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REDUCTIVE GROUPS OVER LOCAL FIELDS

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0. Preliminaries.

0.1. *Introduction.* This is a survey of some aspects of the structure theory of reductive groups over local fields. Since it is mainly intended for "utilizers", the main emphasis will be on statements and examples. The proofs will mostly be omitted, except for short local arguments which may give a better insight in the way the theory operates. When proofs are available in the literature (which is not always the case!), references will be given; references to **[8]** are often conditional, as explained in §1.5.

We shall not try to give the historical background of the results exposed here. Let us merely recall that the theory was initiated by N. Iwahori and H. Matsumoto **[15]**, who were considering split semisimple groups, that quasi-split and classical groups were later on studied by H. Hijikata **[13]**, and that, in the generality given here, most results are due to F. Bruhat and the author **[6]**, **[7]**, **[8]**, **[9]**. For further information, one may consult the introduction of **[8]**.

0.2. *Notations.* The following notations will be used throughout the paper: K denotes a field endowed with a nontrivial discrete valuation ω, the value group $\omega(K^\times)$ ($\subset \boldsymbol{R}$) is also called Γ, $\mathfrak{o}$ represents the ring of integers, $\mathfrak{p} = \pi\mathfrak{o}$ with $\pi \in \mathfrak{o}$ its prime ideal and $\bar{K} = \mathfrak{o}/\mathfrak{p}$ the residue field. *We always assume K complete and $\bar{K}$ perfect.* We consider an algebraic group G defined over K whose neutral component G° is reductive, and call S a maximal K-split torus of G, N (resp. Z) the normalizer (resp. the centralizer) of S in G, ${}^v\tilde{W}$ the finite group $N(K)/Z(K)$ (as usual, ?(K) stands for the group of rational points of ? over K), $X^* = X^*(S) = \mathrm{Hom}_K(S, \mathrm{Mult})$ (resp. $X_* = X_*(S) = \mathrm{Hom}_K(\mathrm{Mult}, S)$) the group of characters (resp. cocharacters) of S, V the real vector space $X_* \otimes \boldsymbol{R}$, $\Phi = \Phi(G, S) \subset X^*$ the set of roots of G relative to S, ${}^vW$ the Weyl group of the root system Φ which we identify with a normal subgroup of ${}^v\tilde{W}$ (equal to ${}^v\tilde{W}$ if $G = G^\circ$) and U_a, for $a \in \Phi$, the unipotent subgroup of G° normalized by S and corresponding to the root a (i.e., the group called $U_{(a)}$ in [3, 5.2]).

1. The apartment of a maximal split torus and the affine root system.

1.1. *The split case.* As a motivation for what follows, we first consider the case where G° is *split*, that is, where S is a maximal torus of G. Then, the groups U_a are K-isomorphic to the additive group. Indeed, the choice of a "Chevalley basis" in the Lie algebra of G determines a system of K-isomorphisms χ_a: Add $\to U_a$ (an "épinglage") satisfying the commutation relations of Chevalley [10, p. 27]. Since K is a local field, its additive group is filtered and so are the groups $U_a(K)$, "par transport de structure". The terms of those filtrations are conveniently indexed by affine functions on V: for $a \in \Phi$ and $\gamma \in \Gamma$, $a + \gamma$ is such a function and we set

(1) $$X_{a+\gamma} = \chi_a(\omega^{-1}[\gamma, \infty]).$$

If we transform the Chevalley basis by Ad s for an element $s \in S(K)$, the system (χ_a) is replaced by $(\chi_a') = (\chi_a \circ a(s))$ and, setting $X'_{a+\gamma} = \chi_a'(\omega^{-1}[\gamma, \infty])$, we have

(2) $$X'_{a+\gamma} = X_{a+\gamma+\omega(a(s))}.$$

Thus, the terms of the filtrations of the groups $U_a(K)$ are unchanged but their indexation has undergone a translation. The same conclusion holds for an arbitrary change of Chevalley basis (one just has to replace s by a rational element of the image of S in the adjoint group).

We may express that conclusion in a more invariant way as follows. There exist an affine space A under V, a system Φ_{af} of affine functions on A and a mapping $\alpha \mapsto X_\alpha$ of Φ_{af} onto a set of subgroups of $G(K)$ with the following property: to every Chevalley basis, there corresponds a point $0 \in A$ such that Φ_{af} consists of all functions

(3) $$\alpha : x \mapsto a(x - 0) + \gamma \qquad (x \in A; a \in \Phi, \gamma \in \Gamma)$$

and that, if (χ_a) denotes the "épinglage" associated with the given basis, the group X_α corresponding to the function (3) is given by (1). The group $S(K)$ operates by translations on A in such a way that, for $s \in S(K)$, we have

(4) $$s^{-1}X_\alpha s = X_{\alpha \circ s}.$$

From (2) it follows that the translation $\nu(s) \in V$ of A induced by s (i.e., defined by $s(x) = x + \nu(s)$ for $x \in A$) is given by

(5) $$a(\nu(s)) = -\,\omega(a(s)) \quad \text{for every } a \in \Phi.$$

More generally, the normalizer $N(K)$ of $S(K)$ in $G(K)$ operates on A by affine transformations in such a way that (4) holds for any $s \in N(K)$.

1.2. *The apartment $A(G, S, K)$.* Our purpose is to generalize the above results to an arbitrary group G in the following form: *to G, S, K, we want to associate an* *affine space $A = A(G, S, K)$ under V on which $N(K)$ operates, a system $\Phi_{\mathrm{af}} = \Phi_{\mathrm{af}}(G, S, K)$ of affine functions on A and a mapping $\alpha \mapsto X_\alpha$ of Φ_{af} onto a set of subgroups of $G(K)$, such that the relation* 1.1(4) *holds for $s \in N(K)$, that the vector parts $v(\alpha)$ of the functions $\alpha \in \Phi_{\mathrm{af}}$ are the elements of Φ, and that, for $a \in \Phi$, the groups X_α with $v(\alpha) = a$ form a filtration of $U_a(K)$.*

We first proceed with the construction of the space A; the set Φ_{a} and the X_α's will be defined in §§1.6 and 1.4. The relations (5) show us the way. The group $X^*(Z)$ of K-rational characters of Z can be identified with a subgroup of finite index of X^*. Let $\nu: Z(K) \to V$ be the homomorphism defined by

$$\chi(\nu(z)) = -\,\omega(\chi(z)) \quad \text{for } z \in Z(K) \text{ and } \chi \in X^*(Z), \tag{1}$$

and let Z_c denote the kernel of ν. Then, $\Lambda = Z(K)/Z_c$ is a free abelian group of rank $\dim S = \dim V$. The quotient $\tilde{W} = N(K)/Z_c$ is an extension of the finite group ${}^v\tilde{W}$ by Λ. Therefore, there is an affine space A ($= A(G, S, K)$) under V and an extension of ν to a homomorphism, which we shall also denote by ν, of N in the group of affine transformations of A. If G is semisimple, the system (A, ν) is canonical, that is, unique up to unique isomorphism. Otherwise, it is only unique up to isomorphism, but one can, following G. Rousseau **[19]**, "canonify" it as follows: calling $\mathscr{D}G^\circ$ the derived group of G° and S_1 the maximal split torus of the center of G°, one takes for A the direct product of $A(\mathscr{D}G^\circ, G^\circ \cap S, K)$ (which is canonical) and $X_*(S_1) \otimes \boldsymbol{R}$ .The affine space A is called the *apartment of S* (relative to G and K). The group $N(K)$ operates on A through $\tilde{W}$.

1.3. *Remark.* Since $V = \mathrm{Hom}(X^*, \boldsymbol{R}) = \mathrm{Hom}(X^*(Z), \boldsymbol{R})$, the groups $\mathrm{Hom}(X^*, \Gamma)$ and $\mathrm{Hom}(X^*(Z), \Gamma)$ are lattices in V and one has

$$\mathrm{Hom}(X^*, \Gamma) \subset \nu(Z(K)) = \Lambda \subset \mathrm{Hom}(X^*(Z), \Gamma). \tag{1}$$

If G is connected and split, both inclusions are equalities, but in general they can be proper. Suppose for instance that $G = R_{L/K}$ Mult, where L is a separable extension of K of degree n, and let Γ_1 be the value group of L. The group $X^*(Z)$ is generated by the norm homomorphism $N_{L/K}$, hence has index n in X^*. On the other end, Λ is readily seen to be equal to $n \cdot \mathrm{Hom}(X^*(Z), \Gamma_1)$. In particular, the first (resp. the second) inclusion (1) is an equality if and only if the extension L/K is unramified (resp. totally ramified). A semisimple example is provided by $G = SU_3$ with splitting field L; exactly the same conclusions as above hold with $n = 2$ (indeed, in that case $Z = R_{L/K}$ Mult). One can prove that *the first inclusion* (1) *is an equality whenever G splits over an unramified extension of K.*

1.4. *Filtration of the groups $U_a(K)$.* Let $a \in \Phi$ and $u \in U_a(K) - \{1\}$. It is known (cf. [3, §5]) that the intersection $U_{-a}uU_{-a} \cap N$ consists of a single element $m(u)$ whose image in ${}^v\tilde{W}$ is the reflection r_a associated with a, from which follows that $r(u) = \nu(m(u))$ is an affine reflection whose vector part is r_a. Let $\alpha(a, u)$ denote the affine function on A whose vector part is a and whose vanishing hyperplane is the fixed point set of $r(u)$ and let Φ' be the set of all affine functions whose vector part belongs to Φ. For $\alpha \in \Phi'$, we set $X_\alpha = \{u \in U_a(K) \mid u = 1 \text{ or } \alpha(a, u) \geqq \alpha\}$. The following results are fundamental.

1.4.1. *For every α as above, X_α is a group.*

1.4.2. *If $\alpha, \beta \in \Phi'$, the commutator group (X_α, X_β) is contained in the group generated by all $X_{p\alpha+q\beta}$ for $p, q \in \boldsymbol{N}^*$ and $p\alpha + q\beta \in \Phi'$.*

Clearly, the X_α's with $v(\alpha) = a$ form a filtration of $U_a(K)$. We denote by $X_{\alpha+}$ the union of all $X_{\alpha+\varepsilon}$ for $\varepsilon \in \boldsymbol{R}$, $\varepsilon > 0$ (of course, $X_{\alpha+} = X_{\alpha+\varepsilon}$ for ε sufficiently small). From 1.4.2, it follows that $X_{\alpha+}$ is a normal subgroup of X_α, and we set $\bar{X}_\alpha = X_\alpha/X_{\alpha+}$. Thus, the $\bar{X}_\alpha$'s, for $v(\alpha) = a$, are the quotients of the filtration of $U_a(K)$ in question. It is obvious that for $n \in N(K)$, one has $n^{-1}X_\alpha n = X_{\alpha\circ\nu(n)}$.

1.5. *About proofs, references and generalizations.* Let us identify A with V via the choice of an "origin" 0, and, for every $a \in \Phi$ and $u \in U_a(K)$, set $\varphi_a(u) = \alpha(a, u) - a$ ($\in \boldsymbol{R}$). The assertions 1.4.1 and 1.4.2 essentially mean that the system of functions $(\varphi_a)_{a\in\Phi}$ *is a valuation of the root datum* $(Z(K); (U_a(K))_{a\in\Phi})$, as defined in [**8**, 6.2]. That fact itself is roughly equivalent with (actually somewhat stronger than) the existence of a certain BN-pair in the group generated by all $U_a(K)$ (cf. [**8**, 6.5 and 6.2.3(e)]), and with the existence of the affine building of G over K (cf. §2 below and [**8**, §7]). Those results have been announced in [**6**], [**7**] and [**8**, 6.2.3(c)]), but complete proofs by the same authors have not yet appeared (though the case of classical groups is completely handled in [**8**, §10], and quasi-split groups are essentially taken care of by [**8**, 9.2.3]). In the meantime, proofs of closely related results have been published by H. Hijikata [**14**] and by G. Rousseau [**19**].

In the sequel, quite a few statements will be followed by references to [**8**]; this will usually mean that the quoted section of [**8**] contains a proof of the statement in question *once* 1.4.1 *and* 1.4.2 *are admitted.*

For the sake of simplicity we have assumed that ω is discrete and $\bar{K}$ perfect. In fact, much of what we shall say until §3.3 remains valid (with suitable reformulations) without those assumptions, provided that 1.4.1 and 1.4.2 hold, and this has been shown to be always the case except possibly if char $\bar{K} = 2$ for some groups G whose semisimple part has factors of exceptional type and relative rank $\leqq 2$ (cf. [**8**, §10], [**25**] and [**19**]).

1.6. *The affine root system* Φ_{af}. For every affine function α on A whose vector part $a = v(\alpha)$ belongs to Φ, one has an obvious inclusion $\bar{X}_{2\alpha} \hookrightarrow \bar{X}_\alpha$ (if $2a \notin \Phi$, we set $\bar{X}_{2\alpha} = \{1\}$) and the quotient $\bar{X}_\alpha/\bar{X}_{2\alpha}$ has a natural structure of vector space over $\bar{K}$ (cf. 3.5.1) whose dimension is finite and will be denoted by $d(\alpha)$. In particular, if char $\bar{K} = p$, $\bar{X}_\alpha$ is a p-group. An affine function α such that $a = v(\alpha) \in \Phi$ is called an *affine root* of G (relative to S and K) if $d(\alpha) \neq 0$, that is, if X_α is not contained in $X_{\alpha+\varepsilon}\cdot U_{2a}(K)$ ($= X_{\alpha+\varepsilon}$ if $2a \notin \Phi$) for any strictly positive constant ε. We denote by $\Phi_{\mathrm{af}}(G, S, K) = \Phi_{\mathrm{af}}$ the *affine root system of* G, i.e., the set of all its affine roots. Note that if $2a \notin \Phi$, one has $\alpha(a, u) \in \Phi_{\mathrm{af}}$ for every $u \in U_a(K) - \{1\}$; in particular, if Φ is reduced, $\Phi_{\mathrm{af}} = \{\alpha(a, u) \mid a \in \Phi, u \in U_a(K) - \{1\}\}$.

1.7. *Half-apartments, chambers, affine Weyl group.* For every affine function α such that $a = v(\alpha) \in \Phi$, we denote by A_α the set $\alpha^{-1}([0, \infty))$, by ∂A_α its boundary $\alpha^{-1}(0)$ and by r_α the affine reflection whose vector part is the reflection r_a (cf. 1.4) and whose fixed hyperplane is ∂A_α. The sets A_α (resp. ∂A_α) for $\alpha \in \Phi_{\mathrm{af}}$ are called the *half-apartments* (resp. the *walls*) of A, and the *chambers* are defined as the connected components of the complement in A of the union of all walls. The facets of the chambers are also called the *facets* of A; thus, the chambers are the facets of maximum dimension. If G is quasi-simple the facets (and in particular the chambers) are simplices, if G is semisimple they are polysimplices (i.e., direct products of simplices) and in general they are direct products of a polysimplex and a real affine space.

The group W generated by all r_α with $\alpha \in \Phi_{af}$ is called the *Weyl group* of the affine root system Φ_{af}. (If G is not semisimple, this is a slight abuse of language since W depends not only on Φ_{af} but also on the subspace of V generated by $X_*(\mathscr{D}G^\circ \cap S)$, where $\mathscr{D}G^\circ$ denotes the derived group of G°.) If G is semisimple, W is the affine Weyl group of a reduced root system (cf. [**5**, VI. 2.1]) whose elements are proportional to those of Φ, *but which is not necessarily proportional to* Φ, *even if* Φ *is reduced* (cf. the examples in §§1.15, 1.16).

Clearly, Φ_{af} is stable by the group $\tilde{W} = \nu(N(K))$ (cf. §1.2). It follows that the half-apartments, the walls and the chambers are permuted by $\tilde{W}$, and that W is a normal subgroup of $\tilde{W}$.

1.8. *Bases, local Dynkin diagram, characteristic dimensions.* The Weyl group W is simply transitive on the set of all chambers (i.e., it permutes the chambers transitively and the stabilizer of a chamber in W is reduced to the identity). Let C be a chamber and let $L_0, \cdots, L_l$ be the walls bounding C. For $i \in \{0, \cdots, l\}$, let α_i be the unique affine root such that $L_i = \partial A_{\alpha_i}$ and $\frac{1}{2}\alpha_i \notin \Phi_{af}$. The set $\{\alpha_i \mid i = 0, \cdots, l\}$ is called the *basis* of Φ_{af} associated to C.

Let a_i be the vector part of α_i and let us introduce in the dual of V a positive definite scalar product (,) invariant by the (ordinary) Weyl group ${}^vW$. To Φ_{af}, one associates a (*local*) *Dynkin diagram* $\Delta = \Delta(\Phi_{af})$ obtained as follows:

The elements α_i of a basis are represented by dots v_i, called the *vertices* of the diagram;

if $2\alpha_i \in \Phi_{af}$, the vertex v_i is marked with a cross;

two distinct vertices v_i, v_j are joined by an empty, a simple, a double, a triple or a fat segment (*edge* of the diagram) according as the integer $\lambda_{ij} = 4(a_i, a_j)^2/(a_i, a_i)(a_j, a_j)$ equals 0, 1, 2, 3 or 4 (in the latter case, a_j is a positive multiple of $-a_i$);

if $\lambda_{ij} = 2$ or 3 (which implies that $(a_i, a_i) \neq (a_j, a_j)$) or if $\lambda_{ij} = 4$ and $a_j \neq -a_i$, the edge joining v_i and v_j is oriented by an arrow pointing toward the vertex representing the "shortest" of the two roots a_i and a_j.

Since the chambers are permuted simply transitively by W,

1.8.1. *the Dynkin diagram does not depend, up to* canonical *isomorphism, on the choice of the chamber* C.

It is easily seen that the system (A, Φ_{af}) is determined up to isomorphism by the Dynkin diagram Δ and the dimension of A (i.e., the relative rank of G). The Coxeter diagram underlying the Dynkin diagram—i.e., deduced from it by disregarding the crosses and arrows—is the Coxeter diagram of W, hence the Coxeter diagram of an affine reflection group (cf. [**5**, V.3.4, and VI.4.3], where our "diagrams" are called "graphes").

Conversely, consider any Coxeter diagram which is the diagram of an affine reflection group, orient all double and triple edges and possibly some fat ones, and mark some vertices (possibly none) with a cross. Then, *the diagram thus obtained is the local Dynkin diagram* Δ *of some group* G *over some field* K *if and only if, for every vertex* v *marked with a cross, all edges having* v *as an extremity are double or fat and none of them is oriented away from* v.

The necessity of the condition is obvious. As for the sufficiency, the classification of §4 even shows that for any given *locally compact* local field K, every diagram satisfying the above condition is the local Dynkin diagram of some semisimple group G over K: indeed, it is an easy matter to list all irreducible diagrams in question, and one verifies readily that they all appear in the tables of §4. Note that the above statement, or alternatively the tables of §4, provide the classification of all affine root systems, for a suitable "abstract" definition of such systems, which the interested reader will have no difficulty to formulate (cf. also [**8**, 1.4], where the affine root systems are called "échelonnages", and, for the reduced case, [**17**]).

If the vertex v of Δ represents the affine root α, we set $d(v) = d(\alpha) + d(2\alpha)$ $(= d(\alpha)$ if $2\alpha \notin \Phi_{\mathrm{af}})$, where the function d is defined as in §1.6. The integer $d(v)$ of course depends not only on Δ and v but on the group G itself. In the tables of §4, the value of $d(v)$ is indicated for every v whenever it is not equal to 1. *If G is split or if the residue field $\bar{K}$ is algebraically closed, all $d(v)$ are equal to* 1.

1.9. *Root system attached to a point of A and special points.* For $x \in A$, we denote by Φ_x the subset of Φ consisting of the vector parts of all affine roots vanishing in x, and by W_x the group generated by all reflections r_α for $\alpha \in \Phi_{\mathrm{af}}$ and $\alpha(x) = 0$ (cf. §1.7). To x, we also associate as follows a set I_x of vertices of the local Dynkin diagram Δ: there is an element w of the Weyl group W which carries x in the closure of the "fundamental chamber" C and one sets $I_x = \{v_i \mid wx \notin L_i\}$, with the notations of §1.8; that I_x is independent of the choice of w follows from well-known properties of Coxeter groups: cf., e.g., [**5**, V. 3.3, Proposition 1]. The objects Φ_x, W_x, I_x depend only on the facet F containing x and will also be denoted by Φ_F, W_F, I_F.

The set Φ_x is a (not necessarily closed) subroot system of Φ whose Weyl group is the vector part of W_x and whose (ordinary) Dynkin diagram is obtained by deleting from Δ the vertices belonging to I_x and all edges containing such a vertex. The set I_x has a nonempty intersection with every connected component of Δ and, conversely, every set of vertices with that property is the set I_x for some x.

The point x is called *special* for Φ_{af} if every element of the root system Φ is proportional to some element of Φ_x, that is, if Φ and Φ_x have the same Weyl group. When it is so, W is the semidirect product of W_x by the group of all translations contained in W; similarly, if G is connected, $\tilde{W}$ is the semidirect product of W_x by $\nu(Z(K)) = Z(K)/Z_c$ (cf. 1.2).

The fact for a point x to be special can be recognized from the set of vertices I_x as follows. A vertex of the Coxeter diagram of an irreducible affine reflection group is called *special* if by deleting from the diagram that vertex and all adjoining edges, one obtains the Coxeter diagram of the corresponding finite (spherical) reflection group. (Equivalently: such a diagram being the Coxeter diagram underlying the extended Dynkin diagram—"graphe de Dynkin complété" in the terminology of [**5**]—of a reduced root system, the special vertices are the vertex representing the minimum root and all its transforms by the automorphisms of the diagram.) Clearly, such vertices exist. Now, $x \in A$ is special if and only if I_x consists of one special vertex out of each connected component of Δ. In particular, special points always exist. In the tables of §4, the special vertices are marked with an *s* or an *hs* ("hyperspecial points": see below).

1.10. *Behaviour under field extension and hyperspecial points.* Let K_1 be a Galois extension of K with Galois group $\mathrm{Gal}(K_1/K) = \Theta$, and let S_1 be a maximal K_1-split torus of G containing S and defined over K. Such a torus exists for instance in the following cases:

if G is quasi-split over K (obvious!);

if K_1 is the maximal unramified extension of K [**6**(c), 3, Corollaire 1];

if the residue field $\bar{K}$ is finite and K_1/K is unramified.

(The latter condition is necessary as is shown by the following example due to Serre: suppose that Θ has even order and no subgroup of index 2, and that G is the norm one group of a division quaternion algebra; then G splits over K_1 but none of its maximal tori does.) Let $A_1 = A(G, S_1, K_1)$ be the apartment of S_1 and let $\Phi_{1\mathrm{af}} = \Phi_{\mathrm{af}}(G, S_1, K_1)$ be the corresponding affine root system. The Galois group Θ operates on A_1 ("canonified" as in §1.2) "par transport de structure", and A can be identified with the fixed point set A_1^Θ.

That identification is not quite obvious. To characterize it, we have to describe an operation of $N(K)$ on A_1^Θ (cf. 1.2). First observe that A_1^Θ clearly is an affine space under V. Let now $n \in N(K)$, let N_1 be the normalizer of S_1 in G and let ν_1 be the canonical homomorphism of $N_1(K_1)$ into the group of affine transformations of A_1. Since the conjugate ${}^nS_1$ is a maximal K_1-split torus of Z, there exists $z \in Z(K_1)$ with $n' = nz^{-1} \in N_1(K_1)$. Upon multiplying z by a suitable element of $(Z \cap N_1)(K_1)$, one may choose it so that $\nu_1(n')$ stabilizes A_1^Θ. Let now $\nu(z)$ be the element of V defined by the relation 1.2(1) where ω must be replaced by the valuation of K_1. Then $n = n'z$ operates on A_1^Θ through $\nu_1(n') \circ \nu(z)$. That this action is independent of the choices made and indeed defines an operation of $N(K)$ on A_1^Θ is best seen by using the "building" of G over K_1 defined in §2: that building contains A_1^Θ and is operated upon by $G(K_1)$, hence by $N(K)$, and one verifies that $N(K)$ stabilizes A_1^Θ and operates on it as described above. Note that, more generally, the results of §2.6 show that if S_1' is *any* maximal K_1-split torus of G containing S, A can be naturally identified with an affine subspace of $A(G, S_1', K_1)$; much of what we shall say here extends to that situation.

1.10.1. *If K_1/K is unramified, Φ_{af} consists of all nonconstant restrictions $\alpha|_A$, with $\alpha \in \Phi_{1\mathrm{af}}$.*

That is no longer true in general when K_1/K is ramified. An obvious example is provided by the case where G is split over K. Then, $S_1 = S$, $A_1 = A$, and if we identify A with V as in §1.1, we have $\Phi_{\mathrm{af}} = \{a + \gamma \mid a \in \Phi,\ \gamma \in \Gamma\}$ and $\Phi_{1\mathrm{af}} = \{a + \gamma \mid a \in \Phi, \gamma \in \Gamma_1\}$, where Γ_1 denotes the value group of K_1.

From 1.10.1, it follows readily that

1.10.2. *If K_1/K is unramified, every point of A which is special for $\Phi_{1\mathrm{af}}$ is also special for Φ_{af}.*

The above example shows that that assertion becomes false without the assumption on K_1/K. A point $x \in A$ is called *hyperspecial* if there exist K_1, S_1 as above such that K_1/K is unramified, that G splits over K_1 and that x is special for $\Phi_{1\mathrm{af}}$. Then, it is easily seen, using 1.10.2, that the same holds for any Galois unramified splitting field K_1 of G and any choice of S_1 (assuming that such a torus exists). More intrinsic characterizations of the hyperspecial points will be given in 3.8.

If G is quasi-split and splits over an unramified extension of K, hyperspecial points do exist. Indeed, take for K_1 the minimum splitting field of G and (obligatorily) $S_1 = Z$, let $a_1, \cdots, a_l$ be a basis of the root system $\Phi(G, S_1)$ invariant by Θ and choose $\alpha_1, \cdots, \alpha_l \in \Phi_{1\mathrm{af}}$ so that $v(\alpha_i) = a_i$ and that $\{\alpha_1, \cdots, \alpha_l\}$ is stable by Θ (the possibility of such a choice readily follows from the description of A and Φ_{af} given in §1.1). Then, the equations $\alpha_1 = \cdots = \alpha_l = 0$ define an affine subspace of A_1 invariant by Θ (in fact a single point if G is semisimple), and every point invariant by Θ in that subspace belongs to A and is clearly hyperspecial.

Suppose G is quasi-simple. We say that a vertex v of the local Dynkin diagram is *hyperspecial* (with respect to G) if the points $x \in A$ such that $I_x = \{v\}$ are hyperspecial (a property which depends only on v obviously). In the tables of §4, hyperspecial vertices are marked with an *hs*.

Let now K_1 be the maximal unramified extension of K. The group G is said to be *residually quasi-split* over K if there is a chamber of A_1 stable by $\mathrm{Gal}(K_1/K)$, and hence meeting A. We say that G is *residually split* if A_1 is fixed by $\mathrm{Gal}(K_1/K)$, that is, if G has the same rank over K and over K_1, i.e., if $S_1 = S$. For an explanation of the terminology and another definition, cf. 3.5.2.

1.10.3. *If the residue field $\bar{K}$ is finite, G is residually quasi-split. If $\bar{K}$ is algebraically closed, G is residually split.*

By a well-known result of R. Steinberg, if $\bar{K}$ is algebraically closed, G is quasi-split. From that, it follows that:

1.10.4. *Every residually split group is quasi-split.*

If $\bar{K}$ is finite and, more generally, if G is residually quasi-split, G has a "natural splitting field". Indeed, there is a smallest unramified extension K' of K on which G is residually split, namely the smallest splitting field of S_1 (which does not depend on the choice of that torus), and the group G, being quasi-split over K', has a smallest splitting field K'' over K'. The field K'' can also be characterized among all splitting fields of G over K as the unique one for which the pair consisting of the degree $[K'':K]$ and the ramification index $e(K''/K)$ is minimal for the lexicographic ordering.

1.11. *Absolute and relative local Dynkin diagram; the index.* In this section, K_1 denotes the maximal unramified extension of K, and A_1, S_1, $\Phi_{1\mathrm{af}}$ have the same meaning as in §1.10. As in the classical, "global" situation (cf. [**22**] and the references given there), one associates to G, K, S_1 (in fact, to G, K alone: cf. §2.4) a *local index* consisting of

the Dynkin diagram Δ_1 of $\Phi_{1\mathrm{af}}$ (*absolute local Dynkin diagram*),

the action of $\Theta = \mathrm{Gal}(K_1/K)$ on Δ_1 "par transport de structure", and

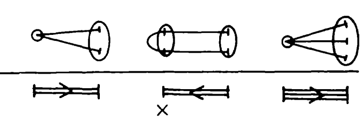
a Θ-invariant set of vertices of Δ_1, called the *distinguished vertices.*

The latter are characterized as follows: to define $\Phi_{1\mathrm{af}}$, one uses a chamber C_1 of A_1 whose closure contains a chamber of A (such a C_1 exists by 1.10.1), and then, the distinguished vertices are those representing the elements of the basis of $\Phi_{1\mathrm{af}}$ associated to C_1 (§1.8) whose restriction to A is not constant.

Residually quasi-split and residually split groups can be characterized as follows in terms of the index. The group G is residually quasi-split if and only if the orbits of Θ in the set of all nondistinguished vertices are unions of full connected components of Δ_1, and G is residually split if and only if Θ operates trivially on Δ_1, all vertices of Δ_1 are distinguished and the smallest splitting field of the connected center of G is totally ramified.

The index of G determines its relative local Dynkin diagram $\Delta = \Delta(G, S, K)$ and the integers $d(v)$ (cf. §1.8) uniquely. We shall indicate an easy algorithm which allows us, in most cases, to deduce the latter from the former. First of all, there is a canonical bijective correspondence $v \mapsto O(v)$ between the vertices of Δ and the orbits of Θ in the set of distinguished vertices of Δ_1. For every vertex v (resp. every pair $\{v, v'\}$ of vertices) of Δ, let $\bar{\Delta}_{1,v}$ (resp. $\bar{\Delta}_{1,vv'}$) denote the subdiagram of Δ_1 obtained by removing from it all vertices not belonging to $O(v)$ (resp. $O(v) \cup O(v')$) and all edges containing such vertices, and let $\Delta_{1,v}$ (resp. $\Delta_{1,vv'}$) be the subdiagram of $\bar{\Delta}_{1,v}$ (resp. $\bar{\Delta}_{1,vv'}$) consisting of its connected components which contain at least one distinguished vertex. Then, $\Delta_{1,v}$, together with the action of $\Theta \cong \mathrm{Gal}(\bar{K}_1/\bar{K})$ on it and the set of distinguished orbits it contains, is the index (in the sense of [3] and [22]) of a semisimple group of relative rank one over $\bar{K}$, the integer $d(v)$ is half the total number of absolute roots of that group and v is marked with a cross in Δ if and only if the relative root system of the group in question has type BC_1 (if $\bar{K}$ is finite—or more generally if all vertices of $\Delta_{1,v}$ are distinguished—that means that $\Delta_{1,v}$ is a disjoint union of diagrams of type A_2).

As for the edge of Φ joining v and v', its type is determined by $\Delta_{1,vv'}$, $O(v)$ and $O(v')$. If no connected component of $\Delta_{1,vv'}$ meets both $O(v)$ and $O(v')$, then v and v' are joined by an "empty edge". Otherwise, Θ permutes transitively the connected components of $\Delta_{1,vv'}$ and the result can be described in terms of any one of them, say $\Delta^\circ_{1,vv'}$. If the latter has only two vertices $v_1 \in O(v)$ and $v'_1 \in O(v')$, then v and v' are joined in Δ in the same way as v_1 and v'_1 in $\Delta^\circ_{1,vv'}$. Thus, we may assume that $\Delta^\circ_{1,vv'}$ has at least three vertices. Suppose first that $\Delta^\circ_{1,vv'}$ is not a full connected component of Δ_1. Then, there is an "admissible index" (i.e., an index appearing in the tables of [22]) of relative rank 2 whose underlying Dynkin diagram is $\Delta^\circ_{1,vv'}$ and whose distinguished orbits are $O(v) \cap \Delta^\circ_{1,vv'}$ and $O(v') \cap \Delta^\circ_{1,vv'}$; indeed, it follows from the assertions 3.5.2 below that to $\{v,v'\}$ is canonically associated a quasi-simple group defined over a certain extension of $\bar{K}$ and having such an index. The relative Dynkin diagram corresponding to that index, which can be computed by simple explicit formulae given in [**22**, 2.5], provides the nature of the edge joining v and v' in Δ. The following table gives the result in the case where all vertices of $\Delta^\circ_{1,vv'}$ are distinguished (e.g., in the case where the residue field $\bar{K}$ is finite); in the first row, which represents $\Delta^\circ_{1,vv'}$, the sets $O(v) \cap \Delta^\circ_{1,vv'}$ and $O(v') \cap \Delta^\circ_{1,vv'}$ are circled:

$\Delta^\circ_{1,vv'}$	
corresponding edge in Δ	

There remains to consider the case where $\Delta^\circ_{1,vv'}$ is a full connected component of Δ_1, which means that v, v' are the two vertices of the local Dynkin diagram of a quasi-simple factor of relative rank 1 of G (cf. §1.12). Here we shall restrict ourselves to the case where all vertices are distinguished and simply refer the reader to the tables of §4 which give Δ in all the cases that can occur.

1.12. *Reduction to the absolutely quasi-simple case*; *restriction of scalars.* We shall now indicate how the local Dynkin diagram—with the attached integers $d(v)$—and the index of an arbitrary group G can be deduced from those of related absolutely simple groups.

First of all, those data are the same for G and for the adjoint group of G°. Thus, we may assume that G is connected and adjoint, hence is a direct product of K-simple groups. Then, the Dynkin diagram—with the $d(v)$ attached—and the index of G are the disjoint unions of the Dynkin diagrams and the indices of its simple factors.

There remains to consider the case where G is K-simple, which means [3, 6.21] that $G = R_{L/K}\, H$, where L is a separable extension of K, H is an absolutely simple group defined over L and $R_{L/K}$ denotes, as usual, the restriction of scalars. We shall, more generally, assume that $G = R_{L/K}\, H$ for an arbitrary reductive group H; this allows us to decompose the extension L/K into its unramified and its totally ramified parts and to handle the two cases separately.

If L/K is totally ramified, the index, the local Dynkin diagram and the integers $d(v)$ are the same for G, K as for H, L.

If L/K is unramified, the index of G, K consists of $[L: K]$ copies of the index of H, L permuted transitively by $\mathrm{Gal}(K_1/K)$ whose operation on the whole diagram is "induced up" from the operation of $\mathrm{Gal}(K_1/L)$ on one copy, the relative local Dynkin diagram of G, K is the same as that of H, L, and the integers $d(v)$ are $[L: K]$ times as big.

1.13. *The case of simply connected groups.* In §1.7, we have seen that the Weyl group W of Φ_{af} is a normal subgroup of $\tilde{W} = N(K)/Z(K)$. *When G is semisimple and simply connected, one has $W = \tilde{W}$.* In this and in other instances, *nonsimply connected groups behave with respect to the "local theory" in a way similar to nonconnected groups with respect to the classical theory.*

1.14. *Example. General linear groups.* Let D be a finite dimensional central division algebra over K. The unique extension of the valuation ω to D will also be denoted by ω. Suppose that $G = \mathrm{GL}_{n,D}$, the algebraic group defined by $G(L) = \mathrm{GL}_n(D \otimes L)$ for any K-algebra L, and take for S the "group of invertible diagonal matrices with central entries", that is, the split torus whose group of rational points $S(K)$ consists of all diagonal matrices $\mathrm{Diag}(s_1, \cdots, s_n)$ with $s_i \in K^\times$. The homomorphisms e_i: Mult $\to S$ defined by

$$e_i(t) = \mathrm{Diag}(1, \cdots, 1, t^{-1}, 1, \cdots, 1)^1$$

with the coefficient t^{-1} in the ith place ($i = 1, \cdots, n$) form a basis of X_* and hence of $V = X_* \otimes \boldsymbol{R}$. If $(a_i)_{1 \leq i \leq n}$ is the dual basis in the dual of V, the relative roots of G are the characters $a_{ij} = a_j - a_i$ $(i \neq j)$, the group $U_{a_{ij}}(K)$ consists of the matrices

$$u_{ij}(d) = 1 + ((g_{rs})) \quad \text{with } g_{rs} = \delta_r^i\, \delta_s^j\, d \quad (d \in D),$$

and $N(K)$ is the group of all invertible monomial matrices

$$n(\sigma; d_1, \cdots, d_n) = ((g_{ij})) \quad \text{with } g_{ij} = \delta_i^{\sigma(j)}\, d_j,$$

where σ is a permutation of $\{1, \cdots, n\}$ and $d_i \in D^\times$ $(= D - \{0\})$. For $d \in D^\times$, one has, with the notations of 1.4,

$$(1) \qquad m(u_{ij}(d)) = u_{ji}(-d^{-1})u_{ij}(d)u_{ji}(-d^{-1}) = n(\sigma; d_1, \cdots, d_n),$$

where σ is the transposition of i and j, $d_j = d$, $d_i = -d^{-1}$ and $d_k = 1$ for $k \neq i, j$. We may identify the apartment A with V in such a way that

$$\nu(n(\sigma; d_1, \cdots, d_n))\left(\sum_{i=1}^n v_i e_i\right) = \sum_{i=1}^n v_i' e_i \quad \text{with } v'_{\sigma(i)} = v_i + \omega(d_i).$$

[1] To avoid confusion, we adhere to the notations of [8, §10] which, unfortunately, impose this somewhat unnatural choice of the basis (e_i) (and, consequently, of (a_i)). This remark also applies to §§1.15 and 1.16.

From (1) and the definition of $\alpha(a, u)$, it now follows that $\alpha(a_{ij}, u_{ij}(d)) = a_{ij} + \omega(d)$. Thus

$$\Phi_{\mathrm{af}} = \{a_{ij} + \gamma \mid i, j \in \{1, \cdots, n\}, i \neq j, \gamma \in \Gamma\}, \tag{2}$$

and the filtration of $U_{a_{ij}}(K)$ by the groups X_α with $v(\alpha) = a_{ij}$ (cf. §1.4) is the image of the natural filtration of D by the isomorphism $d \mapsto u_{ij}(d)$. In particular, for any $\alpha \in \Phi_{\mathrm{af}}$, the integer $d(\alpha)$ of §1.6 is equal to the dimension of the residual algebra of D over $\bar{K}$. The description of the walls and half-apartments is readily deduced from (2). The chambers are prisms with simplicial bases, one of them, call it C, being defined by the inequalities $a_1 < a_2 < \cdots < a_n < a_1 + \omega(\pi_1)$, where π_1 denotes a uniformizing element of D. The corresponding basis consists of the affine roots $a_{i,i+1}$ $(i = 1, \cdots, n-1)$ and $a_{n1} + \omega(\pi_1)$, and we see that the local Dynkin diagram is a cycle of length n (affine diagram of type A_{n-1}). The special points are all the points of the one-dimensional facets of the chambers, that is, all the points $\sum v_i e_i$ where $v_i - v_1$ is an integral multiple of $\omega(\pi_1)$ for all i; they are hyperspecial if and only if $D = K$.

1.15. *Example. Quasi-split special unitary groups in odd dimension.* Let L be a separable quadratic extension of K. The valuation of L extending ω will also be called ω, and we denote by π_1 a uniformizing element of L, by Γ_1 the value group $\omega(L^\times) = Z \cdot \omega(\pi_1)$ and by τ the nontrivial K-automorphism of L. Let n be a strictly positive integer and set $I = \{\pm 1, \cdots, \pm n\}$. In L^{2n+1}, we consider the hermitian form

$$h\colon ((x_{-n}, \cdots, x_n), (y_{-n}, \cdots, y_n)) \mapsto \sum_{i \in I} x_i^\tau y_{-i} + x_0^\tau y_0. \tag{1}$$

Suppose that G is the algebraic group $SU(h)$ and let the torus S be defined by $S(K) = \{\mathrm{Diag}\,(d_{-n}, \cdots, d_n) | d_i \in K \text{ and } d_{-i} d_i = d_0 = 1 \text{ for all } i\}$. The homomorphisms e_i: Mult $\to S$ $(i = 1, \cdots, n)$ defined by $e_i(t) = \mathrm{Diag}(d_{-n}, \cdots, d_n)$ with $d_{-i} = t$, $d_i = t^{-1}$, $d_j = 1$ for $j \neq \pm i$ form a basis of X_*. If we denote by $(a_i)_{1 \leq i \leq n}$ the dual basis and if we set $a_{-i} = -a_i$ and $a_{ij} = a_i + a_j$, we have $\Phi = \{a_{ij} \mid i, j \in I, j \neq \pm i\} \cup \{a_i, 2a_i \mid i \in I\}$. For c, $d \in L$ such that $c^\tau c + d + d^\tau = 0$ and $i, j \in I$ with $j \neq \pm i$, we define the following elements of $G(K)$:

$u_{ij}(c) = 1 + ((g_{rs}))$ with $g_{-j,i} = c^\tau$, $g_{-i,j} = -c$ and all other $g_{rs} = 0$,

$u_i(c, d) = 1 + ((g_{rs}))$ with $g_{-i,0} = -c^\tau$, $g_{-i,i} = d$, $g_{0i} = c$ and all other $g_{rs} = 0$.

Then, $U_{a_{ij}}(K) = \{u_{ij}(c) \mid c \in L\}$, $U_{a_i}(K) = \{u_i(c, d) \mid c, d \in L, c^\tau c + d + d^\tau = 0\}$ and $U_{2a_i}(K) = \{u_i(0, d) \mid d \in L, d + d^\tau = 0\}$. The group $N(K)$ consists of all matrices of determinant one of the form $n(\sigma; d_{-n}, \cdots, d_n) = ((g_{ij}))$ with $g_{ij} = \delta_i^{\sigma(j)} d_j$, where σ is a permutation of $I \cup \{0\} = \{-n, \cdots, n\}$ which fixes 0 and preserves the partition of I in pairs $(-i, i)$, and the d_i's are elements of L such that $d_{-i}^\tau d_i = 1$ for all i.

For $c \in L$, one has, with the notations of 1.14,

$$\begin{aligned} m(u_{ij}(c)) &= u_{-i,-j}(-c^{-1}) u_{ij}(c) u_{-i,-j}(-c^{-1}) \\ &= n(\sigma; d_{-n}, \cdots, d_n) \end{aligned} \tag{2}$$

where σ is the permutation $(i, -j)(j, -i)$, $d_{-i} = c^{-1}$, $d_{-j} = -(c^\tau)^{-1}$, $d_j = -c$, $d_i = c^\tau$ and all other d_r are equal to 1. Similarly, for c, d as above with $c \neq 0$ (and hence $d \neq 0$),

$$\begin{aligned} m(u_i(c, d)) &= u_{-i}(-cd^{-1}, (d^\tau)^{-1}) u_i(c, d) u_{-i}(-c(d^\tau)^{-1}, (d^\tau)^{-1}) \\ &= n(\sigma; d_{-n}, \cdots, d_n) \end{aligned} \tag{3}$$

where σ is the transposition $(i, -i)$, $d_{-i} = (d^{\tau})^{-1}$, $d_0 = -d^{\tau}d^{-1}$, $d_i = d$ and all other d_r are equal to 1.

We may identify the apartment A with V in such a way that, for $v_1, \cdots, v_n \in \boldsymbol{R}$ and setting $v_{-i} = -v_i$, one has

$$(4) \qquad \nu(n(\sigma; d_{-n}, \cdots, d_n))\left(\sum_{i=1}^{n} v_i e_i\right) = \sum_{i=1}^{n} v_i' e_i \quad \text{with } v'_{\sigma(i)} = v_i + \omega(d_i).$$

From (2), (3), (4) and the definition of $\alpha(a, u)$, it follows that

for $c \neq 0$, $\alpha(a_{ij}, u_{ij}(c)) = a_{ij} + \omega(c)$,

for c, d as above and $c \neq 0$, $\alpha(a_i, u_i(c, d)) = a_i + \frac{1}{2}\omega(d)$,

for $d \in L^{\times}$ with $d + d^{\tau} = 0$, $\alpha(2a_i, u_i(0, d)) = 2a_i + \omega(d)$.

Setting $\Gamma' = \{\omega(d) | d \in L^{\times}, d + d^{\tau} = 0\}$, we see that for $\gamma \in \Gamma$ (resp. $\gamma \in \Gamma'$) $a_{ij} + \gamma$ (resp. $2a_i + \gamma$) is an affine root for all i, j. Furthermore, the filtration of $U_{a_{ij}}(K)$ (resp. $U_{2a_i}(K)$) by the subgroups X_{α} is the image of the natural filtration of L (resp. its intersection with the subgroup $\{d | d \in L, d + d^{\tau} = 0\}$) by the isomorphism $c \mapsto u_i(c)$ (resp. $d \mapsto u_i(0, d)$). In particular, the corresponding values of the integer $d(\alpha)$ of §1.6 are $d(2a_i + \gamma) = 1$ and $d(a_{ij} + \gamma) = 1$ or 2 according as L/K is ramified or not.

To determine under which condition $a_i + \frac{1}{2}\gamma \in \Phi_{\mathrm{af}}$, we first note that, with the notations of §1.4, $X_{a_i+\gamma/2} = \{u_i(c, d) | c^{\tau}c + d + d^{\tau} = 0, \omega(d) \geqq \gamma\}$. By definition, $a_i + \frac{1}{2}\gamma \in \Phi_{\mathrm{af}}$ if and only if $X_{a_i+\gamma/2} \not\subset X_{a_i+\gamma/2+\varepsilon} \cdot U_{2a_i}$ for every strictly positive ε. That means that there exists $c \in L$ such that

$$(5) \qquad \gamma = \sup\{\omega(d) \mid c^{\tau}c + d + d^{\tau} = 0\}.$$

More precisely, an easy computation shows that, with the notations of §1.6, the group $\bar{X}_{a_i+\gamma/2}/\bar{X}_{2a_i+\gamma}$ is isomorphic to the residue field of L or is trivial according as whether or not γ is given by (5) for some c; thus, we see that, in the first case (i.e., when $a_i + \frac{1}{2}\gamma \in \Phi_{\mathrm{af}}$), $d(a_i + \frac{1}{2}\gamma) = 2$ or 1 according as L/K is unramified or ramified. If we set $\delta = \sup\{\omega(d) \mid d \in L, d + d^{\tau} + 1 = 0\}$, a real number which is strictly negative if L/K is ramified and char $\bar{K} = 2$, and $= 0$ otherwise, the right-hand side of (5) can be written $\omega(c^{\tau}c) + \delta = 2\omega(c) + \delta$, and we conclude that

$$\Phi_{\mathrm{af}} = \{a_{ij} + \gamma \mid i, j \in I, j \neq \pm i, \gamma \in \Gamma_1\} \cup \{2a_i + \gamma \mid i \in I, \gamma \in \Gamma'\}$$
$$\cup \{a_i + \tfrac{1}{2}\gamma \mid i \in I, \gamma \in 2\Gamma_1 + \delta\}.$$

Let us show that

$$(6) \qquad \text{if } L/K \text{ is ramified, } \delta \notin \Gamma'.$$

Indeed, assume the contrary and let $x, y \in L$ be such that $x + x^{\tau} + 1 = y + y^{\tau} = 0$ and $\omega(x) = \omega(y) = \delta$. Upon multiplying y by a suitable unit of K, we may assume that $xy^{-1} + 1 \equiv 0 \pmod{\pi_1}$, but then $(x + y) + (x + y)^{\tau} + 1 = 0$ and $\omega(x + y) > \delta$, which contradicts the maximality of δ.

In view of (6), one of the following holds:

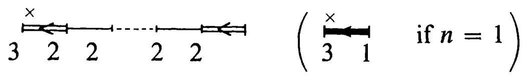
$$(7) \qquad L/K \text{ is unramified and } \Gamma = \Gamma_1 = \Gamma';$$

$$(8) \qquad L/K \text{ is ramified, } \Gamma = 2\Gamma_1 \text{ and } \Gamma' = 2\Gamma_1 + \delta + \omega(\pi_1).$$

In both cases, $\Gamma' \cup (2\Gamma_1 + \delta) = \Gamma_1$; therefore, the walls are the vanishing sets of the affine functions $a_{ij} + \gamma$ and $2a_i + \gamma$, with $\gamma \in \Gamma_1$, and the inequalities $0 < a_1 <$

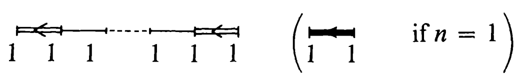
$a_2 < \cdots < a_n < \frac{1}{2}\,\omega(\pi_1)$ define a chamber. The corresponding basis is $\{a_1, a_{-1,2}, \cdots, a_{-n+1,n}, 2a_{-n} + \omega(\pi_1)\}$ if δ is an even multiple of $\omega(\pi_1)$ and $\{a_{-n} + \frac{1}{2}\,\omega(\pi_1), a_{-n+1,n}, \cdots, a_{-1,2}, 2a_1\}$ otherwise. It follows from (7), (8) that in the first case, $2a_1$ is an affine root if and only if L/K is unramified, and in the second case (where L/K is necessarily ramified) $2a_{-n} + \omega(\pi_1)$ is never an affine root. As a result, we see that, whatever the value of δ, the local Dynkin diagram, together with the attached integers $d(v)$ (cf. §1.8) are

(9)



or

(10)



according as L/K is unramified or ramified.

A point $v = \sum_{i=1}^{n} v_i e_i \in A$ is special if and only if either $v_i \in \Gamma_1$ for all i or $v_i - \frac{1}{2}\omega(\pi_1) \in \Gamma_1$ for all i. It is hyperspecial if and only if L/K is unramified and $v_i \in \Gamma_1$ for all i, which means that I_v consists of the vertex at the right end of the diagram (9).

1.16. *Example. Quasi-split but nonsplit orthogonal groups.* Let L be a separable quadratic extension of K and let n be an integer $\geqq 2$. In the space $K^n \oplus L \oplus K^n$, viewed as a $(2n + 2)$-dimensional vector space over K, we consider the quadratic form

$$q\colon (x_{-n}, \cdots, x_n) \mapsto \sum_{i=1}^{n} x_{-i}x_i + N_{L/K}x_0 \qquad (x_0 \in L;\, x_i \in K \text{ for } i \neq 0)$$

(where $N_{L/K}\colon L \to K$ denotes the norm), and we suppose that G is the orthogonal group $O(q)$. The elements of $G(R)$, for any K-algebra R, are conveniently represented by $(2n + 1) \times (2n + 1)$ matrices $((g_{ij}))_{-n \leqq i,\, j \leqq n}$ where $g_{ij} \in R$ if both i and j are not zero, $g_{0j} \in L \otimes_K R$ if $j \neq 0$, $g_{i0} \in \mathrm{Hom}_K(L, R)$ if $i \neq 0$, and $g_{00} \in \mathrm{Hom}_K(L, L) \otimes_K R$. For S, we take the group of diagonal matrices $\mathrm{Diag}(d_{-n}, \cdots, d_n)$ with $d_{-i}d_i = 1$ for $1 \leqq i \leqq n$ and $d_{00} = \mathrm{id}$. The characters a_i: $\mathrm{Diag}(d_{-n}, \cdots, d_n) \mapsto d_{-i}$ for $1 \leqq i \leqq n$ form a basis of $X^*(S)$ and if we set $a_{-i} = -a_i$, $a_{ij} = a_i + a_j$ and $I = \{\pm 1, \cdots, \pm n\}$, we have

$$\Phi = \{a_{ij} \mid i, j \in I,\ j \neq \pm i\} \cup \{a_i \mid i \in I\},$$

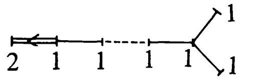
a root system of type B_n.

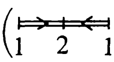
Here, we shall simply describe the affine root system Φ_{af} and the local Dynkin diagram without giving the details of the calculations, which can be found, in a more general setting (covering also the groups handled in the previous section) in [**8**, 10.1]. Calling again Γ_1 the value group of L, one has, for a suitable identification of A and V,

$$\Phi_{\mathrm{af}} = \{a_{ij} + \gamma \mid i, j \in I, j \neq \pm i, \gamma \in \Gamma\} \cup \{a_i + \gamma \mid i \in I, \gamma \in \Gamma_1\}.$$

If the extension L/K is unramified, the inequalities $0 < a_1 < \cdots < a_n < \omega(\pi) - a_{n-1}$ define a chamber, the corresponding basis of Φ_{af} is $\{a_1, a_{-1,2}, \cdots,$

$a_{-n+1,n}, a_{-n+1,-n} + \omega(\pi)\}$ and the local Dynkin diagram, together with the attached integers $d(v)$, is

(1) 

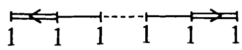
(

if $n = 2$).

The special vertices are the two endpoints on the ramified side of the diagram (the two endpoints of the diagram if $n = 2$); both correspond to hyperspecial points of A. If L/K is ramified, the inequalities $0 < a_1 < \cdots < a_n < \frac{1}{2}\omega(\pi)$ define a chamber, the corresponding basis is

$$\{a_1, a_{-1,2}, \cdots, a_{-n+1,n}, a_{-n} + \tfrac{1}{2}\omega(\pi)\}$$

and the local Dynkin diagram is



The special vertices are the two endpoints of the diagram and they do not correspond to hyperspecial points. Note that in the unramified case, the Weyl group W is an affine reflection group of type B_n, whereas in the ramified case, it is of type C_n.

2. The building.

2.1. *Definitions.* The *building* $\mathscr{B} = \mathscr{B}(G, K)$ *of* G *over* K can be constructed by "gluing together" the apartments of the various maximal K-split tori of G. More precisely, a definition of $\mathscr{B}$ is provided by the following statement where by "$G(K)$-set", we mean a set with a left action of $G(K)$ on it.

Let $A = A(G, K)$ *be given as in* §1.2. *Then, there exists one and, up to unique isomorphism, only one* $G(K)$*-set* $\mathscr{B}$ *containing* A *and having the following properties*: $\mathscr{B} = \bigcup_{g\in G(K)} gA$, *the group* $N(K)$ *stabilizes* A *and operates on it through* ν (*cf.* §1.2) *and for every affine root* α, *the group* X_α *of* §1.4 *fixes the half-apartment* $A_\alpha = \alpha^{-1}([0, \infty))$ *pointwise.*

(N.B. The "canonicity" of the building $\mathscr{B}$ is the same as that of A: cf. §1.2.)

The proof roughly goes as follows. We assume that G is semisimple (which is no essential restriction). Modulo 1.4.1 and 1.4.2—as explained in §1.5—the existence of $\mathscr{B}$ is proved in [**8**, 7.4]. It is then clear that there is a "universal" $G(K)$-set $\tilde{\mathscr{B}}$ with the given properties, which is obtained by taking the quotient of the direct product $G(K) \times A$ by a certain equivalence relation. The canonical mapping of $\tilde{\mathscr{B}}$ in the building $\mathscr{I}$ defined in [**8**, 7.4.2] is obviously surjective, and it is also injective because, as is readily verified, the stabilizer of a point of $\tilde{\mathscr{B}}$ contains the stabilizer of its image in $\mathscr{I}$. Thus, $\mathscr{I}$ maps onto any $G(K)$-set $\mathscr{B}$ with the required properties and, using [**8**, 7.3.4], one shows that the stabilizers of the points of A cannot be bigger in $\mathscr{B}$ than they are in $\mathscr{I}$ without "eating more of $N(K)$" than they are allowed to by the prescribed action of $N(K)$ on A.

The sets gA with $g \in G(K)$ are called the *apartments* of the building. The apartment gA can be identified with "the" apartment of the maximal split torus ${}^gS$. That gives a one-to-one correspondence between the apartments of $\mathscr{B}$ and the maximal K-split tori of G: indeed, gA is the only apartment stable by ${}^gS(K)$ (the proof of [**8**, 2.8.11] shows that) and ${}^gN(K)$, which determines ${}^gS$, is the stabilizer of

gA in $G(K)$. "In most cases", gA can also be characterized as the fixed-point set in $\mathscr{B}$ of the group of units ${}^gS^\circ = \{s \in {}^gS(K) | \omega(\chi(s)) = 0$ for all characters $\chi \in X^*({}^gS)\}$, but that is not always true (for more precise statements, cf. §3.6).

In this context, it is worthwhile to note also that the half-apartment A_α (for $\alpha \in \Phi_{\text{af}}$) is *never* the fixed-point set of the group X_α: indeed, if $\mathscr{B}$ is metrized in the way described below (§2.3), there is a constant c such that for every point $x \in A_\alpha$ at distance d of the wall ∂A_α, the whole ball with center x and radius cd is pointwise fixed by X_α (cf. [**8**, 7.4.33]).

If S_1 is the maximal split torus of the center of G° and if $G_1, \cdots, G_m$ are the almost simple factors of G°, the building $\mathscr{B}$ is canonically isomorphic with the direct product of the buildings $\mathscr{B}(S_1, K)$ (which is an affine space) and $\mathscr{B}(G_i, K)$ $(i = 1, \cdots, m)$. If G is K-anisotropic (i.e., if $S = \{1\}$), $\mathscr{B}$ consists of a single point. If $G = R_{L/K}\, H$, where L is a separable extension of K and H is a reductive group over L, the buildings $\mathscr{B}(G, K)$ and $\mathscr{B}(H, L)$ are canonically isomorphic.

2.2. *Affine structures, facets, retractions, topology and other canonical structures on* $\mathscr{B}$. Since the stabilizer $N(K)$ of A in $G(K)$ preserves its affine structure and its partition in facets, each apartment gA of $\mathscr{B}$ (with $g \in G(K)$) is endowed with a natural structure of real affine space and a partition in facets. Those structures agree on intersections. Indeed,

2.2.1. *If A' and A'' are two apartments, there is an element of* $G(K)$ *which maps A' onto A'' and fixes the intersection $A' \cap A''$ pointwise; furthermore, $A' \cap A''$ is a closed convex union of facets in A' (hence also in A'')* [**8**, 7.4.8].

From that, we deduce a partition of $\mathscr{B}$ in *facets*, among which those which are open in apartments are called *chambers*. In particular, if G° is quasi-simple (resp. semisimple), $\mathscr{B}$ is a simplicial (resp. polysimplicial) complex.

Given two facets of $\mathscr{B}$, there is an apartment containing them both [**8**, 7.4.18]. In particular, given two points $x, y \in \mathscr{B}$, there is an apartment which contains them and it follows from 2.2.1 that, for $t \in [0, 1] \subset \boldsymbol{R}$, the point $(1-t)x + ty$, which is well defined in any such apartment, is independent of it. The set $\{(1-t)x + ty \mid t \in [0, 1]\}$ is called the *geodesic segment* joining x and y in $\mathscr{B}$.

Let A' be an apartment and let $C \subset A'$ be a chamber. For every apartment containing C, there is a unique isomorphism of affine spaces of that apartment onto A' which fixes C pointwise. In view of 2.2.1, all those isomorphisms can be glued together in a mapping $\rho_{A';C}\colon \mathscr{B} \to A'$ called the *retraction of $\mathscr{B}$ onto A' with center C*. Clearly, geodesic segments are mapped by $\rho_{A';C}$ onto broken lines (connected unions of finitely many geodesic segments).

The building $\mathscr{B}$ is commonly endowed with a topology invariant by $G(K)$ which is most naturally defined via the metric considered below (2.3), but which can also be more canonically defined as the weakest topology such that all $\rho_{A';C}$ are continuous. *If the residue field $\bar{K}$ is finite, that topology makes $\mathscr{B}$ into a locally compact space* and coincides with the "CW-topology" (that is, the quotient topology of the natural topology of the disjoint union of all apartments). Otherwise, it is strictly weaker than the latter. In all cases, the topological space *$\mathscr{B}$ is contractible*; indeed, for every point $x \in \mathscr{B}$, the mappings $\varphi_t\colon \mathscr{B} \to \mathscr{B}$ defined by $\varphi_t(y) = tx + (1-t)y$ form a homotopy from the identity to the retraction of $\mathscr{B}$ onto $\{x\}$ [**8**, 7.4.20].

A subset of $\mathscr{B}$ is called *bounded* if its image by some retraction $\rho_{A';C}$ is bounded, in which case its image by every such retraction is bounded, as is easily seen. As usual, a subset H of $G(K)$ is called *bounded* if for every K-regular function f on G, the set $\omega(f(H))$ is bounded from below. The action of $G(K)$ on $\mathscr{B}$ is bounded and "proper" in the following sense: in the mapping $(g, b) \mapsto (gb, b)$ of $G(K) \times \mathscr{B}$ in $\mathscr{B} \times \mathscr{B}$, bounded subsets of $G(K) \times \mathscr{B}$—i.e., subsets of products of bounded sets—are mapped onto bounded sets, and the inverse images of bounded subsets of $\mathscr{B} \times \mathscr{B}$ are bounded. If $\bar{K}$ is finite, "bounded" becomes synonymous with "relatively compact"; in particular, the action of $G(K)$ on $\mathscr{B}$ is proper in the usual sense.

2.3. *Metric and simplicial decomposition.* In various questions, buildings play for p-adic reductive groups the same role as the symmetric spaces in the study of noncompact real simple Lie groups (cf. [**24**, §5] and the references given there). This section shows some aspects of the analogy; cf. also [**18**, 5.32]. Note that, unlike those introduced in §2.2, the structures considered here are *not canonical*, at least when G is not semisimple.

Let us choose in V a scalar product invariant under the Weyl group ${}^vW$. If G is quasi-simple, such a scalar product is unique up to a scalar factor, and there are various "natural" ways of normalizing it (Killing form, prescription of the length of short coroots, etc.). Canonical choices are also possible—componentwise—if G is semisimple, but not in general. From the scalar product in question, one deduces a Euclidean distance on A, hence, through the action of $G(K)$, on any apartment. From 2.2.1, it follows that two points x, y of $\mathscr{B}$ have the same distance $d(x, y)$ in all apartments containing them, and the properties of the retractions $\rho_{A';C}$ described in §2.2 readily imply that the building $\mathscr{B}$ endowed with the distance function d: $\mathscr{B} \times \mathscr{B} \to \boldsymbol{R}_+$ is a *complete metric space* [**8**, 2.5]. The associated topology coincides with that defined in §2.2. Again using the retractions $\rho_{A';C}$ one shows [**8**, 3.2.1] that d satisfies the following inequality, where $x, y, z, m \in \mathscr{B}$ and $d(x, m) = d(y, m) = \frac{1}{2} d(x, y)$:

$$d(x, z)^2 + d(y, z)^2 \geqq 2d(m, z)^2 + \tfrac{1}{2}d(x, y)^2.$$

In Riemannian geometry, that inequality characterizes the spaces with negative sectional curvatures (hence is valid in noncompact irreducible symmetric spaces!); as in the Riemannian case, it can be used here to prove the following *fixed-point theorem*:

2.3.1. *A bounded group of isometries of $\mathscr{B}$ has a fixed point* [**8**, 3.2.4]. Interesting applications are provided by Galois groups ("Galois descent" of the building) and by bounded subgroups of $G(K)$ (cf. §3.2).

In some applications (cf., e.g., [**2**]), it is useful to dispose of a simplicial decomposition of $\mathscr{B}$ invariant under $G(K)$. To obtain it, it suffices to choose a simplicial decomposition of A invariant under $N(K)$ and finer than the partition in facets—it is easily seen that such a decomposition always exists—and to carry it over to all apartments by means of the $G(K)$-action. If G is semisimple, one can more directly use the canonical barycentric subdivision of the partition of B in polysimplical facets. If G is quasi-simple, that partition itself meets the requirements.

2.4. *Dynkin diagram; special and hyperspecial points.* Let C be a chamber of $\mathscr{B}$. Starting from any apartment containing C, we can, following §1.8, define a local Dynkin diagram $\Delta(G, C)$ which, in view of 2.2.1, does not depend, up to unique isomorphisms, on the choice of the apartment. If C' is another chamber, 1.8.1, applied to any apartment containing C and C', provides an isomorphism $\varphi_{C'C}$:

$\Delta(G,C) \to \Delta(G, C')$ which, again by 2.2.1, is independent of the apartment in question. All those isomorphisms are coherent: if C, C', C'' are three chambers, one has $\varphi_{C''C} = \varphi_{C''C'} \circ \varphi_{C'C}$. Thus, we can talk about *the* local Dynkin diagram $\Delta(G) = \Delta(G, K)$ of G over K, a diagram which is well-defined up to unique isomorphisms. The same is true of the absolute local Dynkin diagram (§1.11), which is nothing else but the diagram $\Delta(G, K_1)$ of G over the unramified closure K_1 of K, and of the local index (§1.11).

The definitions of §§1.9 and 1.10 can be immediately transposed to arbitrary points x and arbitrary facets F of the building $\mathscr{B}$: one chooses an apartment containing x or F, uses the definition under consideration and deduces from 2.2.1 that the result is independent of the apartment chosen. Thus, to every point x (resp. facet F) of $\mathscr{B}$ is canonically associated a set I_x (resp. I_F) of vertices of $\Delta(G)$ and a root system Φ_x (resp. Φ_F), the latter being only defined up to noncanonical isomorphisms. We can also talk about *special* and *hyperspecial* points of $\mathscr{B}$. The criterion in terms of I_x for a point x to be special (last paragraph of §1.9) remains of course valid. A necessary condition for the existence of hyperspecial points is that G split over an unramified extension of K; that condition is also sufficient if G is quasi-split.

To every vertex v of the diagram $\Delta(G)$ is attached an integer $d(v)$: the definition given in §1.8 made reference to an apartment A but the result is independent of its choice, always by 2.2.1. If the residue field $\bar{K}$ is finite, isomorphic with $\boldsymbol{F}_q$, the number $d(v)$ can be interpreted as follows: a facet F of codimension one and "type v", that is, such that I_F is the complement of v in the set of all vertices of $\Delta(G)$, is contained in the closure of exactly $q^{d(v)} + 1$ chambers (cf. §3.5).

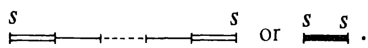
2.5. *Action of* (Aut G)(K) *on* $\mathscr{B}$ *and* Δ; *conjugacy classes of special and hyperspecial points.* The group (Aut G)(K) of all K-automorphisms of G and, in particular, the group $G_{\mathrm{ad}}(K)$ of rational points of the adjoint group G_{ad} of G°, act on $\mathscr{B}$ and on the local Dynkin diagram $\Delta = \Delta(G)$ "par transport de structure". Through the canonical homomorphism int: $G \to \mathrm{Aut}\, G$, that gives an action of $G(K)$ on $\mathscr{B}$ and on Δ. The action of $G(K)$ on $\mathscr{B}$ provided by the definition of $\mathscr{B}$ as a $G(K)$-set coincides with this one if G is semisimple but not in general; however, the induced actions on Δ are always the same. We call $\Xi = \Xi(G, K)$ the image of $G(K)$ in Aut Δ.

If G is semisimple and simply connected, it operates trivially on Δ, *i.e.*, $\Xi = \{1\}$ (another illustration of the "philosophy" of §1.13).

Suppose G connected. Then, Ξ is also the image of $Z(K)$ in Aut Δ, and it can be computed as follows. We denote by $\tilde{G}$ a simply connected covering of the derived group of G, by $\tilde{S}$ the maximal split torus of $\tilde{G}$ whose image in G is contained in S, by $\tilde{Z}$ the centralizer of $\tilde{S}$ in $\tilde{G}$, by S_1 the maximal subtorus of S which is central in G and by Z_s the image of $\tilde{Z}(K)$ in $Z(K)$. Then, $S_1(K)$, Z_s and $Z_c = \{z \in Z(K) | \omega(\chi(z)) = 0 \text{ for all } \chi \in X^*(Z)\}$ (cf. §1.2) are normal subgroups of $Z(K)$ and their product $S_1(K) \cdot Z_s \cdot Z_c$ is the kernel of the action of $Z(K)$ on Δ; thus $\Xi = Z(K)/(S_1(K) \cdot Z_s \cdot Z_c)$. If G is quasi-split—in particular if $\bar{K}$ is algebraically closed—Z and $\tilde{Z}$ are tori and the computation of Ξ is particularly easy. Note that, in most interesting cases, the subgroup Ξ of Aut Δ is uniquely determined by the underlying "abstract" group.

Two facets F and F' of $\mathscr{B}$ are in the same orbit of $G(K)$—for any one of the two actions of $G(K)$ on $\mathscr{B}$ described above—if and only if I_F and $I_{F'}$ (cf. §2.4) are in the same orbit of Ξ. In particular, *if G is semisimple and simply connected, the orbits of $G(K)$ in the set of special points of $\mathscr{B}$ are in canonical one-to-one correspondence with the sets of vertices of Δ consisting of one special vertex out of each connected component.*

Suppose G semisimple. If G is K-split, the group $G_{\mathrm{ad}}(K)$ permutes transitively the special points of $\mathscr{B}$: that is an immediate consequence of Proposition 2 in [**5**,VI.2.2]. A case analysis shows that, for *any* semisimple G, $G_{\mathrm{ad}}(K)$ permutes transitively the special points except possibly if the Coxeter diagram underlying Δ has a connected component of the form



Suppose now that G is quasi-simple and that the Coxeter diagram in question is one of those above. Then, obvious necessary conditions for $G_{\mathrm{ad}}(K)$ (and even $(\mathrm{Aut}\ G)(K)$) to permute transitively the special points are the existence of an automorphism of Δ permuting its two special vertices, and the equality of the numbers $d(v)$ attached to them. One verifies that if the residue field $\bar{K}$ is finite, those conditions are also sufficient.

For arbitrary G, if $\mathscr{B}$ has hyperspecial points, the facets consisting of such points—hence the points themselves if G is semisimple—are permuted transitively by $G_{\mathrm{ad}}(K)$.

2.6. *Behaviour under field extensions.*

The buildings behave functorially with respect to Galois extensions.

More precisely, for every Galois extension K_1 of K, we can consider the building $\mathscr{B}(G, K_1)$, on which the Galois group $\mathrm{Gal}(K_1/K)$ acts naturally (in the nonsemisimple case, one has to "canonify" the apartments—and hence $\mathscr{B}$—as described in §1.2), and there is a *unique* system of injections

$$\iota_{K_2K_1}: \mathscr{B}(G, K_1) \to \mathscr{B}(G, K_2) \qquad (K_1,K_2 \text{ Galois extensions of } K \text{ with } K_1 \subset K_2)$$

with the following properties:

the image of $\iota_{K_2K_1}$ is pointwise fixed by $\mathrm{Gal}(K_2/K_1)$;

the restriction of $\iota_{K_2K_1}$ to any apartment of $\mathscr{B}(G, K_1)$ is an affine mapping into an apartment of $\mathscr{B}(G, K_2)$;

$\iota_{K_2K_1}$ is $G(K_1)$-covariant;

if $K_1 \subset K_2 \subset K_3$, one has has $\iota_{K_3K_1} = \iota_{K_3K_2} \circ \iota_{K_2K_1}$.

The last property allows us to identify coherently every $\mathscr{B}(G, K_1)$ with its image by every $\iota_{K_2K_1}$.

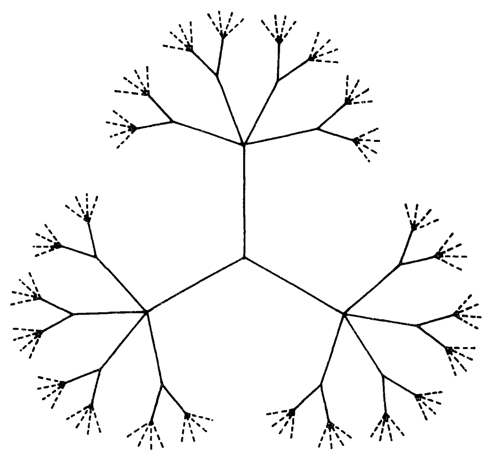
2.6.1. *If K_1/K is unramified* (*or even tamely ramified*: *cf.* [**19**]), *$\mathscr{B}$ is the fixed point set of* $\mathrm{Gal}(K_1/K)$ *in $\mathscr{B}(G, K_1)$ and the apartment $A = A(G, S, K)$ is the intersection of $\mathscr{B}$ with the apartment $A(G, S_1, K_1)$ of any maximal K_1-split K-torus S_1 of G containing S.* Still assuming that K_1/K is unramified, one deduces from 1.10.2 that *a point x of $\mathscr{B}$ which is special in $\mathscr{B}(G, K_1)$ is also special in $\mathscr{B}$*; *if furthermore G is split over K_1, the point x is hyperspecial.*

If K_1/K is wildly ramified, the fixed point set of $\mathrm{Gal}(K_1/K)$ in $\mathscr{B}(G, K_1)$ may be strictly bigger than $\mathscr{B}$: it then looks like the building $\mathscr{B}$ "covered with barbs". Suppose for example that G is split and is not a torus, that $K = \mathbf{Q}_2$ and that K_1 is totally ramified over K and different from K (which implies that K_1/K is wildly ramified). The apartment $A = A(G, S, K)$ of $\mathscr{B}$ is also an apartment of $\mathscr{B}(G, K_1)$.

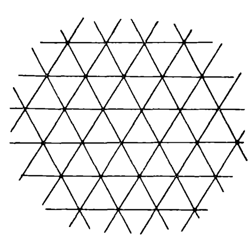
Let F be a facet of codimension one of A with respect to K_1 which is not a facet with respect to K (cf. the example following 1.10.1). By the last assertions of §§1.8 and 2.4, there is exactly one chamber of $\mathscr{B}(G, K_1)$ not contained in A and whose closure contains F; it must of course be fixed by $\mathrm{Gal}(K_1/K)$ and cannot be contained in $\mathscr{B}$ since F is not a facet of $\mathscr{B}$.

For a proof of the above results and a more detailed analysis of the situation, cf. **[19]**.

2.7. *Example. Groups of relative rank* 1. The building of a semisimple group of relative rank 1 is a contractible simplicial complex of dimension 1, i.e., *a tree*. All its vertices are special points. If $\bar{K} \cong F_q$ and if d, d' are the integers $d(v)$ attached to the two vertices of the Dynkin diagram, each edge of the tree has one vertex of order $q^d + 1$ and one vertex of order $q^{d'} + 1$ (cf. §2.4). Consider for instance the special orthogonal group of a nonsplit quadratic form in 5 variables over Q_2: here, $d = 1$, $d' = 2$, and the building looks as suggested by the picture below. In that case, the vertices of order 5 are hyperspecial and the others are not.



2.8. *Example.* SL_3 *and* GL_3. Suppose that $G = \mathrm{SL}_3$. The building $\mathscr{B} = \mathscr{B}(G, K)$ is a 2-dimensional simplicial complex whose maximal simplices are equilateral triangles, for the metric introduced in §2.3. The apartments are Euclidean planes triangulated in the familiar way:



To picture the building itself, one must imagine it "ramifying" along every edge, each edge belonging to $q + 1$ triangles if $q = \mathrm{card}\ \bar{K}$. The link of each vertex in $\mathscr{B}$

is the "spherical building" of $SL_3(\bar{K})$, that is, the "flag complex" of a projective plane over $\bar{K}$, a picture of which can be found in **[24]** or **[26]** for the special case where $\bar{K} \cong \boldsymbol{F}_2$. All vertices of $\mathscr{B}$ are hyperspecial.

The building $\mathscr{B}(GL_3, K)$ is the direct product of $\mathscr{B}$ and an affine line.

2.9. *Example. General linear groups.* We adopt the hypotheses and the notations of §1.14. In particular, D denotes a finite dimensional central division algebra over K and $G = GL_{n,D}$ (thus $G(K) = GL_n(D)$). Then, the building $\mathscr{B} = \mathscr{B}(G, K)$ can be interpreted as the set $\mathscr{N}$ of all "additive norms" in D^n, that is, of all functions $\varphi: D^n \to \boldsymbol{R} \cup \{+\infty\}$ such that

$$\varphi(x + y) \geqq \inf\{\varphi(x), \varphi(y)\} \qquad (x, y \in D^n),$$
$$\varphi(xd) = \varphi(x) + \omega(d) \qquad (x \in D^n, d \in D).$$

More precisely, if we identify the apartment A with V as in §1.14, the mapping $A \to \mathscr{N}$ which maps $\sum_{i=1}^n v_i e_i$ onto the norm

$$(x_1, \cdots, x_n) \mapsto \inf\{\omega(x_i) - v_i | i = 1, \cdots, n\}$$

extends—of course uniquely—to an isomorphism of $G(K)$-sets $\mathscr{B} \to \mathscr{N}$, where $G(K)$ operates on $\mathscr{N}$ by $(g\varphi)(x) = \varphi(g^{-1}x)$. A norm φ is special—i.e., corresponds to a special point of B—if and only if there is a basis $(b_i)_{1 \leqq i \leqq n}$ of the vector space D^n and a real number f such that

$$\varphi\Big(\sum_{i=1}^n b_i d_i\Big) = f + \inf\{\omega(d_i) | i = 1, \cdots, n\} \qquad (d_i \in D);$$

φ is hyperspecial if and only if it is special and $D = K$.

A similar interpretation of $\mathscr{B}(SL_{n,D}, K)$ can be found in **[8**, p. 238]. The space $\mathscr{N}$ has been first considered by O. Goldman and N. Iwahori **[12]**.

2.10. *Example. Special unitary groups.* Let L, ω, π_1, Γ_1, τ and δ have the same meaning as in §1.15. In particular, L is a separable quadratic extension of K and $\delta = \sup\{\omega(d) \mid d \in L,\ d + d^\tau = 1\}$. Let E be a finite dimensional vector space over L endowed with a nondegenerate hermitian form h relative to τ, and suppose that $G = SU(h)$. Then, the building $\mathscr{B}$ of G over K, which is also, by the way, the building of $U(h)$, can be interpreted as the set $\mathscr{N}_h$ of all additive norms $\varphi: E \to \boldsymbol{R} \cup \{+\infty\}$ satisfying the inequalities

$$\omega(h(x, x)) \geqq 2\varphi(x) - \delta \qquad (x \in E),$$
$$\omega(h(x, y)) \geqq \varphi(x) + \varphi(y) \qquad (x, y \in E),$$

and maximal with that property (cf. **[8**, p. 239] for a more general result).

Suppose further that $E = L^{2n+1}$ and that h is as in 1.15(1). Then, the identification of $\mathscr{B}$ and $\mathscr{N}_h$ can be described more explicitly as follows: with the notations of §1.15, the mapping $A \to \mathscr{N}_h$ which maps $\sum_{i=1}^n v_i e_i$ onto the norm

$$(x_{-n}, \cdots, x_n) \mapsto \inf\{\omega(x_i) - v_i,\ \omega(x_{-i}) + v_i,\ \omega(x_0) - \delta | 1 \leqq i \leqq n\}$$

extends uniquely to an isomorphism of $G(K)$-sets $\mathscr{B} \to \mathscr{N}_h$. A norm $\varphi \in \mathscr{N}_h$ is special—i.e., corresponds to a special point of $\mathscr{B}$—if and only if there is a basis $(b_i)_{1 \leqq i \leqq n}$ of E with respect to which h has the form 1.15(1) and a constant $f \in \frac{1}{2}\Gamma_1$ such that, for $x_i \in L$,

$$(1)\qquad \varphi\Big(\sum_{i=1}^{n} b_i x_i\Big) = \inf\{\omega(x_i) - f,\ \omega(x_{-i}) + f,\ \omega(x_0) - \delta \mid 1 \leqq i \leqq n\}$$

(one can then choose the basis (b_i) so that $f = 0$ or $\frac{1}{2}\omega(\pi_1)$). The norm φ is hyperspecial if L/K is unramified and if there is a basis (b_i) of E such that (1) holds for $f = 0$.

3. Stabilizers and centralizers. *From now on, G is assumed to be connected.*

3.1. *Notations; a BN-pair.* For every algebraic extension K_1 of K with finite ramification index and every subset Ω of the building $\mathscr{B}(G, K_1)$, we denote by $G(K_1)^\Omega$ the group of all elements of $G(K_1)$ fixing Ω pointwise. If Ω is reduced to a point x, we also write $G(K_1)^x$ for $G(K_1)^\Omega$. Note that if F is a facet of $\mathscr{B}(G, K_1)$ and if x is a point of F "in general position", one has $G(K_1)^F = G(K_1)^x$. The stabilizers $G(K)^x$ of special (resp. hyperspecial) points $x \in \mathscr{B}$ are called *special* (resp. *hyperspecial*) *subgroups* of $G(K)$.

We recall that if G is semisimple and simply connected, the group $\tilde{W} = N(K)/Z(K)$ coincides with the Weyl group W of the affine root system Φ_{af}. As before, we set $\mathscr{B} = \mathscr{B}(G, K)$.

3.1.1. *Suppose that $\tilde{W} = W$. Then $G(K)^F = G(K)^x$ for every facet F of $\mathscr{B}$ and every $x \in F$. Furthermore, if C is a chamber of $A = A(G, S)$, the pair $(G(K)^C, N(K))$ is a BN-pair (or Tits system: cf.* **[5]**, **[23]**) *in $G(K)$ with Weyl group W.* In that case, the groups $G(K)^x$ for $x \in \mathscr{B}$ are called the *parahoric subgroups* of $G(K)$ (cf. **[8]**), but we shall avoid using that terminology here in order not to prejudge of its most suitable extension to the nonsimply connected case. An alternative construction of the building $\mathscr{B}$ starting from the above BN-pair (which can be defined independently of the building, as we shall see) and using the parahoric subgroups defined by means of that BN-pair is given in [**8**, §2].

Let Ω be a nonempty subset of the apartment A whose projection on the building of the semisimple part of G (cf. last paragraph of 2.1) is bounded. For any root $a \in \Phi$, let $\alpha(a, \Omega)$ denote the smallest affine root whose vector part is a and which is positive on Ω. Let Φ' be the set of all nondivisible roots—i.e., all roots $a \in \Phi$ such that $\frac{1}{2}a \notin \Phi$—and let Φ'^+ (resp. Φ'^-) be the set of all nondivisible roots which are positive (resp. negative) with respect to a basis of Φ, arbitrarily chosen. Set $N(K)^\Omega = N(K) \cap G(K)^\Omega$ and let Z_c and X_α be defined as in §§1.2 and 1.4. Then one has the following group-theoretical description of $G(K)^\Omega$ (cf. [**8**,6 .4.9, 6.4.48, 7.4.4]):

If $X^\pm(\Omega)$ denotes the group generated by all $X_{\alpha(a,\Omega)}$ with $a \in \Phi'^\pm$, the product mapping $\prod_{a\in\Phi'^\pm} X_{\alpha(a,\Omega)} \to X^\pm(\Omega)$ is bijective for every ordering of the factors of the product and one has $G(K)^\Omega = X^-(\Omega) \cdot X^+(\Omega) \cdot N(K)^\Omega$. If Ω contains an open subset of A, the product mapping $\prod_{a\in\Phi'} X_{\alpha(a,\Omega)} \times Z_c \to G(K)^\Omega$ is bijective for every ordering of the factors of the product.

3.2. *Maximal bounded subgroup.* For every nonempty subset Ω of $\mathscr{B}$, $G(K)^\Omega$ is a bounded subgroup of $G(K)$ (cf. §2.2). If the residue field $\bar{K}$ is finite, $G(K)^\Omega$ is even compact and, in what follows, "maximal bounded" can be replaced by "maximal compact".

From 2.3.1, one easily deduces that:

every bounded subgroup of $G(K)$ is contained in a maximal one and every maximal bounded subgroup is the stabilizer $G(K)^x$ of a point x of $\mathscr{B}$.

It is now clear that *if x belongs to a facet of minimal dimension of $\mathscr{B}$, $G(K)^x$ is a maximal bounded subgroup of $G(K)$, in particular, special subgroups are maximal bounded subgroups.* From 3.1.1, it follows that the above two statements give a complete description of the maximal bounded subgroups in the simply connected case:

if G is semisimple and simply connected, the maximal bounded subgroups of $G(K)$ are precisely the stabilizers of the vertices of the building $\mathscr{B}$; they form $\prod_{i=1}^{r} (l_i + 1)$ conjugacy classes, where $l_1, \cdots, l_r$ denote the relative ranks of the quasi-simple factors of G.

For an analysis of the nonsimply connected case, cf. [**8**, 3.3.5].

3.3. *Various decompositions.* Let C be a chamber of $A = A(G, S)$. We identify A with the vector space V in such a way that 0 becomes a special point contained in the closure of C; in particular, *$G(K)^0$ is a special subgroup* of $G(K)$. Set $D = \boldsymbol{R}_+^* \cdot C$ (a "vector chamber") and $B = G(K)^C$; if $\bar{K}$ is finite or, more generally, if G is residually quasi-split, and if G is simply connected, B is an *Iwahori subgroup* of $G(K)$ (cf. §3.7). Let U^+ be the group generated by all U_a for which $a|_C$—and hence $a|_D$—is positive and let Y be the "intersection of V and $\tilde{W}$", that is, the group of all translations of A contained in $\tilde{W}$; thus, Y is the image of $Z(K)$ by the homomorphism ν of §1.2. Set $Y_+ = Y \cap \bar{D}$ (closure of D) and $Z(K)_+ = \nu^{-1}(Y_+)$, a subsemigroup of $Z(K)$.

3.3.1. *Bruhat decomposition.* One has $G(K) = BN(K)B$ and the mapping $BnB \mapsto \nu(n)$ ($n \in N(K)$) is a bijection of the set $\{BgB \mid g \in G(K)\}$ onto $\tilde{W}$.

If $n \in N(K)$ and $\nu(n) = w$, we also write $BnB = BwB$, as usual. If $\bar{K} = \boldsymbol{F}_q$, the cardinality q_w of BwB/B (used for instance in [**1**]) is given by the following formula in terms of the integers $d(v)$ of §1.8: set $w = r_1 \cdots r_l w_0$, where $(r_1, \cdots, r_l)$ is a reduced word in the Coxeter group W and $w_0(C) = C$, and let v_i be the vertex of Δ representing r_i; then $q_w = q^d$ with $d = \sum_{i=1}^{l} d(v_i)$. In particular, we have another interpretation of $d(v)$: $q^{d(v)} = q_{r(v)}$ where $r(v)$ denotes the fundamental reflection corresponding to the vertex v of Δ.

More generally, for any $\bar{K}$, the quotient BwB/B has a natural structure of "perfect variety" over $\bar{K}$, in the sense of Serre [Publ. Math. I.H.E.S. **7** (1960), 1.4], and, as such, it is isomorphic to a $\bar{K}$-vector space of dimension $\sum_{i=1}^{l} d(v_i)$, with the above notations.

3.3.2. *Iwasawa decomposition.* One has $G(K) = G(K)^0 Z(K) U^+(K)$ and the mapping $G(K)^0 z U^+(K) \mapsto \nu(z)$ ($z \in Z(K)$) is a bijection of $\{G(K)^0 g U^+(K) | \, g \in G(K)\}$ onto Y.

3.3.3. *Cartan decomposition.* One has $G(K) = G(K)^0 Z(K) G(K)^0$ and the mapping $G(K)^0 z G(K)^0 \mapsto \nu(z)$ ($z \in Z(K)_+$) is a bijection of $\{G(K)^0 g G(K)^0 | g \in G(K)\}$ onto Y_+.

In particular, we see that if $\bar{K}$ is finite, the convolution algebra of all functions $G(K) \mapsto \boldsymbol{C}$ with compact support which are bi-invariant under $G(K)^0$ (Hecke algebra) has a canonical basis indexed by Y_+. *That algebra is commutative.*

For the proofs and some generalizations of the above results, cf. [**8**, §4].

3.4. *Some group schemes.* The results of this section and the next are special cases of results which will be established in [**9**].

It is well known that the maximal bounded subgroups of $\mathrm{SL}_n(K)$ are the group $\mathrm{SL}_n(\mathfrak{o})$ and its conjugates under $\mathrm{GL}_n(K)$. It is natural to ask whether, more generally, the maximal bounded subgroups of $G(K)$ can always be interpreted as the groups of units of some naturally defined $\mathfrak{o}$-structures on G. A positive answer is provided by the statement 3.4.1 below. In this section, we denote by G_{ss} the derived group of G and by $\mathrm{pr}_{\mathrm{ss}}$ the canonical projection $\mathscr{B}(G, K_1) \to \mathscr{B}(G_{\mathrm{ss}}, K_1)$ (cf. the last paragraph of 2.1) for any K_1.

3.4.1. *If Ω is a nonempty subset of an apartment of $\mathscr{B}$ whose projection* $\mathrm{pr}_{\mathrm{ss}}(\Omega)$ *is bounded, there is a smooth affine group scheme $\mathscr{G}_\Omega$ over $\mathfrak{o}$, unique up to unique isomorphism, with the following properties*:

the generic fiber $\mathscr{G}_{\Omega, K}$ of $\mathscr{G}_\Omega$ is G;

for every unramified Galois extension K_1 of K with ring of integers $\mathfrak{o}_{K_1}$, the group $\mathscr{G}_\Omega(\mathfrak{o}_{K_1})$ is equal to $G(K_1)^\Omega$ (cf. 3.1), *where Ω is identified with its canonical image in the building $\mathscr{B}(G, K_1)$ (cf.* 2.6).

Clearly, $\mathscr{G}_\Omega$ depends only on the closed convex hull of $\mathrm{pr}_{\mathrm{ss}}(\Omega)$.

The following two statements are easy consequences of the definitions.

3.4.2. *If G is split, the group schemes $\mathscr{G}_x$ associated to the special points x of $\mathscr{B}$ are the Chevalley group schemes with generic fiber G.*

3.4.3. *Let K_1 be an unramified Galois extension of K with ring of integers $\mathfrak{o}_{K_1}$, let $\Omega \subset \mathscr{B}$ be as above and let Ω_1 be the canonical image of Ω in $\mathscr{B}_1 = \mathscr{B}(G, K_1)$* (2.6). *Then $\mathscr{G}_{\Omega_1}$ is the group scheme over $\mathfrak{o}_{K_1}$ deduced from $\mathscr{G}_\Omega$ by change of base. Conversely, let $\mathscr{G}$ be a smooth group scheme over $\mathfrak{o}$ with generic fiber G and suppose that, by change of base from $\mathfrak{o}$ to $\mathfrak{o}_{K_1}$, $\mathscr{G}$ becomes a group scheme $\mathscr{G}_{\Omega_1}$ with $\Omega_1 \subset \mathscr{B}_1$ as in* 3.4.1; *then* $\mathrm{pr}_{\mathrm{ss}}(\Omega_1)$ *is stable by* $\mathrm{Gal}(K_1/K)$, *and if it is pointwise fixed by* $\mathrm{Gal}(K_1/K)$, *hence can be identified with a subset of* $\mathscr{B}(G_{\mathrm{ss}}, K)$ (*cf.* 2.6.1) *whose inverse image by* $\mathrm{pr}_{\mathrm{ss}}$ *in $\mathscr{B}$ we denote by Ω, one has $\mathscr{G} = \mathscr{G}_\Omega$.*

If Ω' is any nonempty subset of the closure of Ω, the inclusion homomorphisms $G(K_1)^\Omega \to G(K_1)^{\Omega'}$, for K_1 as in 3.4.1, define a morphism of group schemes $\mathscr{G}_\Omega \to \mathscr{G}_{\Omega'}$ which we denote by $\rho_{\Omega'\Omega}$. We represent by $\bar{G}_\Omega$ and $\bar{\rho}_{\Omega'\Omega}$ the algebraic group defined over $\bar{K}$ and the $\bar{K}$-homomorphism obtained from $\mathscr{G}_\Omega$ and $\rho_{\Omega'\Omega}$ by reduction mod $\mathfrak{p}$.

3.4.4. *The reduction homomorphism $\mathscr{G}_\Omega(\mathfrak{o}) = G(K)^\Omega \to \bar{G}_\Omega(\bar{K})$ is surjective.*

3.5. *Reduction* mod $\mathfrak{p}$. Let Ω be as in 3.4. Our next purpose is to investigate the group $\bar{G}_\Omega$. We assume, without loss of generality, that $\Omega \subset A(G, S)$. Then, the well-defined split torus scheme whose generic fiber is S is a closed subscheme of $\mathscr{G}_\Omega$, and its reduction mod $\mathfrak{p}$, called $\bar{S}$, is a maximal $\bar{K}$-split torus of $\bar{G}_\Omega$. The character group of $\bar{S}$ is canonically isomorphic with the character group X^* of S and will be identified with it; similarly, we identify the cocharacter group of $\bar{S}$ with X_*. The neutral component $\bar{G}^0$ of $\bar{G}$ possesses a unique Levi subgroup containing $\bar{S}$, which we denote by $\bar{G}_\Omega^{\mathrm{red}}$; it is defined over $\bar{K}$. We suppose Ω convex.

Let F be a facet meeting Ω and of maximal dimension with that property. Since F satisfies all the conditions imposed on Ω, the reductive group $\bar{G}_F^{\mathrm{red}}$ is defined and it also contains $\bar{S}$. One shows that *the identity map of $\bar{S}$ onto itself extends uniquely to an isomorphism $\bar{G}_\Omega^{\mathrm{red}} \to \bar{G}_F^{\mathrm{red}}$*; if $F \subset \Omega$, that is nothing else but the restriction of $\bar{\rho}_{\Omega F}$ to $\bar{G}_\Omega^{\mathrm{red}}$. In the sequel, we shall be mainly concerned with the group $\bar{G}_F^{\mathrm{red}}$.

The notion of *coroot associated with a root a* is usually defined for split groups (cf. [**11**, XX, 2.8], [**20**, §1]). In view of the next statement, we extend it as follows to arbitrary reductive groups: if $2a$ is not a root, we simply take the coroot associated with a in the split subgroup of maximal rank defined in [**3**, §7]; if $2a$ is a root, we define the coroot associated with a as being twice the coroot associated with $2a$ (X_* being written additively).

3.5.1. *The root system of* $\bar{G}_F^{\mathrm{red}}$ *with respect to* $\bar{S}$ *is the system* Φ_F (*cf.* 1.9); *in particular, its Dynkin diagram is obtained from the local Dynkin diagram* $\Delta(G, K)$ *by deleting the vertices belonging to* I_F (*cf.* 1.9) *and all edges containing such vertices. The coroot associated with a root* $a \in \Phi_F$ *is the same for* $\bar{G}_F^{\mathrm{red}}$ *as for* G. *If* $\bar{U}_a$ *denotes the unipotent subgroup of* $\bar{G}_F^{\mathrm{red}}$ *corresponding to* a, *the group* $\bar{U}_a(\bar{K})$ *is nothing else but the group* $\bar{X}_\alpha$ *of* §1.4, *where* α *is the affine root vanishing on* F *and whose vector part is* a.

Applying that to the unramified closure of K, one gets the following immediate consequence.

3.5.2. *The index of* $\bar{G}_F^{\mathrm{red}}$ *over* $\bar{K}$, *in the sense of* [**3**] *and* [**22**], *is obtained from the local index of* G *by deleting from* Δ_1 *all vertices belonging to the orbits* $O(v)$ *with* $v \in I_F$ (*the notations are those of* §1.11) *and all edges containing such vertices. In particular, if* G *is residually quasi-split* (*resp. residually split*), $\bar{G}_F^{\mathrm{red}}$ *is quasi-split* (*resp. split*). *When* F *is a chamber, then* G *is residually quasi-split* (*resp. residually split*) *if and only if* $\bar{G}_F^{\mathrm{red}}$ *is a torus* (*resp. a split torus*).

If G is simply connected, the group $\bar{G}_\Omega$ is connected. In general, the group of components of $\bar{G}_\Omega$ is easily computed when one knows the group $\Xi_1 = \Xi(G, K_1)$ (cf. §2.5), where K_1 is the maximal unramified extension of K. Here, we shall give the result only in the case of a facet.

3.5.3. *The group of components of* $\bar{G}_F$ *is canonically isomorphic with the intersection of the stabilizers of the orbits* $O(v)$ *with* $v \in I_F$ *in the group* Ξ_1. *A component is defined over* $\bar{K}$ *if and only if the corresponding element of* Ξ_1 *is centralized by* $\mathrm{Gal}(K_1/K)$. *If* $\bar{K}$ *is finite, every component of* $\bar{G}_F$ *which is defined over* $\bar{K}$ *has a* $\bar{K}$-*rational point* (*by Lang's theorem*).

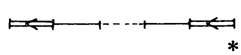
The groups $\bar{G}_F^{\mathrm{red}}$ give an insight into the geometry of the building through the following statement:

3.5.4. *The link of* F *in* B *is canonically isomorphic with the spherical building of* $\bar{G}_F^{\mathrm{red}}$ *over* $\bar{K}$, *i.e. the "building of* $\bar{K}$-*parabolic subgroups" of* $\bar{G}_F^{\mathrm{red}}$ (*cf.* [**23**, 5.2]).

The groups $\bar{G}_F^{\mathrm{red}}$ also provide an alternative definition of the integers $d(v)$ of §1.8. Suppose F is of codimension one and let v be the complement of I_F in the set of all vertices of Δ. Then, $\bar{G}_F^{\mathrm{red}}$ has semisimple $\bar{K}$-rank 1 and $d(v)$ *is the dimension of its maximal unipotent subgroups, or, equivalently, the dimension of the variety* $\bar{G}_F^\circ/\bar{P}_F$, *where* $\bar{P}_F$ *is a minimal* $\bar{K}$-*parabolic subgroup of* $\bar{G}_F^\circ$, *the neutral component of* $\bar{G}_F$. This, together with 3.5.4, implies the interpretation of $d(v)$ given in 2.4. If G is residually split, $\bar{G}_F^\circ/\bar{P}_F$ is a projective line, hence $d(v) = 1$; in particular, we recover the last statement of §1.8.

While 3.5.2 gives an easy algorithm to determine the *type* of $\bar{G}_F^{\mathrm{red}}$, 3.5.1, applied to the unramified closure of K, actually provides the absolute *isomorphism class* of that group. Here is an immediate application of that. Suppose that G is quasi-simple, simply connected and residually split and that F is a special point. Then, $\bar{G}_F^{\mathrm{red}}$ is a simply connected quasi-simple group except if the local Dynkin diagram is the following one:

(1)



and if I_F is the vertex marked with a $*$. Indeed, it is readily verified that in all other cases, Φ_F contains all nonmultipliable relative roots of G, and the assertion follows from [**4**, 2.23 and 4.3]. In the exceptional case above, $\bar{G}_F^{\text{red}}$ is a special orthogonal group, hence not simply connected. Using the fact that in a simply connected group the derived group of the centralizer of a torus is also simply connected, one easily deduces from the preceding result the following more general one. Let us say that a special vertex of the absolute local Dynkin diagram Δ_1 is *good* if it is not the vertex $*$ of a connected component of type (1) of that diagram. Then *if G is semisimple and simply connected and if $\bigcup_{v\in I_F} O(v)$ contains a good special vertex out of each connected component of Δ_1, the derived group of $\bar{G}_F^{\text{red}}$ is simply connected.*

3.6. *Fixed points of groups of units of tori.* Let M be a subgroup of the group of units $S_c = \{s \in S(K) \mid \omega(\chi(s)) = 0 \text{ for all } \chi \in X^*\}$ of S. We wish to find under which condition the apartment $A = A(G, S)$ is the full fixed point set $\mathscr{B}^M$ of M in $\mathscr{B}$. From the properties of the building recalled in §2.2, one deduces that $A = \mathscr{B}^M$ if and only if, for every facet F of A of codimension one, the only chambers containing F in their closure and fixed by M are the two chambers of A with those properties. By 3.5.4, that means that the image $\bar{M}$ of M in $\bar{S}(\bar{K})$ has only two fixed points in the spherical building of $\bar{G}_F^{\text{red}}$ over $\bar{K}$. If a is any one of the two nondivisible roots in Φ_F, that condition amounts to $a(\bar{M}) \neq \{1\}$. Thus, we conclude that:

3.6.1. *A necessary and sufficient condition for A to be the full fixed point set of M in $\mathscr{B}$ is that $a(M) \not\subset 1 + \mathfrak{p}$ for every relative root $a \in \Phi$.*

In particular,

if $\bar{K}$ has at least four elements (resp. if $\bar{K} \cong \boldsymbol{F}_2$) A is always (resp. never) the full fixed point set of S_c in B.

The preceding discussion also gives information on the fixed point set of the group of units $S_{1,c}$ of a nonsplit torus S_1 which becomes maximal split over an unramified Galois extension K_1 of K: one applies 3.6.1 to the action of $S_{1,c}$ on $\mathscr{B}(G, K_1)$ and one goes down to $\mathscr{B}$ by Galois descent, using 2.6.1. In that way, one gets the following result for instance:

If S_1 is an anisotropic torus which becomes maximal split over an unramified Galois extension of K, then $S_1(K)$ has a unique fixed point in the building $\mathscr{B}$.

By contrast, it is easily shown that if S_1 is a maximal torus of $G = \mathrm{SL}_2$ whose splitting field is ramified, then $S_1(K)$ necessarily fixes a chamber of $\mathscr{B}$ and possibily more than one[2]; for a similar torus S_1 in PGL_2, $S_1(K)$ may have a single fixed point in $\mathscr{B}$ and may have more than one.

3.7. *Iwahori subgroups; volume of maximal compact subgroups.* In this section, we suppose G residually quasi-split; remember that that is no assumption if the residue field $\bar{K}$ is finite (1.10.3).

To every chamber C of the building $\mathscr{B}$, we associate as follows a subgroup $\mathrm{Iw}(C)$ of $G(K)$, called the *Iwahori subgroup* corresponding to C: if $\bar{G}_C^\circ$ denotes the neutral component of the algebraic group $\bar{G}_C$ (cf. 3.4), $\mathrm{Iw}(C)$ is the inverse image in $\mathscr{G}_C(\mathfrak{o}) = G(K)^C$ of the group $\bar{G}_C^\circ(\bar{K})$ under the reduction homomorphism $\mathscr{G}_C(\mathfrak{o}) \to \bar{G}_C(\bar{K})$. Clearly, *all Iwahori subgroups of $G(K)$ are conjugate.* From 3.5.2, it follows that $\bar{G}_C^\circ$ is a solvable group, hence is the semidirect product of a torus $\bar{T}$ by a uni-

[2] This answers a question of G. Lusztig.

potent group $\bar{U}$. By general results on smooth group schemes, it follows that Iw(C) is the semidirect product of $\bar{T}(\bar{K})$ by a pronilpotent group $\mathrm{Iw}_u(C)$; if $\bar{K}$ has finite characteristic p, $\mathrm{Iw}_u(C)$ is a pro-p-group, and if $\bar{K}$ is finite, $\bar{T}(\bar{K})$ is of course a finite group, of order prime to p.

If x is a point of the closure of C, the image of $\bar{G}_C^\circ$ by the homomorphism $\bar{\rho}_{xC}$: $\bar{G}_C \to \bar{G}_x$ is a Borel subgroup $\bar{B}$ of $\bar{G}_x$, and the kernel of $\bar{\rho}_{xC}$ is a connected unipotent group. It follows that $\bar{\rho}_{xC}$ maps $\bar{G}_C^\circ(\bar{K})$ onto $\bar{B}(\bar{K})$ and consequently, by 3.4.4, that Iw(C) is the inverse image of $\bar{B}(\bar{K})$ in $G(K)^x = \mathscr{G}_x(\mathfrak{o})$ under the reduction homomorphism. Thus, the Iwahori subgroups of $G(K)$ can also be defined as the inverse images in the stabilizers $G(K)^x$, for $x \in \mathscr{B}$, of the $\bar{K}$-Borel subgroups of the reductions $\bar{G}_x$.

Now, suppose that $\bar{K}$ is finite. Then, $G(K)$ is a unimodular locally compact group of which the Iwahori subgroups are compact open subgroups. Therefore, there is a unique Haar measure μ for which the Iwahori subgroups have volume 1. From the above, it follows that, for any $x \in \mathscr{B}$, the volume of $G(K)^x$ with respect to μ is the index $[\bar{G}_x(\bar{K}): \bar{B}(\bar{K})]$ where $\bar{B}$ is any $\bar{K}$-Borel subgroup of $\bar{G}_x$. If x is "in general position" in the facet F of $\mathscr{B}$ containing it, one has $\bar{G}_x = \bar{G}_F$ and the assertions 3.5.2 and 3.5.3 provide an effective way of computing that volume knowing the local index of G (together with the correspondence $v \mapsto O(v)$ of 1.11), the set I_x of vertices of Δ and the group $\Xi_1 = \Xi(G, K_1)$ (cf. 2.5), where K_1 is the unramified closure of K.

3.8. *Hyperspecial points and subgroups.* From 3.5.1 and 3.5.3, one easily deduces the following characterization of the hyperspecial points of $\mathscr{B}$ defined in §1.10:

3.8.1. *A point x of $\mathscr{B}$ is hyperspecial if and only if the neutral component of the group $\bar{G}_x$ is reductive, in which case $\bar{G}_x$ itself is connected and hence reductive.*

One can also show that the schemes $\mathscr{G}_x$ corresponding to the hyperspecial points x are the only smooth group schemes over $\mathfrak{o}$ with generic fiber G and reductive reduction. Thus, the hyperspecial subgroups of $G(K)$ can be characterized as the groups of units of such group schemes.

3.8.2. *Suppose that $\bar{K}$ is finite and that $G(K)$ possesses hyperspecial subgroups* (*a condition satisfied, for instance, if G is quasi-split and has an unramified splitting field*: *cf.* 1.10); *then, the hyperspecial subgroups of $G(K)$ are among all compact subgroups of $G(K)$, those whose volume is maximum.*

The proof, using §§3.5 and 3.7, is not difficult.

3.9. *The global case.* Let L be a global field. For every finite extension L' of L and every place v of L', we denote by $\mathfrak{o}_{L'}$ (resp. $\mathfrak{o}_v$) the ring of integers of L' (resp. of the completion L'_v). Let H be a reductive linear group defined over L. We suppose H embedded in the general linear group GL_n and, for every L' and v as above, we set $H(\mathfrak{o}_v) = H(L'_v) \cap \mathrm{GL}_n(\mathfrak{o}_v)$. Another way of viewing that group consists in considering the $\mathfrak{o}_L$-group scheme structure $\mathscr{H}_{\mathfrak{o}_L}$ "on H" defined by the standard lattice $\mathfrak{o}_L^n$ in L^n—in more precise terms, $\mathscr{H}_{\mathfrak{o}_L}$ is the schematic closure of H in the standard general linear group scheme $\mathscr{GL}_{n,\mathfrak{o}_L}$—; then $H(\mathfrak{o}_v) = \mathscr{H}_{\mathfrak{o}_L}(\mathfrak{o}_v)$. For any ring R containing $\mathfrak{o}_L$, we denote by $\mathscr{H}_R$ the group scheme over R deduced from $\mathscr{H}_{\mathfrak{o}_L}$ by change of base.

3.9.1. *At almost all finite places v of L, $\mathscr{H}_{\mathfrak{o}_v}$ is the group scheme $\mathscr{H}_x$ associated with a hyperspecial point x of the building $\mathscr{B}(H, L_v)$; hence $H(\mathfrak{o}_v)$ is a hyperspecial subgroup of $H(L_v)$.*

Indeed, let L' be a Galois extension of L over which the group H splits, and let $\mathscr{H}'_{\mathfrak{o}_{L'}}$ be a Chevalley group scheme over $\mathfrak{o}_{L'}$ with generic fiber $\mathscr{H}_{L'}$. Since the group schemes $\mathscr{H}_{\mathfrak{o}_{L'}}$ and $\mathscr{H}'_{\mathfrak{o}_{L'}}$ have the same generic fiber, they "coincide" at almost all places of L'. Since almost all places of L' are unramified over their restrictions to L, the assertion now follows from 3.4.2, 3.4.3 and 2.6.1.

3.10. *Example. General linear groups.* Suppose that $G = \mathrm{GL}_n$. The Iwahori subgroups of $\mathrm{GL}_n(K)$ are the subgroups conjugated to

$$B = \{((g_{ij}))|g_{ii} \in \mathfrak{o}^\times,\ g_{ij} \in \mathfrak{o} \text{ for } i < j \text{ and } g_{ij} \in \mathfrak{p} \text{ for } i \geqq j\}.$$

Let $(b_i)_{1 \leqq i \leqq n}$ be the canonical basis of K^n. For $1 \leqq r \leqq n$, let Λ_r be the lattice in K^n generated by $\{b_i/\pi | i \leqq r\} \cup \{b_i | i > r\}$ and let P_r be the stabilizer of Λ_r in $\mathrm{GL}_n(K)$. Thus, P_r is the group of all matrices whose determinant is a unit and which have the following form

$$\begin{array}{c} \\ r \\ n-r \end{array}\!\!\begin{array}{c} \begin{array}{cc} r & n-r \end{array} \\ \left(\begin{array}{c|c} \mathfrak{o} & \pi^{-1}\mathfrak{o} \\ \hline \pi\mathfrak{o} & \mathfrak{o} \end{array}\right) \end{array},$$

where the notation means that the upper left corner is an $r \times r$ matrix with coefficients in $\mathfrak{o}$, the upper right corner an $r \times (n-r)$ matrix with coefficients in $\pi^{-1}\mathfrak{o}$, etc. The group B is the centralizer in $\mathrm{GL}_n(K)$ of the chamber C described in §1.14. The subgroups P_r are special and every special subgroup is conjugate to any one of them. The P_r's are the stabilizers of the points of $\mathscr{B}$ contained in a one-dimensional facet of C; with the notations of §2.9, the points fixed by P_r correspond to the norms of the form

$$(x_1, \cdots, x_n) \mapsto \inf(\{\omega(x_i) + \omega(\pi) - c | i \leqq r\} \cup \{\omega(x_i) - c | i > r\})$$

for some constant $c \in \boldsymbol{R}$. If $\boldsymbol{v}$ is any such point, the scheme $\mathscr{G}_{\boldsymbol{v}}$ is the Chevalley scheme "on" GL_n defined by the lattice Λ_r. One can describe the scheme $\mathscr{G}_C$, whose group of units is B, by embedding GL_n in GL_{n^2} by means of the sum of n times the standard representation, and considering in K^{n^2} the lattice $\Lambda_1 \oplus \cdots \oplus \Lambda_n$.

Note that B is the stabilizer of any point of C. The corresponding statement for $G = \mathrm{PGL}_n$ is not true. For instance, the image in $\mathrm{PGL}_n(K)$ of the group generated by B and by the linear transformation

$$b_1 \mapsto b_2 \mapsto \cdots \mapsto b_n \mapsto b_1\pi^{-1} \tag{1}$$

is the stabilizer of the "center of gravity" $\boldsymbol{y}$ of the chamber of $\mathscr{B}$ (PGL_n, K) projecting C. That group is also a maximal bounded subgroup of $\mathrm{PGL}_n(K)$. The scheme $\mathscr{G}_{\boldsymbol{y}}$ can be described by means of a lattice in the Lie algebra of $G = \mathrm{PGL}_n$ on which G acts by the adjoint representation. If F denotes the cyclic group of order n generated by the reduction mod $\mathfrak{p}$ of the image of (1) in PGL_n, the group $\bar{G}_{\boldsymbol{y}}$ is the semidirect product of F by a connected group; in particular, its group of components is cyclic of order n.

3.11. *Example. Quasi-split special unitary group in odd dimension.* We take over all hypotheses and notations from 1.15 and denote by $\mathfrak{o}_L$ the ring of integers of L. Let $\lambda \in L$ be such that $\lambda + \lambda^\tau + 1 = 0$ and that $\omega(\lambda) = \delta$ (we recall that δ is defined as $\sup\{\omega(d) \mid d \in L,\ d + d^\tau + 1 = 0\}$). We suppose the uniformizing element π_1 chosen in such a way that $(\lambda\pi_1) + (\lambda\pi_1)^\tau = 0$: if L/K is unramified the possibility of such a choice is obvious and if L/K is ramified it follows from 1.15(6). Let $(b_i)_{-n \leqq i \leqq n}$ be the canonical basis of L^{2n+1}. For $0 \leqq r \leqq n$, we consider the basis $(b_i^{(r)})_{-n \leqq r \leqq n}$ defined by $b_i^{(r)} = b_i/\pi_1$ for $i < -r$, $b_i^{(r)} = b_i$ for $-r \leqq i \leqq 0$ and $b_i^{(r)} = \lambda b_i$ for $i > 0$, and we denote by Λ_r the $\mathfrak{o}_L$-lattice generated by that basis. Note that if $\delta = \omega(\lambda) = 0$, which is always the case except if L/K is ramified and char $\bar{K} = 2$, Λ_r is also generated by the basis $\{b_i/\pi_1 \mid i < -r\} \cup \{b_i | i \geqq -r\}$. The stabilizer P_r of Λ_r in $G(K)$ is also the stabilizer of the point $\boldsymbol{v}_r$ of $V = A \subset \mathscr{B}$ (with the conventions of 1.15) determined by

$$\begin{aligned} a_i(\boldsymbol{v}_r) &= \tfrac{1}{2}\delta & &\text{if } i \leqq r, \\ &= \tfrac{1}{2}(\delta + \omega(\pi_1)) & &\text{if } i > r. \end{aligned}$$

The points $\boldsymbol{v}_r$ are the vertices of the chamber defined by the inequalities $\frac{1}{2}\delta < a_1 < \cdots < a_n < \frac{1}{2}\delta + \frac{1}{2}\omega(\pi_1)$; they correspond, by $\boldsymbol{v} \mapsto I_{\mathfrak{v}}$, to the vertices of the diagrams (9) and (10) of 1.15 in the natural order, from left to right. The scheme $\mathscr{G}_{\mathfrak{v}_r}$ is the $\mathfrak{o}_L$-structure on G defined by the lattice Λ_r.

We shall now briefly investigate the algebraic group $\bar{G}_{\mathfrak{v}_r}$ obtained from $\mathscr{G}_{\mathfrak{v}_r}$ by reduction mod $\mathfrak{p}$. We choose r once and for all and use primed letters to designate the coordinates with respect to the basis $(b_i^{(r)})$. With those coordinates, the hermitian form h is given by

$$\begin{aligned} h((x_i'), (y_i')) = x_0'^{\tau} y_0' &+ \sum_{i=1}^{r} (\lambda^{\tau} x_i'^{\tau} y_{-i}' + \lambda x_{-i}'^{\tau} y_i') \\ &+ \frac{\lambda^{\tau}}{\pi_1} \sum_{i=r+1}^{n} (x_i'^{\tau} y_{-i}' - x_{-i}'^{\tau} y_i'). \end{aligned}$$

We set $\bar{E} = \Lambda_r/\pi\Lambda_r$. That is a $2(2n + 1)$-dimensional vector space over $\bar{K}$ and one shows that the natural morphism $\bar{G}_{\mathfrak{v}_r} \to \mathrm{GL}(\bar{E})$ is a monomorphism; in other words $\bar{G}_{\mathfrak{v}_r}$ can be viewed as a subgroup of $\mathrm{GL}(\bar{E})$. From this point on, we must treat separately the unramified and the ramified case.

First case. L/K is unramified. Then, $\bar{E}$ is also a vector space over the residue field $\bar{L}$ of L. By reduction mod π_1, the antihermitian form $\pi_1 h/\lambda^{\tau}$ becomes the antihermitian form $\bar{h}_1$: $((\bar{x}_i'), (\bar{y}_i')) \mapsto \sum_{i=r+1}^{n} (\bar{x}_i'^{\tau} \bar{y}_{-i}' - \bar{x}_{-i}'^{\tau} \bar{y}_i')$ in $\bar{E}$, with obvious notation. Let $\bar{E}_0$ be the kernel of that form, defined by the equations $\bar{x}_i' = 0$ for $|i| > r$, and let $\Lambda_{r,0}$ be the inverse image of $\bar{E}_0$ in Λ_r. We now consider the restriction of h to $\Lambda_{r,0} \times \Lambda_{r,0}$ which, by reduction mod $\mathfrak{p}$, becomes the hermitian form

$$\bar{h}_2\colon ((\bar{x}_i')_{-r \leqq i \leqq r}, (\bar{y}_i')_{-r \leqq i \leqq r}) \mapsto \bar{x}_0'^{\tau} \bar{y}_0' + \sum_{i=1}^{r} (\lambda^{\tau} \bar{x}_i' \bar{y}_{-i}' + \lambda \bar{x}_{-i}' \bar{y}_i')$$

in $\bar{E}_0$. Finally, $\bar{G}_{\mathfrak{v}_r}$ can be described as the stabilizer of the pair $(\bar{h}_1, \bar{h}_2)$ in the group $R_{\bar{L}/\bar{K}}\,(\mathrm{SL}_{\bar{L}}(\bar{E}))$, that is, the special linear group of the $\bar{L}$-vector space $\bar{E}$ "considered as an algebraic group over $\bar{K}$" by restriction of scalars. Let $\bar{E}_1$ be the subspace of $\bar{E}$ defined by the equations $\bar{x}_i' = 0$ for $-r \leqq i \leqq r$, and let $\bar{\bar{h}}_1$ denote the restriction of the antihermitian form $\bar{h}_1$ to $\bar{E}_1 \times \bar{E}_1$. Then, $\bar{G}_{\mathfrak{v}_r}$ clearly contains the group $SU(\bar{\bar{h}}_1) \times SU(\bar{h}_2)$, which is nothing else but its Levi subgroup $\bar{G}_{\mathfrak{v}_r}^{\mathrm{red}}$ (cf. 3.5). Observe that, in conformity with 3.5.1, the diagram obtained from the diagram (9) of 1.15 by deleting its $(r + 1)$st vertex and the adjoining edges is a diagram of type $BC_r \times C_{n-r}$, which is indeed the type of the relative root system of $SU(\bar{\bar{h}}_1) \times SU(\bar{h}_2)$.

Second case. L/K is ramified. Then, the scalar multiplication by π_1 in Λ_r, reduced mod π, provides an endomorphism $\nu\colon \bar{E} \to \bar{E}$, obviously centralized by $\bar{G}_{\mathfrak{v}_r}$, and whose kernel is equal to its image $\nu(\bar{E})$. The quotient $\bar{\bar{E}} = \bar{E}/\nu(\bar{E})$ is canonically isomorphic with the quotient $\Lambda_r/\pi_1\Lambda_r$ and will be identified with it. From the fact that ν is centralized by $\bar{G}_{v_r}$, it follows that the projection $\bar{E} \to \bar{\bar{E}}$ induces a homomorphism of $\bar{K}$-algebraic groups $\bar{G}_{v_r} \to \mathrm{GL}(\bar{\bar{E}})$ whose kernel is unipotent. Here, we shall only describe the image $\bar{\bar{G}}_{v_r}$ of that homomorphism, leaving as an exercise the determination of the full structure of $\bar{G}_{v_r}$.

By reduction mod π_1, the antihermitian form $\pi_1 h/\lambda^\tau$ becomes the alternating form $\bar{h}_1$: $((\bar{x}_i'), (\bar{y}_i')) \mapsto \sum_{i=-r+1}^{n} (\bar{x}_i' \bar{y}_{-i}' - \bar{x}_{-i}' \bar{y}_i')$ in $\bar{\bar{E}}$. Let $\bar{\bar{E}}_0$ be the kernel of that form, defined by the equations $\bar{x}_i' = 0$ for $|i| > r$, and let $\Lambda_{r,0}$ be the inverse image of $\bar{\bar{E}}_0$ in Λ_r. We now consider the function q: $\Lambda_{r,0} \to K$ defined by $q(x) = h(x, x)$. By reduction, it becomes the quadratic form $\bar{q}$: $\bar{\bar{E}}_0 \to \bar{K}$ given by

$$\bar{q}((\bar{x}_i')_{-r\leq i\leq r}) = \bar{x}_0'^2 - \sum_{i=1}^{r} \bar{x}_{-i}'\bar{x}_i'.$$

Finally, the group $\bar{\bar{G}}_{v_r}$ is the group of all elements of $\mathrm{SL}(\bar{\bar{E}})$ stabilizing $\bar{h}_1$ and inducing on $\bar{\bar{E}}_0$ an element of the (reduced) group $SO(\bar{q})$. Let $\bar{\bar{E}}_1$ be the subspace of $\bar{\bar{E}}$ defined by the equations $\bar{x}_i' = 0$ for $-r < i < r$ and let $\bar{\bar{h}}_1$ denote the restriction of $\bar{h}_1$ to $\bar{\bar{E}}_1 \times \bar{\bar{E}}_1$. Then $\mathrm{Sp}(\bar{\bar{h}}_1) \times SO(\bar{q})$ is a Levi subgroup of $\bar{\bar{G}}_{v_r}$, which is the isomorphic image in that group of the Levi subgroup $\bar{G}_{v_r}^{\mathrm{red}}$ of $\bar{G}_{v_r}$. As in the unramified case, we can test the statement 3.5.1, this time by using the diagram (10) of 1.15 which provides, for $\bar{G}_{v_r}^{\mathrm{red}}$, a root system of type $B_r \times C_{n-r}$.

Note that, also in the unramified case, we could have, instead of the restriction of h to $\Lambda_{r,0} \times \Lambda_{r,0}$, considered its "contraction" q: $\Lambda_{r,0} \to K$ defined by $q(x) = h(x, x)$, thus making the treatment of the two cases still more similar. On the other hand, we have introduced λ in order to reduce the case distinction to a minimum; in the unramified case, as well as if char $\bar{K} \neq 2$, we could have replaced λ by 1 everywhere, thus simplifying the equations somewhat.

3.12. *Example. Quasi-split but nonsplit special orthogonal group.* Now, we take over the hypotheses and notations of §1.16 except that we take for G the *special* orthogonal group $SO(q)$. We shall not, as in §3.11, treat that example in any systematic way. Our only aim here is to give an example of a vertex $\mathbf{v}$ of the building such that $\bar{G}_v$ is not connected. We suppose that L/K is unramified. Let Λ be the lattice $\mathfrak{o}^n \oplus \pi\mathfrak{o}_L \oplus \mathfrak{o}^n$ in $K^n \oplus L \oplus K^n$ where $\mathfrak{o}_L$ is the ring of integers of L. The stabilizer P of Λ in $G(K)$ is also the stabilizer of the point $\mathbf{v} \in V \subset A$ defined by $a_i(\mathbf{v}) = \frac{1}{2}\omega(\pi)$ for $1 \leqq i \leqq n$, which is a vertex of the chamber described in §1.16. In the diagram (1) of §1.16, I_v is the vertex at the extreme left. As in §3.11, one can describe $\bar{G}_v$ as a subgroup of $\mathrm{GL}(\bar{E})$, where $\bar{E}$ is the $\bar{K}$-vector space $\Lambda/\pi\Lambda$. By reduction, the form q, restricted to Λ, becomes the quadratic form $\bar{q} : (\bar{x}_i)_{-n\leq i\leq n} \mapsto \sum_{i=1}^{n} \bar{x}_{-i}\bar{x}_i$ in $\bar{E}$, with obvious notations ($\bar{x}_i$ belongs to the residue field of L if $i = 0$ and to $\bar{K}$ otherwise). Let $\bar{E}_0$ be the two-dimensional kernel of $\bar{q}$ defined by the equations $\bar{x}_i = 0$ for $i \neq 0$, and let $\bar{\bar{q}}$ be the quadratic form in $\bar{\bar{E}} = \bar{E}/\bar{E}_0$ image of $\bar{q}$. Clearly, $\bar{G}_v$ preserves the form $\bar{q}$. Therefore, the projection $\bar{E} \to \bar{\bar{E}}$ induces a $\bar{K}$-homomorphism $\bar{G}_v \to O(\bar{\bar{q}})$. One verifies that that homomorphism is *surjective* (in particular, $\bar{G}_v$ is not connected) and that it maps $\bar{G}_v^{\mathrm{red}}$ isomorphically onto $SO(\bar{\bar{q}})$. If τ denotes the nontrivial K-automorphism of L, the linear transformation $(x_i)_{-n\leq i\leq n} \mapsto (x_n, x_{-n+1}, \cdots, x_{-1}, x_0^\tau, x_1, \cdots, x_{n-1}, x_{-n})$, which belongs to P, provides by reduction an element of $\bar{G}_v(\bar{K})$ which is mapped into $O(\bar{\bar{q}})$ but not into $SO(\bar{\bar{q}})$.

4. Classification.

4.1. *Introduction.* To finish with, we give the classification of simple groups in the case where the residue field $\bar{K}$ is finite, which will be assumed from now on. We recall that, in the characteristic zero case, that classification has been given first by M. Kneser **[16]**. The tables 4.2 and 4.3, together with the comments in §4.5, provide a list of all central isogeny classes of absolutely quasi-simple groups over K. For each type of group, they give the following information, where K_1 denotes the unramified closure of K:

a *name* of the shape ${}^aX$ where the symbol X represents the absolute local Dynkin diagram Δ_1 (1.11) with the notations of **[8**, 1.4.6]—except that our *C-BC* corresponds to the *C-BC*$^{\text{III}}$ of **[8]**—and where a is the order of the automorphism group of Δ_1 induced by Gal(K_1/K); for residually split groups, $a = 1$ and the superscript a is omitted from the notation; note that the index on the right of X is the relative rank over K_1, hence equal to the number of vertices of Δ_1 minus one; primes, double primes, etc. are used to distinguish types of groups which would otherwise have the same name;

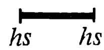
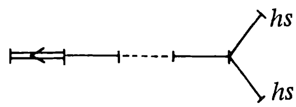
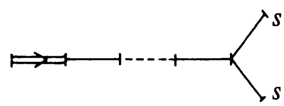
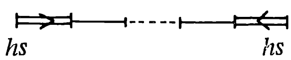
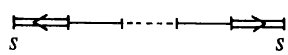
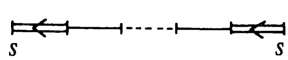
the symbol representing the *affine root system* (or échelonnage) in the notations of **[8]**; in the residually split case, that symbol coincides with the name of the type and is not given separately; note that the right part of the symbol gives the type of the *relative root system* Φ and, in particular, that the index on the right of it is the relative rank over K, hence equal to the number of vertices of the relative local Dynkin diagram Δ minus one;

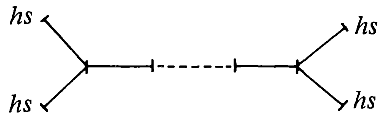
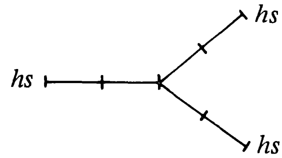
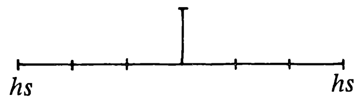
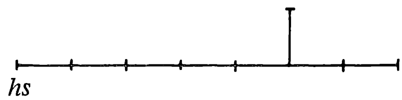
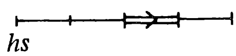
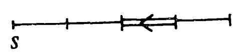
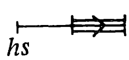
the *local index* (§1.11), the *relative local Dynkin diagram* Δ (§1.8) and *the integers* $d(v)$ attached to its vertices (§1.8); the action of Gal(K_1/K) on Δ_1—through a cyclic group of order a (see above)—is essentially characterized by its orbits in the set of vertices of Δ_1, orbits which are exhibited as follows: the elements of the orbit $O(v)$ corresponding to a vertex v of Δ (1.11) are placed close together on the same vertical line as v (in the few cases, such as ${}^2D_n$, ${}^2D''_{2m}$, etc., where two vertices of Δ are on the same vertical, the correspondence $v \mapsto O(v)$ should be clear from the way the diagrams are drawn); since $\bar{K}$ is finite G is residually quasi-split (1.10.3), hence all vertices of Δ_1 are distinguished except for the unique anisotropic type ${}^dA_{d-1}$ (§1.11), and there is no need for a special notation like the circling of orbits, as in **[22]**; *hyperspecial vertices* (§1.10) are marked with an *hs* and the other *special vertices* (§1.9) with an *s*;

the *index* of the form, in the "usual" sense of [3] and **[22]**; for simplicity, we do not represent that index by a picture but rather by the corresponding symbol in the notation of **[22]**; we recall that that symbol carries, among other, the following information: the *absolute type* of the group, the *absolute rank*, the *relative rank* (already provided by the symbol representing Φ_{af}) and the order of the automorphism group of the ordinary Dynkin diagram induced by the Galois group of the separable closure of K.

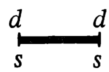
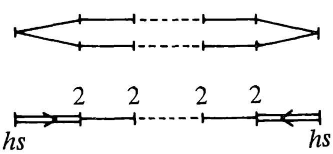
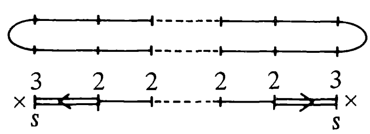
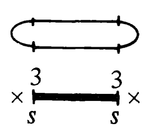
In the case of the inner forms of A_n, the diagrams are, for technical reasons, replaced by explanations in words.

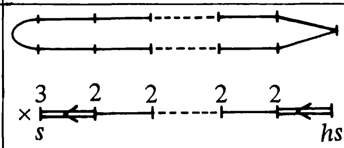
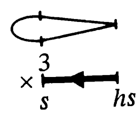
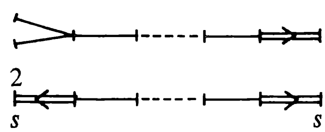
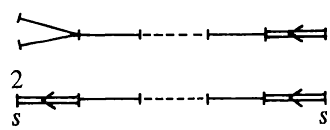
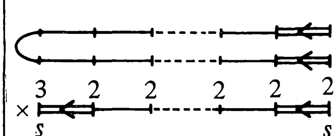
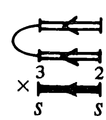
4.2. *Residually split groups.*

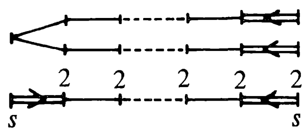
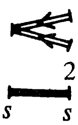
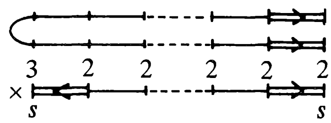
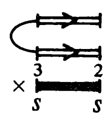
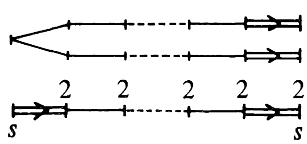
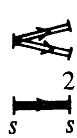
Name	Local Dynkin diagram	Index [22]
A_n $(n \geqq 2)$	A cycle of length $n + 1$ all vertices of which are hyperspecial	${}^1A^{(1)}_{n,n}$
A_1		${}^1A^{(1)}_{1,1}$
B_n $(n \geqq 3)$		$B_{n,n}$
B-C_n $(n \geqq 3)$		${}^2A^{(1)}_{2n-1,n}$
C_n $(n \geqq 2)$		$C^{(1)}_{n,n}$
C-B_n $(n \geqq 2)$		${}^2D^{(1)}_{n+1,n}$
C-BC_n $(n \geqq 2)$		${}^2A^{(1)}_{2n,n}$
C-BC_1		${}^2A^{(1)}_{2,1}$

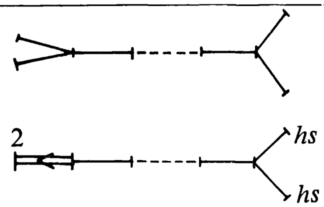
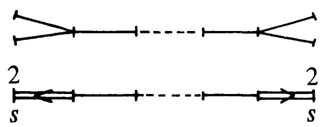
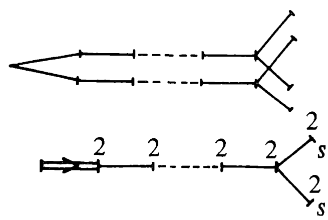
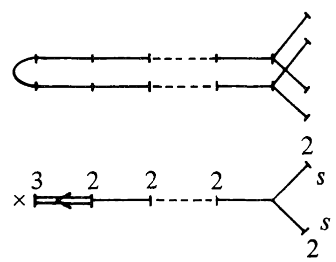
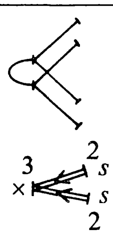
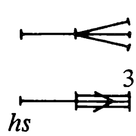
Name	Local Dynkin diagram	Index [22]
D_n ($n \geqq 4$)		$^1D^{(1)}_{n,n}$
E_6		$^1E^0_{6,6}$
E_7		$E^0_{7,7}$
E_8		$E^0_{8,8}$
F_4		$F^0_{4,4}$
F^{I}_4		$^2E^2_{6,4}$
G_2		$G_{2,2}$
G^{I}_2		$^3D_{4,2}$ or $^6D_{4,2}$

4.3. *Nonresidually split groups.*

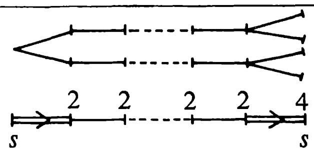
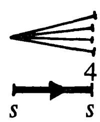
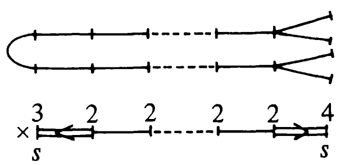
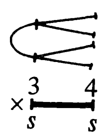
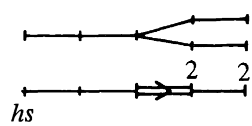
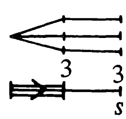
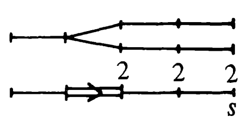
Name	Affine root system (notations of [8, 1.4.6])	Local index and relative local Dynkin diagram.	Index [22]
$^dA_{md-1}$ ($m \geqq 3$, $d \geqq 2$)	A_{m-1}	The absolute local Dynkin diagram Δ_1 is a cycle of length md on which $\mathrm{Gal}(K_1/K)$ acts through a cyclic group of order d generated by a rotation of the cycle. The relative diagram is a cycle of length m all vertices of which are special but not hyperspecial and carry the number d.	$^1A^{(d)}_{md-1,m-1}$
$^dA_{2d-1}$ ($d \geqq 2$)	A_1	Δ_1 is as above, with $m = 2$ Δ is 	$^1A^{(d)}_{2d-1,1}$
$^dA_{d-1}$ ($d \geqq 2$)	$\varnothing$	Δ_1 is as above, with $m = 1$, or, if $d = 2$, consists of a fat segment whose vertices are permuted by $\mathrm{Gal}(K_1/K)$ and $\Delta = \varnothing$	$^1A^{(d)}_{d-1,0}$
$^2A'_{2m-1}$ ($m \geqq 2$)	C_m		$^2A^{(1)}_{2m-1,m}$
$^2A''_{2m-1}$ ($m \geqq 3$)	$C\text{-}BC^{\mathrm{II}}_{m-1}$		$^2A^{(1)}_{2m-1,m-1}$
$^2A''_3$	$C\text{-}BC^{\mathrm{II}}_1$		$^2A^{(1)}_{3,1}$

Name	Affine root system (notations of [8, 1.4.6])	Local index and relative local Dynkin diagram.	Index [22]
${}^2A'_{2m}$ $(m \geqq 2)$	$C\text{-}BC_m^{\mathrm{IV}}$		${}^2A^{(1)}_{2m,m}$
${}^2A'_2$	$C\text{-}BC_1^{\mathrm{IV}}$		${}^2A^{(1)}_{2,1}$
${}^2B_n$ $(n \geqq 3)$	$C\text{-}B_{n-1}$		$B_{n,n-1}$
${}^2B\text{-}C_n$ $(n \geqq 3)$	$C\text{-}BC_{n-1}^{\mathrm{III}}$		${}^2A^{(1)}_{2n-1,n-1}$
${}^2C_{2m-1}$ $(m \geqq 3)$	$C\text{-}BC_{m-1}^{\mathrm{IV}}$		$C^{(2)}_{2m-1,m-1}$
${}^2C_3$	$C\text{-}BC_1^{\mathrm{IV}}$		$C^{(2)}_{3,1}$

Name	Affine root system (notations of [8, 1.4.6])	Local index and relative local Dynkin diagram.	Index [22]
${}^2C_{2m}$ $(m \geqq 2)$	C_m		$C^{(2)}_{2m,\,m}$
${}^2C_2$	A_1		$C^{(2)}_{2,\,1}$
${}^2C\text{-}B_{2m-1}$ $(m \geqq 3)$	$C\text{-}BC^{\mathrm{I}}_{m-1}$		${}^2D^{(2)}_{2m,\,m-1}$
${}^2C\text{-}B_3$	$C\text{-}BC^{\mathrm{I}}_1$		${}^2D^{(2)}_{4,\,1}$
${}^2C\text{-}B_{2m}$ $(m \geqq 2)$	$C\text{-}BC^{\mathrm{III}}_m$		${}^2D^{(2)}_{2m+1,\,m}$
${}^2C\text{-}B_2$	$C\text{-}BC^{\mathrm{III}}_1$		${}^2D^{(2)}_{3,\,1}$

Name	Affine root system (notations of [8, 1.4.6])	Local index and relative local Dynkin diagram.	Index [22]
${}^2D_n$ $(n \geqq 4)$	B_{n-1}		${}^2D^{(1)}_{n,\,n-1}$
${}^2D'_n$ $(n \geqq 4)$	$C\text{-}B_{n-2}$		${}^1D^{(1)}_{n,\,n-2}$
${}^2D''_{2m}$ $(m \geqq 3)$	$B\text{-}C_m$		${}^1D^{(2)}_{2m,\,m}$
${}^2D''_{2m+1}$ $(m \geqq 3)$	$B\text{-}BC_m$		${}^2D^{(2)}_{2m+1,\,m}$
${}^2D''_5$	$B\text{-}BC_2$[3]		${}^2D^{(2)}_{5,\,2}$
${}^3D_4$	G_2		${}^3D_{4,2}$

[3]This "échelonnage" is missing in the table of [8, p. 29].

Name	Affine root system (notations of [8, 1.4.6])	Local index and relative local Dynkin diagram.	Index [22]
${}^4D_{2m}$ ($m \geqq 3$)	$C\text{-}BC_{m-1}^{\mathrm{III}}$		${}^2D_{2m,\,m-1}^{(2)}$
${}^4D_4$	$C\text{-}BC_1^{\mathrm{III}}$		${}^2D_{4,1}^{(2)}$
${}^4D_{2m+1}$ ($m \geqq 3$)	$C\text{-}BC_{m-1}^{\mathrm{I}}$		${}^1D_{2m+1,\,m-1}^{(2)}$
${}^4D_5$	$C\text{-}BC_1^{\mathrm{I}}$		${}^1D_{5,1}^{(2)}$
${}^2E_6$	F_4		${}^2E_{6,4}^2$
${}^3E_6$	G_2^{I}		${}^1E_{6,2}^{16}$
${}^2E_7$	F_4^{I}		$E_{7,4}^5$

4.4. *Interpretation.* We shall now repeat the classification in classical terms. The following enumeration gives a representative of every central isogeny class of absolutely quasi-simple groups over K and, in each case, the name of the corresponding type, with the notations of the first column of the Tables 4.2 and 4.3.

Special linear group SL_m of a d^2-dimensional central division K-algebra ($md \geqq 2$). The type is ${}^dA_{md-1}$.

Special unitary group $SU(h)$ of a hermitian form h in r variables ($r \geqq 3$) with Witt index r' over a quadratic extension L of K. If L/K is ramified, we assume $r \neq 4$ because the case $r = 4$ is more adequately represented by an ordinary special orthogonal group in 6 variables or a quaternionic special orthogonal group in 3 variables according as $r' = 2$ or 1. If the form h is split, the type is ${}^2A'_{r-1}$ if L/K is unramified and $C\text{-}BC_n$ ($r = 2n + 1$) or $B\text{-}C_n$ ($r = 2n$) otherwise. If h is not split, one has $r = 2r' + 2$ and the type is ${}^2A''_{r-1}$ or ${}^2B\text{-}C_{r'+1}$ according as L/K is unramified or ramified.

Special orthogonal group $SO(q)$ of a quadratic form q in r variables ($r \geqq 6$) with Witt index r' over K. If r is even, we denote by L the center of the even Clifford algebra of q which is isomorphic to $K \oplus K$ (form q of discriminant one or Arf invariant zero) or is a quadratic extension of K. If L/K is unramified (in particular if $L = K \oplus K$), we assume $r \neq 6$ because the case $r = 6$ is more adequately represented by a special unitary group in 4 variables. If $r = 2r'$ (resp. $2r' + 1$), the type is $D_{r'}$ (resp. $B_{r'}$). If $r = 2r' + 2$, L is a quadratic extension of K and the type is ${}^2D_{r'+1}$ or $C\text{-}B_{r'}$ according as L/K is unramified or ramified. Finally, if $r = 2r' + 3$ (resp. $2r' + 4$), the type is ${}^2B_{r'+1}$ (resp. ${}^2D'_{r'+2}$).

The *symplectic group* in $2n \geqq 4$ variables is of type C_n.

Special unitary group of a *quaternion* hermitian form in r variables ($r \geqq 2$) relative to the standard involution. The Witt index is always maximal and the type is ${}^2C_r$.

Special orthogonal group $SO(q)$ of a *quaternionic* σ-quadratic form in r variables ($r \geqq 3$) relative to an involution σ of the quaternion algebra whose space of symmetric elements is 3-dimensional (cf. **[21]**, **[23]**; if char $K \neq 2$, σ-quadratic amounts to σ-hermitian and the group is also the *special unitary* group of an *antihermitian* form relative to the standard involution). Let r' be the Witt index of the form and L the center of its "even" Clifford algebra $\mathrm{Cl}(q)$ (cf. **[21]**). If L/K is unramified (in particular if $L \cong K \oplus K$), we assume $r \neq 3$ because the case $r = 3$ is more adequately represented by a special unitary group in 4 variables; if furthermore $r = 2r'$, we also assume $r \neq 4$ because the case $r = 2r' = 4$ is more adequately represented, through the triality principle, by an ordinary special orthogonal group in 8 variables with Witt index 2. One always has $2r' \leqq r \leqq 2r' + 3$. If $r = 2r'$, one has $L \cong K \oplus K$ and the group is of type ${}^2D''_r$. If $r = 2r' + 1$ (resp. $2r' + 2$), L is a quadratic extension of K and the type is ${}^2D''_r$ (resp. ${}^4D_r$) or ${}^2C\text{-}B_{r-1}$ according as L/K is unramified or ramified. If $r = 2r' + 3$, one has $L \cong K \oplus K$ and the type is ${}^4D_r$.

Quasi-split triality D_4. Let L denote the splitting field, which is a cyclic extension of degree 3 or a Galois extension of degree 6 with Galois group $\mathfrak{S}_3$. If L/K is unramified (hence cyclic of degree 3), the type is ${}^3D_4$; otherwise, it is G_2^{I}.

Split exceptional groups. The type has the same name G_2, F_4, E_6, E_7 or E_8 as the absolute type of the group.

Quasi-split groups of type E_6. The type is ${}^2E_6$ or F_4^I according as the quadratic splitting field L is unramified or ramified.

Nonquasi-split groups of type E_6 and E_7. They are the forms of E_6 and E_7 constructed by means of a central division algebra of dimension 9 and 4 respectively; their types are ${}^3E_6$ and ${}^2E_7$.

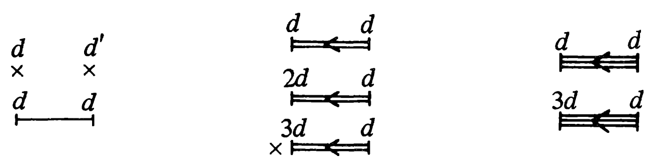
4.5. *Invariants.* All types of groups listed in the Tables 4.2, 4.3 exist over an arbitrary field K with finite residue field. The central isogeny class corresponding to a given name is always unique except in the following cases.

The isogeny classes of type ${}^dA_{md-1}$ for $d \geqq 5$ are classified by the pairs of opposite central division algebras of dimension d^2 over K; their number is therefore $\frac{1}{2}\varphi(d)$, where φ is the Euler function.

The isogeny classes of the types B-C_n, ${}^2B$-C_n, C-B_n, ${}^2C$-B_n, C-BC_n and F_4^I are classified by the ramified quadratic extensions of K, namely the extension always called L in §4.4.

The groups of type G_2^I are classified by the Galois extensions L of K which are either cyclic of degree 3 or noncyclic of degree 6.

4.6. *The classification kit.* The following experimental facts provide a handy way of reconstructing the classification. First note that, except for ${}^dA_{d-1}$, each type in the Tables 4.2 and 4.3 is completely characterized by the local Dynkin diagram and the integers $d(v)$ attached. Now, consider a connected Coxeter diagram of affine type and rank (number of vertices) at least three, attach an integer to all vertices, mark some of them (possibly none) with a cross and orient each double or triple link with an arrow. Then a necessary and sufficient condition for the existence of a semisimple group G having the resulting diagram as its relative local Dynkin diagram with the given integers as $d(v)$ is that all subdiagrams formed by the pairs of vertices belong to one of the following types, representing the ordinary Dynkin diagrams of quasi-split groups of relative rank two:



The group G can furthermore be chosen to be absolutely quasi-simple if and only if the integers $d(v)$ are relatively prime or if the underlying Coxeter diagram is a cycle. As for the types of relative rank one, whose underlying Coxeter diagram is ⊢⊣, they can be obtained as "limit cases" of types of higher ranks, but we shall not elaborate on that.

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