Proceedings of Symposia in Pure Mathematics
Vol. **33** (1979), part 1, pp. 3–27

REDUCTIVE GROUPS

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This contribution contains a review of the theory of reductive groups. Some knowledge of the theory of linear algebraic groups is assumed, to the extent covered in §§1–5 of Borel's report [**2**] in the 1965 Boulder conference.

§§1 and 2 contain a discussion of notion of the "root datum" of a reductive group. This is quite important for the theory of L-groups. Since the relevant results are not too easily accessible in the literature (they are dealt with, in a more general context, in the latter part of the Grothendieck-Demazure seminar [**17**]), it is shown how one can deduce these results from the theory of semisimple groups (which is well covered in the literature). In §§3 and 4 we review facts about the relative theory of reductive groups. There is more overlap with [**2**, §6], which deals with the same material.

§5 contains a discussion of a useful class of Lie groups (the "selfadjoint" ones). We indicate how the familiar properties of these groups can be established, assuming the algebraic theory of reductive groups.

I am grateful to A. Borel for valuable suggestions and to J. J. Duistermaat for comments on the material of §5.

1. Root data and root systems. The notion of root datum (introduced in [**17**, Exposé XXI] under the name of "donnée radicielle") is a slight generalization of the notion of root system, which is quite useful for the theory of reductive groups. Below is a brief discussion of root data. For more details see [loc. cit.]. For the theory of root systems we refer to [**7**].

1.1. *Root data.* A *root datum* is a quadruple $\Psi = (X, \Phi, X^\vee, \Phi^\vee)$ where: X and $X^\vee$ are free abelian groups of finite type, in duality by a pairing $X \times X^\vee \to \mathbf{Z}$ denoted by $\langle\ ,\ \rangle$, Φ and $\Phi^\vee$ are finite subsets of X and $X^\vee$ and there is a bijection $\alpha \mapsto \alpha^\vee$ of Φ onto $\Phi^\vee$. If $\alpha \in \Phi$ define endomorphisms s_α and $s_{\alpha^\vee}$ of X, $X^\vee$, respectively, by

$$s_\alpha(x) = x - \langle x, \alpha^\vee \rangle \alpha, \qquad s_{\alpha^\vee}(u) = u - \langle \alpha, u \rangle \alpha^\vee.$$

Then the following two axioms are imposed:

(RD1) *For all* $\alpha \in \Phi$ *we have* $\langle \alpha, \alpha^\vee \rangle = 2$;

(RD2) *For all* $\alpha \in \Phi$ *we have* $s_\alpha(\Phi) \subset \Phi$, $s_{\alpha^\vee}(\Phi^\vee) \subset \Phi^\vee$.

It follows from (RD1) that $s_\alpha^2 = \mathrm{id}$, $s_\alpha(\alpha) = -\alpha$ (and similarly for $s_{\alpha^\vee}$). It is clear from the definition of a root datum that if $\Psi = (X, \Phi, X^\vee, \Phi^\vee)$ is one, then $\Psi^\vee = (X^\vee, \Phi^\vee, X, \Phi)$ is also one, the *dual* of Ψ.

Let Ψ be as above. Let Q be the subgroup of X generated by Φ and denote by X_0

AMS (MOS) subject classifications (1970). Primary 20G15, 22E15.

the subgroup of X orthogonal to Φ^x. Put $V = Q \otimes \mathbf{Q}$, $V_0 = X_0 \otimes \mathbf{Q}$. Define similarly subgroups $Q^\vee$, $X_0^\vee$ of $X^\vee$ and vector spaces $V^\vee$, $V_0^\vee$.

We say that Ψ is *semisimple* if $X_0 = \{0\}$ and *toral* if Φ is empty.

1.2. Lemma. *$Q \cap X_0 = \{0\}$ and $Q + X_0$ has finite index in X.*

This is contained in [**17**]. We sketch a proof. Define a homomorphism $p: X \to X^\vee$ by

$$p(x) = \sum_{\alpha \in \Phi} \langle x, \alpha^\vee \rangle \alpha^\vee$$

Since $\langle x, p(x)\rangle = \sum_{\alpha\in\Phi} \langle x, \alpha^\vee\rangle^2$ we have $X_0 = \operatorname{Ker} p$.

Next observe that if $\alpha \in \Phi$ we have $p(\alpha) = \frac{1}{2}\langle \alpha, p(\alpha)\rangle\alpha^\vee$, as follows by summation over $\beta \in \Phi$ from the identity

$$\langle \alpha, \beta^\vee\rangle^2\alpha^\vee = \langle \alpha, \beta^\vee\rangle\beta^\vee + \langle \alpha, s_{\alpha^\vee}(\beta^\vee)\rangle s_{\alpha^\vee}(\beta^\vee).$$

This shows that $p \otimes \mathrm{id}$ is a surjection $V \to V^\vee$, whence $\dim V^\vee \leqq \dim V$. By symmetry we then have $\dim V = \dim V^\vee$, whence $Q \cap \operatorname{Ker} p = \{0\}$. The assertion now follows readily.

1.3. *Root systems.* It follows from the proof of 1.2 that $V^\vee$ can be identified with the dual of the vector space V. We write again $\langle\ ,\ \rangle$ for the pairing. Also identify Φ with $\Phi \otimes 1 \subset V$ and assume that $\Phi \neq \varnothing$. We then see that Φ is a root system in V in the sense of [7]. Recall that this means that the following conditions are satisfied:

(RS1) *Φ is finite and generates V, moreover $0 \notin \Phi$;*

(RS2) *for all $\alpha \in \Phi$ there is $\alpha^\vee \in V^\vee$ such that $\langle \alpha, \alpha^\vee\rangle = 2$ and that s_α (defined as before) stabilizes Φ;*

(RS3) *for all $\alpha \in \Phi$ we have $\alpha^\vee(\Phi) \subset \mathbf{Z}$.*

The s_α then generate a finite group of linear transformations of V, the *Weyl group* $W(\Phi)$ of Φ.

If $\Psi = (X, \Phi, X^\vee, \Phi^\vee)$ is a root datum which is not toral, we call the root system $\Phi \subset V$ the *root system of* Ψ. The Weyl group $W(\Phi)$ is identified with the group of automorphisms of X generated by the s_α of 1.1 and with the group of automorphisms of $X^\vee$ generated by the $s_{\alpha^\vee}$.

The following observation is sometimes useful.

1.4. Lemma. *Axiom* (RD2) *is equivalent to*:

(RD2′)(a) *For all $\alpha \in \Phi$ we have $s_\alpha(\Phi) \subset \Phi$;*

(b) *the s_α ($\alpha \in \Phi$) generate a finite group.*

It suffices to prove that (RD2′) implies the second assertion of (RD2). Let α, $\beta \in \Phi$. Then $s_\alpha s_\beta s_\alpha$ and $s_{s_\alpha(\beta)}$ are involutions in the group generated by the s_α. We have by an easy computation,

$$s_{s_\alpha(\beta)} s_\alpha s_\beta s_\alpha(x) = x + (\langle x, {}^t s_\alpha(\beta^\vee)\rangle - \langle x, s_\alpha(\beta)^\vee\rangle) s_\alpha(\beta),$$

where ${}^t s_\alpha$ is the transpose of s_α. Since $\langle s_\alpha(\beta), {}^t s_\alpha(\beta^\vee)\rangle - \langle s_\alpha(\beta), s_\alpha(\beta)^\vee\rangle = \langle \beta, \beta^\vee\rangle - \langle s_\alpha(\beta), s_\alpha(\beta)^\vee\rangle = 0$, we see that the above automorphism of X is unipotent. Since it lies in a finite group it must be the identity. Hence $s_\alpha(\beta)^\vee = {}^t s_\alpha(\beta^\vee)$, and the assertion follows by observing that $s_{\alpha^\vee} = {}^t s_\alpha$.

1.5. *Properties of root systems.* Let $\Phi \subset V$ be a root system. Proofs of the properties reviewed below can be found in [7].

(a) If $\alpha \in \Phi$ and $\lambda\alpha \in \Phi$ then $\lambda = \pm 1, \pm\frac{1}{2}, \pm 2$. The root system Φ is called *reduced* if for all $\alpha \in \Phi$ the only multiples of α in Φ are $\pm\alpha$. To every root system, there belong two reduced root systems, obtained by removing for every $\alpha \in \Phi$ the longer (or shorter) multiples of α.

(b) Φ is the direct sum of root systems $\Phi' \subset V'$ and $\Phi'' \subset V''$ if $V = V' \oplus V''$ and $\Phi = \Phi' \cup \Phi''$. A root system is *irreducible* if it is not the direct sum of two subsystems.

Every root system is a direct sum of irreducible ones.

(c) The only reduced irreducible root systems are the usual ones: $\boldsymbol{A}_n$ $(n \geqq 1)$, $\boldsymbol{B}_n$ $(n \geqq 2)$, $\boldsymbol{C}_n$ $(n \geqq 3)$, $\boldsymbol{D}_n$ $(n \geqq 4)$, $\boldsymbol{E}_6$, $\boldsymbol{E}_7$, $\boldsymbol{E}_8$, $\boldsymbol{F}_4$, $\boldsymbol{G}_2$.

(d) For each dimension n there exists one irreducible nonreduced root system, denoted by $\boldsymbol{BC}_n$ (see below).

EXAMPLES. (i) $\boldsymbol{B}_n$. Take $V = \boldsymbol{Q}^n$ with standard basis $\{e_1, \cdots, e_n\}$. Then $V^\vee = \boldsymbol{Q}^n$, with dual basis $\{e_1^\vee, \ldots, e_n^\vee\}$. We have $\boldsymbol{B}_n = \{\pm e_i \pm e_j \ (i < j) \text{ and } \pm e_i \ (1 \leqq i, j \leqq n)\}$. If $\alpha = \pm e_i \pm e_j$ then $\alpha^\vee = \pm e_i^\vee \pm e_j^\vee$; if $\alpha = \pm e_i$ then $\alpha^\vee = \pm 2 e_i$. The Weyl group $W(\boldsymbol{B}_n)$ consists of the linear transformations which permute the coordinates and change their signs in all possible ways.

The $\alpha^\vee \in V^\vee$ form an irreducible root system of type $\boldsymbol{C}_n$. We have $W(\boldsymbol{C}_n) \simeq W(\boldsymbol{B}_n)$.

(ii) With the same notations we have $\boldsymbol{BC}_n = \{\pm e_i \pm e_j \ (i < j), \pm e_i, \text{ and } \pm 2e_i \ (1 \leqq i, j \leqq n)\}$. Then $W(\boldsymbol{BC}_n) = W(\boldsymbol{B}_n)$.

1.6. *Weyl chambers.* Let $\Phi \subset V$ be a root system. We now view it as a subset of $V_{\boldsymbol{R}} = V \otimes_{\boldsymbol{Q}} \boldsymbol{R}$. A hyperplane H of $V_{\boldsymbol{R}}$ is *singular* if it is orthogonal to an $\alpha^\vee$. A *Weyl chamber* C in $V_{\boldsymbol{R}}$ is a connected component of the complement of the union of the singular hyperplanes. To a Weyl chamber one associates an ordering of the roots: $\alpha > 0 \Leftrightarrow \langle x, \alpha^\vee \rangle > 0$ for all $x \in C$.

$\alpha \in \Phi$ is *simple* (for this ordering) if it is not the sum of two positive roots. The set of simple roots Δ is called a *basis* of Φ. We have the following properties.

(a) The Weyl group $W(\Phi)$ acts simply transitively on the set of Weyl chambers.

(b) The s_α $(\alpha \in \Delta)$ generate $W(\Phi)$. More precisely, $(W, (s_\alpha)_{\alpha \in \Delta})$ is a Coxeter system (see [7]).

(c) Every root is an integral linear combination of simple roots, with coefficients all of the same sign.

(d) Say that Δ is connected if it cannot be written as a disjoint union $\Delta = \Delta' \cup \Delta''$, where $(\Delta' + \Delta'') \cap \Delta = \varnothing$.

Then we have: Φ is irreducible $\Leftrightarrow$ Δ is connected.

A connected Δ leads to a connected Dynkin graph. These are described in [7].

1.7. We collect a few facts about root data to be needed later. First, there is the notion of *direct sum* of root data. This is clear and we skip the definition.

Next we have to say something about morphisms of root data. The following suffices. For more general cases see [7]; see also 2.11(ii) and 2.12.

Let $\Psi = (X, \Phi, X^\vee, \Phi^\vee)$ and $\Psi' = (X', \Phi', (X')^\vee, (\Phi')^\vee)$ be two root data. A homomorphism $f: X' \to X$ is called an *isogeny of* Ψ' *into* Ψ *if*:

(a) f is injective and $\operatorname{Im} f$ has finite index in X,

(b) f induces a bijection of Φ' onto Φ and its transpose ${}^t f$ induces a bijection of $\Phi^\vee$ onto $(\Phi')^\vee$.

Notice that then ${}^t f$ is also an injection $X^\vee \to (X')^\vee$ with finite cokernel. Also, coker f and coker$({}^t f)$ are in duality.

EXAMPLE. Given Ψ, we shall construct a Ψ' and an isogeny of X into X', which we shall call the *canonical isogeny* associated to Ψ.

If L is a subgroup of X we denote by $\tilde{L}$ the largest subgroup containing L such that $\tilde{L}/L$ is finite. Then $L = \tilde{L}$ if and only if L is a direct summand.

Let X_0 and Q be as in 1.1. By 1.2 we can view Φ as a subset of X/X_0. It follows that $\Psi_1' = (X/X_0, \Phi, \tilde{Q}^\vee, \Phi^\vee)$ is a semisimple root datum. Likewise, $\Psi_1'' = (X/\tilde{Q}, \emptyset, X_0, \emptyset)$ is a toral root datum. Put $\Psi' = \Psi_1' \oplus \Psi_2''$. Then the *canonical isogeny* f: $X \to (X/X_0) \oplus (X/\tilde{Q})$ is the canonical homomorphism of X into the right-hand side.

1.8. Let Φ be a root system in the $\boldsymbol{Q}$-vector space V. The $\alpha^\vee$ $(\alpha \in \Phi)$ in the dual $V^\vee$ of V also form a root system $\Phi^\vee$ (the dual or inverse root system). There are finitely many semisimple root data $(X, \Phi, X^\vee, \Phi^\vee)$ where $X \subset V$, $X^\vee \subset V^\vee$. In fact, let Q and $Q^\vee$ be the lattices in V and $V^\vee$ generated by Φ and $\Phi^\vee$, respectively, and define $P = \{x \in V | \langle x, \alpha^\vee \rangle \in \boldsymbol{Z} \text{ for all } \alpha \in \Phi\}$. $P^\vee \subset V^\vee$ is defined similarly. Then $Q \subset P$ and P/Q is a finite group, in duality with $P^\vee/Q^\vee$. An X as above is then contained between P and Q and for each such X there is a unique $X^\vee$ between $P^\vee$ and $Q^\vee$ such that $(X, \Phi, X^\vee, \Phi^\vee)$ is a root datum.

1.9. Let $\Psi = (X, \Phi, X^\vee, \Phi^\vee)$ be a root datum. Assume that its root system $\Phi \subset V$ is *reduced*. Let Δ be a basis of Φ. Then $\Delta^\vee = \{\alpha^\vee | \alpha \in \Delta\}$ is a basis of the dual root system $\Phi^\vee \subset V^\vee$.

We call *based root datum* a sextuple $\Psi_0 = (X, \Phi, \Delta, X^\vee, \Phi^\vee, \Delta^\vee)$, where $(X, \Phi, X^\vee, \Phi^\vee)$ is a root datum with reduced root system Φ and where Δ is a basis of Φ. However, since Δ and $\Delta^\vee$ determine Φ and $\Phi^\vee$ uniquely, it also makes sense to view a based root system as a quadruple $\Psi_0 = (X, \Delta, X^\vee, \Delta^\vee)$. This we shall do.

2. Reductive groups (absolute theory).

2.1. Let G be a connected reductive linear algebraic group. In this section we consider the absolute case, where fields of definition do not come in. So we can view G as a subgroup of some $\mathbf{GL}(n, \Omega)$, Ω an algebraically closed field (see [**2**]). Let S be a subtorus of G. We define the root system $\Phi(G, S)$ of G with respect to S to be the set of nontrivial characters of S which appear when one diagonalizes the representation of S in the Lie algebra $\mathfrak{g}$ of G, S operating via the adjoint representation.

2.2. *The root datum of* G. Fix a maximal torus T of G. We shall associate to the pair (G, T) a root datum $\psi(G, T) = (X, \Phi, X^\vee, \Phi^\vee)$ (also denoted by $\psi(G)$).

X is the group of rational characters $X^*(T)$ of T. This is a free abelian group of finite rank. $X^\vee$ is the group $X_*(T)$ of 1-parameter multiplicative subgroups of T, i.e., the group of homomorphisms (of algebraic groups) $\mathbf{GL}_1 \to T$. Then $X^\vee$ can be put in duality with X by a pairing $\langle\ ,\ \rangle$ defined as follows: if $x \in X^*(T)$, $u \in X_*(T)$, then $x(u(t)) = t^{\langle x, u\rangle}$ $(t \in \Omega^*)$.

We take $\Phi = \Phi(G, T)$, the root system of G with respect to T. To complete the definition we have to describe $\Phi^\vee$. If $\alpha \in \Phi$, let T_α be the identity component of the kernel of α. This is a subtorus of codimension 1. The centralizer Z_α of T_α in G is a connected reductive group with maximal torus T, whose derived group G_α is semisimple of rank 1, i.e., is isomorphic to either $\mathbf{SL}(2)$ or $\mathbf{PSL}(2)$. There is a unique homomorphism $\alpha^\vee\colon \mathbf{GL}_1 \to G_\alpha$ such that $T = (\mathrm{Im}\ \alpha^\vee)T_\alpha$, $\langle \alpha, \alpha^\vee \rangle = 2$. These $\alpha^\vee$ make up $\Phi^\vee$.

The axiom (RD1) is built into the definition of $\alpha^\vee$. We use 1.4 to establish (RD2). Let $n_\alpha \in G_\alpha - T_\alpha$ normalize T_α. Then $n_\alpha^2 \in T_\alpha$ and, s_α being as in 1.1, we have, for $x \in X$, $t \in T$,

$$x(n_\alpha t n_\alpha^{-1}) = t^{s_\alpha(x)}.$$

In fact, working in G_α one shows that there is $u \in X^\vee$ such that the left-hand side equals $t^{x - \langle x, u\rangle \alpha}$. One then shows that $\langle \alpha, u \rangle = 2$ and that $\langle x, u \rangle = 0$ if $\langle x, \alpha^\vee \rangle = 0$. It follows that (RD2′) holds. So $\Phi(G)$ is a root datum. The root system Φ is reduced (for all these facts see [3] or [**14**]).

2.3. To each $\alpha \in \Phi$ there is associated a unique homomorphism of algebraic groups $x_\alpha\colon \boldsymbol{G}_a \to G_\alpha$ such that

$$t x_\alpha(u) t^{-1} = x_\alpha(t^\alpha u) \qquad (t \in T,\ u \in \Omega).$$

Put $U_\alpha = \mathrm{Im}(x_\alpha)$ and let $X_\alpha \in \mathfrak{g}$ be a nonzero tangent vector to U_α. Then

$$\mathfrak{g} = \mathrm{Lie}(T) \oplus \sum_{\alpha \in \Phi} \Omega X_\alpha.$$

Let B be a Borel subgroup containing T. There is a unique ordering of Φ (as in 1.6) such that B is generated by T and the U_α with $\alpha > 0$, and any $B \supset T$ is so obtained. It follows that we can associate to the triple (G, B, T) a based root system $\psi_0(G, B, T) = (X^*(T), \Delta, X_*(T), \Delta^\vee)$ (or $\psi_0(G)$), where Δ is the basis of Φ determined by the ordering associated to B.

2.4. *Isogenies.* An *isogeny* $\phi\colon G \to G'$ of algebraic groups is a surjective rational homomorphism with finite kernel.

Examples. (i) The canonical homomorphism $\mathbf{SL}(2) \to \mathbf{PSL}(2)$ ($\mathbf{PSL}(2)$ is to be viewed as the group of linear transformations of the space of 2×2-matrices of the form $x \mapsto gxg^{-1}$, where $g \in \mathbf{SL}(2)$). If char $\Omega = 2$ this is an isomorphism of abstract groups, but not of algebraic groups.

(ii) Let G be defined over the finite field $\boldsymbol{F}_q$. The Frobenius isogeny $G \to G$ raises all coordinates to the qth power. It is again an isomorphism of groups, but not of algebraic groups.

Let G and G' be connected reductive and let T be a maximal torus of G. *A central isogeny* $\phi\colon G \to G'$ is an isogeny which (with the notations of 2.3) induces an isomorphism in the sense of algebraic groups of U_α onto its image, for all $\alpha \in \Phi$. Equivalently, $d\phi(X_\alpha) \neq 0$ for all $\alpha \in \Phi$ (where $d\phi$ is the induced Lie algebra homomorphism). The image $T' = \phi(T)$ is a maximal torus of G'. We shall say that ϕ is a central isogeny of (G, T) onto (G', T').

Let $f(\phi)$ be the homomorphism $X^*(T') \to X^*(T)$ defined by ϕ.

2.5. Proposition. (i) *If ϕ is a central isogeny then $f(\phi)$ is an isogeny of $\psi(G', T')$ into $\psi(G, T)$;*

(ii) *if ϕ and ϕ' are central isogenies of (G, T) onto (G', T') such that $f(\phi) = f(\phi')$ then there is $t \in T$ with $\phi' = \phi \circ \mathrm{Int}(t)$.*

That $f(\phi)$ has property (a) of 1.7 is equivalent to the fact that ϕ induces a surjection $T \to T'$ with finite kernel. There is a bijection $\alpha \mapsto \alpha'$ of root systems such that $\phi(U_\alpha) = U_{\alpha'}$ or that $d\phi(X_\alpha) = X_{\alpha'}$ (choosing $X_{\alpha'}$ properly). We then have $\mathrm{Ad}(\phi(t))X_{\alpha'} = \alpha(t)X_{\alpha'}$, whence $f(\phi)(\alpha') = \alpha$. Then ${}^t f(\phi)(\alpha') = (\alpha')^\vee$, as follows for example from the equality ${}^t s_\alpha = s_{\alpha^\vee}$ established in the proof of 1.8. This proves (i).

Let Δ be a basis of Φ. One knows that the U_α with $\alpha \in \Delta$ together with T generate G. So an isogeny ϕ is completely determined by its restriction to T and to the U_α ($\alpha \in \Delta$). Since $f(\phi)$ determines $T \to T'$, the only freedom one has when $f(\phi)$ is given, is in the choice of the isomorphisms $U_\alpha \xrightarrow{\sim} U_{\alpha'}$ ($\alpha \in \Delta$). The assertion of (ii) then readily follows.

2.6. Let ϕ be a central isogeny of (G, T) onto (G', T'). Then $\mathrm{Ker}\,\phi$ lies in T. It is a finite group isomorphic to $\mathrm{Hom}(X/\mathrm{Im}\,f(\phi), \Omega^*)$. Let p be the characteristic exponent of Ω. Then this kernel is isomorphic to the p-regular part of $X/\mathrm{Im}\,f(\phi)$. It follows that there is a factorization of ϕ: $G \xrightarrow{\pi} G/\mathrm{Ker}\,\phi \xrightarrow{\rho} G'$, where π is the canonical homomorphism and where ρ is an isomorphism if $p = 1$ and ρ is a purely inseparable isogeny if $p > 1$ (i.e., such that ρ is an isomorphism of groups). Let $\mathfrak{t}$ be the Lie algebra of T; we have $\mathrm{Ker}(d\phi) \subset \mathfrak{t}$. Now $\mathfrak{t}$ can be identified with $X^\vee \otimes_{\mathbf{Z}} \Omega$. It follows that $\mathrm{Ker}(d\phi)$ is isomorphic to the kernel of $f(\phi)^\vee \otimes \mathrm{id}$: $X^\vee \otimes_{\mathbf{Z}} \Omega \to (X')^\vee \otimes_{\mathbf{Z}} \Omega$, which is isomorphic to $((X')^\vee/p(X')^\vee + \mathrm{Im}\,f(\phi)^\vee) \otimes_{\mathbf{F}_p} \Omega$. Hence $\mathrm{Ker}(d\phi) = 0$ if and only if $\mathrm{Coker}(f(\phi))$ (which is dual to $\mathrm{Coker}({}^t f(\phi))$) has order prime to p.

If $p > 1$ then $\mathrm{Ker}(d\phi)$ is a central restricted subalgebra of $\mathfrak{g}$, which is stable under $\mathrm{Ad}(G)$. Let $G/\mathrm{Ker}(d\phi)$ be the quotient of G by $\mathrm{Ker}\,d\phi$ (see [3, p. 376].) It follows that we can factor ϕ, $G \xrightarrow{\sigma} G/\mathrm{Ker}\,(d\phi) \xrightarrow{\tau} G'$, where σ is the canonical (central) isogeny of [loc. cit.]. These remarks imply that we can factorize ϕ as follows:

$$G = G_0 \xrightarrow{\pi_0} G_1 \xrightarrow{\pi_1} G_2 \xrightarrow{\pi_2} \cdots \longrightarrow G_{s-1} \xrightarrow{\pi_{s-1}} G_s = G', \tag{1}$$

where $\phi = \pi_{s-1} \circ \cdots \circ \pi_0$. Put $\phi_i = \pi_{s-1} \circ \cdots \circ \pi_i$ ($i \geqq 1$). Then $G_1 = G/\mathrm{Ker}\,\phi$, $G_{i+1} = G_i/\mathrm{Ker}(d\phi_i)$ ($1 \leqq i \leqq s - 1$) and the π_i are canonical isogenies.

Also, if

$$\begin{array}{ccc} G & \xrightarrow{\phi} & G' \\ \downarrow & & \downarrow \\ \tilde{G} & \xrightarrow{\tilde{\phi}} & \tilde{G}' \end{array}$$

is a commutative diagram of isogenies, we can arrange the factorizations of ϕ and $\tilde{\phi}$ such that there is a diagram with commuting squares

$$\begin{array}{ccccccccc} G = G_0 & \longrightarrow & G_1 & \longrightarrow & \cdots & \longrightarrow & G_{s-1} & \longrightarrow & G_s = G' \\ \downarrow & & \downarrow & & & & \downarrow & & \downarrow \\ \tilde{G} = \tilde{G}_0 & \longrightarrow & \tilde{G}_1 & \longrightarrow & \cdots & \longrightarrow & \tilde{G}_{s-1} & \longrightarrow & \tilde{G}_s = \tilde{G}' \end{array} \tag{2}$$

Notice that the vertical arrows are uniquely determined once the first one is given.

2.7. LEMMA. *Let ϕ and ϕ_1 be central isogenies of (G, T) onto (G', T') and (G_1', T_1'), respectively. Assume that* $\mathrm{Im}\,f(\phi) = \mathrm{Im}\,f(\phi_1)$. *Then (G', T') and (G_1', T_1') are isomorphic.*

Let $G'' \subset G' \times G_1'$ be the image of the homomorphism $g \mapsto (\phi(g), \phi_1(g))$. Let $\psi\colon G \to G''$ be the induced homomorphism. Then ψ is a central isogeny of connected reductive groups and so are the projections $\pi_1\colon G'' \to G'$, $\pi_2\colon G'' \to G_1'$. A straightforward check (working in tori) shows that Ker $\pi_i = \{e\}$, Ker $d\pi_i = \{0\}$ $(i = 1, 2)$. Hence π_1 and π_2 are isomorphisms, whence the lemma.

2.8. LEMMA. *Let Ψ' be a root datum and let f be an isogeny of Ψ' into $\psi(G, T)$. Then there exist a pair (G', T') and a central isogeny ϕ of (G, T) onto (G', T') such that $\Psi' = \psi(G', T'), f(\phi) = f$.*

From the knowledge of f we can recover successively the groups G_i figuring in (1). This allows one to define G'. We omit the details.

2.9. THEOREM. (i) *For any root datum Ψ with reduced root system there exist a connected reductive group G and a maximal torus T in G such that $\Psi = \psi(G, T)$. The pair (G, T) is unique up to isomorphism;*

(ii) *let $\Psi = \psi(G, T), \Psi' = \psi(G', T')$. If f is an isogeny of Ψ' into Ψ there exists a central isogeny ϕ of (G, T) onto (G', T') with $f(\phi) = f$. Two such ϕ differ by an automorphism* Int(t) $(t \in T)$ *of G.*

Let f be the canonical isogeny $\Psi \to \Psi'$ of 1.7. Using 2.8 we see that it suffices to prove the existence statement of (i) for the two cases that Ψ is semisimple or toral. The second case $(\Phi = \phi)$ is easily dealt with: take for G the torus $T = \mathrm{Hom}(X, \Omega^*)$. In the semisimple case the statement follows from the existence theorem of the theory of semisimple groups which can be dealt with using the theory of Chevalley groups. (See **[18]** or **[6**, part A]. The uniqueness statement of (i) is part of (ii). To prove (ii) one first reduces to the case that f is an isomorphism (using 2.7 and 2.8).) In the case that G is semisimple the statement of (ii) is Chevalley's fundamental isomorphism theorem, proved, e.g., in **[10**, Exposé 24], or in **[14**, Chapter XI]. The case of a torus G is easy. In the general case, there are central isogenies $G_1 \times S \to G$, $G_1' \times S' \to G'$, where G_1 and G_1' are the derived groups of G and G', and where S and S' are tori, such that the corresponding isogenies of root data are just the canonical ones of 1.7 (see also 2.15).

Now f defines an isogeny f_1 of $\psi(G_1' \times S')$ into $\psi(G \times S)$ and we may assume that there exists a central isogeny $\phi_1\colon G_1 \times S \to G_1' \times S'$ with $f(\phi_1) = f_1$.

We can then complete the diagram like (2), with $G_1 \times S$, $G_1' \times S'$, G, G' instead of G, $\tilde{G}$, G', $\tilde{G}'$, respectively, and with ϕ_1 as first arrow. The right-hand arrow, which is uniquely determined by ϕ_1, is then the required isomorphism. The last point of (ii) follows from 2.5 (ii).

2.10. REMARKS. (i) In the semisimple case the existence statement of 2.9(i) is due to Chevalley [Séminaire Bourbaki, Exposé 219, 1960–1961]. He constructs a group scheme G_0 over $\mathbf{Z}$ such that $G = G_0 \times_{\mathbf{Z}} k$. This construction is also discussed in **[6**, part A].

A generalization of 2.9, where the field k is replaced by a base scheme, is contained in **[17**, Exposé XXIV].

(ii) The result on the existence of central isogenies of reductive groups contained in 2.9(ii) is a special case of one on arbitrary isogenies, which we shall briefly indicate. Let $\phi\colon G \to G'$ be an isogeny of connected reductive groups with $\phi(T) = T'$. Let f be the induced homomorphism $X^*(T') \in X^*(T)$. Let x_α, $x_{\alpha'}$ $(\alpha \in \Phi, \alpha' \in \Phi')$

be as in 2.3. One shows that there is a bijection $\alpha \mapsto \alpha'$ of Φ onto Φ' and a function $q: \Phi \to \{p^n | n \in N\}$ (p the characteristic exponent) such that the x_α and $x_{\alpha'}$ can be so normalized that $\phi(x_\alpha(t)) = x_{\alpha'}(t^{q(\alpha)})$. It follows that

$$f(\alpha') = q(\alpha)\alpha, \qquad {}^t f(\alpha^\vee) = q(\alpha)(\alpha')^\vee.$$

If ϕ is a central isogeny then all $q(\alpha)$ are 1. If ϕ is the Frobenius isogeny of 2.4 then f is multiplication by q and all $q(\alpha)$ are q.

Such an f is called a *p-morphism* in [**17**, Exposé XXI]. The analogue in question of 2.9(ii) is obtained by assuming f to be a p-morphism and admitting in the conclusion an arbitrary isogeny ϕ. The proof can be given along similar lines, reducing to the case of an isomorphism. For semisimple G the result is due to Chevalley [**10**, Expose 23].

EXAMPLE OF A p-MORPHISM. $p = 2$, G is semisimple of type $\boldsymbol{B}_2$ and, with the notations of the example in 1.5, we have $f(e_1) = e_1 + e_2, f(e_2) = e_1 - e_2$.

A classification of the possible p-morphisms can be found, e.g., in [**17**, Exposé XXI, p.71].

2.11. Let $\phi: G \to G'$ be a homomorphism of connected reductive algebraic groups. Let T and T' be maximal tori in G, G' with $\phi(T) \subset T'$. Assume that Im ϕ is a *normal* subgroup of G'. We shall briefly describe the relation between the root data $\psi(G, T) = (X, \Phi, X^\vee, \Phi^\vee)$ and $\psi(G', T') = (X', \Phi, (X')^\vee, (\Phi')^\vee)$. Let $f: X' \to X$ be the dual of $\phi: T \to T'$. In general, f is neither injective nor surjective.

Put $\Phi_1 = \Phi \cap \operatorname{Im} f$, $\Phi_2 = \Phi - \Phi_1$. Then $\Phi = \Phi_1 \cup \Phi_2$ is a decomposition into orthogonal subsets (i.e., $\langle \Phi_1, \Phi_2^\vee \rangle = \langle \Phi_2, \Phi_1^\vee \rangle = 0$).

Likewise, if $\Phi_2' = \Phi' \cap \operatorname{Ker} f$, $\Phi_1' = \Phi' - \Phi_2'$, then $\Phi' = \Phi_1' \cup \Phi_2'$ is a decomposition into orthogonal subsets. There is a bijection $\alpha \mapsto \alpha'$ of Φ_1 onto Φ_1' and a function $q : \Phi_1 \to \{p^n | n \in N\}$, such that for $\alpha \in \Phi_1$ we have

$$f(\alpha') = q(\alpha)\alpha, \qquad {}^t f(\alpha^\vee) = q(\alpha)\,(\alpha')^\vee.$$

Moreover $f(\alpha) = 0$ if $\alpha \in \Phi_2'$ and ${}^t f(\alpha) = 0$ if $\alpha \in \Phi_2$.

2.12. It follows readily from 2.9(i) that if Ψ_0 is a based root datum with reduced root system there exists a triple (G, B, T) as in 2.3 with $\psi_0(G, B, T) = \Psi_0$, which is unique up to isomorphism. There is no canonical isomorphism of one such triple onto another one. In fact, based root systems have nontrivial automorphisms.

In this connection the following results should be mentioned. Let $\psi_0(G, B, T) = \psi_0(G)$. For each $\alpha \in \Delta$ fix an element $u_\alpha \neq e$ in the group U_α. The following is then an easy consequence of 2.5(ii).

2.13. PROPOSITION. Aut $\psi_0(G)$ *is isomorphic to the group* $\operatorname{Aut}(G, B, T, \{u_\alpha\}_{\alpha \in \Delta})$ *of automorphisms of G which stabilize B, T and the set of* u_α.

2.14. COROLLARY. *There is a split exact sequence*

$$\{1\} \longrightarrow \operatorname{Int}(G) \longrightarrow \operatorname{Aut}(G) \longrightarrow \operatorname{Aut} \psi_0(G) \longrightarrow \{1\}.$$

In fact, an isomorphism as in 2.13 defines a splitting. Any two such splittings differ by an automorphism $\operatorname{Int}(t)$ $(t \in T)$.

2.15. Let G be a connected reductive group, with a maximal torus T. Put $\psi(G, T) = (X, \Phi, X^\vee, \Phi^\vee)$.

We have a decomposition $G = G' \cdot S$, where G' is the derived group of G (which is semisimple) and where S is a torus, viz. the identity component of the center C of G. We have $T = T' \cdot S$, where T' is a maximal torus of G'.

We use the notations of 1.1 and 1.7.

The following facts can be checked without difficulty.

(a) $\psi(G', T') = (X/X_0, \Phi, \tilde{Q}^\vee, \Phi^\vee)$ (we view Φ as a subset of X/X_0, as we may by 1.2).

(b) The character group $\mathrm{Hom}_{\mathrm{alg.gr.}}(C, \Omega^*)$ is X/Q, and we have $X^*(S) = X/\tilde{Q}$, $X^\vee(S) = X_0^\vee$.

(c) The isogeny $G' \times S \to G$ defines the canonical isogeny of (G, T) (see 1.7). This fact was already used in the proof of 2.5(ii).

It follows that G is semisimple if and only if $\psi(G, T)$ is semisimple. In that case we say that G is *adjoint* if $X = Q$ and *simply connected* if $X = P$ (notation of 1.8). From 2.5(ii), using what was said in 1.8, we see that a semisimple group G is adjoint (resp. simply connected) if and only if a central isogeny ϕ: $G \to G'$ (resp. ψ: $G' \to G$) is an isomorphism.

In the case of a general reductive G we have the following facts.

(d) The derived group G' is adjoint $\Leftrightarrow X = Q \oplus X_0 \Leftrightarrow X^\vee = P^\vee \oplus X_0^\vee$.

(e) G' is simply connected $\Leftrightarrow P \subset X + (X_0 \otimes Q) \Leftrightarrow \tilde{Q}^\vee = Q^\vee$.

(f) The center of G is connected $\Leftrightarrow Q = \tilde{Q} \Leftrightarrow P^\vee \subset X^\vee + (X_0^\vee \otimes Q)$.

3. Reductive groups (relative theory). Here we let a ground field $k \subset \Omega$ come into play. We denote by $\bar{k}$ the algebraic closure of k in Ω and by k_s its separable closure.

A linear algebraic group G which is defined over k will be called a *k-group*. We then denote by $G(k)$ the group of its k-rational points (and not by G_k, as in **[2]**). If A is a k-algebra, we denote by $G(A)$ the group $\mathrm{Hom}_k(k[G], A)$ (see **[2]**).

3.1. *Forms of algebraic groups* **[16**, III, §1]. Let G and G' be k-groups. G' is said to be a *k-form* of G if G and G' are isomorphic over Ω.

EXAMPLE. $k = \boldsymbol{R}$. Then $\boldsymbol{U}(n)$ is an $\boldsymbol{R}$-form of $\mathbf{GL}(n)$.

To describe k-forms one proceeds as follows. The k-group G is completely determined by the group $G(k_s)$ of k_s-rational points. This means the following: if $G \to \mathbf{GL}(n)$ is an isomorphism of G onto a closed subgroup of $\mathbf{GL}(n)$, everything being defined over k, then the subgroup $G(k_s)$ of $\mathbf{GL}(n, k_s)$ determines G, up to k-isomorphism. The fact that G is defined over k is reflected in an action of the Galois group $\Gamma_k = \mathrm{Gal}(k_s/k)$ on $G(k_s)$. The k-forms G' of G can be described as follows (up to k-isomorphism). We have $G'(k_s) = G(k_s)$ and there is a continuous function $c: s \mapsto c_s$ of Γ_k to the group of k_s-automorphisms of G (the Galois group being provided with the Krull topology and the second group with the discrete topology), satisfying

$$(*) \qquad c_{st} = c_s \cdot s(c_t) \qquad (s, t \in \Gamma_k),$$

such that the action of Γ_k on $G'(k_s)$ (denoted by $(s, g) \mapsto s * g$) is obtained by "twisting" the original action with c: $(s * g) = c_s(s \cdot g)$. G' is k-isomorphic to G if and only if there exists an automorphism c such that $c_s = c^{-1} \cdot sc$.

We say that G' is an *inner form* of G if all c_s are inner automorphisms.

If C is a group on which Γ_k acts, the continuous functions $s \mapsto c_s$ of Γ_k to C which satisfy $(*)$ are called 1-cocycles of Γ_k with values in C. The equivalence classes of these cocycles for the relation: $(c_s) \sim (c'_s)$ if and only if there is $c \in C$ such that $c'_s = c^{-1} \cdot c_s \cdot (sc)$, form the 1-cohomology set $H^1(k, C)$. It has a privileged element 1, coming from the constant function $c_s = e$.

3.2. *Reductive k-groups.* Now let G be a connected reductive k-group. It is said to be *quasi-split* if it contains a Borel subgroup which is defined over k (this is a very restrictive property). G is *split* (over k) if it has a maximal torus which is defined over k and k-split. In this case G is quasi-split.

EXAMPLE. $G = SO(F)$ (see [**2**, pp. 15–16]). This is quasi-split but not split if and only if the dimension n of the underlying vector space is even and the index equals $\frac{1}{2}n - 1$.

From the splitting of 2.14 one concludes that G is an inner form of a quasi-split group.

Now let B be a Borel subgroup of G and $T \subset B$ a maximal torus, both defined over k_s. Let $\psi_0(G) = (X, \Delta, X^\vee, \Delta^\vee)$ be the based root datum defined by (G, B, T). If $s \in \Gamma_k$ there is $g_s \in G(k_s)$ such that

$$\operatorname{int}(g_s)(sB) = B, \quad \operatorname{int}(g_s)(sT) = T.$$

Then $\operatorname{int}(g_s) \circ s$ defines an automorphism of T depending only on s (since the coset Tg_s is uniquely determined). This automorphism determines an automorphism $\mu_G(s)$ of X, permuting the elements of Δ (since $\operatorname{int}(g_s) \circ s$ fixes B). It is easy to check that μ_G defines a homomorphism $\mu_G\colon \Gamma_k \to \operatorname{Aut} \psi_0(G)$. Let G' be a k-form of G. Then $\mu_G = \mu_{G'}$ if and only if G and G' are inner forms of each other.

3.3. *Restriction of the base field* [**22**, 1.3]. Let $l \subset k_s$ be a finite separable extension of k. Let G be an l-group. Then there exists a k-group $H = R_{l/k}G$ characterized by the following property [**2**, 1.4]: for any k-algebra A we have $H(A) = G(A \otimes_k l)$. In particular, $H(k) = G(l)$. Let Σ be the set of k-isomorphisms $l \to k_s$. We then have $H(k_s) = G(k_s)^\Sigma$. The action of $\Gamma_k = \operatorname{Gal}(k_s/k)$ on $H(k_s)$ is as follows.

If $\phi \in G(k_s)^\Sigma$ is a function on Σ with values in $G(k_s)$, and $s \in \Gamma_k$, then, for $\sigma \in \Sigma$,

$$(s \cdot \phi)(\sigma) = \phi(s \cdot \sigma).$$

$R_{l/k}G$ is obtained from G by *restriction of the ground field* from l to k. If G is connected or reductive then so is $R_{l/k}\, G$.

Now let G be connected and reductive. Fix B and T (defined over k_s) as in 3.2 and let $\psi_0(G)$ be the based root datum defined by (G, B, T). Then $H = R_{l/k}G$ contains the Borel subgroup $B_1 = B^\Sigma$ and the maximal torus $T_1 = T^\Sigma$. The based root datum $\psi_0(R_{l/k}G)$ (relative to B_1 and T_1) is then $\psi_0(G)^\Sigma$. The action of Γ_k on the lattice X^Σ is like before: if $s \in \Gamma_k$, $\phi \in X^\Sigma$ then $(s \cdot \phi)(\sigma) = \phi(s \cdot \sigma)$.

3.4. *Anisotropic reductive groups.* A connected reductive k-group G is called *anisotropic* (over k) if it has no nontrivial k-split k-subtorus.

EXAMPLES. (i) Let F be a nondegenerate quadratic form on a k-vector space (char $k \neq 2$). Let $G = SO(F)$ be the special orthogonal group of F (the identity component of the orthogonal group $O(F)$). It is anisotropic over k if and only if F does not represent 0 over k (the proof is given in [**2**, p. 13]).

(ii) If k is a locally compact (nondiscrete) field then G is anisotropic over k if and only if $G(k)$ is *compact*.

(iii) If k is any field then G is anisotropic if and only if $G(k)$ has no unipotent elements $\neq e$ and the group of its k-rational characters $\operatorname{Hom}_k(G, \mathbf{GL}_1)$ is trivial.

3.5. *Properties of reductive k-groups.* We next review the properties of reductive groups. The reference for these is [5].

Let G be a connected reductive k-group. Let S be a maximal k-split torus of G, i.e., a k-subtorus of G which is k-split and maximal for these properties. Any two such tori are conjugate over k, i.e., by an element of $G(k)$. Their dimension is called the *k-rank* of G.

The root system $\Phi(G, S)$ of G with respect to S (see 2.1) is called the *relative root system* of G (notation ${}_k\Phi$ or ${}_k\Phi(G)$). This is indeed a root system in the sense of [7], lying in the subspace V of $X^*(S) \otimes \mathbf{Q}$ spanned by ${}_k\Phi$. Its Weyl group is the *relative Weyl group* of G (notation ${}_kW$ or ${}_kW(G)$). Let $N(S)$ and $Z(S)$ denote normalizer and centralizer of S in G; these are k-subgroups. Then $N(S)/Z(S)$ operates on ${}_k\Phi$ and in V. In fact it can be identified with ${}_kW$. Any coset of $N(S)/Z(S)$ can be represented by an element in $N(S)\,(k)$.

$Z(S)$ is a connected reductive k-group. Its derived group $Z(S)'$ is a semisimple k-group which is *anisotropic.* To a certain extent G can be recovered from $Z(S)'$ and the relative root system ${}_k\Phi$ (for details see [19]). There is a decomposition of the Lie algebra $\mathfrak{g}$ of G:

$$\mathfrak{g} = \mathfrak{g}_0 + \sum_{\alpha \in {}_k\Phi} \mathfrak{g}_\alpha$$

where for $\alpha \in X^*(S)$ we have defined $\mathfrak{g}_\alpha = \{X \in \mathfrak{g} \mid \mathrm{Ad}(s)X = s^\alpha X,\ s \in S\}$. Then $\mathfrak{g}_0$ is the Lie algebra of $Z(S)$. If $\alpha \in {}_k\Phi$ there is a unique unipotent k-subgroup U_α of G normalized by S, such that its Lie algebra is $\mathfrak{g}_\alpha$.

In the absolute case ($k = \Omega$) S is a maximal torus, Φ is the ordinary root system and the U_α are as in 2.3. If G is split over k then S is a maximal torus of G and ${}_k\Phi$ coincides with the absolute root system Φ.

In the general case ${}_k\Phi$ need not be reduced, nor is $\dim \mathfrak{g}_\alpha = \dim U_\alpha$ always 1.

3.6. *Parabolic subgroups.* Recall that a parabolic subgroup P of an algebraic group G is a closed subgroup such that G/P is a projective variety. Equivalently, P is parabolic if P contains a Borel subgroup of G.

Now let G be as in 3.5. Then the minimal parabolic k-subgroups of G are conjugate over k. If P is one, there is a maximal k-split torus of G such that P is the semidirect product of k-groups $P = Z(S) \cdot R_u(P)$ ($R_u(P)$ denotes the unipotent radical). There is an ordering of ${}_k\Phi$ such that P is generated by $Z(S)$ and the U_α of 3.5 with $\alpha > 0$. The minimal parabolic k-subgroups containing a given S correspond to the Weyl chambers of ${}_k\Phi$. They are permuted simply transitively by the relative Weyl group.

Fix an ordering of ${}_k\Phi$ and let ${}_k\Delta$ be the basis of ${}_k\Phi$ defined by it. For any subset $\theta \subset {}_k\Delta$ denote by P_θ the subgroup of G generated by $Z(S)$ and the U where $\alpha \in {}_k\Phi$ is a linear combination of the roots of ${}_k\Delta$ in which all roots not in θ occur with a coefficient $\geqq 0$. Then $P_{{}_k\Delta} = G$, $P_\phi = P$ and $P_\theta \supset P$.

The P_θ are the *standard parabolic subgroups* of G containing P. Any parabolic k-subgroup Q of G is k-conjugate to a unique P_θ. If S_θ is the identity component of $\bigcap_{\alpha \in \theta}(\mathrm{Ker}\ \theta)$ then S_θ is a k-split torus of G and we have $P_\theta = Z(S_\theta) \cdot R_u(P_\theta)$, a semidirect product of k-groups. The unipotent radical $R_u(P_\theta)$ is generated by the U_α where α is a positive root which is not a linear combination of elements of θ.

Let Q be any parabolic k-subgroup of G, with unipotent radical V (which is defined over k). A *Levi subgroup* of Q is a k-subgroup L such that Q is the semidirect product of k-groups $Q = L \cdot V$. It follows from the above that such L exist. Two Levi subgroups of Q are k-conjugate. If A is a maximal k-split torus in the centre of L, then $L = Z(A)$. If A is any k-split subtorus of G then there is a parabolic k-subgroup Q of G with Levi subgroup L. Two such Q are not necessarily k-conjugate (as they are when A is a maximal k-split torus). Two parabolic k-subgroups Q_1 and Q_2 are *associated* if they have Levi subgroups which are k-conjugate. This defines an equivalence relation on the set of parabolic k-subgroups.

If Q_1 and Q_2 are two parabolic k-subgroups, then $(Q_1 \cap Q_2) \cdot R_u(Q_1)$ is also a parabolic k-subgroup, contained in Q_1. It is equal to Q_1 if and only if there is a Levi subgroup of Q_1 containing a Levi subgroup of Q_2. Q_1 and Q_2 are called *opposite* if $Q_1 \cap Q_2$ is a Levi subgroup of Q_1 and Q_2.

3.7. *Bruhat decomposition of* $G(k)$. Let P and S be as in 3.5 and put $U = R_u(P)$. If $w \in {}_kW$ denote by n_w a representation in $N(S)(k)$. The *Bruhat decomposition* of $G(k)$ asserts that $G(k)$ is the disjoint union of the double cosets $U(k)n_wP(k)$ $(w \in {}_kW)$.

One can phrase this in a more precise way. If $w \in {}_kW$ there exist two k-subgroups U'_w, U''_w of U such that $U = U'_w \times U''_w$ (product of k-varieties) and that the map $U'_w \times P \to Un_wP$ sending (x, y) onto xn_wy is an isomorphism. We then have

$$(G/P)(k) = G(k)/P(k) = \bigcup_{w \in {}_kW} \pi(U'_w(k)),$$

where π is the projection $G \to G/P$.

If $k = \Omega$ this gives a cellular decomposition of the projective variety G/P.

If $\theta \in {}_k\Delta$ let W_θ be the subgroup of ${}_kW$ generated by the reflections defined by the $\alpha \in {}_k\Delta$. If $\theta, \theta' \in {}_k\Delta$ there is a bijection of double cosets

$$P_\theta(k)\backslash G(k)/P_{\theta'}(k) \simeq W(\theta)\backslash_k W/W(\theta').$$

Let Σ be the set of generators of ${}_kW$ defined by ${}_k\Delta$. The above assertions (except for the algebro-geometric ones) then all follow from the fact that $(G(k), P(k), Z(S)(k), \Sigma)$ is a Tits system in the sense of **[7]**.

3.8. *The Tits building.* Let G be the connected reductive k-group. We define a simplicial complex $\mathscr{B}$, the (simplicial) *Tits building of* (G, k), as follows.

The vertices of $\mathscr{B}$ are the maximal nontrivial parabolic k-subgroups of G. A set $(P_1, \cdots, P_n)$ of distinct vertices determines a simplex of $\mathscr{B}$ if and only if $P = P_1 \cap \cdots \cap P_n$ is parabolic. In that case, the P_i are uniquely determined by P. It follows that the simplices of $\mathscr{B}$ correspond to the nontrivial parabolic k-subgroups of G. Let σ_P be the simplex defined by P. Then σ_P is a face of $\sigma_{P'}$ if and only if $P' \subset P$. The maximal simplices correspond to minimal k-parabolics. These simplices are called *chambers*. A codimension 1 face of a chamber is a *wall*. Two chambers are *adjacent* if they are distinct and have a wall in common. One shows that any two chambers σ, σ' can be joined by a gallery, i.e., a set of chambers $\sigma = \sigma_0, \sigma_1, \cdots, \sigma_s = \sigma'$, such that σ_i and σ_{i+1} are adjacent $(0 \leqq i < s)$.

It is clear that $G(k)$ operates on $\mathscr{B}$.

One can show (using a concrete geometric realization of the abstract simplicial complex $\mathscr{B}$) that $\mathscr{B}$ has the homotopy type of a bouquet of spheres.

For more details about buildings see **[20]**.

3.9. EXAMPLES. (i) The preceding results apply when $k = \Omega$, the absolute case. In particular, we then have the properties of parabolics of 3.6 and the Bruhat decomposition of 3.7.

(ii) $G = \mathbf{GL}(n)$ (k arbitrary). This is indeed a reductive k-group. Its Lie algebra $\mathfrak{g}$ is the Lie algebra of all $n \times n$-matrices.

Let S be the subgroup of diagonal matrices. This is a maximal k-split torus which is also a maximal torus of G (in the absolute sense). Let $e_i \in X = X^*(S)$ map $s \in S$ onto its ith diagonal element. The e_i form a basis of X. The root system $\Phi = \Phi(G, S)$, which coincides with the relative root system ${}_k\Phi$, consists of the $e_i - e_j \in X$ with $i \neq j$. One checks that the root datum of G is given by $X = \boldsymbol{Z}^n$, $X^\vee = \boldsymbol{Z}^n$, $\Phi = \{e_i - e_j\}_{i \neq j}$, $\Phi^\vee = \{e_i^\vee - e_j^\vee\}_{i=j}$, where $(e_i^\vee)$ is the basis of X dual to (e_i).

The subgroup B of all upper triangular matrices is a minimal parabolic k-subgroup. It is a Borel subgroup. Its unipotent radical U is the group of all upper triangular matrices with ones in the diagonal. The basis Δ of Φ defined by B is $(e_i - e_{i+1})_{1 \leqq i \leqq n-1}$. The Weyl group W (which coincides with the relative Weyl group ${}_kW$) is isomorphic to the symmetric group $\mathfrak{S}_n$, viewed as the group of permutations of the basis (e_i).

The parabolic subgroups $P \supset B$ are the groups of block matrices

$$\begin{pmatrix} A_{11} & A_{12} & \cdots & A_{1s} \\ 0 & A_{22} & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & \cdots & A_{ss} \end{pmatrix}$$

where A_{ij} is an $n_i \times n_j$-matrix with $n_1 + \cdots + n_s = n$, the A_{ii} being nonsingular. Its unipotent radical consists of these matrices where $A_{ii} = 1$ ($1 \leqq i \leqq s$). The subgroup of P of matrices with $A_{ij} = 0$ for $j > i$ is a Levi subgroup of P. The center of L consists of those elements of S at which the elements of Δ different from one of the $e_{n_i} - e_{n_i+1}$ ($1 \leqq i \leqq s$) are trivial. Hence with the notations of 3.6, we have $P = P_\theta$, where $\theta \in \Delta$ is the complement of the set of these roots.

A more geometric description of parabolic subgroups is as follows.

Let $V = \Omega^n$. A *flag* in V is a sequence $0 = V_0 \subset V_1 \subset \cdots \subset V_s = V$ of distinct subspaces of V. A *k-flag* is one where all V_i are defined over k, i.e., have a basis consisting of vectors in k^n.

G operates on the set of all flags. The parabolic subgroups of G are then the isotropy groups of flags. One sees that there is a bijection of the set of all parabolic subgroups of G onto the set of all flags, under which k-subgroups correspond to k-flags.

If P is a parabolic subgroup, then the points of G/P can be viewed as the flags of the same type as P (i.e., such that the subspaces of the flags have a constant dimension).

The Tits building of (G, k) can then also be described in terms of flags: The simplices correspond to the nontrivial k-flags (i.e., those with $s > 1$). If σ_f is the simplex defined by the flag f, then σ_f is a face of $\sigma_{f'}$ if and only if f' refines f (in the obvious sense). The chambers correspond to the maximal flags ($s = n$, $\dim V_i = i$) and the vertices of $\mathscr{B}$ are described by the nontrivial k-subspaces of V. We see that the combinatorial structure of $\mathscr{B}$ pictures the incidences in the projective space $\boldsymbol{P}_{n-1}(k)$.

The smallest nontrivial special case is $n = 3$, $k = \boldsymbol{F}_2$. Here $\mathscr{B}$ is a graph with 14 vertices and 21 edges (drawn in [**20**, p. 210]).

(iii) Let char $k \neq 2$. Let V be a vector space over k (in the sense of algebraic geometry). Let F be a nondegenerate quadratic form on V which is defined over k. With respect to a suitable basis of $V(k)$ we have

$$F(x_1, \cdots, x_n) = x_1x_n + x_2x_{n-1} + \cdots + x_qx_{n-q+1} + F_0(x_{q+1}, \cdots, x_{n-q}),$$

where F_0 is anisotropic over k (i.e., does not represent zero nontrivially). The index q of F is the dimension of the maximal isotopic subspaces of $V(k)$.

Let $G = SO(F)$ be the special orthogonal group of F. It is a connected semisimple k-group. A maximal k-split torus S in G is given by the matrices of the form

$$\operatorname{diag}(t_1, \cdots, t_q, 1, \cdots, 1, t_1^{-1}, \cdots, t_q^{-1}).$$

Then $Z(S)$ is the direct product of S and the anisotropic k-group $SO(F_0)$.

For a description of a minimal parabolic k-subgroup and the determination of the relative root system ${}_k\Phi$ we refer to [**2**, p. 16]. The latter is of type B_q if $2q \neq n$ and of type D_q otherwise. If $q < [n/2]$ there are always subgroups U_α of dimension > 1 (notations of 3.5).

A geometric description of parabolic k-subgroups similar to the one for $\mathbf{GL}(n)$ can be given. They are in this case the isotropy groups of *isotropic* k-*flags* in V, i.e., flags all of whose subspaces are isotropic with respect to F.

4. Special fields. Let G be a k-group. In this section we discuss some special features for particular k.

4.1. ***R** and **C***. If G is a $\boldsymbol{C}$-group then $G(\boldsymbol{C})$ has a canonical structure of complex Lie groups. The latter is connected if and only if G is Zariski-connected (this can be deduced from Bruhat's lemma, compare 4.2).

Now let $k = \boldsymbol{R}$. Then $G(\boldsymbol{R})$ is canonically a Lie group.

4.2. LEMMA. (i) $G(\boldsymbol{R})$ *is compact if and only if the identity component* G^0 *is a reductive anisotropic* $\boldsymbol{R}$-*group*;

(ii) $G(\boldsymbol{R})$ *has finitely many connected components.*

(i) is easily established. As to (ii), it suffices to prove this if G is connected reductive. In that case one reduces the statement, via Bruhat's lemma, to the case that G is either compact or a torus. In these cases the assertion is clear.

$G(\boldsymbol{R})$ need not be connected if G is Zariski-connected, as one sees in simple cases (e.g., $G = \mathbf{GL}(n)$).

If G is a $\boldsymbol{C}$-group then the real Lie group $R_{\boldsymbol{C}/\boldsymbol{R}}(G)(\boldsymbol{R})$ (see 3.3) is that defined by the complex Lie group $G(\boldsymbol{C})$.

4.3. *Finite fields.* Let $k = \boldsymbol{F}_q$ and let $\bar{k}$ be an algebraic closure. F denotes the Frobenius automorphism $x \mapsto x^q$ of $\bar{k}/k$. The basic result here is Lang's theorem [**6**, p. 171].

4.4. THEOREM. *If G is a connected k-group then $g \mapsto g^{-1}(Fg)$ is a surjective map of $G(\bar{k})$ onto itself.*

Using that G is an inner form of a quasi-split k-group (see 3.1) one deduces that *a connected reductive k-group is quasi-split.* A complete classification of simple k-groups can then be given.

Before continuing with local and global fields, we must say a little about group schemes over rings.

4.5. *Groups over rings.* If G is a k-group, the product and inversion are described (see [**2**, p. 4]) by morphisms of algebraic varieties $\mu: G \times G \to G$, $\rho: G \to G$, which in turn are given by homomorphisms of k-algebras $\mu^*: k[G] \to k[G] \otimes_k k[G]$ and $\rho^*: k[G] \to k[G]$. These have a number of properties (which we will not write down) expressing the group axioms. We thus obtain a description of the notion of linear algebraic group in terms of the coordinate algebra.

The fact that k is a field does not play any role in this description.

Replacing k by a commutative ring $\mathfrak{o}$, we get a notion of "linear algebraic group G over $\mathfrak{o}$", which is habitually called "affine group scheme G over $\mathfrak{o}$", which we abbreviate to $\mathfrak{o}$-group. It can be viewed as a functor, cf. [**2**, p. 4]. We write $G(\mathfrak{o})$ for the group of $\mathfrak{o}$-points of G (i.e., the value of the functor at $\mathfrak{o}$).

Let $\mathfrak{o}[G]$ be its algebra. If $\mathfrak{o}'$ is an $\mathfrak{o}$-algebra we have, by base extension, an $\mathfrak{o}'$-group $G \times_{\mathfrak{o}} \mathfrak{o}'$, with algebra $\mathfrak{o}[G] \otimes_{\mathfrak{o}} \mathfrak{o}'$.

Let $\mathfrak{m}$ be a maximal ideal of $\mathfrak{o}$ and put $k(\mathfrak{m}) = \mathfrak{o}/\mathfrak{m}$; this is an $\mathfrak{o}$-algebra.

DEFINITION. The $\mathfrak{o}$-group G has *good reduction* at $\mathfrak{m}$ if $G \times_{\mathfrak{o}} k(\mathfrak{m})$ is a $k(\mathfrak{m})$-group. G is *smooth* if it has good reduction at all maximal ideals $\mathfrak{m}$.

EXAMPLE OF BAD REDUCTION. $\mathfrak{o} = \boldsymbol{Z}$, G is the group of matrices

$$\begin{pmatrix} a & b \\ 2b & a \end{pmatrix}$$

with $a^2 - 2b^2 = 1$. Then $\boldsymbol{Z}[G] = \boldsymbol{Z}[X, Y]/(X^2 - 2Y^2 - 1)$ and $\boldsymbol{Z}[G] \otimes \boldsymbol{F}_2 \simeq \boldsymbol{F}_2[X, Y]/(X^2)$, which cannot be the coordinate ring of a linear algebraic group over $\boldsymbol{F}_2$, since it contains nilpotent elements.

Now let G be a k-group and let $\mathfrak{o}$ be a subring of k. We shall say that G is *definable over* $\mathfrak{o}$ if there exists a smooth $\mathfrak{o}$-group G_0 such that $G \simeq G_0 \times_{\mathfrak{o}} k$. By abuse of notation, we sometimes write, if $\mathfrak{o}'$ is an $\mathfrak{o}$-algebra, $G(\mathfrak{o}')$ for $G_0(\mathfrak{o}')$. One can also define when an algebraic variety over k is definable over $\mathfrak{o}$; it is clear how to do this.

EXAMPLE. By a theorem of Chevalley a complex connected semisimple group is definable over $\boldsymbol{Z}$ [**6**, A, §4].

4.6. *Local fields.* Let k be a local field. We denote by $\mathfrak{o}$ its ring of integers and by $\mathfrak{m}$ the maximal ideal of $\mathfrak{o}$. The residue field $\mathfrak{o}/\mathfrak{m}$ is denoted by F.

A profound study of reductive groups over local field has been made by Bruhat and Tits. So far, only part of this has been published [**9**]. For a résumé see [**8**]. In Tits' contribution [**21**] in these PROCEEDINGS more details are given about the Bruhat-Tits theory. In particular, he discusses the building of a reductive k-group and the theory of maximal compact subgroups. Here we mention only a few results.

4.7. LEMMA. *Let G be a connected reductive k-group. Then there is an unramified extension l of k such that G is quasi-split over l.*

This can be deduced from the fact that a maximal unramified extension of k is a field of dimension $\leqq 1$ (see [**16**, p. II-11]).

The group $G(k)$ of k-rational points is a locally compact topological group (even a Lie group over k). It is a compact group if and only if the identity component G^0 is a reductive anisotropic k-group. It is shown in the Bruhat-Tits theory that if G is connected and simply connected simple k-group, there is a finite-dimensional division algebra D with center k such that $G(k) \simeq \mathbf{SL}(1, D)$.

If G is connected and reductive and is definable over $\mathfrak{o}$ (which is always the case if G is k-split, see [**8**, p.31]), then $G(\mathfrak{o})$ is a compact subgroup of $G(k)$. There is a reduction map $G(\mathfrak{o}) \to G(F)$, which is *surjective.*

4.8. *Global fields.* Now let k be a global field. If v is a valuation of k, let k_v be the corresponding completion of k. If v is nonarchimedean, $\mathfrak{o}_v$ denotes the ring of integers of k_v. If S is a nonempty finite set of valuations of k, containing all the archimedean ones, denote by $\mathfrak{o}_S$ the ring of elements of k which are integral outside S. It is a Dedekind ring.

Let G be a connected reductive k-group.

4.9. LEMMA. (i) *There is an S such that G is definable over $\mathfrak{o}_S$;*

(ii) *$G \times_k k_v$ is quasi-split for almost all v.*

(i) is easily established. Let C be the group of inner automorphisms of G; it is a semisimple k-group. We identify it with its group of k_s-rational points. There is $\gamma \in H^1(k, C)$ such that G, twisted by a cocycle c from γ (see 3.1), is quasi-split. This is another way of saying that G is an inner form of a quasi-split group. Now c defines a principal homogeneous space C_c of C over k, i.e., an algebraic variety over k, on which C acts simply transitively, the action being defined over k (see [**16**, p. I-58]). We have $\gamma = 1$ if and only if C_c has a k-rational point. To prove (ii) it now suffices to show that the image of γ in $H^1(k_v, C \times_k k_v)$ is trivial for almost all v, or that C_c has a k_v-rational point for almost all v.

Let v be nonarchimedean such that $C \times_k k_v$ and $C_c \times_k k_v$ are definable over $\mathfrak{o}_v$, say $C \times_k k_v = C_0 \times_{\mathfrak{o}_v} k_v$, $C_c \times_k k_v = C_{c,0} \times_{\mathfrak{o}_v} k_v$. Assume furthermore that the reduced group $C_0 \times_{\mathfrak{o}_v} F_v$, over the residue field F_v, is a connected F_v-group. These conditions are satisfied for almost all v. By 4.4, it follows that $C_{c,0} \times_{\mathfrak{o}_v} F_v$ has an F_v-rational point. A version of Hensel's lemma then gives that $C_{c,0}$ has an $\mathfrak{o}_v$-rational point, which shows that C has a k_v-rational point. This implies (ii), as we have seen.

4.10. *Adelization.* Let A be the adele ring of k. It is a k-algebra, so the group of A-points $G(A)$ of G is defined. Let $G \hookrightarrow \mathbf{GL}_n$ be an embedding over k. Then $g \mapsto (g, g^{-1})$ maps $G(A)$ bijectively onto a closed subset of $A^{n^2+1} \oplus A^{n^2+1}$. Endowed with the induced topology, $G(A)$ is a locally compact group, the *adele group* of G. It has $G(k)$ as a discrete subgroup. The topology on $G(A)$ is independent of the choice of the embedding $G \hookrightarrow \mathbf{GL}(n)$. An alternative way to define $G(A)$ is as follows. Let S_0 have the property of 4.9(i). For each finite set of valuations $S \supset S_0$, the group

$$G(A_S) = \prod_{v \notin S} G(\mathfrak{o}_v) \times \prod_{v \in S} G(k_v)$$

is a locally compact group. If $S \subset S'$ then $G(A_S) \subset G(A_{S'})$. $G(A)$ can also be defined as the limit group $G(A) = \text{inj lim}_{S \supset S_0} G(A_S)$ (this is independent of the choice of S_0).

For each v, we have an injection $G(k_v) \to G(A)$.

EXAMPLES. (a) $G = \mathbf{GL}(1)$. Then $G(A)$ is the group of idèles (the units of A).

(b) $k = \boldsymbol{Q}$, $G = \mathbf{SL}(n)$. One checks that

$$G(A) = \mathbf{SL}(n, \boldsymbol{R}) \cdot \Big(\prod_p \mathbf{SL}(n, \boldsymbol{Z}_p)\Big) \cdot G_{\boldsymbol{Q}},$$

from which one sees that there is a surjective continuous map $G(A)/G(\boldsymbol{Q}) \to \mathbf{SL}(n, \boldsymbol{R})/\mathbf{SL}(n, \boldsymbol{Z})$. More precisely, if one defines, for a positive integer N, the consequence subgroup $\Gamma(N)$ of $\mathbf{SL}(n, \boldsymbol{Z})$ by $\Gamma(N) = \{\gamma \in \mathbf{SL}(n, \boldsymbol{Z}) \mid \gamma \equiv 1 (\mathrm{mod}\ N)\}$, then

$$G(A)/G(\boldsymbol{Q}) = \text{proj lim } \mathbf{SL}(n, \boldsymbol{R})/\Gamma(N).$$

(c) Let G be a $\boldsymbol{Q}$-group and let $G \hookrightarrow \mathbf{GL}(n)$ be an embedding (over $\boldsymbol{Q}$). Fix a lattice L in $\boldsymbol{Q}^n$ and let Γ be the subgroup of $G(\boldsymbol{Q})$ of elements stabilizing L.

There exists a connection, similar to that of the previous examples, between $G(A)/G(\boldsymbol{Q})$ and $G(\boldsymbol{R})/\Gamma$ (see [1]).

The main results about $G(A)/G(k)$ are as follows (G a connected reductive k-group). Let X be the group of k-rational characters of G. For each $\chi \in X$ define a character $|\chi|: G(A) \to \boldsymbol{R}^*$ by $|\chi|((g_v)) = \prod_v |\chi(g_v)|_v$, where $|\ |_v$ is an absolute value, normalized so as to satisfy the product formula. Let $G(A)^0$ be the intersection of kernels of the $|\chi|$, for $\chi \in X$.

The product formula shows that $G(k) \subset G(A)^0$.

4.11. THEOREM. (i) *$G(A)^0/G(k)$ has finite invariant volume;*

(ii) (*G semisimple*) *$G(A)^0/G(k)$ is compact if and only if G is anisotropic over k.*

This is a consequence of reduction theory, due to Borel and Harish-Chandra for number fields and to Harder for function fields (see [1] and [11]). Notice, that by restriction of the ground field, it suffices to prove this for $k = \boldsymbol{Q}$ or $k = \boldsymbol{F}_q(T)$.

5. A class of Lie groups. In this section we discuss a class of Lie groups close to the groups of real points of reductive $\boldsymbol{R}$-groups. This is the class of groups occurring in Wallach's paper in these PROCEEDINGS (see also [12]). We shall indicate briefly how the properties of these groups can be deduced from the algebraic properties of reductive groups, discussed above.

We shall say that an algebraic group $\boldsymbol{G}$ defined over a field of characteristic zero is reductive if its identity component G^0 (in the Zariski topology) is so.

5.1. Let G be a Lie group, with Lie algebra $\mathfrak{g}$. Its identity component is denoted by G^0. We denote by ${}^0G$ the intersection of the kernels of all continuous homomorphisms $G \to \boldsymbol{R}^*_+$. Then ${}^0G$ is a closed normal subgroup and $G/{}^0G$ is a vector group.

A *split component* of G is a vector subgroup V of G such that $G = {}^0G \cdot V$, ${}^0G \cap V = \{e\}$.

We assume henceforth that G possesses the following properties:

(1) *There is a reductive $\boldsymbol{R}$-group $\boldsymbol{G}$ and a morphism $\nu: G \to \boldsymbol{G}(\boldsymbol{R})$ with finite kernel whose image is an open subgroup of $\boldsymbol{G}(\boldsymbol{R})$.*

It follows that ν induces an isomorphism of $\mathfrak{g}$ onto the set of real points of Lie $\boldsymbol{G}$. We shall often identify $\mathfrak{g}$ and $\nu(\mathfrak{g})$. It also follows that $\nu(G)^0 = \boldsymbol{G}(\boldsymbol{R})^0$ and that G^0 has finite index in G (since this is so for $\boldsymbol{G}(\boldsymbol{R})$, see 4.2(ii), and ker ν is finite).

(2) *The image of G in the automorphism group of $\mathfrak{g}_C = \mathfrak{g} \otimes_{\boldsymbol{R}} \boldsymbol{C}$ lies in the image of the identity component $\boldsymbol{G}^0$ of $\boldsymbol{G}$.*

The main reason to allow for finite coverings of linear groups is to include the metaplectic group and all connected semisimple groups with finite center. The main use of (2) is to insure that G acts trivially on the center of the universal enveloping algebra of $\mathfrak{g}$.

5.2. Let θ_0 be the automorphism of $\mathbf{GL}(n, \boldsymbol{R})$ which sends g to ${}^t g^{-1}$. Let $\mathfrak{S}$ (resp. S_0) be the set of real symmetric (resp. and positive nondegenerate) $n \times n$-matrices. Then $\exp: x \mapsto \exp x = 1 + x + x^2/2! + \cdots$ is an isomorphism of $\mathfrak{S}$ onto S_0. Given $s \in S_0$, there is a unique analytic subgroup of $\mathbf{GL}(n, \boldsymbol{R})$, isomorphic to $\boldsymbol{R}$, contained in S_0, and passing through s. It is contained in any θ_0-stable Lie subgroup of $\mathbf{GL}(n, \boldsymbol{R})$ with finitely many connected components which contains s.

5.3. Lemma. *Let $\boldsymbol{G} \subset \mathbf{GL}(n, \boldsymbol{C})$ be an $\boldsymbol{R}$-subgroup stable under θ_0.*

(i) *Let $s \in S_0 \cap \boldsymbol{G}(\boldsymbol{R})$. Then there exists a θ_0-stable $\boldsymbol{R}$-split torus $\boldsymbol{S}$ of $\boldsymbol{G}$ such that $s \in \boldsymbol{S}(\boldsymbol{R})^0$;*

(ii) *let $X \in \mathfrak{S} \cap \mathrm{Lie}(\boldsymbol{G})$. There is a θ_0-stable $\boldsymbol{R}$-split subtorus of $\boldsymbol{G}$ whose Lie algebra contains X;*

(iii) *$\boldsymbol{G}$ is reductive.*

The element s generates an infinite subgroup of G, whose Zariski closure is a torus with the required properties. (ii) follows from (i), applied to $\exp x$. Let $\boldsymbol{U}$ be the unipotent radical of $\boldsymbol{G}$, let $s \in \boldsymbol{G}(\boldsymbol{R})$. Then s and $(\theta_0 s)s^{-1}$ are unipotent. By (i) the last element is also semisimple, which implies $s = 1$. Hence $\boldsymbol{U} = \{1\}$. This proves (iii).

5.4. By definition, a Cartan involution of $\mathbf{GL}(n, \boldsymbol{R})$ is an automorphism conjugate to θ_0 by an inner automorphism. Let G and $\boldsymbol{G}$ be as in 5.1. Let $\boldsymbol{G} \subset \mathbf{GL}(n, \boldsymbol{C})$ be an embedding over $\boldsymbol{R}$. Then $\nu(G)$ is stable under some Cartan involution θ of $\mathbf{GL}(n, \boldsymbol{R})$. In other words, we may assume, after conjugation, that $\nu(G)$ is stable under θ_0 (in which case it is said to be selfadjoint) [1], [15]. Let $\mathfrak{k}$ (resp. $\mathfrak{S}$) be the fixed point (resp. -1 eigenspace) of θ in $\mathfrak{g}$, and K the inverse image in G of the fixed point set of θ in $\nu(G)$.

5.5. Proposition. *The automorphism θ of $\mathfrak{g}$ extends uniquely to an automorphism of G whose fixed point set is K. The map μ: $(k, x) \mapsto k \cdot \exp x$ is an isomorphism of analytic manifolds of $K \times \mathfrak{S}$ onto G.*

The automorphisms of G thus defined are the Cartan involutions of G. They form one conjugacy class with respect to inner automorphisms by elements of G^0. The decomposition $G = K \cdot S$ ($S = \exp \mathfrak{S}$) is a Cartan decomposition of G.

After conjugation, we may assume that $\theta = \theta_0$. If $G = \mathbf{GL}(n, \boldsymbol{R})$, then $K = \boldsymbol{O}(n)$, $S = S_0$ and our assertion follows from the polar decomposition of real matrices. Assume now that ν is the identity. Write $g \in G$ as a product $g = k \cdot s$ where $k \in \boldsymbol{O}(n)$, $s \in S_0$. Then $s^2 = (\theta_0\, g)^{-1} \cdot g \in G$, and the unique 1-parameter subgroup in S_0 through s^2 (see 5.2) is contained in G. In this group, there is a unique element with square s^2, which must then be equal to s. Thus $s \in G$, hence also $k \in G$. This implies that μ is surjective. Injectivity follows from the uniqueness of the polar decomposition. The decomposition $\mathfrak{g} = \mathfrak{k} + \mathfrak{S}$ implies that the tangent map at any point is bijective; hence μ is an analytic isomorphism. Thus G is the direct product of K and a euclidean space.

This proves the proposition when ν is the identity. Let $\tilde{G}$ be the simply connected group with Lie algebra $\mathfrak{g}$, $\tilde{K}$ the analytic subgroup of $\tilde{G}$ with Lie algebra $\mathfrak{k}$ and $\pi: \tilde{G} \to G^0$ the natural projection. Since the fundamental group of K is that of G^0, the group $\tilde{K}$ is the universal covering of K; hence $\ker \pi \subset \tilde{K}$. The automorphism θ

of $\mathfrak{g}$ extends to one of $\tilde{G}$, which fixes $\tilde{K}$ pointwise, hence acts trivially on ker π, and goes down to an automorphism of G^0. The result for $\nu(G)$ implies that $G = K \cdot S = K \cdot G^0$. Since ker ν acts trivially on $\mathfrak{g}$, hence on G^0, $\theta: G^0 \to G^0$ extends obviously to an automorphism of G which fixes K pointwise. The remaining assertions are then obvious.

5.6. COROLLARY. (i) *K is a maximal compact subgroup of G;*
(ii) *K meets every connected component of G.*

G is the topological product of K by a connected space S, whence (ii). The first assertion follows from the fact that every $s \neq 1$ in S generates an infinite discrete subgroup.

Fix a Cartan involution θ of $\boldsymbol{GL}(n, \boldsymbol{R})$ stabilizing $\nu(\boldsymbol{G})$. We also denote by θ the Cartan involution of G defined in 5.5.

5.7. Let C be the center of G. It has again the properties (1), (2) and it is θ-stable. The group corresponding to $\boldsymbol{G}$ is the center of $\boldsymbol{G}$. The subset corresponding to S is now a vector group V. It is, in fact, *the* maximal θ-stable vector subgroup contained in C. Let G_1 be the derived group of G.

5.8. LEMMA. (i) *${}^0G = KG_1$ and V is a split component of G;*
(ii) *${}^0G$ has the properties* (1), (2) *and is θ-stable.*

KG_1 is a θ-stable closed normal subgroup of G, contained in ${}^0G$. The Lie algebra $\mathfrak{g}$ is the direct sum of those of KG_1 and of V, which implies (i). As to (ii), for the algebraic group of (1) we take the Zariski closure of $\nu({}^0G)$ in $\boldsymbol{G}$. Its identity component differs from $\boldsymbol{G}^0$ only in its center. This implies (2), and the final assertion is clear.

5.9. *Parabolic subgroups.* A parabolic subalgebra $\mathfrak{p}$ of $\mathfrak{g}$ is a subalgebra such that $\mathfrak{p}_C$ is the Lie algebra of a parabolic $\boldsymbol{R}$-subgroup of $\boldsymbol{G}^0$. A *parabolic subgroup P* of G is the normalizer in G of a parabolic subalgebra (which then is the Lie algebra of P). The parabolic subgroups of G correspond to the parabolic $\boldsymbol{R}$-subgroups of $\boldsymbol{G}^0$.

Let $\boldsymbol{P}$ be a parabolic $\boldsymbol{R}$-subgroup of $\boldsymbol{G}^0$. Let $\boldsymbol{N}$ be its unipotent radical. Put $\boldsymbol{L} = \boldsymbol{P} \cap \theta\boldsymbol{P}$.

5.10. LEMMA. (i) *$\boldsymbol{L}$ is a Levi subgroup of $\boldsymbol{P}$;*
(ii) *the Lie algebra of $\boldsymbol{G}$ is the direct sum of those of $\boldsymbol{N}$, $\theta\boldsymbol{N}$ and $\boldsymbol{L}$.*

5.3(iii) shows that $\boldsymbol{L}$ is reductive. $\boldsymbol{LN}$ is a parabolic $\boldsymbol{R}$-subgroup of $\boldsymbol{G}^0$ contained in $\boldsymbol{P}$ (see 3.6) with unipotent radical $\boldsymbol{N}$, hence equal to $\boldsymbol{P}$. This proves (i). Then (ii) follows by using that $\boldsymbol{P}$ and $\theta\boldsymbol{P}$ are opposite parabolics.

5.11. Let $\boldsymbol{S}$ be the maximal $\boldsymbol{R}$-split torus in the center of $\boldsymbol{L}$. Put $A = \nu^{-1}(\boldsymbol{S}(\boldsymbol{R}))^0$, $N = \nu^{-1}(\boldsymbol{N}(\boldsymbol{R}))^0$. These are subgroups of G. Since A is a θ-stable vector group, we have $\theta a = a^{-1}$ for all $a \in A$. Let P be the parabolic subgroup of G defined by $\boldsymbol{P}$ and put $L = P \cap \theta P$, $M = {}^0L$. Then L is the centralizer of A in G. Also, L and M are θ-stable.

5.12. PROPOSITION. (i) *L satisfies* (1), (2) *of* 5.1. *A is a split component of L and of P;*
(ii) *$(m, a, n) \mapsto man$ defines an analytic diffeomorphism of $M \times A \times N$ onto P.*

Let $\boldsymbol{H}$ be the centralizer of $\boldsymbol{S}$ in $\boldsymbol{G}$. Then $\nu(L) \subset \boldsymbol{H}$. Moreover, the identity component $\boldsymbol{H}^0$ is reductive and is equal to $\boldsymbol{H} \cap \boldsymbol{G}^0$ (the last point because centralizers of tori in Zariski-connected algebraic groups are connected, see [**3**, p. 271]). It follows that (1) and (2) hold for L. The last point of (i) is easy. From $\boldsymbol{P} = \boldsymbol{LN}$ we conclude that $P = LN = MAN$. Now (ii) readily follows.

The decomposition of 5.12(ii) is called the *Langlands decomposition* of P. There is a similar decomposition of the Lie algebra of P: $\mathfrak{p} = \mathfrak{m} + \mathfrak{a} + \mathfrak{n}$.

A *parabolic pair* in G is a pair (G, A) where P is a parabolic subgroup of G and A is as above.

5.13. *Minimal parabolic subgroups.* Now assume that P is a minimal parabolic subgroup. Then $\boldsymbol{P}$ is a minimal parabolic $\boldsymbol{R}$-subgroup of $\boldsymbol{G}^0$. In that case the derived group of $\boldsymbol{L}$ is an anisotropic semisimple $\boldsymbol{R}$-group. It follows that M is compact. We then must have $M \subset K$ (recall that M is θ-stable), so $M = K \cap P$. Let Φ be the root system of $(\boldsymbol{G}, \boldsymbol{S})$ (see 3.5). If $\alpha \in \Phi$ put $\mathfrak{g}_\alpha = \{X \in \mathfrak{g} \mid \mathrm{Ad}(a)X = a^\alpha X, a \in A\}$. Then we also have

$$\mathfrak{g}_\alpha = \{X \in \mathfrak{g} \mid [H, X] = d\alpha(H)X, H \in \mathfrak{a}\}$$

($\mathfrak{a}$ being the Lie algebra of A). Also, there is an ordering of Φ such that $\mathfrak{n} = \sum_{\alpha>0} \mathfrak{g}_\alpha$, $\theta\mathfrak{n} = \sum_{\alpha<0} \mathfrak{g}_\alpha$ ($\mathfrak{n}$ and $\theta\mathfrak{n}$ are the Lie algebras of N and θN).

5.14. Lemma. (i) *We have direct sum decompositions*

$$\mathfrak{g} = \mathfrak{a} + \mathfrak{m} + \mathfrak{n} + \theta\mathfrak{n}, \qquad \mathfrak{g} = \mathfrak{k} + \mathfrak{a} + \mathfrak{n};$$

(ii) $\mathfrak{a}$ *is a maximal commutative subalgebra of* $\mathfrak{s}$.

The first decomposition follows by using 5.10(ii). It then follows that $\mathfrak{k}$ is the direct sum of $\mathfrak{m}$ and the space of all $X + \theta X\,(X \in \mathfrak{n})$. Hence $\mathfrak{k} \cap \mathfrak{n} = \{0\}$, $\mathfrak{k} + \mathfrak{n} = \mathfrak{m} + \mathfrak{n} + \theta\mathfrak{n}$. This gives the second decomposition. We also get that $\mathfrak{s}$ is the direct sum of $\mathfrak{a}$ and the space of all $X - \theta X\,(X \in \mathfrak{n})$. Since $\mathfrak{a}$ commutes with no nonzero element of $\mathfrak{n} + \theta\mathfrak{n}$, the assertion of (ii) follows.

From the above we see that Φ is also the root system of the symmetric pair (G, K) (see [**13**, Chapter VII]).

5.15. Proposition (Iwasawa decomposition). $(k, a, n) \mapsto kan$ *is an analytic diffeomorphism of* $K \times A \times N$ *onto* G.

Let ϕ be the map of the statement.

(a) im ϕ is closed. AN is a closed subgroup of G and G/AN is compact (because G/P and M are compact). Let π be the projection $G \to G/AN$. Since K is compact, $\mathrm{im}(\pi \circ \phi)$ is closed. Hence so is $\mathrm{im}\,\phi = \pi^{-1}\,\mathrm{im}(\pi \circ \phi)$.

(b) im ϕ meets all components of G, since k does (see 4.6(ii)).

(c) The tangent map $d\phi$ is bijective at any point (k, a, n). This follows from the direct sum decomposition $\mathfrak{g} = \mathfrak{k} + \mathfrak{a} + \mathfrak{n}$.

(a), (b) and (c) imply that im ϕ is open and closed and meets all components. Hence ϕ is surjective. To finish the proof, it suffices to show that ϕ is injective. It is enough to prove that $kan = a_1$ implies $a = a_1$, $k = n = e$. Now if this is so we have $\theta n = a^2 n a_1^{-2}$. The image under ν of the last element is unipotent. It then follows that $a^2 = a_1^2$, $a = a_1$, whence $\theta n \in N \cap \theta N = \{e\}$.

5.16. Corollary. *For any parabolic subgroup P of G we have $G = KP$.*

Let W be the Weyl group of the root system Φ. This is the relative Weyl group of $(\boldsymbol{G}^0, \boldsymbol{S})$ (see 3.5), i.e.,

$$W = N_{\boldsymbol{G}^0(\boldsymbol{R})}(\boldsymbol{S})/Z_{\boldsymbol{G}^0(\boldsymbol{R})}(\boldsymbol{S}).$$

5.17. LEMMA. *M meets all components of G.*

It suffices to prove this for the case that $\nu(G) = \boldsymbol{G}^0(\boldsymbol{R})$. In that case it follows from Bruhat's lemma that the connected components of $\boldsymbol{G}^0(\boldsymbol{R})$ all meet $N = N_{\boldsymbol{G}^0(\boldsymbol{R})}(\boldsymbol{S})$. Let $\alpha \in \Phi$ be a simple root (for the order defined by $\boldsymbol{P}$), $\boldsymbol{U}_{(\alpha)}$ the unipotent subgroup whose Lie algebra is $(\mathfrak{g}_\alpha)_C + (\mathfrak{g}_{2\alpha})_C$ (where the second term is zero if $2\alpha \notin \Phi$), and $\boldsymbol{U}_{(-\alpha)} = \theta(\boldsymbol{U}_{(\alpha)})$. It is known that $\boldsymbol{U}_{(\alpha)}(\boldsymbol{R}) \cdot \boldsymbol{U}_{(-\alpha)}(\boldsymbol{R}) \cdot \boldsymbol{U}_{(\alpha)}(\boldsymbol{R})$ contains an element of N representing the reflection in W defined by α. It follows that $N \subset Z_{\boldsymbol{G}^0(\boldsymbol{R})}(\boldsymbol{S})G^0$, which implies the assertion.

It follows from the lemma that $W = N_G(A)/Z_G(A)$.

5.18. LEMMA. $W \simeq (K \cap N_G(A))/M$.

Let $g = kan \in N_G(A)$. Then also $(\theta n)^{-1}a^2 n \in N_G(A)$. Let $w_0 \in N_{\boldsymbol{G}^0(\boldsymbol{R})}(\boldsymbol{S})$ represent the element of maximal length of W. Then $\theta n^{-1} = w_0 n_1 w_0^{-1}$, for some $n_1 \in N$. We then have $n_1 w_0^{-1} a^2 n \in w_0^{-1} N_G(A)$. The uniqueness statement of Bruhat's lemma then implies that $n_1 = e$, whence $n = e$. This implies the assertion.

From 5.18 we see that W is the Weyl group of the symmetric spaces G/K ([**13**, p. 244]).

5.19. PROPOSITION (CARTAN DECOMPOSITION). *We have $G = KAK$.*

This follows from 5.5 and the following lemma. Here S is as in 5.3 (observe that K normalizes S).

5.20. LEMMA. $S = \bigcup_{k \in K} kAk^{-1}$.

It suffices to prove this for $\nu(G)$, i.e., when $\nu = \mathrm{id}$. Let $s \in S$. Then s lies in a θ-stable $\boldsymbol{R}$-split torus $\boldsymbol{S}_1 \subset \boldsymbol{G}^0$. Since $\boldsymbol{S}$ is a maximal $\boldsymbol{R}$-split torus in $\boldsymbol{G}^0$ (because $\boldsymbol{P}$ is a minimal parabolic $\boldsymbol{R}$-subgroup of $\boldsymbol{G}^0$, see 3.5) we have that $\boldsymbol{S}_1$ is conjugate to a subtorus of $\boldsymbol{S}$ by an element of $\boldsymbol{G}^0(\boldsymbol{R})$. By 5.17 we may take this element to be in $\boldsymbol{G}^0(\boldsymbol{R})^0$, hence in G. So there is $g \in G$ with $a_1 = g^{-1}sg \in A$. Writing $g = kan$ we obtain

$$n^{-1}a^{-2} \cdot \theta n \cdot a_1 = a_1 n^{-1} \cdot a^{-2} \cdot \theta n.$$

Using again the uniqueness statement of Bruhat's lemma, as in the proof of 5.18, we see that a_1 commutes with n. It follows that s is conjugate to an element of A via K, which is what had to be proved.

5.21. We finally give a brief elementary discussion of the geometric properties of the symmetric space G/K. We identify it with S (cf. 5.5). It is a homogeneous space for G, the action being given by $(x, s) \mapsto x \cdot s = xs(\theta x)^{-1}$. If $x \in \mathfrak{s}$ define $\|X\|^2 = \mathrm{Tr}(X^2)$. This defines a K-invariant Euclidean distance on $\mathfrak{s}$. The exponential map exp defines a diffeomorphism of $\mathfrak{s}$ onto S. Its inverse is denoted by log.

Define a Euclidean metric $d(\ ,\ )$ on A by $d(a, b) = \|\log a - \log b\|$. This determines a structure of Euclidean affine space on A.

We may and shall assume that $\nu = \mathrm{id}$.

5.22. LEMMA. *Let* $s, t \in S$.

(i) *There is* $x \in G$ *such that* $x \cdot s$ *and* $x \cdot t$ *lie in* A;

(ii) *if* x' *is another element with the property of* (i) *then there is* $n \in G$ *normalizing* A *such that* $x^{-1}nx'$ *fixes* s *and* t.

(i) follows from 5.20. To prove (ii) it is sufficient to assume $x' = e$. Then $s, t \in A$. Put $x \cdot s = a$, $x \cdot t = b$. It now follows that $a^{-1/2}xs^{1/2}$ and $b^{-1/2}xt^{1/2}$ lie in K, from which one concludes that st^{-1} and ab^{-1} are conjugate in G. The uniqueness part of Bruhat's lemma then implies that these elements are conjugate by an element of the Weyl group W, from which (ii) follows.

5.23. LEMMA. *If* $x \in G$, $x \cdot A = A$ *then* x *normalizes* A *in* G.

Apply 5.22(ii), taking $x' = e$, s a regular element of A, $t' = e$. It follows that we may assume $x \in K$ and $xs(\theta x)^{-1} = xsx^{-1} = s$. Since s is regular, x centralizes A. The assertion follows.

5.24. The translates $x \cdot A$ of A in S are called *apartments* in S. It follows from 5.23 that for any apartment $\mathscr{A}$ there is a unique structure of Euclidean affine space on $\mathscr{A}$ such that any bijection $A \to \mathscr{A}$ of the form $a \mapsto x \cdot a$ is an isomorphism of such spaces.

5.22(i) shows that for any two elements $s, t \in S$ there is an apartment $\mathscr{A}$ containing them. It follows from 5.22(ii) that, if $s \neq t$, the line in $\mathscr{A}$ containing s and t, together with its structure of 1-dimensional affine space, is *independent* of the choice of $\mathscr{A}$. We call such lines *geodesics* in S.

It now also makes sense to speak of the *geodesic segment* $[st]$, and of the midpoint of $[st]$.

It also follows that there is a unique G-invariant function d on $S \times S$ whose restriction to $A \times A$ is the function of 5.21.

5.25. PROPOSITION. (i) *d is a distance on* S;

(ii) *if* $s, t, u \in S$, $d(s, t) = d(s, u) + d(u, t)$ *then* u *lies on the segment* $[st]$;

(iii) *a closed sphere* $\{x \in X \mid d(x, a) \leqq r\}$ *is compact in* X.

It suffices to prove this for the case $G = \mathbf{GL}(n, \boldsymbol{R})$.

A proof of (i) and (ii) is given in the appendix to this section. The proof of (iii) is easy.

5.26. PROPOSITION. *For each* $s \in S$ *there is a unique involutorial analytic diffeomorphism* σ_s *of* S *with the following properties*:

(a) σ_s *is an isometry* (*for* d),

(b) s *is the only fixed point of* σ_s,

(c) σ_s *stabilizes all geodesics through* s.

We have $\sigma_{x \cdot s} = x \circ \sigma_s \circ x^{-1}$.

We may take $x = e$. The geodesic through e and $\exp X$ consists of the $\exp(\xi X)$ ($\xi \in \boldsymbol{R}$). Observing that $d(e, \exp(\xi X))$ is proportional to $|\xi|$ it follows that the only possibility for σ_e is the map $t \mapsto t^{-1}$. That this satisfies our requirements is clear. The final statement follows from the rest.

5.27. LEMMA. *Let* s, s' *be distinct points of* S, *let* m *be the midpoint* $[ss']$. *Let* t *be* *a point of S not lying on the geodesic through s and s'. Then $d(t, m) < \frac{1}{2}d(t, s) + \frac{1}{2}d(t, s')$.*

Let $\sigma = \sigma_m$, then $\sigma s = s'$. We have

$$2d(t, m) = d(t, \sigma(t)) \leqq d(t, s) + d(\sigma(t), s) = d(t, s) + d(t, s').$$

If the extreme terms are equal we have, by 5.25(ii) that s lies on $[t, \sigma(t)]$. Then so does $s' = \sigma s$, and s, s', t lie on a geodesic, which is contrary to the assumption. The inequality follows.

If C is a subset of X we denote by $I(C)$ the subgroup of the group of isometries of S whose elements stabilize C.

5.28. LEMMA. *If C is compact the group $I(C)$ has a fixed point in S.*

Let $r = \inf_{x \in X} \sup_{y \in C} d(x, y)$. Then $F = \{x \in S \mid \sup_{y \in C} d(x, y) = r\}$ is the intersection of the decreasing family of sets $F_n = \{x \in S \mid \sup_{y \in C} d(x, y) \leqq r + 1/n\}$ ($n = 1, 2, \cdots$). Since these are nonempty and compact (by 5.25 (iii)) it follows that C is nonempty.

Suppose $a, b \in F$, $a \neq b$ and let m be the midpoint of $[ab]$. We then have by 5.27, for each $y \in C$, $d(m, y) < \frac{1}{2}d(a, y) + \frac{1}{2}d(b, y) = r$, which is impossible. Hence F consists of only one point. It is clearly fixed by $I(C)$.

5.29. THEOREM. *Let M be a compact subgroup of G. Then M fixes a point of S.*

This follows by applying 5.28 to an orbit of M in S.

5.30. COROLLARY. *M is conjugate to a subgroup of K.*

5.31. COROLLARY. *All maximal compact subgroups of G are conjugate.*

5.32. REMARKS. (1) In our discussion of the symmetric space S we wanted to stress, more than is usually done, the analogy with the Bruhat-Tits building $\mathscr{B}$ of a p-adic reductive group. We mention a few features of this analogy.

(a) It is clear from our discussion that S can be obtained, like $\mathscr{B}$, by gluing together apartments (see [**21**, 2.1]).

(b) We have introduced metric and geodesics in S in the same way as is done in the case of $\mathscr{B}$ (see [**9**, 2.5] and [**21**, 2.3]). In the case of $\mathscr{B}$ an important role is played in the discussion of the metric, by the retractions onto an apartment [loc. cit., 2.2]. Such retractions can also be introduced in S (an example is the map ρ used in the appendix).

(c) The fixed point Theorem 5.29 has a counterpart for $\mathscr{B}$ [**8**, 3.2.4].

(d) I owe the proof of 5.29, using the strong convexity property 5.27, to J. J. Duistermaat. 5.29 can also be proved via the argument used in [**8**] (see also [**21**, 2.3]) to prove its counterpart for $\mathscr{B}$. This requires the inequality (m is the midpoint of $[x, y]$)

$$d(x, z)^2 + d(y, z)^2 \geqq 2d(m, z)^2 + \tfrac{1}{2}d(x, y)^2,$$

which can also be established in our situation (e.g. by using that the exponential map $\mathfrak{s} - S$ increases distances).

Appendix. Proof of 5.25 for $G = \mathbf{GL}(n, \boldsymbol{R})$. A is now the group of all diagonal matrices with positive entries. It suffices to show:

if $a, b \in A$, $s \in S$, then $d(a, b) \leqq d(a, s) + d(s, b)$ equality holding if and only if s lies on the segment $[ab]$.

As is well known, $s \in S$ being given there is a unique $\rho(s) \in A$ and a unique upper triangular unipotent matrix $u(s)$ such that $s = {}^t u(s)\rho(s)u(s)$. If $a \in A$, we have $\rho(asa) = \rho(a)^2\rho(s)$. We shall prove the following.

LEMMA. *$d(\rho(s), e) \leqq d(s, e)$, equality holding if and only if $s \in A$.*

From the lemma it also follows that, for all $a \in A$, $d(\rho(s), a) \leqq d(s, a)$. Hence $d(a, b) \leqq d(a, \rho(s)) + d(\rho(s), b) \leqq d(a, s) + d(s, b)$, proving the triangular inequality. The case of equality is easily dealt with.

It remains to prove the lemma. Let $a_1 \geqq a_2 \geqq \cdots \geqq a_n$ be the eigenvalues of s and $b_1 \geqq b_2 \geqq \cdots \geqq b_n$ those of $\rho(s)$. The lemma asserts that

$$\sum_{i=1}^{n} (\log b_i)^2 \leqq \sum_{i=1}^{n} (\log a_i)^2.$$

Results of this kind are known, they can be found, e.g., in H. Weyl, Ges. Abh. Bd. IV, p. 390. We use Weyl's method. Let $g = \rho(s)^{1/2}\, u(s)$, so $s = {}^t g \cdot g$. There is a vector $v \in \boldsymbol{R}^n$ with $gv = b^{1/2}\, v$. Then $(,)$ denoting the standard inner product in $\boldsymbol{R}^n$,

$$b_1(v, v) = (v, sv) \leqq a_1(v, v),$$

whence $b_1 \leqq a_1$. Applying a similar argument, working in the exterior powers of $\boldsymbol{R}^n$, we see that $b_1 b_2 \cdots b_p \leqq a_1 a_2 \cdots a_p$ $(1 \leqq p \leqq n - 1)$, and, of course, $b_1 b_2 \cdots b_n = a_1 a_2 \cdots a_n$. We may, and shall, assume that $a_1 a_2 \cdots a_n = 1$.

Let $V \in \boldsymbol{R}^n$ be the subspace of vectors with coordinate sum 0.

Let $\alpha_i = e_i - e_{i+1}$ $(1 \leqq i \leqq n - 1$, (e_i) is the canonical basis), and define $\omega_i \in V$ by $(\omega_i, e_j) = \delta_{ij}$ $(1 \leqq i, j \leqq n - 1)$. Then ω_i is the projection onto V of $e_1 + \cdots + e_i$. Let $x = (\log a_1, \log a_2, \cdots, \log a_n)$, $y = (\log b_1, \log b_2, \cdots, \log b_n)$. We then know that

$$(x, \alpha_i) \geqq 0, \quad (y, \alpha_i) \geqq 0, \quad (x - y, \omega_i) \geqq 0 \qquad (1 \leqq i \leqq n - 1),$$

and we have to prove that $(y, y) \leqq (x, x)$. Now $(x, x) - (y, y) = (x - y, x - y) + 2(y, x - y)$. Since y is a linear combination, with positive coefficients, of the ω_i and $x - y$ is a similar combination of the α_i, we have that $(y, x - y) \geqq 0$. The inequality which we have to prove now becomes obvious.

It is clear that we can only have $(x, x) = (y, y)$ if $x = y$. In that case we have $\mathrm{Tr}(s) = \mathrm{Tr}(\rho(s))$. But it is immediate that if $u(s) \neq e$, we must have

$$\mathrm{Tr}({}^t u(s)\rho(s)u(s)) > \mathrm{Tr}\, \rho(s).$$

So $\mathrm{Tr}(s) = \mathrm{Tr}\, \rho(s)$ implies $\rho(s) = s$. This finishes the proof of the lemma.

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