\} \\ =& \{1_G b^i ac^i, \;\; 1_G b^i bcc^i, \;\; 1_G b^{i+1} cb^i \; : \; i \geq 0, \; a \in A_0\} \\ =& \{1_G (b^i ac^i), \;\; 1_G 1 \; : \; i \geq 0, \; a \in A_0\} \\ =& \{b^i ac^i \; : \; i \geq 0, \; a \in A_0\} \cup \{1_G\}, \end{aligned}$$

note that $b^iac^i = a\theta^i, \;\; i \geq 0$, so $b^iac^i \in G$ for any $i \geq 0$. Thus, the set

$$\{b^iac^i \; : \; i \geq 0, \; a \in A_0\} = \{a\theta^i \; : \; i \geq 0, \; a \in A_0\} = \bigcup_{i \geq 0} A_0\theta^i$$

generates G. ■

Proposition 4.12 *If $BR(G, \theta)$ is finitely presented then G is finitely generated.*

Theorem 4.13 *$BR(G, \theta)$ is finitely presented if and only if G can be defined by a presentation $< A \mid \mathfrak{R} >$, where A is finite and*

$$\mathfrak{R} = \bigcup_{k \geq 0} \overline{\mathfrak{R}}\theta^k = \{u\theta^k = v\theta^k : k \geq 0, \;\; (u = v) \in \overline{\mathfrak{R}}\}$$

for some finite set of relations $\overline{\mathfrak{R}} \subseteq A^* \times A^*$.

The proofs of these last two results can be found in [2]. Except for Proposition 4.12, these results were generalized for monoids in [1], we will now state these results.

Proposition 4.14 *Let M be a monoid, $\sigma : M \longrightarrow U(M)$ a morphism. The Bruck-Reilly extension $BR(M,\sigma)$ is finitely generated if and only if there exists a finite subset A_0 of M, such that M is generated by the set $A = \bigcup_{k\geq 0} A_0\sigma^k$.*

Proposition 4.15 *Let M be a monoid and $\sigma : M \longrightarrow U(M)$ a morphism. If $BR(M,\sigma)$ is finitely presented and M is generated by a set A, then M is defined by the presentation $< A \mid \mathfrak{R} >$ where $\mathfrak{R} = \bigcup_{k\geq 0} \overline{\mathfrak{R}}\sigma^k$, for some finite set of relations $\overline{\mathfrak{R}}$.*

Proposition 4.16 *Let M be a finitely generated monoid defined by a presentation $< A \mid \mathfrak{R} >$ where A is finite and $\mathfrak{R} = \bigcup_{k\geq 0} \overline{\mathfrak{R}}\sigma^k$, for some finite set of relations $\overline{\mathfrak{R}}$. Then $BR(M,\sigma)$ is finitely presented.*

Finally, we relate finite presentability as an inverse monoid with finite presentability as a monoid, in Bruck-Reilly extensions of groups. We will follow the proof given in [2].

Theorem 4.17 *Let $S = BR(G,\theta)$ be a Bruck-Reilly extension of a group G. Then S is finitely presented as an inverse monoid if and only if S is finitely presented as a monoid.*

Proof. $S = BR(G,\theta)$ is an inverse monoid, then a monoid presentation for S also defines it when considered as an inverse monoid presentation. Hence, if S is finitely presented as a monoid it is also finitely presented as an inverse monoid.

Conversely, suppose that S is finitely presented as an inverse monoid. Let $< A' \mid \mathfrak{R} >$ be a monoid presentation for G, by Proposition 4.2, we have

$$S \cong < A',\ b,\ c \mid \mathfrak{R},\ bc = 1,\ ba = (a\theta)b,\ ac = c(a\theta),\ (a \in A') > .$$

So S admits an inverse monoid presentation

$$< A,\ b,\ c \mid \mathfrak{T}' >,$$

for some finite set $A \subseteq A'$ and some finite set of defining relations $\mathfrak{T}'$. The relation $bc = 1$ holds in S so c is the inverse of b in S. Since our presentation for S is an inverse monoid presentation, applying Tietze Transformations (T4) (removing the generator c, substituting all occurrences of c in $\mathfrak{T}'$ by b^{-1}), we obtain the inverse monoid presentation for S

$$< A,\ b \mid \mathfrak{T} >$$

where, obviously, $\mathfrak{T}$ is a finite set. We know that the following relations hold in S:

$$bb^{-1} = 1, \qquad aa^{-1} = a^{-1}a = 1, \quad ba = (a\theta)b, \quad ab^{-1} = b^{-1}(a\theta), \qquad (a \in A),$$

so, applying Tietze Transformations (T1), we obtain the inverse monoid presentation for S

$$\begin{array}{rl} < A,\ b \mid & \mathfrak{T},\ aa^{-1} = a^{-1}a = 1,\ bb^{-1} = 1, \\ & ba = (a\theta)b,\ ab^{-1} = b^{-1}(a\theta),\ (a \in A) >, \end{array}$$

and, by Remark 3, we know that a monoid presentation for S is

$$\begin{array}{rl} < A,\ A^{-1},\ b,\ b^{-1} \mid & \mathfrak{T},\ aa^{-1} = a^{-1}a = 1,\ bb^{-1} = 1,\ ba = (a\theta)b, \\ & ab^{-1} = b^{-1}(a\theta),\ ww^{-1}w = w,\ ww^{-1}zz^{-1} = zz^{-1}ww^{-1}, \\ & \qquad (a \in A,\ w, z \in (A \cup A^{-1} \cup \{b, b^{-1}\})^*) > . \end{array}$$

Note that from the relations $aa^{-1} = a^{-1}a = 1$, for all $a \in A$, we can deduced the relations $\alpha\alpha^{-1} = \alpha^{-1}\alpha = 1$, for all $\alpha \in (A \cup A^{-1})^*$. We have seen that for any w in $(A \cup A^{-1} \cup \{b, b^{-1}\})^*$, we have $w = (b^{-1})^i \alpha b^j$, for some $i, j \geq 0$, and $\alpha \in (A \cup A^{-1})^*$, as a consequence of the relations

$$bb^{-1} = 1, \quad ba = (a\theta)b, \quad ab^{-1} = b^{-1}(a\theta), \quad (a \in A).$$

Then

$$\begin{aligned} ww^{-1} &= b^{-i}\alpha b^j (b^{-i}\alpha b^j)^{-1} \\ &= b^{-i}\alpha b^{-j}\alpha^{-1} b^i \\ &= b^{-i}\alpha\alpha^{-1} b^i & (bb^{-1} = 1) \\ &= b^{-i} b^i, & (aa^{-1} = a^{-1}a = 1,\ \forall a \in A) \end{aligned}$$

it follows that

$$\begin{aligned} ww^{-1}w &= b^{-i} b^i b^{-i} \alpha b^j \\ &= b^{-i}\alpha b^j & (bb^{-1} = 1) \\ &= w. \end{aligned}$$

Similarly, for $z \in (A \cup A^{-1} \cup \{b, b^{-1})^*$, we have $zz^{-1} = b^{-k}b^k$ for some $k \geq 0$, so we obtain

$$ww^{-1}zz^{-1} = b^{-i}b^ib^{-k}b^k = \begin{cases} b^{-i}b^{i-k}b^k = b^{-i}b^{i-k+k} = b^{-i}b^i & \text{if } i \geq k \\ b^{-i}b^{-k+i}b^k = b^{-i-k+i}b^k = b^{-k}b^k & \text{if } i < k \end{cases}$$

$$zz^{-1}ww^{-1} = b^{-k}b^kb^{-i}b^i = \begin{cases} b^{-k}b^{-i+k}b^i = b^{-k-i+k}b^i = b^{-i}b^i & \text{if } i \geq k \\ b^{-k}b^{k-i}b^i = b^{-k}b^{k-i+i} = b^{-k}b^k & \text{if } i < k, \end{cases}$$

hence $ww^{-1}zz^{-1} = zz^{-1}ww^{-1}$. This shows that the relations

$$ww^{-1}w = w, \quad ww^{-1}zz^{-1} = zz^{-1}ww^{-1}, \quad (w, z \in (A \cup A^{-1} \cup \{b, b^{-1}\})^*),$$

can be deduced from the relations

$$aa^{-1} = a^{-1}a = 1, \quad bb^{-1} = 1, \quad ba = (a\theta)b, \quad ab^{-1} = b^{-1}(a\theta), \quad (a \in A),$$

so, applying Tietze Transformations (T2), we know that the presentation

$$\begin{array}{r} < A,\ A^{-1},\ b,\ b^{-1} \mid \ \mathfrak{T},\ aa^{-1} = a^{-1}a = 1,\ bb^{-1} = 1,\ ba = (a\theta)b, \\ ab^{-1} = b^{-1}(a\theta),\ (a \in A) > \end{array}$$

defines S as a monoid. Since A and $\mathfrak{T}$ are finite, we conclude that S is finitely presented as a monoid. ■

3 Some Results on Clifford Semigroups

Let Y be a semilattice, i.e. a commutative semigroup of idempotents, and $\{G_\alpha : \alpha \in Y\}$ a set of groups indexed by Y, such that $G_\alpha \cap G_\beta = \emptyset$ for $\alpha \neq \beta$. We will represent by 1_α the identity of the group G_α, $\alpha \in Y$. Suppose that for all $\alpha \geq \beta$ in Y, where

$$\alpha \geq \beta \quad \Leftrightarrow \quad \beta = \beta\alpha = \alpha\beta,$$

there exists a morphism $\phi_{\alpha,\beta} : G_\alpha \longrightarrow G_\beta$ such that

$$\begin{array}{ll} \forall \alpha \in Y & \phi_{\alpha,\alpha} = id_{G_\alpha}, \\ \forall \alpha, \beta, \gamma \in Y, \quad \alpha \geq \beta \geq \gamma, & \phi_{\alpha,\beta}\phi_{\beta,\gamma} = \phi_{\alpha,\gamma}, \end{array}$$

where id_{G_α} is the identity map in G_α. Define a multiplication in $S = \bigcup_{\alpha \in Y} G_\alpha$ by the rule

$$xy = (x\phi_{\alpha,\alpha\beta})(y\phi_{\beta,\alpha\beta}), \qquad x \in G_\alpha, \ y \in G_\beta,$$

for any $\alpha, \beta \in Y$. This multiplication is associative, see [6, Section 4.1]. We will denote this semigroup by $\mathcal{S}(Y, G_\alpha, \phi_{\alpha,\beta})$ and say that it is a *strong semilattice of groups.*

A semigroup S is called a *Clifford Semigroup* if there exists a unary operation $x \mapsto x^{-1}$ on S, with the properties:

$$\begin{aligned} \forall x, y \in S \quad (x^{-1})^{-1} = x, \quad xx^{-1}x = x, \quad xx^{-1} = x^{-1}x, \\ (xx^{-1})(yy^{-1}) = (yy^{-1})(xx^{-1}). \end{aligned}$$

Proposition 4.18 *Let S be a semigroup. The following statements are equivalent:*

(i) $S\ is\ a\ Clifford\ semigroup$;
(ii) $S\ is\ a\ strong\ semilattice\ of\ groups$;
(iii) $S\ is\ regular\ and\ its\ idempotents\ commute\ with\ all\ elements\ in\ S$;
(iv) $S\ is\ regular\ and\ each\ \mathcal{D}-class\ of\ S\ contains\ exactly\ one\ idempotent.$

For a proof see for example [6, Theorem 4.2.1]. From (iii) we know that a Clifford semigroup is an inverse semigroup.

Proposition 4.19 *Let $S = \mathcal{S}(Y, G_\alpha, \phi_{\alpha,\beta})$ be a Clifford monoid. The group of units of S is G_e, where e is the identity of the semilattice Y.*

Proof. Let a be a unit of S, there exists $b \in S$ such that $ab = 1$. We know that $a \in G_\alpha$ and $b \in G_\beta$, for some $\alpha, \beta \in Y$, then $ab \in G_{\alpha\beta}$, by the multiplication defined in S, so $1 \in G_{\alpha\beta}$, hence 1 must be the identity of the group $G_{\alpha\beta}$. Let γ be an arbitrary element of Y and $x \in G_\gamma$ arbitrary. We know that $x1 = x$, but $G_\gamma G_{\alpha\beta} \subseteq G_{\gamma\alpha\beta}$, so

$$x \in G_\gamma, \ \ x \in G_{\gamma\alpha\beta} \quad \Rightarrow \quad G_\gamma \cap G_{\gamma\alpha\beta} \neq \emptyset \quad \Leftrightarrow \quad \gamma = \gamma(\alpha\beta),$$

similarly we obtain $(\alpha\beta)\gamma = \gamma$, so $(\alpha\beta)$ is the identity of Y. It follows that

$$\alpha(\alpha\beta) = \alpha \quad \Leftrightarrow \quad (\alpha\alpha)\beta = \alpha \quad \Leftrightarrow \alpha\beta = \alpha,$$

so a belongs to $G_{\alpha\beta}$, hence the group of units is contained in $G_{\alpha\beta}$, and

$$\forall y \in G_{\alpha\beta} \;\; \exists y^{-1} \in G_{\alpha\beta} \;:\; yy^{-1} = y^{-1} = 1_{\alpha\beta} = 1,$$

so all elements of $G_{\alpha\beta}$ are units of S. We conclude that $G_{\alpha\beta}$ is the group of units of S, where $\alpha\beta$ is the identity of Y. ■

Proposition 4.20 *Let $S = \mathcal{S}(Y, G_\alpha, \phi_{\alpha,\beta})$ be a Clifford semigroup. In S we have $\mathcal{H} = \mathcal{L} = \mathcal{R} = \mathcal{D}$. Moreover the $\mathcal{D}$-classes of S are the groups G_α, $\alpha \in Y$.*

Proof. Let D_x represent the $\mathcal{D}$-class of x in S. By Proposition 4.18 (*iv*), we know that the $\mathcal{D}$-classes of S are the sets D_e, with $e = ee \in S$. Let D_e be any $\mathcal{D}$-class in S and x an arbitrary element of D_e. There exists $z \in S$ such that $x \;\mathcal{R}\; z \;\mathcal{L}\; e$, then

$$\begin{array}{rcll} x \;\mathcal{R}\; z & \Leftrightarrow & xx^{-1} = zz^{-1} & (S \text{ inverse}) \\ & \Leftrightarrow & xx^{-1} = z^{-1}z, & (S \text{ Clifford}) \\ z \;\mathcal{L}\; e & \Leftrightarrow & z^{-1}z = e^{-1}e & \\ & \Leftrightarrow & z^{-1}z = ee^{-1}, & \end{array}$$

so $x\mathcal{R}e$ and, similarly, $x\mathcal{L}e$, it follows that $x\mathcal{H}e$. Then $D_e \subseteq H_e$, where H_e represents the $\mathcal{H}$-class of e, and we may conclude that

$$\mathcal{D} = \mathcal{R} = \mathcal{L} = \mathcal{H}$$

in S. Let $\alpha \in Y$ arbitrary, and $x, y \in G_\alpha$. We have

$$x = x1_\alpha = x(yy^{-1}), \qquad xy = x(y),$$

so $xy\mathcal{R}x$ and similarly we can see that $xy\mathcal{L}y$, then $x\mathcal{D}y$. It follows, by what we have just seen, that $x\mathcal{H}y$. Thus, for any $x \in G_\alpha$ we have $G_\alpha \subseteq H_x$. Now consider $x, y \in S$ such that $x\mathcal{H}y$, and suppose that $x \in G_\beta$ and $y \in G_\gamma$ for some $\beta, \gamma \in Y$ with $\beta \neq \gamma$. From $\mathcal{H} = \mathcal{R}$ we obtain

$$x \; \mathcal{H} \; y \quad \Leftrightarrow \quad xx^{-1} = yy^{-1},$$

then

$$\begin{aligned}(xy)(xy)^{-1} &= (xy)(y^{-1}x^{-1}) = x(yy^{-1})x^{-1} = x(xx^{-1})x^{-1} \\ &= x(x^{-1}x)x^{-1} = (xx^{-1})(xx^{-1}) = xx^{-1}\end{aligned}$$

so $xy\mathcal{R}x$, this is equivalent to $xy\mathcal{H}x$. We know that $xy \in G_{\beta\gamma}$, and, by what we have seen above, we have $G_{\beta\gamma} \subseteq H_{xy}$ and $G_\beta \subseteq H_x$. From $H_{xy} = H_x$, we obtain $G_{\beta\gamma}, G_\beta \subseteq H_x = D_x$, but each $\mathcal{D}$-class contains exactly one idempotent, so

$$1_{\beta\gamma} = 1_\beta \quad \Rightarrow \quad G_{\beta\gamma} \cap G_\beta \neq \emptyset \quad \Leftrightarrow \quad \beta\gamma = \beta.$$

Similarly we can see that $\beta\gamma = \gamma$, hence $\gamma = \beta$, that contradicts our assumption. We conclude that the groups G_α, $\alpha \in Y$, are the $\mathcal{D}$-classes of S. ■

We can obtain a presentation for the Clifford semigroup $S = \mathcal{S}(Y, G_\alpha; \phi_{\alpha,\beta})$ in terms of the presentations for the groups G_α, $\alpha \in Y$, in the following way:

Proposition 4.21 *Suppose that the group G_α, $\alpha \in Y$, is defined by the semigroup presentation $< A_\alpha \mid \mathfrak{R}_\alpha >$, with $A_\alpha \cap A_\beta = \emptyset$ for $\alpha \neq \beta$. Let*

$$A = \bigcup_{\alpha \in Y} A_\alpha, \qquad \mathfrak{R} = \bigcup_{\alpha \in Y} \mathfrak{R}_\alpha,$$

and $1_\alpha \in A_\alpha^*$ be a word representing the identity of G_α. Then

$$\begin{aligned}< \; A \;\; \mid \;\; \mathfrak{R}, \quad 1_\alpha 1_\beta = 1_\beta 1_\alpha, \quad 1_\gamma a = a 1_\gamma = a\phi_{\sigma,\gamma} \\ (\alpha, \beta, \gamma, \sigma \in Y, \quad \alpha \neq \beta, \quad \sigma > \gamma, \quad a \in A_\sigma) \;\; >\end{aligned}$$

is a presentation for the Clifford semigroup S.

For a proof see [7, Theorem 5.1]. If S is a Clifford monoid, then $1_\xi = 1$ for some $\xi \in Y$. So, to obtain a monoid presentation for S, in terms of the presentations of the groups G_α, $\alpha \in Y$, we just need to add the relation $1_\xi = 1$ to the presentation given in this last result.

Note: From the presentation given above, we can see that if Y is finite and, for all $\alpha \in Y$, the group G_α is finitely presented (generated) then S is finitely presented (generated). The next result shows that this is a necessary and sufficient condition.

Theorem 4.22 *If the Clifford monoid S is finitely presented (generated), then every group G_α, $\alpha \in Y$, is finitely presented (generated).*

PROOF. Suppose that S is finitely presented (generated). Since $S = \bigcup_{\alpha\in Y} G_\alpha$ and $G_\alpha \cap G_\beta = \emptyset$ for $\alpha \neq \beta$, the group G_α is a maximal subgroup of S for all $\alpha \in Y$. Then, by Proposition 2.13, the index of G_α in S equals the number of $\mathcal{H}$-classes in the $\mathcal{R}$-class of G_α, $\alpha \in Y$. But, in Proposition 4.19, we have seen that G_α is a $\mathcal{D}$-class and an $\mathcal{H}$-class of S, so there is exactly one $\mathcal{H}$-class in the $\mathcal{R}$-class of G_α, then G_α has index one, for all $\alpha \in Y$. It follows, from Theorem 3.8 (3.7), that G_α is finitely presented (generated), for all $\alpha \in Y$. ■

An alternative proof of this result can be found in [3, Theorem 6.1]. Note that if S is finitely generated then Y must be finite, see [6, Theorem 4.5.3].

4 Bruck-Reilly Extensions of Clifford Monoids

4.1 Properties

Given a Clifford monoid $S = \mathcal{S}(Y, G_\alpha, \phi_{\alpha,\beta})$, let e be the identity of the semilattice Y, and θ a morphism from S into G_e. Considering the Bruck-Reilly extension $BR(S,\theta)$, since S is inverse, we know that $BR(S,\theta)$ is an inverse monoid. Let

$(m,a,n),(p,b,q) \in \mathbb{N}_0 \times S \times \mathbb{N}_0$ arbitrary, by Proposition 4.1, we know that

$$(m,a,n)\ \mathcal{D}_{BR(S,\theta)}\ (p,b,q) \quad \Leftrightarrow \quad a\ \mathcal{D}_S\ b,$$

and we have seen that a is $\mathcal{D}$ related to b in S if and only if they belong to the same group G_α, for some $\alpha \in Y$. So, the $\mathcal{D}$-classes of $BR(S,\theta)$ are the sets $\mathbb{N}_0 \times G_\alpha \times \mathbb{N}_0$ with $\alpha \in Y$.

Note that these $\mathcal{D}$-classes are not groups, since, given $x = xx \in S$ and $m,n \in \mathbb{N}_0$ with $m \neq n$, the triples (m,x,m) and (n,x,n) are two, different, idempotents in $BR(S,\theta)$ that belong to the same $\mathcal{D}$-class.

We will now see that the result in Theorem 4.17, also holds for Bruck-Reilly extensions of Clifford monoids.

Theorem 4.23 Let S be a Clifford monoid, θ a morphism from S into $U(S)$ and $BR(S,\theta)$ the Bruck-Reilly extension of S. Then $BR(S,\theta)$ is finitely presented as an inverse monoid if and only if it is finitely presented as a monoid.

Proof. A monoid presentation for $BR(S,\theta)$ also defines it when considered as an inverse monoid presentation, so, if $BR(S,\theta)$ is finitely presented as a monoid it is finitely presented as an inverse monoid.

Conversely, suppose that $BR(S,\theta)$ is finitely presented as an inverse monoid. Given a monoid presentation, $< Q \mid \mathfrak{R} >$, for S, by Theorem 4.2, $BR(S,\theta)$ is defined by the monoid presentation

$$< Q,\ b,\ c \mid \mathfrak{R},\ bc = 1,\ ba = (a\theta)b,\ ac = c(a\theta),\ (a \in Q) > .$$

Since $BR(S,\theta)$ is finitely presented as an inverse monoid it admits an inverse monoid presentation

$$< A,\ b,\ c \mid \mathfrak{T} >$$

for some finite set $A \subseteq S$ and some finite set of defining relations $\mathfrak{T}$. We know that $bc = 1$ in $BR(S,\theta)$ (by the presentation given above), so $bcb = b$ and $cbc = c$, i.e. c is the inverse of b in $BR(S,\theta)$, hence, applying Tietze Transformations (T4) we know that the presentation

$$< A,\ b \mid \mathfrak{T}' >$$

where $\mathfrak{T}'$ is the set $\mathfrak{T}$ with the occurrences of c substituted by b^{-1}, defines $BR(S,\theta)$ as an inverse monoid. By the first presentation given for $BR(S,\theta)$, we know that the following relations hold in $BR(S,\theta)$:

$$bb^{-1} = 1, \quad ba = (a\theta)b, \quad ab^{-1} = b^{-1}(a\theta), \quad (a \in A),$$

so we can add them to the presentation of $BR(S,\theta)$. $A \subseteq S$ and S is a Clifford semigroup so for any $a_1, a_2 \in A$ we have

$$a_1(a_2a_2^{-1}) = (a_2a_2^{-1})a_1, \qquad a_1a_1^{-1} = a_1^{-1}a_1, \qquad a_1 = a_1a_1^{-1}a_1,$$

adding these relations to the presentation of $BR(S,\theta)$ we obtain the following presentation for it:

$$\begin{aligned} < A,\ b \mid \ \ &\mathfrak{T}',\ bb^{-1} = 1,\ aa^{-1} = a^{-1}a,\ ba = (a\theta)b,\ ab^{-1} = b^{-1}(a\theta), \\ &a = aa^{-1}a,\ \ a(a_1a_1^{-1}) = (a_1a_1^{-1})a,\ \ (a, a_1 \in A) > . \end{aligned}$$

We have been applying Tietze Transformations (T1) and these operations do not change the type of structure defined by the presentation, so this presentation still defines $BR(S,\theta)$ as an inverse monoid. Now, by Remark 3, we know that a monoid presentation for $BR(S,\theta)$ is

$$\begin{aligned} < A,\ A^{-1}, b,\ b^{-1} \ \ \mid \ \ &\mathfrak{T}',\ bb^{-1} = 1, && (4.1)\\ &ba = (a\theta)b, \quad ab^{-1} = b^{-1}(a\theta), && (4.2)\\ &aa^{-1} = a^{-1}a, && (4.3)\\ &a = aa^{-1}a, && (4.4)\\ &a(a_1a_1^{-1}) = (a_1a_1^{-1})a, && (4.5)\\ &w = ww^{-1}w, \quad ww^{-1}zz^{-1} = zz^{-1}ww^{-1}, && (4.6)\\ &\quad (a, a_1 \in A),\ \ w, z \in (A \cup A^{-1} \cup \{b, b^{-1}\})^*) > . \end{aligned}$$

Note that, for any $a, a_1 \in A$

$$\begin{aligned} a^{-1}b^{-1} &= (ba)^{-1} && (BR(S,\theta) \text{ inverse})\\ &= ((a\theta)b)^{-1} && (4.2)\\ &= b^{-1}(a\theta)^{-1} && (BR(S,\theta) \text{ inverse})\\ &= b^{-1}(a^{-1}\theta), && (\theta \text{ morphism}) \end{aligned}$$

$$ba^{-1} = (ab^{-1})^{-1} \underset{(4.2)}{=} (b^{-1}(a\theta))^{-1} = (a^{-1}\theta)b^{-1},$$

$$a^{-1}(a_1a_1^{-1}) = (a_1a_1^{-1}a)^{-1} \underset{(4.5)}{=} (aa_1a_1^{-1})^{-1} = (a_1a_1^{-1})a^{-1},$$

this last relation implies, from (4.3), that $a^{-1}(a_1^{-1}a_1) = (a_1a_1^{-1})a^{-1}$, we also have

$$a^{-1} \underset{(4.4)}{=} (aa^{-1}a)^{-1} = a^{-1}aa^{-1}.$$

So, the relations

$$\begin{array}{lll} xb^{-1} = b^{-1}(x\theta), & bx = (x\theta)b, & \\ x(yy^{-1}) = (yy^{-1})x, & x = xx^{-1}x, & (x, y \in A \cup A^{-1}), \end{array}$$

are a consequence of (4.2), (4.5), (4.3), (4.4) and (4.6). Then, like in Theorem 4.17, given $w \in (A \cup A^{-1} \cup \{b, b^{-1}\})^*$, the relations (4.1) and (4.2) imply that $w = b^{-i}a_1a_2 \dots a_kb^j$, for some $i, j \geq 0$ and some $a_1, a_2, \dots, a_k \in A \cup A^{-1}$ and we get

$$\begin{array}{rll} ww^{-1} & = b^{-i}a_1a_2 \dots a_kb^j(b^{-i}a_1a_2 \dots a_kb^j)^{-1} & \\ & = b^{-i}a_1a_2 \dots a_kb^jb^{-j}a_k^{-1} \dots a_2^{-1}a_1^{-1}b^i & \\ & = b^{-i}a_1a_2 \dots a_ka_k^{-1} \dots a_2^{-1}a_1^{-1}b^i & (4.1) \\ & = b^{-i}(a_ka_k^{-1})a_1a_2 \dots a_{k-1}a_{k-1}^{-1} \dots a_2^{-1}a_1^{-1}b^i & (4.5) \\ & \dots & \\ & = b^{-i}(a_ka_k^{-1}) \dots (a_1a_1^{-1})b^i & \end{array}$$

then

$$\begin{array}{rll} ww^{-1}w & = b^{-i}(a_ka_k^{-1}) \dots (a_1a_1^{-1})b^ib^{-i}a_1a_2 \dots a_kb^j & \\ & = b^{-i}(a_ka_k^{-1}) \dots a_1a_1^{-1}a_1a_2 \dots a_kb^j & (4.1) \\ & = b^{-i}(a_ka_k^{-1}) \dots (a_2a_2^{-1})a_1a_2 \dots a_kb^j & (4.4) \\ & = b^{-i}a_1(a_ka_k^{-1}) \dots (a_2a_2^{-1})a_2 \dots a_kb^j & (4.5) \\ & = b^{-i}a_1(a_ka_k^{-1}) \dots (a_3a_3^{-1})a_2a_3 \dots a_kb^j & (4.4) \\ & \dots & \\ & = b^{-i}a_1a_2 \dots a_kb^j & \\ & = w. & \end{array}$$

Thus, the relation $w = ww^{-1}w, \quad w \in (A \cup A^{-1} \cup \{b, b^{-1}\})^*$ is a consequence of the relations (4.1) to (4.5), so we can remove it from the presentation. Given $z \in (A \cup A^{-1} \cup \{b, b^{-1}\})^*$, we have $z = b^pa'_1a'_2 \dots a'_sb^l$ for some $p, l \geq 0$ and some $a'_1, a'_2, \dots, a'_s \in A \cup A^{-1}$, then writing zz^{-1} in the same form we wrote ww^{-1}, we obtain

$$ww^{-1}zz^{-1} = b^{-i}(a_ka_k^{-1}) \dots (a_1a_1^{-1})b^ib^{-p}(a'_sa'^{-1}_s) \dots (a'_1a'^{-1}_1)b^p,$$

if $i > p$ this becomes

$$\begin{aligned} ww^{-1}zz^{-1} &= b^{-i}(a_k a_k^{-1})\dots(a_1 a_1^{-1})b^{i-p}(a'_s {a'_s}^{-1})\dots(a'_1 {a'_1}^{-1})b^p \\ &= b^{-i}(a_k a_k^{-1})\dots(a_1 a_1^{-1})((a'_s {a'_s}^{-1})\dots(a'_1 {a'_1}^{-1}))\theta^{i-p}b^{i-p}b^p \quad (4.2) \\ &= b^{-i}(a_k a_k^{-1})\dots(a_1 a_1^{-1})(a'_s {a'_s}^{-1})\theta^{i-p}\dots(a'_1 {a'_1}^{-1})\theta^{i-p}b^i, \end{aligned}$$

since $i > p$ and $a'_t {a'_t}^{-1}$ is an idempotent for any $t = 1, \dots, s$, the morphism θ^{i-p} maps this element to the identity of S, so

$$ww^{-1}zz^{-1} = b^{-1}((a_k a_k^{-1})\dots(a_1 a_1^{-1})1)b^i = b^{-1}(a_k a_k^{-1})\dots(a_1 a_1^{-1})b^i = ww^{-1},$$

similarly we have

$$\begin{aligned} zz^{-1}ww^{-1} &= b^{-p}(a'_s {a'_s}^{-1})\dots(a'_1 {a'_1}^{-1})b^p b^{-i}(a_k a_k^{-1})\dots(a_1 a_1^{-1})b^i \\ &= b^{-p}(a'_s {a'_s}^{-1})\dots(a'_1 {a'_1}^{-1})b^{-i+k}(a_k a_k^{-1})\dots(a_1 a_1^{-1})b^i \\ &= b^{-p}b^{-i+p}((a'_s {a'_s}^{-1})\dots(a'_1 {a'_1}^{-1}))\theta^{-i+p}(a_k a_k^{-1})\dots(a_1 a_1^{-1})b^i \quad (4.2) \\ &\quad \dots \\ &= b^{-i}(a_k a_k^{-1})\dots(a_1 a_1^{-1})b^i \\ &= ww^{-1}. \end{aligned}$$

If $p > i$, repeating the arguments that we have just use for $i > p$, we obtain

$$ww^{-1}zz^{-1} = zz^{-1} = zz^{-1}ww^{-1}.$$

If $i = p$ then

$$\begin{aligned} ww^{-1}zz^{-1} &= b^{-i}(a_k a_k^{-1})\dots(a_1 a_1^{-1})b^i b^{-i}(a'_s {a'_s}^{-1})\dots(a'_1 {a'_1}^{-1})b^i \\ &= b^{-i}(a_k a_k^{-1})\dots(a_1 a_1^{-1})(a'_s {a'_s}^{-1})\dots(a'_1 {a'_1}^{-1})b^i && (4.1) \\ &= b^{-i}(a'_s {a'_s}^{-1})\dots(a'_1 {a'_1}^{-1})(a_k a_k^{-1})\dots(a_1 a_1^{-1})b^i && (4.5) \\ &= zz^{-1}ww^{-1}. && (4.1) \end{aligned}$$

Hence, the relation $ww^{-1}zz^{-1} = zz^{-1}ww^{-1}$ for any $w, z \in (A \cup A^{-1} \cup \{b, b^{-1}\})^*$ is a consequence of relations (4.1) to (4.5) so we can remove it from the presentation. We obtain the following monoid presentation for $BR(S, \theta)$:

$$\begin{aligned} < A,\ A^{-1}, b,\ b^{-1} \mid\ & \mathfrak{T}', \quad bb^{-1}, \quad ba = (a\theta)b, \\ & ab^{-1} = b^{-1}(a\theta), \quad aa^{-1} = a^{-1}a, \quad a = aa^{-1}a, \\ & a(a_1 a_1^{-1}) = (a_1 a_1^{-1})a, \quad (a, a_1 \in A) > \end{aligned}$$

since A and $\mathfrak{T}'$ are finite we conclude that $BR(S, \theta)$ is finitely presented as a monoid. ■

4.2 Clifford monoid that is the union of two copies of the same group

Let Y be the semilattice

$$\begin{tikzcd} \bullet\, 1 \arrow[d, dash] \\ \bullet\, 0 \end{tikzcd}$$

and G be a group. Let $G_0 = \{g_0 : g \in G\}$ and $G_1 = \{g_1 : g \in G\}$ be two copies of G and define a map $\phi_{1,0} : G_1 \longrightarrow G_0,\ g_1 \mapsto g_0,$ that maps an element of G_1 to its copy in G_0, $\phi_{1,0}$ is clearly an isomorphism. Let S be the Clifford monoid $\mathcal{S}(Y; \{G_0, G_1\}, \phi_{1,0})$, and θ a homomorphism from S into G_1. Note that 1 is the identity of Y, so G_1 is the group of units of S.

Claim 10 *For all $g \in G$ the morphism θ maps g_0 and g_1 to the same element in G_1.*

Proof. For any $g \in G$ we have

$$\begin{aligned}
(g_0 g_1)\theta &= (g_0(g_1\phi_{1,0}))\theta && \text{(multiplication in } S) \\
&= (g_0 g_0)\theta && \text{(def. } \phi_{1,0}) \\
&= (g_0)\theta(g_0)\theta && (\theta \text{ morphism})
\end{aligned}$$

then

$$\begin{aligned}
& (g_0 g_1)\theta = (g_0)\theta(g_1)\theta && (\theta \text{ morphism}) \\
\Leftrightarrow\ & (g_0)\theta(g_0)\theta = (g_0)\theta(g_1)\theta && \text{(by above)} \\
\Leftrightarrow\ & ((g_0)\theta)^{-1}(g_0)\theta(g_0)\theta = ((g_0)\theta)^{-1}(g_0)\theta(g_1)\theta && (G_1 \text{ group}) \\
\Leftrightarrow\ & (g_0)\theta = (g_1)\theta. && (G_1 \text{ group})
\end{aligned}$$

■

Consider the Bruck-Reilly extension $BR(S, \theta)$ of S. The $\mathcal{D}$-classes of $BR(S, \theta)$ are the sets $D_0 = \mathbb{N}_0 \times G_0 \times \mathbb{N}_0$ and $D_1 = \mathbb{N}_0 \times G_1 \times \mathbb{N}_0$.

Theorem 4.24 *The $\mathcal{D}$-classes D_1 and D_0 are Bruck-Reilly extensions of groups, in particular $BR(S,\theta)$ is a disjoint union of two Bruck-Reilly extensions of groups.*

Proof. Let θ_1 be the restriction of θ to G_1. For any $(m, g_1, n), (p, h_1, q) \in D_1$ we have

$$(m, g_1, n)(p, h_1, q) = (m - n + t, (g_1)\theta^{t-n}(h_1)\theta^{t-p}, q + p - t),$$

where $t = max(n, p)$, but $(g_1)\theta^k,\ (h_1)\theta^k \in G_1$, for all $k \in \mathbb{N}_0$, hence

$$(m, g_1, n)(p, h_1, q) = (m - n + t, (g_1)\theta_1^{t-n}(h_1)\theta_1^{t-p}, q + p - t),$$

so D_1 is the Bruck-Reilly extension $BR(G_1, \theta_1)$. Define a map

$$\theta_0 : G_0 \longrightarrow G_0, \quad g_0 \mapsto ((g_0)\theta)_0,$$

where $((g_0)\theta)_0$ is the copy of $(g_0)\theta$ in G_0. We can think of θ_0 as a composition of θ with $\phi_{1,0}$, so θ_0 is a morphism. Let $(m, g_0, n), (p, h_0, n) \in D_0$ be arbitrary and suppose, without loss of generality, that $n > p$. Then

$$\begin{aligned}
(m, g_0, n)(p, h_0, n) &= (m,\ g_0(h_0\theta^{n-p}),\ q - p + n) \\
&= (m,\ g_0((h_0\theta^{n-p})\phi_{1,0}),\ q - p + n) \\
&= (m,\ g_0(h_0\theta^{n-p})_0,\ q - p + n) \\
&= (m,\ g_0((h_0\theta^{n-p-1})\theta)_0,\ q - p + n) \\
&= (m,\ g_0(((h_0\theta^{n-p-1})_0)\theta)_0,\ q - p + n) \qquad \text{(Claim 10)} \\
&= (m,\ g_0(((h_0\theta^{n-p-1})_0)\theta_0),\ q - p + n) \qquad \text{(def. } \theta_0\text{)} \\
&\quad \cdots \\
&= (m,\ g_0(h_0\theta_0^{t-p}),\ q - p + n),
\end{aligned}$$

so D_0 is the Bruck-Reilly extension $BR(G_0, \theta_0)$. Since D_0 and D_1 are the $\mathcal{D}$-classes of $BR(S, \theta)$, we clearly have $BR(S, \theta) = D_0 \cup D_1$ and $D_0 \cap D_1 = \emptyset$. ■

Define a map $\zeta : D_1 \longrightarrow D_0,\ (m, g_1, n) \mapsto (m, g_0, n)$. It is clear that ζ is well-defined, one-one and onto. Let $(m, g_1, n), (p, h_1, q) \in D_1$ be arbitrary, suppose, without loss of generality, that $n > p$, then

$$((m, g_1, n)(p, h_1, q))\zeta = (m,\ g_1(h_1\theta^{n-p}),\ q{-}p{+}n)\zeta = (m,\ (g_1(h_1\theta^{n-p}))_0,\ q{-}p{+}n)$$

and

$$(m, g_1, n)\zeta(p, h_1, q)\zeta = (m, g_0, n)(p, h_0, q) = (m,\ g_0(h_0\theta_0^{n-p}),\ q - p + n),$$

but we have

$$\begin{aligned} &(g_1(h_1\theta^{n-p}))_0 = (g_1(h_1\theta^{n-p}))\phi_{1,0} = (g_1)\phi_{1,0}(h_1\theta^{n-p})\phi_{1,0} \\ &= g_0((h_0\theta^{n-p})\phi_{1,0}) = g_0(h_0\theta^{n-p})_0 = g_0(h_0\theta_0^{n-p}), \end{aligned}$$

so $((m, g_1, n)(p, h_1, q))\zeta = (m, g_1, n)\zeta(p, h_1, q)\zeta$, i.e. ζ is a morphism. It follows that D_0 is isomorphic to D_1.

Now we define a map $\eta : BR(S, \theta) \longrightarrow D_1$, $(m, g_i, n) \mapsto (m, g_1, n)$, where $i \in \{0, 1\}$. This map is obviously onto. Let $(m, g_i, n), (p, h_j, q) \in BR(S, \theta)$ be arbitrary, where $i, j \in \{0, 1\}$. Suppose, without loss of generality, that $n > p$, then

$$\begin{aligned} &((m, g_i, n)(p, h_j, q))\eta = (m,\ g_i(h_j\theta^{n-p}),\ q - p + n)\eta \\ &= (m,\ g_i(h_1\theta^{n-p}),\ q - p + n)\eta = (m,\ (g_i(h_1\theta^{n-p}))_1,\ q - p + n), \end{aligned}$$

if $i = 1$ we obtain

$$(m,\ (g_1(h_1\theta^{n-p}))_1,\ q - p + n) = (m,\ g_1(h_1\theta^{n-p}),\ q - p + n),$$

and, if $i = 0$ we obtain

$$\begin{aligned} (m,\ (g_0(h_1\theta^{n-p}))_1,\ q - p + n) &= (m,\ (g_0((h_1\theta^{n-p})\phi_{1,0}))_1,\ q - p + n) \\ &= (m,\ (g_0((h_1\theta^{n-p})\phi_{1,0}))\phi_{1,0}^{-1},\ q - p + n) \\ &= (m,\ (g_0)\phi_{1,0}^{-1}((h_1\theta^{n-p})\phi_{1,0})\phi_{1,0}^{-1},\ q - p + n) \\ &= (m,\ g_1(h_1\theta^{n-p}),\ q - p + n), \end{aligned}$$

so η is an epimorphism, since

$$(m, g_i, n)\eta(p, h_j, q)\eta = (m, g_1, n)(p, h_1, q) = (m, g_1(h_1\theta^{n-p}), q - p + n).$$

Note that η restricted to D_1 is the identity map, i.e. $\eta_{|D_1} = id_{D_1}$, and $\eta_{|D_0}$ is an isomorphism, the inverse of ζ. We have just proved the following:

Theorem 4.25 *The $\mathcal{D}$-classes of $BR(S, \theta)$ are isomorphic, and the $\mathcal{D}$-class D_1 is a homomorphic image of $BR(S, \theta)$.*

In section 4.2 we gave necessary and sufficient conditions for a Bruck-Reilly extension of a group to be finitely presented. Since the $\mathcal{D}$-classes of $BR(S,\theta)$ are Bruck-Reilly extensions of groups, if we relate the finite presentability of $BR(S,\theta)$ with the finite presentability of its $\mathcal{D}$-classes, we can apply the results in section 4.2 to know when $BR(S,\theta)$ is finitely presented. We will then look for connections between the presentation of $BR(S,\theta)$ and the presentations of its $\mathcal{D}$-classes.

Theorem 4.26 *$BR(S,\theta)$ is finitely generated if and only if D_0 and D_1 are finitely generated.*

Proof. Suppose that $BR(S,\theta)$ is finitely generated. By Proposition 4.14 we know that there exists a finite set $M \subseteq G_0 \cup G_1$ such that $G_0 \cup G_1$ is generated by $\bigcup_{k\geq 0} M\theta^k$. If we multiply two elements of S, the only way of obtaining an element of G_1 is if those two elements belong to G_1, so G_1 is generated by the set

$$(\bigcup_{k\geq 0} M\theta^k) \cap G_1 = (M \cap G_1) \cup (\bigcup_{k>0} M\theta^k).$$

Let $M' = M \cap G_1$ and $M^0 = M \cap G_0$, the generating set of G_1 becomes

$$M' \cup (\bigcup_{k>0} M'\theta^k) \cup (\bigcup_{k>0} M^0\theta^k) = M' \cup (\bigcup_{k>0} M'\theta_1^k) \cup (\bigcup_{k>0} M^0\theta^k),$$

denote this set by A. By Proposition 4.2, we know that D_1 is defined by the presentation

$$< A,\ b,\ c \mid bc = 1,\quad ba = (a\theta_1)b,\quad ac = c(a\theta_1),\quad (a \in A)\ > .$$

Let a be an arbitrary element of M', by the defining relations of D_1, we know that for any $k > 0$ the relation $a\theta_1^k = b^k a c^k$ holds in D_1 (proof of Proposition 4.11). So we can write any element from $\bigcup_{k>0} M'\theta_1^k$ as a product of elements in $M' \cup \{b, c\}$, thus $M' \cup \{b, c\} \cup (\bigcup_{k>0} M^0\theta^k)$ generates D_1. We have

$$\bigcup_{k>0} M^0\theta^k = \bigcup_{k\geq 0}(M^0\theta)\theta^k = \bigcup_{k\geq 0}(M^0\theta)\theta_1^k$$

and again we can write the elements $(a\theta)\theta_1^k$, with $a \in M^0$ and $k \geq 0$, in the form $b^k(a\theta)c^k$, so

$$\bigcup_{k\geq 0}(M^0\theta)\theta_1^k \subseteq (M^0\theta \cup \{b,c\})^*$$

hence, D_1 is generated by $M' \cup (M^0\theta) \cup \{b,c\}$, where M' and M^0 (hence $M^0\theta$) are finite, thus D_1 is finitely generated. It follows, from Theorem 4.25, that D_0 is finitely generated.

Conversely, suppose that D_1 and D_0 are finitely generated. Then, since $BR(S,\theta)$ is the disjoint union of D_1 and D_0, we know that $BR(S,\theta)$ is finitely generated, see [3, Proposition 3.1]. ■

Theorem 4.27 *If $BR(S,\theta)$ is finitely presented then D_0 and D_1 are finitely presented.*

Proof. Suppose that $BR(S,\theta)$ is defined by the monoid presentation $< A \mid \mathfrak{R} >$, where A and $\mathfrak{R}$ are finite. In particular, $BR(S,\theta)$ is finitely generated, hence, by Theorem 4.26, D_1 is finitely generated. Let B be a finite generating set for D_1.

D_1 is a subsemigroup of $BR(S,\theta)$ so for every word $b \in B$ there exists a word $w_b \in A^+$ such that $b = w_b$ holds in $BR(S,\theta)$. Also, for every $b \in B$, $b\zeta (= b\eta^{-1}_{|D_0})$ belongs to D_0, so there exists a word $w_{b\zeta} \in A^+$ such that $b\zeta = w_{b\zeta}$ holds in $BR(S,\theta)$.

Claim 11 *For every word x in D_1 there exists $u_x \in \{w_b : b \in B\}^+$ such that $x = u_x$ in $BR(S,\theta)$.*

Proof. Let $x \in D_1$ be arbitrary, D_1 is generated by B so $x \equiv b_1b_2\ldots b_r$ for some $b_i \in B$, $i = 1,\ldots,r$. Then

$$x \equiv b_1b_2\ldots b_r = w_{b_1}w_{b_2}\ldots w_{b_r}$$

holds in $BR(S,\theta)$, so there exists $u_x \equiv w_{b_1}w_{b_2}\dots w_{b_r} \in \{w_b : b \in B\}^+$ such that $x = u_x$ in $BR(S,\theta)$. ■

Claim 12 *For every word $v \in D_0$ there exists $u_v \in \{w_{b\zeta} : b \in B\}^+$ such that the relation $v = u_v$ holds in $BR(S,\theta)$.*

Proof. Let $v \in D_0$ arbitrary, then $v\eta \in D_1$, then, by Claim 11, there exists $u_{v\eta} \in \{w_b : b \in B\}^+$ such that $v\eta = u_{v\eta}$ in $BR(S,\theta)$. It follows that

$$v\eta = u_{v\eta} \equiv w_{b_1}w_{b_2}\dots w_{b_m} = b_1b_2\dots b_m,$$

for some $w_{b_1}, w_{b_2}, \dots, w_{b_m} \in \{w_b : b \in B\}^+$, so $v\eta = b_1b_2\dots b_m$ holds in $BR(S,\theta)$. But $v \in D_0$ and $\eta_{|D_0}$ is an isomorphism, so we obtain

$$\begin{array}{rl} & v\eta \equiv v\eta_{|D_0} = b_1b_2\dots b_m \\ \Rightarrow & (v\eta_{|D_0})\eta_{|D_0}^{-1} = (b_1b_2\dots b_m)\eta_{|D_0}^{-1} \\ \Leftrightarrow & v = (b_1b_2\dots b_m)\zeta \\ \Leftrightarrow & v = (b_1)\zeta(b_2)\zeta\dots(b_m)\zeta. \end{array}$$

Hence $v = (b_1)\zeta(b_2)\zeta\dots(b_m)\zeta = w_{(b_1)\zeta}w_{(b_2)\zeta}\dots w_{(b_m)\zeta}$. ■

Claim 13 *$A\eta$ generates D_1.*

Proof. Let $w \in D_1$ be arbitrary. Since D_1 is a homomorphic image of $BR(S,\theta)$, there exists $u \in BR(S,\theta)$ such that $w = u\eta$. Since $BR(S,\theta)$ is generated by A we can write $w \equiv a_1a_2\dots a_n$, for some $a_1, a_2, \dots, a_n \in A$. Hence

$$w = u\eta \equiv (a_1a_2\dots a_n)\eta \equiv (a_1)\eta(a_2)\eta\dots(a_n)\eta \quad \in (A\eta)^*,$$

so $A\eta$ generates D_1. ■

Given a relation $u = v$ in $BR(S,\theta)$, saying that this relation holds in D_1 is equivalent to say that $u\eta = v\eta$, since D_1 is a homomorphic image of $BR(S,\theta)$

by the map η.

We will see that D_1 is defined by the monoid presentation

$$< A \mid \mathfrak{R}, \quad w_b = w_{b\zeta} \quad (b \in B) >,$$

where $a \in A$ represents the generator $a\eta$ of D_1.

Let $u = v$ be any relation in $\mathfrak{R}$, $u = v$ holds in $BR(S,\theta)$, then, since η is a morphism, we have $u\eta = v\eta$, hence $u = v$ holds in D_1, and we conclude that $\mathfrak{R}$ holds in D_1.

Let $b \in B$ be arbitrary, $w_{b\zeta} = b\zeta$ holds in $BR(S,\theta)$, so $(w_{b\zeta})\eta = (b\zeta)\eta$. Then

$$(w_{b\zeta})\eta = (b\zeta)\eta \equiv (b\zeta)\eta_{|D_0} \equiv (b\eta^{-1}_{|D_0})\eta_{|D_0} \equiv b = w_b \equiv (w_b)id_{D_1} \equiv (w_b)\eta,$$

so $(w_{b\zeta})\eta = (w_b)\eta$ holds in $BR(S,\theta)$, hence $w_{b\zeta} = w_b$ holds in D_1, for all $b \in B$.

Now let $x = y$ be an arbitrary relation holding in D_1, we have $x, y \in BR(S,\theta)$ and $x\eta = y\eta$. Suppose that $x, y \in D_1$, then

$$x\eta = y\eta \quad \Leftrightarrow \quad xid_{D_1} = yid_{D_1} \quad \Leftrightarrow \quad x = y,$$

so $x = y$ holds in $BR(S,\theta)$, hence $x\eta = y\eta$ is a consequence of $\mathfrak{R}$. Suppose that $x, y \in D_0$, then

$$x\eta = y\eta \quad \Leftrightarrow \quad x\eta_{|D_0} = y\eta_{|D_0} \quad \Leftrightarrow \quad x = y$$

since $\eta_{|D_0}$ is one-one, so $x\eta = y\eta$ is a consequence of $\mathfrak{R}$. Finally, suppose that $x \in D_0$ and $y \in D_1$. By Claim 12, there exists $u_x \in \{w_{b\zeta} : b \in B\}^+$ such that $x = u_x$ holds in $BR(S,\theta)$ (hence, is a consequence of $\mathfrak{R}$) . Let $u_x \equiv w_{(b_1)\zeta} w_{(b_2)\zeta} \ldots w_{(b_r)\zeta}$, for some $b_1, b_2, \ldots b_r \in B$, we have

$$\begin{aligned} & x\eta = y\eta \quad \Leftrightarrow \quad (u_x)\eta = y\eta \\ \Leftrightarrow \quad & (w_{(b_1)\zeta} w_{(b_2)\zeta} \ldots w_{(b_r)\zeta})\eta = y\eta \\ \Leftrightarrow \quad & (w_{b_1} w_{b_2} \ldots w_{b_r})\eta = y\eta, \qquad\qquad (w_{b\zeta} = w_b, \quad \forall b \in B) \end{aligned}$$

and the elements y and $w_{b_1} w_{b_2} \ldots w_{b_r}$ belong to D_1, so we are back in the first case. Thus any relation in D_1 is a consequence of $\mathfrak{R}$ and of the relations $w_b = w_{b\zeta}$, for all $b \in B$, hence

$$< A \mid \mathfrak{R}, \quad w_b = w_{b\zeta} \quad (b \in B) >$$

defines D_1. The sets A, $\mathfrak{R}$ and B are finite so D_1 is finitely presented. Since D_0 is isomorphic to D_1 we conclude that D_0 is finitely presented. ■

This result can be generalized for semigroups, that are not Bruck-Reilly extensions, in the following way:

Theorem 4.28 *Let S be a semigroup, T_1 and T_2 be isomorphic semigroups such that $S = T_1 \cup T_2$ and $T_1 \cap T_2 = \emptyset$. Suppose that $T \cong T_1$ is a homomorphic image of S. If S is finitely presented then T is finitely presented.*

Proof. We can rewrite T_1 and T_2 in the following way:

$$T_i = \{t_i : t \in T\}, \qquad i = 1, 2.$$

T is an homomorphic image of S, so there exists a morphism $\eta : S \longrightarrow T,\ t_i \mapsto t$. Let $< A \mid \mathfrak{R} >$ be a presentation for S, where A and $\mathfrak{R}$ are finite. Let $x \in T$ be arbitrary, since T is a homomorphic image of S, there exists $y \in A^+$, such that $x = y\eta$. Writing y as a product of letters from A, say $y \equiv a_1a_2 \dots a_n$, we obtain

$$x = (a_1a_2 \dots a_n)\eta = (a_1)\eta(a_2)\eta \dots (a_n)\eta,$$

so $A\eta$ generates T. Thus T is finitely generated. For any generator x of S there exists $a_1, a_2 \in A^*$, with $a_1 \in T_1$, $a_2 \in T_2$, and $a \in A\eta$ such that

$$x\eta = a_1\eta = a_2\eta = a.$$

We will see that the presentation

$$< A \mid \mathfrak{R}, \quad a_1 = a_2, \quad (a \in A\eta) \ >,$$

where $x \in A$ represents the generator element $x\eta$ of T, defines the semigroup T. Note that saying that a relation $x = y$ holds in T is equivalent to say that $x\eta = y\eta$. Let $u = v$ be an arbitrary relation in $\mathfrak{R}$, η is a morphism so $u\eta = v\eta$, hence $u = v$ holds in T. For all $a \in A\eta$ we have

$$a_1\eta = a, \quad a_2\eta = a \quad \Rightarrow \quad a_1\eta = a_2\eta,$$

so $a_1 = a_2$ holds in T, for all $a \in A\eta$. Let $x = y$ be an arbitrary relation holding in T, $x, y \in A^+$. Suppose that $x, y \in T_i$, for some $i \in \{1, 2\}$, then

$$x\eta = y\eta \quad \Leftrightarrow \quad x\eta_{|T_i} = y\eta_{|T_i} \quad \Leftrightarrow \quad x = y,$$

since $\eta_{|T_i}$ is a bijection, $i \in \{1, 2\}$. Hence, the relation $x = y$ holds in S, so it is a consequence of $\mathfrak{R}$. Suppose that $x \in T_1$ and $y \in T_2$. The word x belongs to S so we can write it as a product of generators of S, say $x \equiv a_{i_1} a_{i_2}^{(1)} \dots a_{i_r}^{(r)}$, $i_1, i_2, \dots, i_r \in \{1, 2\}$, and we know that for every generator a_i of S, a is a generator of T. So, applying the relation $a_1 = a_2$, $a \in A\eta$ to all elements in the decomposition of x that belong to T_1, we obtain

$$x \equiv a_{i_1} a_{i_2}^{(1)} \dots a_{i_r}^{(r)} = a_2 a_2^{(1)} \dots a_2^{(r)},$$

and, since $a_2 a_2^{(1)} \dots a_2^{(r)}$ and y belong to T_2, we are back in the first case. Hence $x = y$ in T is a consequence of the relations $\mathfrak{R}$ and $a_1 = a_2$, $a \in A\eta$. We conclude that T is defined by the presentation above, and since A and $\mathfrak{R}$ are finite it follows that T is finitely presented. ■

Note: In general, a subsemigroup of a finitely presented semigroup need not be finitely presented.

Returning to the Bruck-Reilly extension $BR(S, \theta)$, we will see that the converse of Theorem 4.27 holds.

Theorem 4.29 *If D_1 is finitely presented then $BR(S, \theta)$ is finitely presented.*

Proof. Suppose that D_1 is finitely presented, then D_0 is also finitely presented. Let B_0 and B_1 be finite generating sets for D_0 and D_1, respectively. Since $BR(S, \theta)$ is the disjoint union of D_0 and D_1, the set $B_0 \cup B_1$ generates $BR(S, \theta)$, see [3, Proposition 3.1], so this monoid is finitely generated. D_1 is a finitely presented Bruck-Reilly extension of the group G_1 so, by Theorem 4.13, G_1 can be defined by a presentation $< A_1 \mid \mathfrak{R}_1 >$ where A_1 is finite and

$$\mathfrak{R}_1 = \bigcup_{k\geq 0} \overline{\mathfrak{R}_1}\theta_1^k = \{u\theta_1^k = v\theta_1^k : k \geq 0, \quad (u=v) \in \overline{\mathfrak{R}_1}\}$$

with $\overline{\mathfrak{R}_1} \subseteq A_1^* \times A_1^*$ finite. D_0 is a finitely presented Bruck-Reilly extension of the group G_0, isomorphic to D_1, so we can consider a presentation for it that is a copy of the presentation of D_1, let it be $< A_0 \mid \mathfrak{R}_0 >$. From Proposition 4.21 we obtain the following presentation for the Clifford monoid S:

$$< A_0,\ A_1 \mid \mathfrak{R}_0, \quad \mathfrak{R}_1, \quad 1_0 a_1 = a_1 1_0 = a_0, \quad 1_0 1_1 = 1_1 1_0, \quad 1_1 = 1, \quad (a_1 \in A_1) >,$$

then, by Proposition 4.2, we have

$$\begin{aligned} BR(S,\theta) \cong < A_0,\ A_1,\ b,\ c \quad \mid \quad & \mathfrak{R}_0, \quad \mathfrak{R}_1, \quad 1_0 1_1 = 1_1 1_0, \quad 1_1 = 1, \\ & bc = 1, \quad 1_0 x_0 = x_0 1_0 = x_1, \quad ba = (a\theta)b, \\ & ac = c(a\theta), \quad (a \in A_0 \cup A_1, \quad x_1 \in A_1) \ > . \end{aligned}$$

Let $u = v$ be an arbitrary relation in $\overline{\mathfrak{R}_1}$ and $k \geq 0$, then

$$u\theta_1^k = v\theta_1^k \quad \Leftrightarrow \quad u\theta^k = v\theta^k$$

since θ_1 coincides with θ in G_1 and $\overline{\mathfrak{R}_1}$ is a set of relations in G_1. We know that for all $u \in A_1$ the relation $u\theta^k = b^k u c^k$ is a consequence of the relations

$$ba = (a\theta)b, \quad ac = c(a\theta), \quad bc = 1, \quad (a \in A_0 \cup A_1),$$

hence, $u\theta^k = v\theta^k$ is a consequence of

$$\overline{\mathfrak{R}_1}, \quad ba = (a\theta)b, \quad ac = c(a\theta), \quad bc = 1, \quad (a \in A_0 \cup A_1),$$

so the set $\mathfrak{R}_1$ can be replaced by $\overline{\mathfrak{R}_1}$ in the presentation of $BR(S,\theta)$. Consider now an arbitrary relation $u = v$ in $\overline{\mathfrak{R}_0}$, and $k \geq 0$ arbitrary. Then

$$\begin{array}{lll}
 & u\theta_0^k = v\theta_0^k \quad \Leftrightarrow & \\
\Leftrightarrow & (u\theta_0^{k-1})\theta_0 = (v\theta_0^{k-1})\theta_0 & \\
\Leftrightarrow & ((u\theta_0^{k-1})\theta)_0 = ((v\theta_0^{k-1})\theta)_0 & (\text{def. } \theta_0) \\
\Leftrightarrow & (((u\theta_0^{k-2})\theta)_0)\theta)_0 = (((v\theta_0^{k-2})\theta)_0)\theta)_0 & (\text{def. } \theta_0) \\
\Leftrightarrow & (((u\theta_0^{k-2})\theta)\theta)_0 = (((v\theta_0^{k-2})\theta)\theta)_0 & (\text{Claim 10}) \\
\Leftrightarrow & ((u\theta_0^{k-2})\theta^2)_0 = ((v\theta_0^{k-2})\theta^2)_0 & \\
 & \cdots & \\
\Leftrightarrow & (u\theta^k)_0 = (v\theta^k)_0 & \\
\Leftrightarrow & (b^k u c^k)_0 = (b^k v c^k)_0, &
\end{array}$$

this last step is a consequence of the relations

$$ba = (a\theta)b, \quad ac = c(a\theta), \quad bc = 1, \quad (a \in A_0 \cup A_1).$$

If $k = 0$ the words $b^k uc^k$ and $b^k vc^k$ belong to D_0, so the relation $(b^k uc^k)_0 = (b^k vc^k)_0$ is equivalent to $u = v$. If $k > 0$ then $b^k uc^k$ and $b^k vc^k$ belong to D_1 and we have

$$(b^k uc^k)_0 = (b^k vc^k)_0 \quad \Leftrightarrow \quad (b^k uc^k)\phi_{1,0} = (b^k vc^k)\phi_{1,0},$$

since $\phi_{1,0}$ is a morphism this is a consequence of $b^k uc^k = b^k vc^k$. Hence $u\theta_0^k = v\theta_0^k$ is a consequence of the relations

$$\overline{\mathfrak{R}_0}, \quad ba = (a\theta)b, \quad ac = c(a\theta), \quad bc = 1, \quad (a \in A_0 \cup A_1).$$

So $BR(S,\theta)$ is defined by the presentation

$$\begin{aligned}
< A_0,\ A_1,\ b,\ c \ \mid\ & \overline{\mathfrak{R}_0},\ \ \overline{\mathfrak{R}_1},\ \ 1_0 1_1 = 1_1 1_0,\ \ 1_1 = 1, \\
& bc = 1,\ \ 1_0 x = x 1_0,\ \ ba = (a\theta)b, \\
& ac = c(a\theta),\ \ (a \in A_0 \cup A_1,\ \ x \in A_1)\ >,
\end{aligned}$$

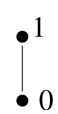
and, since A_0, A_1, $\overline{\mathfrak{R}_0}$ and $\overline{\mathfrak{R}_1}$ are finite, it follows that $BR(S,\theta)$ is finitely presented. ■

In conclusion, we have:

Theorem 4.30 *Let G be a group, $G_i = \{g_i : g \in G\}$, $i = 0, 1$, two copies of G and S the Clifford monoid $\mathcal{S}(\{0,1\};\{G_0, G_1\}, \phi_{1,0})$, where $\phi_{1,0} : G_1 \longrightarrow G_0$, $g_1 \mapsto g_0$. Let $BR(S,\theta)$ be a Bruck-Reilly extension of S. Then, the $\mathcal{D}$-classes of $BR(S,\theta)$ are Bruck-Reilly extensions of the groups G_0 and G_1 and $BR(S,\theta)$ is finitely presented if and only if its $\mathcal{D}$-classes are finitely presented.*

4.3 Clifford monoid that is the union of two groups linked by the morphism $\phi_{1,0} : x \mapsto 1_0$

Let Y be the semilattice



and G_0, G_1 be any two groups. Define a map $\phi_{1,0} : G_1 \longrightarrow G_0$, $x \mapsto 1_0$ where 1_0 is the identity of G_0. For any $x, y \in G_1$ we have

$$(xy)\phi_{1,0} = 1_0, \qquad x\phi_{1,0}y\phi_{1,0} = 1_0 1_0 = 1_0,$$

so $\phi_{1,0}$ is a morphism. Let S be the Clifford monoid $\mathcal{S}(Y;\{G_0, G_1\}, \phi_{1,0})$ and θ any homomorphism from S into its group of units, G_1. Consider the Bruck-Reilly extension of S, $BR(S,\theta)$.

Like in section 4.2, we can see that the $\mathcal{D}$-class, $\mathbb{N}_0 \times G_1 \times \mathbb{N}_0$, of $BR(S,\theta)$, is the Bruck-Reilly extension $BR(G_1, \theta_1)$, where θ_1 is the restriction of θ to G_1. We will denote this Bruck-Reilly extension by D_1. Define a map in G_0

$$\theta_0 : G_0 \longrightarrow G_0, \quad x \mapsto 1_0,$$

like we did to $\phi_{1,0}$, we can see that θ_0 is a morphism. Let D_0 be the Bruck-Reilly extension $BR(G_0, \theta_0)$. Given any two elements $(m, g, n), (p, h, q)$ in $\mathbb{N}_0 \times G_0 \times \mathbb{N}_0$, and supposing, without loss of generality, that $n > p$, their multiplication in $BR(S,\theta)$ is

$$\begin{aligned}(m,g,n)(p,h,q) &= (m,\ g(h\theta^{n-p}),\ q-p+n)\\ &= (m,\ g((h\theta^{n-p})\phi_{1,0}),\ q-p+n) = (m,\ g1_0,\ q-p+n),\end{aligned}$$

and multiplying these two elements in D_0, we obtain

$$(m,g,n)(p,h,q) = (m,\ g(h\theta_0^{n-p}),\ q-p+n)$$
$$= (m,\ g(1_0\theta_0^{n-p-1}),\ q-p+n) = (m,\ g1_0,\ q-p+n),$$

so we can think in the $\mathcal{D}$-class of $BR(S,\theta)$, $\mathbb{N}_0 \times G_0 \times \mathbb{N}_0$, as the Bruck-Reilly extension D_0. We have proved the following:

Theorem 4.31 *$BR(S,\theta)$ is the disjoint union of its $\mathcal{D}$-classes, D_0 and D_1, and these are Bruck-Reilly extensions of groups.*

Now we will, as in section 4.2, relate the finite presentability of $BR(S,\theta)$ with the finite presentability of its $\mathcal{D}$-classes.

Theorem 4.32 *$BR(S,\theta)$ is finitely generated if and only if its $\mathcal{D}$-classes, D_0 and D_1, are finitely generated.*

Proof. Suppose that $BR(S,\theta)$ is finitely generated, then, repeating the arguments we have used in the proof of Theorem 4.26, we can see that D_1 is finitely generated. Suppose that $BR(S,\theta)$ is generated by the finite set M, since $BR(S,\theta)$ is the disjoint union of D_0 and D_1, M is the disjoint union of M_0 and M_1, where $M_i = M \cap D_i$, $i \in \{0,1\}$. Let (m,g,n) be an arbitrary element of D_0, we can write it as a product of elements in $M_0 \cup M_1$, and in this product we must have at least one element of M_0, since a product of elements of D_1 clearly belongs to D_1. Define a set

$$M_1' = \{(m,1_0,n)\ :\ \exists\ x \in G_1 \text{ such that } (m,x,n) \in M_1\}.$$

Claim 14 *$M_0 \cup M_1'$ generates D_0.*

Proof. Let (m,g,n) be an arbitrary element of D_0, since $M_0 \cup M_1$ generates $BR(S,\theta)$, we can write (m,g,n) as a product of k elements in this set, for some

$k \in \mathbb{N}$. We will show, by induction on k, that (m,g,n) can be written as a product of elements in $M_0 \cup M_1'$.

If $k=1$ the element (m,g,n) must belong to M_0, since $M_1 \subseteq D_1$.

Suppose that for all $k \le l$, (m,g,n) can be written as a product of elements from $M_0 \cup M_1'$. Let $k=l+1$, we have

$$(m,g,n) = (m_1,g_1,n_1)\dots(m_l,g_l,n_l)(m_{l+1},g_{l+1},n_{l+1}),$$

for some $(m_1,g_1,n_1),\dots,(m_{l+1},g_{l+1},n_{l+1}) \in M_0 \cup M_1$. Let (p,h,q) be the product of the first l elements in this decomposition, i.e.

$$(p,h,q) = (m_1,g_1,n_1)\dots(m_l,g_l,n_l).$$

· If $(p,h,q) \in D_0$ then, by the hypothesis of induction, it can be written as a product of elements from $M_0 \cup M_1'$. If $(m_{l+1},\ g_{l+1},\ n_{l+1}) \in M_0$, all the elements of our product belong to $M_0 \cup M_1'$. If $(m_{l+1},\ g_{l+1},\ n_{l+1}) \in M_1$ we have

$$(p,h,q)(m_{l+1},\ g_{l+1},\ n_{l+1}) = (p-q+t,\ (h\theta^{t-q})(g_{l+1}\theta^{t-m_{l+1}}),\ n_{l+1}-m_{l+1}+t),$$

where $t = max(q, m_{l+1})$, if $q = m_{l+1}$ we obtain

$$(p,hg,n_{l+1}) = (p,h(g\phi_{1,0}),n_{l+1}) = (p,h1_0,n_{l+1}) = (p,h,q)(m_{l+1},1_0,n_{l+1}),$$

that is in the form we wanted, if $q > m_{l+1}$ we obtain

$$\begin{aligned}&(p,\ h(g\theta^{q-m_{l+1}}),\ n_{l+1}-m_{l+1}+q) = (p,\ h((g\theta^{q-m_{l+1}})\phi_{1,0})),\ n_{l+1}-m_{l+1}+q)\\ &= (p,\ h1_0,\ n_{l+1}-m_{l+1}+q) = (p,h,q)(m_{l+1},\ 1_0,\ n_{l+1}),\end{aligned}$$

if $q < m_{l+1}$ the product becomes

$$(p-q+m_{l+1},\ (g\theta^{m_{l+1}-q})h, n_{l+1})$$

but this belongs to D_1, so this case cannot happen.

· If $(p,h,q) \in D_1$ then $(m_{l+1},\ g_{l+1},\ n_{l+1})$ must belong to M_0. Suppose that (m_i,g_i,n_i) is the first element, counting from the right, in the product

$$(p,h,q) = (m_1,g_1,n_1)\dots(m_l,g_l,n_l)$$

that belongs to M_1, and define

$$(p', h', q') = (m_1, g_1, n_1) \dots (m_{i-1}, g_{i-1}, n_{i-1}).$$

The elements

$$(m_{i+1}, g_{i+1}, n_{i+1}),\ \dots,\ (m_l, g_l, n_l),\ (m_{l+1}, g_{l+1}, n_{l+1})$$

belong to M_0, so their product is in D_0. Since $(p, h, q) \in D_1$, and we removed from this product only elements of M_0, the product $(p', h', q')(m_i, g_i, n_i)$ must be in D_1. The element (m_i, g_i, n_i) is in M_1, so we have

Claim 15 *The product*

$$(m_i, g_i, n_i) \dots (m_l, g_l, n_l)(m_{l+1}, g_{l+1}, n_{l+1})$$

belongs to D_0.

Proof. Let $(\alpha, u, \beta) \in BR(S, \theta)$, $(\gamma, v, \iota) \in D_0$ and $(\delta, r, \vartheta) \in D_1$ be such that

$$(\alpha, u, \beta)(\delta, r, \vartheta),\quad (\delta, r, \vartheta)(\gamma, v, \iota) \in D_1, \qquad \text{and} \qquad (\alpha, u, \beta)(\delta, r, \vartheta)(\gamma, v, \iota) \in D_0.$$

We have

$$(\delta, r, \vartheta)(\gamma, v, \iota) \in D_1 \quad \Leftrightarrow \quad \vartheta > \gamma,$$

then, if $\delta > \beta$ we have

$$\begin{aligned}(\alpha, u, \beta)(\delta, r, \vartheta)(\gamma, v, \iota) &= (\alpha - \beta + \delta,\ (u\theta^{\delta-\beta})r,\ \vartheta)(\gamma, v, \iota)\\ &= (\alpha - \beta + \delta,\ (u\theta^{\delta-\beta})r(v\theta^{\vartheta-\gamma}),\ \iota - \gamma + \vartheta) \in D_1,\end{aligned}$$

if $\delta = \beta$ then

$$\begin{aligned}(\alpha, u, \beta)(\delta, r, \vartheta)(\gamma, v, \iota) &= (\alpha, ur, \vartheta)(\gamma, v, \iota)\\ &= (\alpha,\ ur(v\theta^{\vartheta-\gamma}),\ \iota - \gamma + \vartheta) \in D_1,\end{aligned}$$

finally, if $\delta < \beta$, we must have $u \in G_1$, and we know that $\vartheta - \delta + \beta > \gamma$, it follows that

$$\begin{aligned}(\alpha, u, \beta)(\delta, r, \vartheta)(\gamma, v, \iota) &= (\alpha,\ u(r\theta^{\beta-\delta}),\ \vartheta - \delta + \beta)(\gamma, v, \iota)\\ &= (\alpha,\ u(r\theta^{\beta-\delta})(v\theta^{(\vartheta-\delta+\gamma)-\gamma}),\ \iota - \gamma + \vartheta - \delta + \beta) \in D_1,\end{aligned}$$

so, in all cases we have a contradiction. We conclude that, since

$$(p', h', q') \in D_0 \cup D_1, \qquad (m_i, g_i, n_i) \in D_1,$$
$$(m_{i+1}, g_{i+1}, n_{i+1}) \dots (m_l, g_l, n_l)(m_{l+1}, g_{l+1}, n_{l+1}) \in D_0$$

and

$$(p', h', q')(m_i, g_i, n_i) \in D_1,$$
$$(p', h', q')(m_i, g_i, n_i)(m_{i+1}, g_{i+1}, n_{i+1}) \dots (m_l, g_l, n_l)(m_{l+1}, g_{l+1}, n_{l+1}) \in D_0,$$

we must have

$$(m_i, g_i, n_i)(m_{i+1}, g_{i+1}, n_{i+1}) \dots (m_l, g_l, n_l)(m_{l+1}, g_{l+1}, n_{l+1}) \in D_0.$$

■

By the induction hypothesis, the product

$$(m_i, g_i, n_i) \dots (m_l, g_l, n_l)(m_{l+1}, g_{l+1}, n_{l+1})$$

can be written as a product of elements in $M_0 \cup M_1'$. Note that if $i = 1$ the case is equivalent to the case "$(p, h, q) \in D_0$" since we can read the multiplication from right to left, so this product contains at most l elements. If $(p', h', q') \in D_0$ then, by the induction hypothesis, it can be written as a product of elements in $M_0 \cup M_1'$. If $(p', h', q') \in D_1$ we repeat the process we used for (p, h, q), with the product

$$(m_i, g_i, n_i) \dots (m_{l+1}, g_{l+1}, n_{l+1})$$

written as a product of elements in $M_0 \cup M_1'$.

We conclude that $M_0 \cup M_1'$ generates D_0 ■

Since M_1 and M_0 are finite we know that $M_0 \cup M_1'$ is finite, hence D_0 is finitely generated.

Conversely, suppose that D_0 and D_1 are finitely generated. Then $BR(S, \theta)$ is finitely generated, by [3, Proposition 3.1]. ■

Theorem 4.33 *If D_1 and D_0 are finitely presented then $BR(S,\theta)$ is finitely presented.*

Proof. Suppose that D_1 and D_0 are finitely presented. Like in Theorem 4.29, we obtain the following presentation for $BR(S,\theta)$:

$$\begin{aligned} < A_0,\ A_1,\ b,\ c \ \mid\ & \mathfrak{R}_0,\ \ \mathfrak{R}_1,\ \ 1_0 1_1 = 1_1 1_0,\ \ 1_1 = 1, \\ & bc = 1,\ \ 1_0 x = x 1_0,\ \ ba = (a\theta)b, \\ & ac = c(a\theta),\ \ (a \in A_0 \cup A_1,\ \ x \in A_1)\ >, \end{aligned}$$

where A_0 and A_1 are finite sets generating G_0 and G_1 respectively,

$$\mathfrak{R}_0 = \bigcup_{k \geq 0} \overline{\mathfrak{R}_0}\theta_0^k = \{u\theta_0^k = v\theta_1^k : k \geq 0,\ \ (u = v) \in \overline{\mathfrak{R}_0}\}$$

where $\overline{\mathfrak{R}_0}$ is a finite subset of $A_0^* \times A_0^*$, and

$$\mathfrak{R}_1 = \bigcup_{k \geq 0} \overline{\mathfrak{R}_1}\theta_1^k = \{u\theta_1^k = v\theta_1^k : k \geq 0,\ \ (u = v) \in \overline{\mathfrak{R}_1}\},$$

where $\overline{\mathfrak{R}_1}$ is a finite subset of $A_1^* \times A_1^*$. Since θ_1 is the restriction of θ to G_1 we can see, like on Theorem 4.29, that the relations in $\mathfrak{R}_1$ are a consequence of the relations

$$\overline{\mathfrak{R}_1},\quad bc = 1,\quad ba = (a\theta)b,\quad ac = c(a\theta),\quad (a \in A_0 \cup A_1).$$

Let $u\theta_0^k = v\theta_0^k$ be an arbitrary relation in $\mathfrak{R}_0$. If $k = 0$ the relation $u = v$ belongs to $\overline{\mathfrak{R}_0}$, and if $k > 0$ we obtain

$$u\theta_0^k = v\theta_0^k \quad \Leftrightarrow \quad 1_0 = 1_0,$$

by definition of θ_0. So, for $k > 0$, the sets of relations $\overline{\mathfrak{R}_0}\theta^k$ and $\overline{\mathfrak{R}_1}\theta^k$ are redundant. It follows that $BR(S,\theta)$ is defined by the presentation

$$\begin{aligned} < A_0,\ A_1,\ b,\ c \ \mid\ & \overline{\mathfrak{R}_0},\ \ \overline{\mathfrak{R}_1},\ \ 1_0 1_1 = 1_1 1_0,\ \ 1_1 = 1, \\ & bc = 1,\ \ 1_0 x = x 1_0,\ \ ba = (a\theta)b, \\ & ac = c(a\theta),\ \ (a \in A_0 \cup A_1,\ \ x \in A_1)\ >, \end{aligned}$$

hence, it is finitely presented. ■

Theorem 4.34 *If $BR(S,\theta)$ is finitely presented then its $\mathcal{D}$-classes are finitely presented.*

Proof. Suppose that $BR(S,\theta)$ is finitely presented. By Propositions 4.14 and 4.15 we know that S is defined by the presentation $< A \mid \mathfrak{R} >$ where

$$A = \bigcup_{k\geq 0} \overline{A}\theta^k \quad \text{and} \quad \mathfrak{R} = \bigcup_{k\geq 0} \overline{\mathfrak{R}}\theta^k,$$

for a finite subset $\overline{A}$ of S and some finite set of relations $\overline{\mathfrak{R}} \subseteq A^* \times A^*$. A can be written in the form $A_0 \cup A_1$ where $A_i \subseteq G_i,\ i = 1, 2$.

Claim 16 *G_1 is generated by A_1, subject to the set of relations*

$$(\bigcup_{k>0} \overline{\mathfrak{R}}\theta^k) \cup \{u = v : (u = v) \in \overline{R},\quad u, v \in A_1^*\}.$$

Proof. Since the only way of obtaining an element of G_1 is as a product of elements of G_1, the set A_1 generates G_1. For all $u \in A^*$ and $k > 0$, $u\theta^k$ belongs to G_1, and this group is generated by A_1, so $u\theta^k$ is a product of elements in A_1. Let $u\theta^k = v\theta^k$ be any relation in $\bigcup_{k>0} \overline{\mathfrak{R}}\theta^k$, this relation holds in S and $u\theta^k$, $v\theta^k$ belong to A_1^*, so $u\theta^k = v\theta^k$ must hold in G_1. Similarly,

$$\{u = v : (u = v) \in \overline{R},\quad u, v \in A_1^*\}$$

holds in G_1. Consider now $x = y$ an arbitrary relation in G_1, this relation holds in S, since G_1 is a subgroup of S, so it is a consequence of $\mathfrak{R}$. The words x, y belong to A_1^*, so $x = y$ must be a consequence of

$$(\bigcup_{k>0} \overline{\mathfrak{R}}\theta^k) \cup \{u = v : (u = v) \in \overline{R},\quad u, v \in A_1^*\},$$

since if we have a relation $\alpha = \beta$ where the word α contains a letter from A_0, then $\alpha \in G_0$, and we cannot obtain a relation involving elements of G_1 as a consequence of this relation. It follows that G_1 is defined by the monoid presentation

$$
\begin{aligned}
< A_1 \quad | \quad & u = v, \qquad ((u = v \in \overline{\mathfrak{R}}, \quad u, v \in A_1^*) \\
& x\theta^k = y\theta^k, \qquad (k > 0, \quad (x = y) \in \overline{\mathfrak{R}}) \;\; > .
\end{aligned}
$$

■

From Theorem 4.32 we know that $BR(G_1, \theta_1)$ is finitely generated, then, applying Proposition 4.2, there exists a finite subset of A_1, say A_1', such that $BR(G_1, \theta_1)$ is defined by the presentation

$$
\begin{aligned}
< A_1', \; b, \; c \quad | \quad & u = v, \qquad ((u = v) \in \overline{\mathfrak{R}}, \quad u, v \in {A_1'}^*) \\
& x\theta^k = y\theta^k, \qquad (k > 0, \quad (x = y) \in \overline{\mathfrak{R}}) \\
& bc = 1, \quad ba = (a\theta_1)b, \quad ac = c(a\theta_1), \quad (a \in A_1') \;\; > .
\end{aligned}
$$

The relations $x\theta^k = y\theta^k$, with $k > 0$ and $(x = y) \in \overline{\mathfrak{R}}$ can be rewritten as follows:

$$
x\theta^k = y\theta^k \quad \Leftrightarrow \quad (x\theta)\theta^{k-1} = (y\theta)\theta^{k-1} \quad \Leftrightarrow \quad (x\theta)\theta_1^{k-1} = (y\theta)\theta_1^{k-1},
$$

and we know that $(y\theta)\theta_1^{k-1} = b^{k-1}(y\theta)c^{k-1}$ is a consequence of the relations

$$
bc = 1, \quad ba = (a\theta_1)b, \quad ac = c(a\theta_1), \quad (a \in A_1')
$$

so $x\theta^k = y\theta^k$ is a consequence of these relations and of the relations

$$
x\theta = y\theta, \qquad (x = y) \in \overline{\mathfrak{R}},
$$

hence

$$
\begin{aligned}
D_1 \cong < A_1', \; b, \; c \quad | \quad & u = v, \qquad ((u = v) \in \overline{\mathfrak{R}}, \quad u, v \in {A_1'}^*) \\
& x\theta = y\theta, \qquad ((x = y) \in \overline{\mathfrak{R}}) \\
& bc = 1, \quad ba = (a\theta_1)b, \quad ac = c(a\theta_1), \quad (a \in A_1') \;\; > .
\end{aligned}
$$

Thus D_1 is finitely presented.

Let $< Q \mid \mathfrak{T} >$ be a finite presentation defining $BR(S, \theta)$. By Theorem 4.32, we know that D_0 is generated by $Q_0 \cup Q_1'$, where $Q_i = Q \cap D_i$, $i = 1, 2$, and

$$Q_1' = \{q' : q \in Q_1\},$$

where q represents the element (m, g, n) generating D_1 and q' represents the element $(m, 1_0, n)$ in D_0. Define a map

$$Q_0 \cup Q_1 \longrightarrow Q_0 \cup Q_1', \qquad q \mapsto \begin{cases} q & \text{if } q \in Q_0 \\ q' & \text{if } q \in Q_1 \end{cases}$$

We know that if $q_0 \in Q_0$, $q_1 \in Q_1$ and $q_0 q_1 \in D_0$, then $q_0 q_1 = q_0 q_1'$. Let $\phi : (Q_0 \cup Q_1)^* \longrightarrow (Q_0 \cup Q_1')^*$ be the natural homomorphism defined by the map above. Note that ϕ is a bijection, so ϕ^{-1} exists and is an isomorphism. For any $q' \in Q_1'$, q' belongs to $BR(S, \theta)$, and this semigroup is generated by $Q_0 \cup Q_1$, so we can write q' as a product of elements from $Q_0 \cup Q_1$, i.e.

$$\forall\, q' \in Q_1' \quad \exists\, q'\psi \in (Q_0 \cup Q_1)^* \quad : \quad q' = q'\psi \ \text{ in } BR(S, \theta).$$

We note that, since $q' \in D_0$, the word $q'\psi$ must contain at least one letter from Q_0. Let $q'\psi = q_0 q_1 \dots q_n$, where $q_i \in Q_0 \cup Q_1$, and suppose, without loss of generality, that $q_0 \in Q_0$, then

$$\begin{aligned} (q'\psi)\phi &= (q_0 q_1 \dots q_n)\phi = q_0\phi q_1\phi \dots q_n\phi \\ &= q_0 q_1' \dots q_n' = q_0 q_1 \dots q_n = q'\psi, \end{aligned}$$

so the relations $q' = q'\psi = (q'\psi)\phi$ hold in $BR(S, \theta)$, and $q', (q'\psi)\phi \in D_0$, hence, the relation $q' = ((q'\psi)\phi$ holds in D_0. Now we will see that the presentation

$$< Q_0,\ Q_1' \quad | \quad u\phi = v\phi, \qquad ((u = v) \in \mathfrak{T}, \quad u, v \in D_0) \tag{4.7}$$

$$q' = (q'\psi)\phi, \qquad (q' \in Q_1') \ > \tag{4.8}$$

defines D_0. We have already seen that the relation (4.8) holds in D_0, so let $u = v$ be an arbitrary relation in $\mathfrak{T}$, with $u, v \in D_0$. Then $u = v$ holds in D_0 and ϕ is a morphism, so $u\phi = v\phi$ holds in D_0.

Claim 17 *Let $w_1, w_2 \in D_0$. If w_2 can be obtained from w_1 by using relations from $\mathfrak{T}$, then $w_2\phi$ can be obtained from $w_1\phi$ by using relations* (4.7).

Proof. Suppose that the relation $w_1 = w_2$ is a consequence of $\mathfrak{T}$, with $w_1, w_2 \in D_0$. Without loss of generality suppose that w_2 is obtained from w_1 by using one relation from $\mathfrak{T}$. Then there exists $\alpha, \beta \in (Q_0 \cup Q_1)^*$ and a relation $(u = v) \in \mathfrak{T}$, with $u, v \in D_0$, such that $w_1 \equiv \alpha u \beta$, $w_2 \equiv \alpha v \beta$, it follows that

$$\begin{aligned} w_1\phi &\equiv (\alpha u \beta)\phi \equiv \alpha\phi u\phi \beta\phi, \\ w_2\phi &\equiv (\alpha v \beta)\phi \equiv \alpha\phi v\phi \beta\phi, \end{aligned}$$

where $\alpha\phi, \beta\phi \in (Q_0 \cup Q_1')^*$ and $u\phi = v\phi$ is a relation in (4.7), so $w_2\phi$ can be obtained from $w_1\phi$ by using one relation from (4.7). We conclude that if $w_1 = w_2$ is a relation in D_0 that is a consequence of $\mathfrak{T}$, then $w_1\phi = w_2\phi$ is a consequence of (4.7). ■

Let $x = y$ be any relation in D_0, $x, y \in (Q_0 \cup Q_1')^*$. First suppose that x and y are a product of letters in Q_0 and Q_1', say $x \equiv x_1 x_2 \dots x_n$, $y \equiv y_1 y_2 \dots y_r$, then

$$\begin{aligned} x &\equiv x_1 x_2 \dots x_n = (x_1 x_2 \dots x_n)\phi^{-1}, \\ y &\equiv y_1 y_2 \dots y_r = (y_1 y_2 \dots y_r)\phi^{-1}, \end{aligned}$$

it follows that $(x_1 x_2 \dots x_n)\phi^{-1} = (y_1 y_2 \dots y_r)\phi^{-1}$ holds in $BR(S, \theta)$, so it is a consequence of $\mathfrak{T}$, then, by Claim 17, the relation

$$((x_1 x_2 \dots x_n)\phi^{-1})\phi = ((y_1 y_2 \dots y_r)\phi^{-1})\phi \quad \Leftrightarrow \quad x = y$$

is a consequence of (4.7). If x, y are a product of letters from Q_1', we know that $x = x\psi$ and $y = y\psi$, then the relation $x\psi = y\psi$ holds in $BR(S, \theta)$, so it is a consequence of $\mathfrak{T}$. By Claim 17 the relation $(x\psi)\phi = (y\psi)\phi$ is a consequence of (4.7), and by relations (4.8), we have

$$x = (x\psi)\phi = (y\psi)\phi = y,$$

so $x = y$ is a consequence of (4.7) and (4.8). Thus

$$< \ Q_0, \ Q_1' \ \mid \ (4.7), \ \ (4.8) \ >$$

defines D_0, and we conclude that D_0 is finitely presented. ■

4.4 Bruck-Reilly extension determined by the morphism that maps all elements to the identity

Let S be a Clifford monoid $\mathcal{S}(Y; G_\alpha, \phi_{\alpha,\beta})$, and θ the morphism that maps all elements of S to its identity, 1. Let T be the Bruck-Reilly extension of S defined by θ. We will represent by e the identity of the semilattice Y.

As in the last two sections, we will investigate the $\mathcal{D}$-classes of this Bruck-Reilly extension, and look for conditions for its finite presentability.

Theorem 4.35 *The $\mathcal{D}$-classes of the Bruck-Reilly extension $T = BR(S, \theta)$ are Bruck-Reilly extensions of groups.*

PROOF. We have seen that the $\mathcal{D}$-classes of $BR(S, \theta)$ are the sets $\mathbb{N}_0 \times G_\alpha \times \mathbb{N}_0$ with $\alpha \in Y$. Let θ_α be the morphism that maps all elements of G_α to its identity, 1_α, $\alpha \in Y$. Define a multiplication in $\mathbb{N}_0 \times G_\alpha \times \mathbb{N}_0$, $\alpha \in Y$, by the rule

$$(m, g, n)(p, h, q) = (m - n + t,\ (g\theta_\alpha^{t-n})(h\theta_\alpha^{t-p}),\ q - p + t),$$

where $t = max(n, p)$. Let $\alpha \in Y$, and (m, g, n), $(p, h, q) \in \mathbb{N}_0 \times G_\alpha \times \mathbb{N}_0$, be arbitrary. Suppose, without loss of generality, that $p > n$, then multiplying these elements, using the multiplication defined above, we obtain

$$(m - n + p,\ (g\theta_\alpha^{p-n})h,\ q) = (m - n + p,\ 1_\alpha h,\ q) = (m - n + p,\ h,\ q),$$

and multiplying this elements in $BR(S, \theta)$, we obtain

$$\begin{aligned} &(m - n + p,\ (g\theta^{p-n})h,\ q) = (m - n + p,\ 1h,\ q) \\ = &(m - n + p,\ (1\phi_{e,\alpha})h,\ q) = (m - n + p,\ 1_\alpha h,\ q) = (m - n + p, h, q), \end{aligned}$$

note that the identity of S, 1, belongs to G_e, since e is the identity of Y, and $\phi_{e,\alpha}$ is a morphism so it must map the identity of G_e to the identity of G_α. Since the multiplications above coincide we can conclude that the $\mathcal{D}$-classes of $BR(S, \theta)$ are the Bruck-Reilly extensions, $BR(G_\alpha, \theta_\alpha)$, of the groups G_α, $\alpha \in Y$. ■

Theorem 4.36 *If Y is finite and $BR(G_\alpha, \theta_\alpha)$ is finitely generated, for all $\alpha \in Y$, then T is finitely generated.*

PROOF. Suppose that for all $\alpha \in Y$, $BR(G_\alpha, \theta_\alpha)$ is generated by the finite set B_α, then, by [3, Proposition 3.1], T is generated by the set $\bigcup_{\alpha\in Y} B_\alpha$, since Y is finite we conclude that T is finitely generated. ■

Theorem 4.37 *If Y is finite and its $\mathcal{D}$-classes, $BR(G_\alpha, \theta_\alpha)$, are finitely presented, then T is finitely presented.*

PROOF. Suppose that $BR(G_\alpha, \theta_\alpha)$ is finitely presented, $\alpha \in Y$, then by Theorem 4.13, we know that G_α is defined by the presentation

$$< \ A_\alpha, \mid \bigcup_{k\geq 0} \mathfrak{R}_\alpha \theta_\alpha^k \ >,$$

where A_α is finite and $\mathfrak{R}_\alpha$ is a finite set of relations in $A_\alpha^* \times A_\alpha^*$. Then, using the presentations given in Propositions 4.2 and 4.21, we obtain the following presentation for T:

$$\begin{aligned} < \bigcup_{\alpha\in Y} A_\alpha,\ b,\ c \ \mid\ & \bigcup_{k\geq 0} \mathfrak{R}_\alpha\theta_\alpha^k, \quad 1_\xi 1_\beta = 1_\beta 1_\xi, \quad 1_\gamma x = x 1_\gamma = x\phi_{\sigma,\gamma}, \\ & (\alpha, \xi, \beta, \gamma, \sigma \in Y,\ \ \xi \neq \beta,\ \ \sigma > \gamma,\ \ x \in A_\sigma) \\ & bc = 1, \quad ba = (a\theta)b, \quad ac = c(a\theta), \quad (a \in \bigcup_{\alpha\in Y} A_\alpha) \ > . \end{aligned}$$

For $k > 0$, the set of relations

$$\mathfrak{R}_\alpha\theta_\alpha^k = \{u\theta_\alpha^k = v\theta_\alpha^k \ : \ (u = v) \in \mathfrak{R}_\alpha\}$$

is reduced to the relation $1_\alpha = 1_\alpha$, so these relations are redundant and we can remove them from the presentation. Then

$$\begin{aligned} T \cong < \bigcup_{\alpha\in Y} A_\alpha,\ b,\ c \ \mid\ & \mathfrak{R}_\alpha, \quad 1_\xi 1_\beta = 1_\beta 1_\xi, \quad 1_\gamma x = x 1_\gamma = a\phi_{\sigma,\gamma}, \\ & (\alpha, \xi, \beta, \gamma, \sigma \in Y,\ \ \xi \neq \beta,\ \ \sigma > \gamma,\ \ x \in A_\sigma) \\ & bc = 1, \quad ba = b, \quad ac = c, \quad (a \in \bigcup_{\alpha\in Y} A_\alpha) \ >, \end{aligned}$$

with A_α, $\mathfrak{R}_\alpha$ finite for all $\alpha \in Y$, and Y finite, so T is finitely presented. ■

The converse of this results does not follow directly like in the other cases, since now we have an arbitrary number of groups. Another way of finding conditions for T to be finitely presented is to relate the presentation of T with the presentation of S, and in fact we have:

Theorem 4.38 *T is finitely presented (generated) if and only if S is finitely presented (generated).*

PROOF. Suppose that T is finitely presented. By Propositions 4.14 and 4.15, we know that there exists a finite subset, A, of S such that S is generated by $\bigcup_{k\geq 0} A\theta^k$, subject to the relations $\bigcup_{k\geq 0} \mathfrak{R}\theta^k$, where

$$\mathfrak{R} \subseteq (\bigcup_{k\geq 0} A\theta^k)^* \times (\bigcup_{k\geq 0} A\theta^k)^*$$

is finite. For $k > 0$, we have $A\theta^k = 1$ and the defining relations become

$$\begin{aligned} \mathfrak{R}\theta^k &= \{u\theta^k = v\theta^k \ : \ (u = v) \in \mathfrak{R}\} = \\ &= \{1 = 1 \ : \ (u = v) \in \mathfrak{R}\}. \end{aligned}$$

So S is generated by the set A, and the relations $\mathfrak{R}\theta^k$ are redundant for $k > 0$. Hence S is defined by the presentation $< A \mid \mathfrak{R} >$, thus it is finitely presented.

The converse follows from Proposition 4.3. ■

We did not use the fact that S is a Clifford semigroup, so this result holds for the Bruck-Reilly extension of any monoid, determined by the morphism that maps all elements of the monoid to its identity.

Now we are able to prove the converse of Theorems 4.37 and 4.38, and we conclude that $BR(S, \theta)$ is finitely presented if and only if its $\mathcal{D}$-classes are finitely presented if and only if S is finitely presented.

Theorem 4.39 *If T is finitely presented (generated) then its $\mathcal{D}$-classes are finitely presented (generated).*

Proof. Suppose that T is finitely presented (generated), by Theorem 4.38 we know that S is finitely presented (generated), then, by Theorem 4.22, G_α is finitely presented (generated) for all $\alpha \in Y$. It follows, by Proposition 4.3 (4.2), that $BR(G_\alpha, \theta_\alpha)$ is finitely presented (generated) for all $\alpha \in Y$. ■

4.5 Open Problems

These three particular cases of Bruck-Reilly extensions of Clifford monoids are the first steps in the attempt to answer the question:

Question Is a Bruck-Reilly extension of a Clifford monoid always finitely presented if and only if its $\mathcal{D}$-classes are finitely presented?

Some more particular cases that we intend to study, before considering the general case, are:

Question Can the case studied in section 4.2 be generalized for an arbitrary number of copies of the same group? Are these results still true if we consider an arbitrary morphism linking the groups?

Question Considering the Clifford monoid that is the union of an arbitrary number of groups, linked by the morphism

$$\phi_{\alpha,\beta} : G_\alpha \longrightarrow G_\beta, \quad x \mapsto 1_\beta, \qquad \alpha > \beta,$$

can we generalize the results in section 4.3?

Another question that arises from this dissertation is:

Question Is a Bruck-Reilly extension of an inverse monoid finitely presented as an inverse semigroup if and only if it is finitely presented as a semigroup?

In an attempt to answer this question we can start by considering the Bruck-Reilly extension of an inverse semigroup that is finitely presented as a semigroup, since the existence of an inverse semigroup that is finitely presented as an inverse semigroup and not as a semigroup might have some influence in the solution of this problem.

Appendix A

Ideals and Green's Relations

Let S be a semigroup. A non-empty subset A of S is called a *left ideal* of S if $SA \subseteq A$. We call A a *right ideal* of S if $AS \subseteq A$. We say that A is an *ideal* of S if it is both a left and a right ideal.

Let I be a proper ideal of the semigroup S. We define a congruence on S, ρ_I, by the rule

$$x\rho_I y \quad \Leftrightarrow \quad x = y \ \text{ or } \ x, y \in I.$$

The quotient semigroup of S by this congruence is the set

$$S/\rho_I = \{I\} \cup \{\{x\} : x \in S \backslash I\}.$$

Since the element I of S/ρ_I is the zero in this semigroup, we can think of S/ρ_I as $(S\backslash I) \cup \{0\}$ where all product not falling in $S\backslash I$ are zero. This quotient is sometimes called a *Rees quotient*, and is denoted by S/I.

If $a \in S$, the smallest left ideal of S containing a is the set $Sa \cup \{a\} = S^1 a$, where

$$S^1 = \begin{cases} S & \text{if } S \text{ has an identity} \\ S \cup \{1\} & \text{if } S \text{ does not have identity,} \end{cases}$$

we call it the *principal left ideal generated by* a. Similarly we can define *principal right ideal* and *principal ideal generated by* a.

We define five equivalence relations in S, called *Green's Equivalences*, in terms of principal ideals in S. We have, for $a, b \in S$:

$$\begin{array}{rcl} a\ \mathcal{L}\ b & \Leftrightarrow & S^1 a = S^1 b, \\ a\ \mathcal{R}\ b & \Leftrightarrow & aS^1 = bS^1, \\ a\ \mathcal{J}\ b & \Leftrightarrow & S^1 a S^1 = S^1 b S^1, \\ a\ \mathcal{D}\ b & \Leftrightarrow & \exists\ c \in S\ :\ a\ \mathcal{R}\ c\ \mathcal{L}\ b, \\ a\ \mathcal{H}\ b & \Leftrightarrow & a\ \mathcal{L}\ b \text{ and } a\ \mathcal{R}\ b. \end{array}$$

These equivalences are related in the following way

$$\mathcal{H}\ \subseteq\ \mathcal{L},\ \mathcal{R}\ \subseteq\ \mathcal{D}\ \subseteq \mathcal{J}.$$

When we are working with Green's relations in more than one semigroup, instead of saying that a is $\mathcal{L}$ related with b in the semigroup S, we can write $a\ \mathcal{L}_S\ b$. We use a similar notation for all the other Green's relations.

Some properties of the Green's relations are resumed in the following remark, and can be found in [6, Section 2.1].

Remark :
1. Let $a, b \in S$. Then $a\mathcal{L}b$ if and only if there exists $x, y \in S^1$ such that $xa = b,\ yb = a$. Also, $a\mathcal{R}b$ if and only if there exists $u, v \in S^1$ such that $au = b,\ bv = a$.
2. $\mathcal{L}$ is a right congruence and $\mathcal{R}$ is a left congruence.
3. The relations $\mathcal{L}$ and $\mathcal{R}$ commute.
4. If S is regular, then S^1 can be replaced by S in the definition of $\mathcal{R}$, $\mathcal{L}$ and $\mathcal{J}$.

Given an element $a \in S$, we usually represent the $\mathcal{R}$-class of a, i.e. the set of all elements x in S such that $x\mathcal{R}a$, by R_a. For all $a, b \in S$ we have

$$R_a \cap R_b \neq \emptyset \quad \Leftrightarrow \quad a\mathcal{R}b.$$

We use a similar notation for the other relations, and a similar result holds. Finally, we note that :
· each $\mathcal{D}$-class is a union of $\mathcal{L}$-classes and also a union of $\mathcal{R}$-classes;
· the intersection of an $\mathcal{L}$-class with an $\mathcal{R}$-class is either empty or is an $\mathcal{H}$-class.

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