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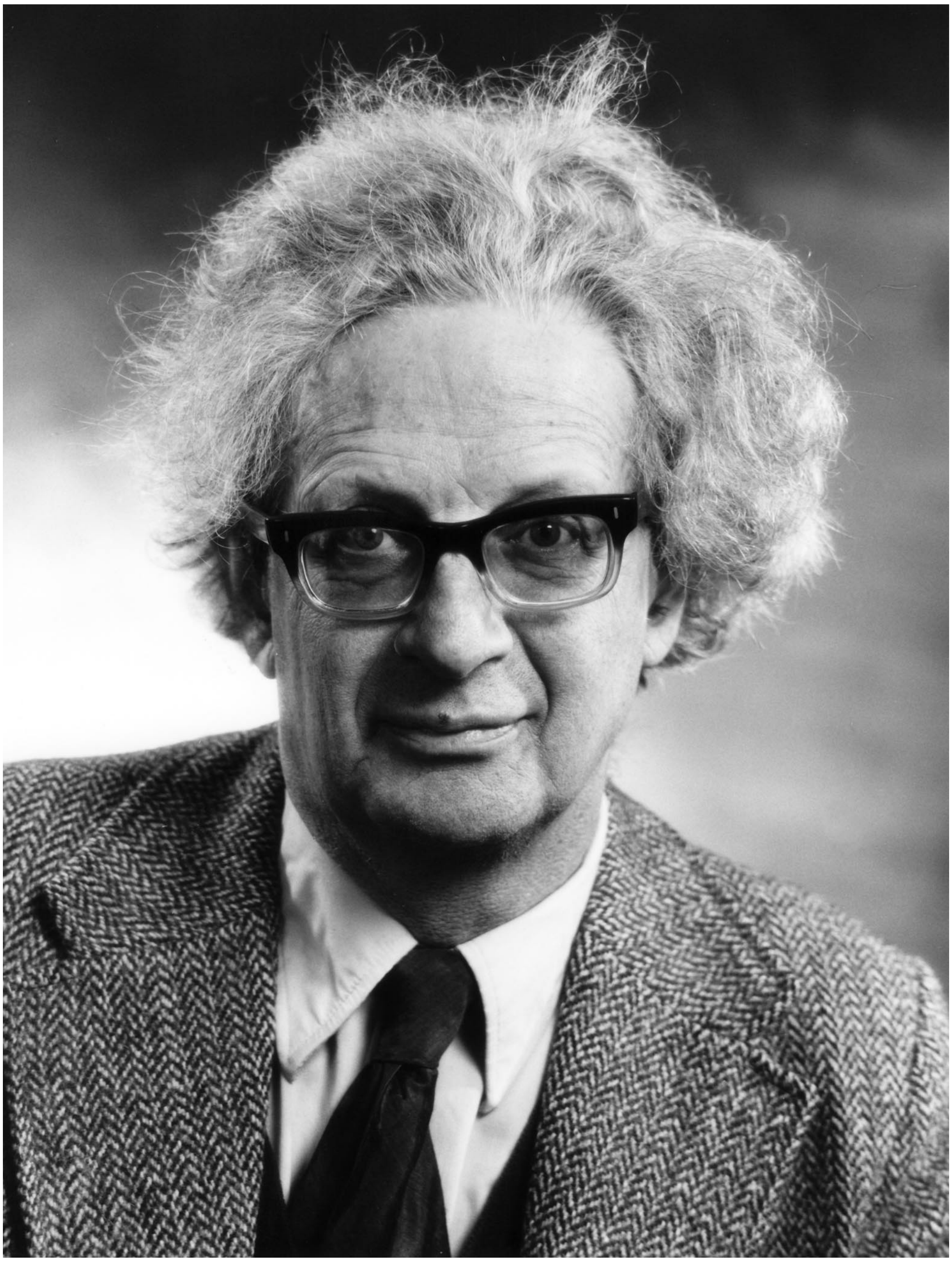

A. Fröhlich

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Elected FRS 1976

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Albrecht Fröhlich was one of the major mathematicians of the latter half of the twentieth century. He will be remembered as one of the few who have succeeded in creating a new subject: he was the creator of Galois module structure, which is now an important branch of algebraic number theory. He died in Cambridge on 8 November 2001, much loved and much honoured. Despite his relaxed persona and happy maturity, his early life was turbulent: he was a Jew, and left school abruptly when he and his family were forced to flee from the Hitler persecution. Thereafter, he completely broke the conventional mould for mathematicians, because he did not attend university till he was very nearly 30 years old and did his most important work when he was nearly 60 years old. He was elected to the Fellowship of the Royal Society in 1976, joining his elder brother Herbert, who had been elected in 1951*.

Ali (as he was always called by his friends) was born in Munich on 22 May 1916; he was the youngest of the three children of Julius and Frida Fröhlich, a Jewish couple who hailed from Rexingen in the Black Forest; his sister, Betti, had been born in Rexingen in 1904 and his brother, Herbert, in 1905. He attended Volksschule, and then the Wittelsbacher Gymnasium from 1926 to 1933; his school reports record that his work in history and religion was of outstanding ('hervorragend') quality and his work in mathematics and science was praiseworthy ('lobenswort'), but his English and Latin were poor. In 1933, Hitler came to power, and life became impossible for Jewish families; the Fröhlichs had made no secret of their origins, and Ali made no secret of his opinions—indeed, he joined a Jewish left-wing

* Herbert's biographical memoir, written by Sir Nevill Mott FRS, appeared in 1992. The brothers had a great deal in common besides a very strong physical resemblance. When one reads the first sentence of Mott's summary, one can see Ali as well as Herbert!

discussion group* and one day walked home in full view of the Nazi offices with the pockets of his shorts stuffed full of pamphlets. A party of Brownshirts beat up Julius and came looking for Ali. The local policeman, who had an apartment above the Fröhlichs and was still a decent human being, had the presence of mind to arrest Ali as an 'enemy of the state'. Ali was released the next morning, but the policeman's wife's reaction when she saw Ali was to ask, 'Are you still here?'. Ali took the hint and left immediately. The people in the French Consulate were very helpful, and by that evening Ali was in Alsace; his father and mother followed soon afterwards.

Ali and his parents remained in Alsace for about a year, waiting for visas to go to Palestine; Ali used to recall that he worked in a factory that made rather bad chocolate. Meanwhile, Herbert had taken his doctorate in 1930, and in 1933 he was Privatdozent in Freiburg; he was deprived of his job, but remained in Germany for a while to try to retrieve some part of the family property; he moved to Leningrad in 1934, but after a year in Russia he was forced to flee again. Herbert finally found a post in 1935 in Bristol, where he remained until 1948. Betti had already married and migrated to Palestine about 10 years earlier (both Herbert and Betti were visiting Munich when the family were forced to flee). Because Betti was already a resident of Palestine, Ali and his parents were permitted to follow her there. Ali lived in Haifa until 1945; he supported himself and his parents, at first by working on the roads and then as an electrician for the Palestine Railways (according to his leaving certificate, he worked as an Artizan Class I (electrical), from April 1942 until the end of October 1945). For a while he was interned, presumably as an enemy alien, but he harboured no resentment.

Once the war in Europe had come to an end, Herbert proposed that Ali should join him in Bristol, to take a degree. Ali was not certain at first whether he should study mathematics or engineering; fortunately, he decided he should become a mathematician. Although Ali's formal education had ended 12 years earlier at the age of 17, Herbert did not find it too difficult to persuade the university to accept Ali as a student for the academic year beginning October 1945: the acting Vice Chancellor, A. M. Tyndall FRS, was Herbert's Head of Department and could reasonably believe that Ali, too, was talented. Indeed, Bristol acted with commendable speed. Herbert wrote formally to the Committee of Deans on 16 April, and on 26 May a letter was sent to Ali in Haifa, telling him that a place had been reserved for him from October. (On 21 June it was recorded that Herbert had agreed to pay Ali's fees if it should be necessary, and Tyndall certified that he was able to do so. It is not recorded whether Herbert was actually made to pay.) The authorities in Palestine were not so helpful; Ali had difficulty in obtaining transport from Palestine to Britain (large numbers of troops were being sent home, and they had priority for the limited shipping available) and indeed the Colonial Office was reluctant to grant a visa and queried the university's acceptance of an 'unqualified candidate'. The Dean replied that it was not the university's practice to consult the Colonial Office on admissions matters, and Ali got his visa, but did not arrive in Bristol till December, too late for the start of the academic year. Ali went to see the Dean, who suggested that because Ali had arrived late he might wish to start his course in the following September; Ali said he wanted to start straight away; the Dean replied, 'I think we can arrange that'. For the first time in years, Ali found himself doing work he totally enjoyed. Despite his apparent lack of

* This group was quite celebrated. Most members were students at the university, and several became academics, including Harry Maor (Harry Obermayer) and Schalom Ben Chorin (Fritz Rosenthal), after whom a street in Munich was recently renamed.

preparation, he passed his first-year examinations after two terms work, with first-class marks.

In 1948 Ali applied to do research under the supervision of Hans Heilbronn (FRS 1951). At first Heilbronn showed some reluctance, remembering the dictum that most mathematicians have done their best work by their early thirties (Ali's age at the time) but he needed little persuasion. The two men appeared very unlike one another: Heilbronn was a shy man, and despite his essential kindliness always appeared forbidding; his duelling scar did not help to dispel this impression. In contrast, everyone who knew him remembers Ali as one of the most approachable of men. In fact they understood each other's experiences (Hans Heilbronn's early history was very similar to Herbert's), and they got along together very well. Heilbronn gave Ali one particularly valuable piece of advice, which, because of its universal application, is worth repeating even though it is recorded (by Ali) in Heilbronn's biographical memoir: 'You must never take too much notice of pessimistic comments from your supervisor, or from any other mathematician, however great'; it is arguable that many scientists do their best work when they are young, because they do not know what is supposed to be impossible! Heilbronn suggested as the theme of Ali's thesis the explicit construction of class 2 extensions of number fields; specifically, he had the idea that one could study ideal class groups, by considering them as modules over the Galois group. Heilbronn himself never published anything on this topic, although he suggested it to other students. (Indeed, he had even mentioned it to Olga Taussky-Todd, who was to be Ali's external examiner; in her 'personal recollections' of Heilbronn, she records that Ali made him 'extremely proud and happy'.) If Heilbronn had any initial doubts about Ali's potential as a research scientist, they were quickly dispelled. Ali submitted the first part of his doctoral thesis for publication in September 1949, and by the beginning of 1951 it was complete. The central principle of this thesis, that one should consider the structure of objects as Galois modules, was seminal to the rest of his work; indeed, he followed this principle with extraordinary constancy and was to be rewarded by a great theorem.

While he was in Bristol, Ali's interests were not confined to mathematics. Ruth Brooks, a young medical student, organized outings for foreign students, and Ali joined one such trip. It was a boat trip, and Ruth and Ali were the only members of the party who were not seasick, so they got to know each other well quite quickly; they were married in 1950 and stayed together until Ali's death. In 1950, Ali applied successfully for a two-year post as an assistant lecturer in Leicester (in those days it was not unusual for an assistant lecturer to be appointed before his thesis was complete); at the end of the first year he was promoted to lecturer, but he moved to Keele in 1952. He became a British citizen in 1951 (he preserved his oath of allegiance, and also the letter asking for a fee of £9 for his certificate of naturalization).

In 1955, encouraged by Ruth, he applied for a readership at King's College, London, and after a tough interview at the hands of H. Davenport FRS, L. J. Mordell FRS and J. G. Semple (Ali recalled that Mordell was particularly fierce) he was appointed. His transformation from Jewish refugee to British academic was complete. He remained at King's until he reached retiring age in 1981, being promoted to a chair in 1962 and serving as Head of Department from 1969.

Ali started a seminar in a very small room in King's, and Ali and Ruth set up house in Wimbledon. During the years that followed, his seminar prospered: between Bob Laxton in the late 1950s and Jan Brinkhuis in the late 1970s, he taught about 20 research students; as his reputation grew, disciples from overseas would come to deliver and listen to seminars in the very small room, to be entertained by Ruth, and to talk to Ali on walks across Wimbledon Common.

The story of Ali's King's College years encapsulates the history of algebraic number theory in Britain during the same period. In 1955, algebraic number theory was unfashionable in this country, and class field theory was a recondite mystery known only to a few. There were indeed two places in Britain where serious algebraic number theory was happening: Cambridge, where J. W. S (Ian) Cassels (FRS 1963) was working on elliptic curves, and King's College, where Ali had inherited Heilbronn's mantle as Britain's proponent of class field theory. Nowadays, largely because of the work of Cassels and of Fröhlich, algebraic number theory is a standard tool, and class field theory has been demystified.

At first Ali and his pupils worked on very varied problems related to class field theory. For instance, in 1961 Bob Laxton wrote his thesis on near-rings and in 1961–62 Ali wrote three papers on non-abelian cohomology; Abraham Lue's thesis in 1965 was related to that subject. In 1966 Ali gave an excellent course (46)* on formal groups, in 1967 Allan McEvett wrote his thesis on Hermitian forms over algebras, and in 1969 Ali published a paper in collaboration with C. T. C. (Terry) Wall FRS on equivariant *K*-theory. Colin Bushnell, who wrote his thesis in 1972, would become an expert on local Langlands theory. All this was distinguished work in the mainstream of mathematics, although it did not yet constitute a major breakthrough.

In this period, one of Ali's really major contributions to mathematics was in 1965, when he and Ian Cassels jointly organized the instructional conference in Brighton. He and Cassels gave preliminary courses, respectively on local and on global algebraic number theory; the main courses, on local class field theory and on global class field theory, were given by J.-P. Serre and by J. Tate. There were also lectures on other necessary or relevant topics, such as cohomology theory, the applications to quadratic forms, and (from Heilbronn) on the classical approach to class field theory. The whole event was meticulously organized: the preliminary lectures were far from conventional, but carefully designed to lead into the main courses; the lectures by Serre and by Tate were magnificent; and competent note-takers (including Bryan Birch and Vernon Armitage) were provided to make an accurate record of what was said. Before Brighton, class field theory was a recondite mystery known only to a few (in Britain, known to very few indeed); after Brighton, it was a standard tool of mathematics, available to any professional. The Brighton Proceedings became a standard text; they would have been reprinted in paperback 30 years later had Academic Press released the copyright. I (B.J.B.) have many memories of that wonderful conference; the worst was correcting the first set of proofs—it was almost Academic Press's first venture into mathematics, and there were errors on nearly every line. The most cheerful was the dinner for speakers and note-takers—I think Ali must have organized it, because everyone was provided with his individual bottle of good French wine.

Brighton marked a milestone in other ways: number theorists got to know one another much better, and in particular the British and French schools moved together; there was a particularly close and enduring association between Ali Fröhlich's group in King's and the group that Jacques Martinet had gathered in Bordeaux. It became natural for French workers in the field to attend conferences in Durham, and to visit Ali in King's, and in turn British workers visited Paris and Bordeaux and the conference centre at Marseilles-Luminy. Ali's seminar prospered; he was elected to the Fellowship of the Royal Society (belatedly I thought) in 1976.

* Numbers in this form refer to the bibliography at the end of the text..

The real breakthrough came in the early 1970s, when two of Ali's interests came together in an extraordinary way. From the time of his thesis, Ali was interested in the structure of modules of algebraic numbers under the action of the Galois group; in that context the most natural question of all was to decide whether or not the integers of a number field have a normal integral basis. Also in the early 1970s, the Langlands conjectures (both local and global), and in particular the understanding of the 'root numbers' occurring in the functional equations of L-functions, were subjects of central interest to algebraic number theorists. In (53) Ali proved the extraordinary and beautiful result that, for tamely ramified quaternion extensions of degree 8 over the rationals, the integers fail to have a normal basis precisely when the root number W is -1. In his introduction Ali acknowledges the earlier work that had led him to this theorem. In 1971 Martinet had given a handy necessary and sufficient condition for such a field to have a normal integral basis; then in 1972 Armitage and Serre had independently given examples of quaternion fields with root number -1, and they noticed that the cases with W equal to -1 were indeed the same as those without integral basis (Serre had the crazy idea, '*trop beau pour être vrai*', that this might be true, and Armitage computed several examples). So, as Ali said, his result was not a total surprise to either Armitage or Serre. What was surprising was the beauty and coherence of Ali's proof, and the remarkable manner in which he was able to exploit this success. In a remarkable series of papers he rapidly established Galois module structure as an important and very vigorous new branch of number theory. Already in 1975, the conference he organized in Durham (with Steve Wilson and Colin Bushnell as assistant organizers, and myself (M.J.T.) acting as dogsbody) celebrated a subject that was heading towards maturity.

With this breakthrough, Ali's international stature was at last properly recognized. He was invited to lecture (69) on Galois module structure at the International Congress of Mathematicians in Vancouver in 1974. He was elected to the Fellowship of the Royal Society in 1976. Also in 1976 he became a Fellow of King's College, London, and was awarded the Senior Berwick Prize by the London Mathematical Society: his paper (57) on Galois module structure was indeed the best paper they published in the relevant period. His little seminar room was full of foreign visitors, notably Leon McCulloh, Steve Ullom and Philippe Cassou-Noguès; these were disciples rather than students. He continued to be highly productive, although not quite at the level of the early 1970s; his pupils and disciples were now working on Galois module structure too, so the subject made very rapid progress. In particular, his fundamental conjecture, connecting the existence of normal bases for tame extensions with the symplectic root numbers, was proved in 1980 by Martin Taylor. He described this period as his 'period of grace'. At the same time his children were growing up, so it was indeed a golden period in his family as well as his professional life.

In 1981 he reached retirement age and relinquished his chair at King's. He was elected to a senior research fellowship at Imperial College and to the Heidelberg Akademie der Wissenschaft. He was also elected to a fellowship at Robinson College, Cambridge, and when Ruth too retired in 1993 they set up house in Barton Road. His mathematical activity was not diminished (about a quarter of his papers were written after he had retired from his chair); he even taught a final research student, David Burns. He enjoyed the atmosphere and freedom of Cambridge, and indeed the most immediate effect of his retirement was that he was able to travel more, to give invited lectures, attend conferences, and generally visit mathematical friends abroad (for Ruth this was a mixed blessing, because he liked to take her everywhere, and she was a busy GP); he also seized the opportunity to write several books. His good friend

Irving Reiner was instrumental in his appointment as Miller Professor in Urbana for 1981–82, and he was Gauss Professor in Göttingen in 1983. He made several extended visits to Cassou-Noguès in Bordeaux and to Jurgen Ritter in Augsburg (1982, 1985, 1989 and 1993), and he attended Tagungen at Oberwolfach in 1983, 1984 (twice), 1988 (twice), 1993 and most remarkably 1999. Special meetings were held in Robinson College to celebrate his 70th and 80th birthdays. In 1992 he was awarded the London Mathematical Society's highest honour, the De Morgan Medal. He received honorary doctorates from Bordeaux in 1986, and from Bristol in 1998.

Ali had a great capacity for enjoyment; as well as mathematics, things that he enjoyed included his family, music, eating (particularly chocolate, despite having worked in a bad chocolate factory), drinking coffee and walking and talking; and notwithstanding his experiences he never lost his affection for German culture. These pleasures could be combined; walking through the Black Forest talking about mathematics on the way to Himbeergeist and Kuchen in a local Gasthof was a treat enjoyed by many visitors to Oberwolfach. It is less easy to list things he did not enjoy, as he had the good sense to avoid them; we rarely heard Ali speak of his time in Palestine, nor did we see him changing a plug. He said he disliked administration ('administration is work, mathematics is pleasure') but he was good at it. He could see what was important, made sure that someone saw to it, and he wasted none of his own time, and as little as possible of other people's, on things that did not matter; he was particularly successful in obtaining funds for his students' research. Ali disliked pretentiousness and pretence, except in jest; but he enjoyed being a 'grand old man' (albeit a very active one) with a house in Cambridge appropriate for an English gentleman; it appealed to his sense of humour. Our happiest memories of him include looking across the circle of the Coliseum and recognizing him with Ruth (because of his shock of white curls he could be recognized from a great distance), and listening to his grandchildren singing to him at his 80th birthday party.

He was a family man through and through, taking great pride and joy in his children (and later grandchildren), even skipping with them across Waterloo Bridge on the way to work. He had great respect for Ruth's work, providing invaluable support and advice; she in turn supported him in all that he did. Despite his mathematical eminence, Ali was a very modest man; he was at times the archetypal absent-minded professor, but was always ready to join his family in laughing at himself. Despite his sense of fun and of the ridiculous, he was a warm and sensitive person; his family were proud of his academic achievements, but remember him first as a wonderful husband, father and grandfather.

Fröhlich's mathematical work

Ali Fröhlich did his best work unusually late in his life. This was not simply because he started late—his great papers, proving the connection between the root number and normal integral bases, were not written until 20 years after his thesis. During those 20 years he had been working in a wide but coherent area of algebra and number theory, seeking to understand Galois actions and proving deep theorems about them. His breakthrough was no accidental coming together of two of his interests, it was a just reward for his deep understanding of a coherent area of mathematics. It is tempting to take paper (53), containing the first news of the breakthrough, as the central point of an account of Ali's mathematical work, and to divide this technical account of his work between 'the early years' before this breakthrough and the 'period of grace' that followed it. This would be the more natural, as the breakthrough happened almost exactly halfway through his mathematical career: Ali published prolifically, mainly between the ages of 34 and 76 years, and (53) was written when he was 55 years old. Most of his papers, including his thesis, stress the study of Galois actions—they belong to a circle of ideas not so very far from (53)—but his work contains diverse strands, and to give an orderly account it seems appropriate to pull them apart. By the nature of Ali's work this will be a bit arbitrary; we have already stressed that his work is unusually tightly woven, and in particular almost all the strands of his earlier work are eventually used in the theory of Galois module structure. We will describe his work more or less in chronological order, but after about 1960 strands are separated. Accordingly, section 3 contains mainstream work on discriminants and on class groups of group rings, but his groundbreaking work on Galois module theory, root numbers and parity problems is reserved for section 4; the work related to Langlands theory, although arising from the work in section 4, has a somewhat different flavour and is discussed in section 5. The distinction between the topics in sections 3, 4 and 5 is rather weakly defined—Ali was advancing on a wide front, and we are separating various parts of the front. Some topics seem more separate, namely his work on quadratic forms and his collaboration with Terry Wall; these and some of his books are described at the end in sections 6, 7 and 8.

1. Class groups of number fields, and class field theory (1–8, 15–19, 29, 30, 88)

Ali's earliest work, stemming from his thesis, concerns the development and interplay between two themes: firstly the study of class groups as modules over Galois groups and the development of the required representation theoretic techniques, and secondly the study of Galois extensions of nilpotency class 2 of a given number field (that is to say, extensions whose Galois group has its commutator subgroup contained in its centre). These two themes are closely related for the following reason: given an abelian extension K of the rationals with known ramification, the knowledge of the maximal class 2 extension of the rationals with prescribed ramification can be used to derive information about the maximal subfield M of the Hilbert class field of K which has the property that $\mathrm{Gal}(M/Q)$ has class 2. This observation may be used to obtain a considerable amount of information about the class group of K.

His thesis is written as two distinct parts, both of which would be major influences on his subsequent work. The first part, published in (1), treats the representations of a finite group G into the group of automorphisms of a finite abelian group A, including a simple classification theory for such representations when the orders of G and A are coprime. In the second part he goes on to apply the preceding work to the case where A is the class group of a ring of integers of a number field with the Galois group G acting on it in the natural way; it seems that this was the first time Galois action on class groups was looked at in such a systematic way, although we are told that Heilbronn had suggested the problem to others. One of the main results of this second part, later published in (2), is a description of the class group of a number field N abelian over a subfield K in terms of the class groups of maximal cyclic sub-extensions of K.

In his paper (3) he develops ideas from his thesis to give, *inter alia*, characterizations of various extensions M of the rationals which are maximal with nilpotency class 2. In the early thirties, A. Scholz (following earlier work of L. Redei and of H. Reichardt) had initiated the study of central extensions of number fields. Class field theory is a powerful tool for describing the abelian extensions of a number field; Ali's idea was to push class field theoretic techniques to obtain information on central extensions of the rationals. He describes presentations of the Galois groups of such maximal extensions *M* in terms of generators and relations, which are derived simply and elegantly from the ramification data of *M*. Subsequently, there was a considerable amount of further work in this area; the work of S. P. Demushkin, of Serre and of I. R. Shafarevich has been particularly influential. For a good account of some of these developments see Koch's article in (73), and see also the concluding chapter of (88).

In (4), (7) and (8), as explained in the first paragraph of this section, he was able to use this work on class 2 extensions to obtain strong divisibility information on the class numbers of number fields in terms of ramification data. His strongest results are obtained for l-power cyclic extensions of the rationals, and may be seen as building on earlier work of Redei (which he extends in (6)) and others on the divisibility properties of class numbers of quadratic fields; a particularly striking result is that the class number of such an l-power cyclic extension L is coprime to l if and only if there is only one prime number that ramifies in L.

He obtained more results of similar type after he had moved to King's in 1955. Also in King's he wrote papers on class field theory with a more classical flavour; notable among these was (17), a big paper that contains, *inter alia*, the classification of quaternion extensions that he later used in (53).

Much later, Ali became conscious that the language of his early papers had become old-fashioned, making them hard to read. He therefore decided to rewrite them, replacing the older style of class field theory with idelic and cohomological techniques. The result was the delightful little book (88).

2. Some abstract algebra (12–14, 23, 24, 26–28, 35)

While he was in Keele, Ali had done some joint work with J. C. Shepherdson on 'Effective procedures in field theory' (10, 11), and after his move to King's College, Ali's interests diversified further. He wrote some good papers on class field theory (for instance, (16) and (17)) learning his craft, but his most original work during his first five or so years at King's was in very abstract algebra. He wrote eight papers, first developing the theory of distributively generated (d.g.) near-rings and then seeking to use this theory to develop a non-abelian homological algebra. Near-fields had been studied in some depth by L. E. Dickson in 1905 and by Hans Zassenhaus in 1935. H. Wielandt had been interested in near-rings in the 1930s, and there were the beginnings of a general theory, for example in work of his pupil G. Betsch and in work of D. W. Blackett in 1953. It seems that Ali's papers were the first systematic study, dealing particularly with d.g. near-rings; Hanna Neumann had realized the importance of the d.g. axiom in a paper published in 1956, in which she tried to use near-rings to solve a problem in the varieties of groups.

We recollect that a near-ring is a ring lacking one distributive law, and a distributively generated near-ring is a near-ring that is generated as an additive group by a multiplicative semigroup of distributive elements. While in general near-rings act on groups as mappings, a d.g. near-ring acts on a group as the mappings generated additively by the endomorphisms of the group. This is the aspect that interested Ali; in (12) and (13) he laid the foundations of the theory, and in (14) he illustrated this theory by determining the near-rings generated by the inner automorphisms of a finite simple non-abelian groups. In (23) and (24) he studied d.g. near-rings in their categorical setting; this work was the basis for the three papers developing a non-abelian homological algebra (26–28). In (27) he commented that this theory has a considerable amount in common with R. Baer's three 1945 papers on 'Representations of groups as quotient groups'; he returned to Baer's theory in (35).

Though Ali wrote no more about near-rings, his papers had considerable influence; his first research student, Bob Laxton, wrote his thesis on this subject, and Ali's friend J. R. Clay, who was one of the most influential workers in the area, often paid tribute to Ali's inspiration.

3. Discriminants, and class groups of group rings (20–22, 31, 33, 38, 39, 66, 67); (32, 36, 37, 40, 49, 54, 56, 62, 63, 71, 89, 103)

The first paper (20) on discriminants, dating from 1960, is not very long, nor is it difficult, but it is important. In it he defines his 'fine discriminant', which is quite clearly the 'correct' notion of discriminant for extensions L of a general number field K; it is an object defined over the ground field that gives a great deal of information about the extension without undue fuss. It has the beautiful property that it is principal if and only if the integers of L have a free basis over the integers of K, and it reduces to the ordinary discriminant when the ground field is the rationals.

The fine discriminant lives in the group of idèles modulo squares of unit idèles rather than the multiplicative group of K itself, so it is not suitable in an elementary context; but Ali uses it regularly in his subsequent work. It is an epitome of Ali's determination to find the correct formulation of fundamental concepts, which would provide him with the tools for his later triumphs.

In subsequent papers, these ideas are refined, related more closely to the module structure of the ideals of the extension field, and extended to general Dedekind domains. In (39) he gives a proper definition of resolvents and discriminants in the context of normal extensions of the quotient fields of general Dedekind domains; at the time, he stated that 'their principal significance lay in their connection with new integral invariants' he had found for such extensions; this seems to refer to (37). However, he was preoccupied by the preparations for the Brighton Conference, and the interactions that followed it, and before the time came to write up this connection in the general case he had made the crucial breakthrough described in the next section and he had made his statement out of date; his resolvents had become an important tool of Galois module structure theory. This theory of resolvents was refined in (66) and applied to Galois module structure in (67).

In parallel with this series of papers on invariants, he wrote a series of papers developing the theory of modules over group rings. The first in this sequence is (32), in which he uses the terminology of Kummer algebras in the tradition of H. Hasse; he followed this by (36), a paper on Galois algebras dedicated to Hasse on the occasion of his 65th birthday. In (37) he developed quite general methods for dealing with non-projective modules over an order such as a group algebra. This paper was ahead of its time from the point of view of arithmetic applications, but came into its own when he initiated a study of wildly ramified extensions in the 1980s. It was the seed for his factorizability theory, initiated in (101) and (107) and further developed by his students Adrian Nelson and David Burns.

Returning to the case of projective modules, the papers (49) and (54) began a study of the class groups (Picard groups) of group rings; these class groups classify the projective modules over a group ring $\mathbb{Z}[G]$ in the same way as the class group of a number field classifies the isomorphism classes of ideals of a ring of integers. In (49) and (54) Ali introduces an idèle-theoretic method to obtain sharp results in case G is a finite abelian p-group. The paper (58), written jointly with his ex-pupils Michael Keating and Steve Wilson, contains calculations for quaternion and for dihedral groups, and in (56) and (62) (the latter written jointly with his friends Irving Reiner and Steve Ullom) they try to give a general method for all finite groups.

Ali developed his general idelic method and published it first in (63), also dedicated to Hasse, this time for his 75th birthday. After recasting his method in dual form, it became the very powerful method for the calculation of class groups known as the Fröhlich Hom-description; it is expounded in the appendix of his magnum opus (71).

4. Galois module theory, root numbers and parity problems (32, 33, 53, 55, 57, 59, 61, 64, 65, 68, 69, 71, 74–77); (79, 80, 82, 97, 98, 101, 102); (105–108)

Each generation produces some mathematicians who make striking contributions and breakthroughs. Ali was one of the very few of his generation to open up a completely new subject area: the arithmetic Galois module theory of rings of integers. About a quarter of his contributions deal directly with this topic, and his papers on class groups, discriminants and Hermitian theory all played a vital supporting role.

It seems that (32), written in about 1960, is the first of his papers that connects the Galois module structure of rings of integers of extensions with arithmetic properties of the base ring, albeit in a rather different context from his major papers; but parity questions are pervasive in his papers on discriminants, from (20) onwards. His life's work suddenly came together at the beginning of the 1970s.

Ali had conjectured ((38), p. 81) that the ideal class $\mathrm{cl}(f(\theta))$ of the Artin conductor of a real character is always a square. In a paper published in 1971 (*Inventiones* 14), Serre produced a counterexample involving a quaternionic extension, but proved Ali's conjecture in case θ was the character of a real representation. Serre conjectured that the Artin root numbers of such orthogonal Galois representations are always $+1$, and the question arose whether symplectic Galois representations could yield Artin root numbers that were -1, and so give interesting examples of zeros of Artin L-functions. As we have already described, Armitage and Serre gave examples of such symplectic representations; for all of their examples, Martinet's criterion could be applied to show that the ring of integers did not have a normal basis. In (53), Ali showed that the ring of integers of a tame Galois extension of the rationals with Galois group H_8 is stably free over the group ring $\mathbb{Z}[H_8]$ precisely when the Artin root number of the irreducible nonlinear representation of the Galois group (which is of course symplectic) is $+1$, and that, subject to the correct conditions, there are infinitely many such extensions, with given ramification, containing a given real quadratic field, with root number $+1$ and infinitely many with root number -1. A little later Ali and Queyrut independently proved Serre's conjecture for orthogonal Galois representations; this proof appeared in their joint paper (55), together with Serre's proof of the function field analogue.

The breakthrough paper (53) was undoubtedly the high point of Ali's mathematical life; it related the algebraic Galois structure of rings of integers to an analytic invariant in an entirely new and sensational way. It thrust him and his subject to the fore on the world stage; in particular, he was invited to present his work at the International Congress of Mathematicians in 1974 (see (74)) and he was awarded the London Mathematical Society's Senior Berwick prize in 1976 for his paper (57), in which he proved (mildly weaker) versions of his results for extensions of the rationals with Galois group H_8, in the context of generalized quaternionic extensions of general number fields. His progress was rapid, and a general picture emerged. Various special families of Galois groups were considered in detail: quaternion groups in (59), and generalized dihedral groups in (68). Simultaneously he was developing general methods: class groups were developed as described in section 3 above; the corresponding tools for determining arithmetic classes in these class groups were developed in (37), (39), (40), (66) and (67). The culmination of this multi-pronged attack on the Galois structure of rings of integers came to full fruition in the famous Fröhlich conjecture, which in its simplest form asserts that the Galois structure of tame rings of integers is determined by the signs of the symplectic Artin root numbers. This conjecture was underpinned by the magisterial paper (71), which drew together numerous different threads of his research into the magnum opus of his research career. The conjecture was to be proved by his pupil Martin Taylor in 1981. His tract (86) contains his definitive account of the theory; it is a pity that it was written in the rather formal style usual in the *Ergebnisse* series.

Ali began a second phase of Galois module theory, namely Hermitian (or quadratic) Galois structure, in which one considers rings of integers endowed with their trace form. Here again he found himself in the position of having, from his earlier work on quadratic forms with a group action, several key ideas and techniques ready for immediate use. He employed a similar strategy to the one he had used so successfully for standard Galois module theory: on the algebraic side he developed a new and powerful theory of Hermitian class groups that classified locally free modules over a group ring which supports a group-invariant form (see especially (83)): the vital new ingredient that he used to classify such structures was his new version of the Pfaffian, which for trace forms is very closely related to his generalized Lagrangian resolvents. On the arithmetic side his work led to his formulation of the second Fröhlich conjecture (proved by Philippe Cassou-Noguès and Martin Taylor in *Ann. Inst. Fourier* **33** (2), 1–17 (1983)), which asserted (conversely) that both global and local tame Artin root numbers could be determined by the Hermitian–Galois structure of the rings of integers endowed with their trace form. The fundamental work is contained in (72) and (76); the book (89), published in 1984 but originating in lecture notes of 1979, contains the statement of his second conjecture, as well as a definitive account of both the algebraic and the arithmetic aspects of his Hermitian class group theory.

The third phase of his interest in Galois module theory concerned the Galois structure of algebraic integers where there is wild ramification. Whereas in the tame case one knows by a result of Emmy Noether that the ring of algebraic integers of a Galois extension is a projective module over the group ring (Ali's tribute (84) to Emmy Noether, written at about this time, is very much to the point), it is not at all clear what limitations or constraints there are on the Galois structure of integers of wildly ramified extensions. In (79) he tried forcing the ring of integers of the extension to be projective by replacing the integral group ring $\mathbb{Z}[G]$ by a maximal order M containing it, but he obtained a more fundamental insight into this hard problem by returning to ideas in (37) and developing the notion of factorizable modules, essentially modules that enjoy certain relations on distinguished sub-modules (97, 101, 107, 108). He felt that the importance of this wild theory was not fully realized, and he would often say that this was the area to which he would introduce a really bright young student! It is certainly an area that is still not well understood.

He was acutely aware that many of his ideas and techniques would be applicable to Galois modules other than rings of integers; indeed, his thesis had dealt with the Galois module structure of ideal class groups, and in the early 1980s Ted Chinburg, one of Ali's disciples, began to use Ali's methods to study the multiplicative group of S-units. Ali's proposed programme of studying multiplicative Galois modules such as units and class groups was launched in (102) and (105). There are well-written accounts of Ali's wild theory and multiplicative theory in the Durham Proceedings (106), which also contains an account of Chinburg's theory.

5. Gauss sums and Langlands theory

Ali was fascinated by Gauss sums and their ubiquity throughout mathematics. His tame Galois module theory had depended on the crucial relationship between Galois Gauss sums and his generalized Lagrange resolvents. This was a truly remarkable relationship: whereas his resolvents had a relatively direct definition in terms of determinants (see (39)), by contrast the Gauss sums (or equivalently the epsilon constants) attached to representations of local Galois or Weil groups have a less direct definition, through inductivity (on virtual representations of degree zero) and explicit formulae for one-dimensional representations. The paper (81), written jointly with Martin Taylor, was inspired by the striking similarity between Gauss sums and resolvents, and it provided a new characterization of tame local Gauss sums in terms of the arithmetic of the local field in question.

Apart from (81), the main papers here (all written after his retirement) are concerned with the problem of understanding Gauss sums within the context of the Langlands programme—at the time when that programme was almost entirely conjectural. The key point is that, under the guise of the 'local constant', the Langlands programme identifies a Galois Gauss sum (of an n-dimensional irreducible representation of the Galois group of a local field F) with a number attached to an irreducible supercuspidal representation of the general linear group $GL(n,F)$. A variant of the programme deals with irreducible representations of $GL(1,D)$, where D is a central F-division algebra of dimension n^2. His two biggest pieces of work in this area, (87) and (93), both written in collaboration with his pupil and colleague Colin Bushnell, are concerned with this aspect ((87) was published as a set of Springer lecture notes, but it is really a research paper).

In the case of $GL(1,D)$ it is easy enough to define an explicit Gauss sum attached to an irreducible representation, just by imitating the classical formulae for $GL(1,F)$, and this Gauss sum is related to the local constant in an easy way. Unfortunately, at that time there was no Langlands correspondence for $GL(1,D)$. However, by comparing the two sorts of Gauss sum they produced in (87) a candidate for the Langlands correspondence for $GL(1,D)$ in case n is not divisible by the residue characteristic of the valuation ring of F. Their suggested correspondence has turned out to be very nearly, but not quite, correct.

For $GL(n,F)$ the notion of Gauss sum is more subtle, and (93) needed one of Ali's best ideas. The normalizer K of a principal order A in the matrix ring $M(n,F)$ is a maximal compact-modulo-centre subgroup of $G = GL(n,F)$. It has many structural features common to the case $n = 1$. By means of an explicit formula, one can attach a Gauss sum to an irreducible representation ρ of K. This Gauss sum may be zero; we say ρ is non-degenerate if the Gauss sum is nonzero. In (93), they showed that if an irreducible supercuspidal representation π of G contains a non-degenerate representation ρ of K, then there is a formula connecting the local constant of π and the Gauss sum of ρ, thereby generalizing part of Tate's thesis (which is the case $n = 1$). This idea works equally well for $GL(m,D)$.

This result has been developed in two ways. Colin Bushnell and his collaborators have used it as the starting point for their investigation of the structure of representations of G, culminating in the Bushnell–Kutzko classification theory. Twenty years later this was still the best way of computing local constants for $GL(n)$! Ali took matters on in a somewhat different way, concentrating on his favourite tamely ramified representations. In (92), (96) and (99) he developed a substantial theory of Gauss sums for tame representations of 'chain groups' K, in many ways parallel to Green's treatment of finite general linear groups. In general, the connection with representations of G is a little loose, but the connection is good for cuspidal representations.

6. Quadratic forms (41, 44, 45, 48, 50, 52, 94)

Ali's earlier work on quadratic forms concerned Grothendieck groups and Witt groups for quadratic and λ-Hermitian modules over rings with an involution, and a group action. Papers (44) and (45) were both written jointly with his pupil Allan McEvett; (44) is a foundational study of quadratic and Hermitian theory for not necessarily commutative rings, in considerable generality; it deals with their associated Grothendieck groups and Witt rings, and also their Morita theory. These ideas are then applied in (45) to the representations of finite groups by automorphisms of a commutative ring with involution, and a detailed study of the associated Grothendieck and Witt theory for group rings; reducing the generality slightly gives results for orthogonal, unitary and symplectic representations over rings with characteristic not 2.

In (48) Ali relates the theory of (44) to the contemporary work of Hyman Bass, Jacques Tits and Terry Wall. In (50) he applies the theory, very beautifully, to a classical arithmetic problem: he studies the Grothendieck groups of quadratic forms over number fields and their rings of integers, and in particular he obtains complete descriptions of the kernel and image of the map induced by the extension from the ring of integers to its field of fractions. In (52) he seeks to develop a theory of invariants for orthogonal representations of a group G over a field K not of characteristic 2, and to obtain some explicit results when G is finite and K is one of the local or global fields of number theory; as he says, the most interesting of the invariants he obtains is his Clifford algebra, which would be a valuable tool in his joint work (60) with Terry Wall on equivariant Brauer groups. At this stage he made his breakthrough in (53), and he was engrossed by his wonderful new theory. This work on quadratic forms would be a valuable basis for the work on Hermitian class groups leading to his second conjecture, but was not really an ingredient used in that work.

He returned to the study of quadratic forms 13 years later, in (94). This paper on so-called Fröhlich twists was inspired by a letter of Serre to Martinet, in which Serre related Hasse–Witt invariants of trace forms to Stiefel–Whitney invariants. Ali showed how such formulae could be obtained for a wider class of forms obtained by twisting Galois invariant forms over the base by the trace form of an extension. This was a fine piece of work, which inspired much more; Martin Taylor gave a Cambridge seminar on the subject, which Ali attended, only two weeks before he died.

7. Collaboration with C. T. C. Wall (47, 51, 60, 109)

Ali was a great inspiration to others, and many mathematicians spent sabbaticals with him; it is therefore surprising that he wrote rather few joint papers.

Ali Fröhlich and Terry Wall came together in the late 1960s, when they were both interested in quadratic forms from a rather similar point of view. Their first paper (47) used the tool of graded categories to extend some of the then recent work of Bass to the equivariant setting. Thereafter they would meet regularly in London, with Terry often being invited back home to Wimbledon. They used these meetings to sketch out a considerable programme of work that could be developed, building on their separate interests; Terry records that these discussions were highly congenial—Ali always seemed to have time to talk! However, they had rather different mathematical styles, and although they made considerable progress on their common problems they found it harder to agree on the best way of expounding their results. The second note (51) deals with the equivariant Brauer group associated to a ring with a group action, and methods for calculating it; it was written by Ali, who included the number-theoretic applications he hoped for. The paper (60) on monoidal categories was written by Terry. The core of this paper constructed the cohomological theory for graded categories, and was conceived as groundwork for a paper to follow (it was Ali's turn to write it) containing details and applications. However, at that stage their interests separated: Ali was fully involved in Galois module structure, and Terry was moving into singularity theory. The paper-to-follow (109), which establishes some fundamental exact sequences involving equivariant Brauer groups, did not appear until about 25 years later, when Terry realized that the conference proceedings were an ideal chance to place on record the progress they had made together and to rescue their ideas from oblivion. It was the last of Ali's publications.

8. Books (42, 46, 73, 86–89, 104)

Altogether, Ali was responsible for eight books: four monographs, two sets of conference proceedings, his lecture notes on formal groups (46) and the textbook on algebraic number theory (104) written jointly with Martin Taylor. The monographs (86–89) were written in a burst immediately after he had retired from his chair, when he took the opportunity to write definitive accounts of his greatest contributions. We have commented on them in the appropriate sections (respectively, sections 4, 5, 1 and 4).

Both sets of conference proceedings were important. The Brighton notes (42) indeed form a historic landmark in twentieth-century number theory, as we stressed in the first half of this memoir. The Durham proceedings (73) were not quite of the same importance, but they were a worthy celebration of the birth of Galois module structure and are still a valuable reference. Ali's own article (74) remains an excellent introduction to the subject, to be read before (89) and (71), and certainly before the rather tough *Ergebnisse* tract (86). Ali's article is flanked by contributions from Tate on local constants and from Serre on Galois representations, there is a big contribution from Martinet, and Ali derived particular pleasure from an exposition by Stark of his (then recent) conjecture.

In the late 1960s, formal groups were all the rage, and Ali gave a course in King's. It was well received and was published as Springer lecture notes; for a while it was the standard text on the subject, for those who did not wish to read the original papers. Ali was particularly pleased with his account of Kummer theory.

One of Ali's very last publications was his textbook of algebraic number theory, written jointly with Martin Taylor. For the authors, it was a happy marriage between Ali's wish to pass on the insights that he had gained over many years, and Martin's wish to write a text that would include many examples and would encourage students to try examples and calculations. It is not for us to comment on how well they succeeded; certainly, the book sold very well!

Envoi

Ali's contributions to mathematics were not only in his published works. While he was in King's he taught 22 doctoral students, from Bob Laxton in 1961 to Jan Brinkhuis in 1981; David Burns in 1990 was his 23rd student. In addition to his students, he inspired many disciples to continue the study of the subject he had founded. He was wonderfully able to inspire others; in the words of Philippe Cassou-Noguès, 'He was the light of my mathematical life'.

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